CONVERGENCE OF TAIL SUM FOR RECORDS

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ABSTRACT. Suppose $\left\{R_n^{(L)}(F) : n \geq 1\right\}$ is the sequence of lower records from a distribution F where F is continuous with $\inf\{x \in \text{supp}(F)\} = 0$. We derive conditions under which logarithm of the tail sum of records, $\sum_{j=n}^{\infty} R_n^{(L)}(F)$, properly centred and scaled, converge weakly. We also prove two results on Π-varying and regularly varying functions, which are of independent interest.

1. INTRODUCTION

Let F be a distribution on $[0, \infty)$ with only possible discontinuity at 0. Further, we assume that $\inf\{x \in \text{supp}(F)\} = 0$. Let $\{X_1, X_2, \dots\}$ be i.i.d. observations from F. Let us define $L_1 = 1$ and set, for $n \geq 2$,

$$L_n = \inf\{k > L_{n-1} : X_k \leq X_{L_{n-1}}\}.$$

Then $\{R_n^{(L)}(F) := X_{L_n} : n \geq 1\}$ is called the sequence of lower records from the distribution F. We will drop F from $R_n^{(L)}(F)$ whenever there is no chance of confusion. Similarly, we may define upper records from a distribution F. The n-th upper record from the distribution F will be denoted by $R_n^{(U)}(F)$. We also define the infinite sums

$$S_F := \sum_{n=1}^{\infty} R_n^{(L)} \quad \text{and} \quad T_n(F) := \sum_{k=n}^{\infty} R_k^{(L)},$$

whenever they are finite.

Under the assumption on F, it is clear that $R_n^{(L)}(F)$ converges almost surely to 0. It was proved by Bose et al. (2003), (see also Iksanov (2004)) that $\sum_{n=1}^{\infty} R_n^{(L)} < \infty$ a.e. if and only if

$$\int_0^1 x \frac{F(dx)}{F(x)} < \infty. \tag{1}$$

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In such a case, the Laplace transform of the sum S_F is given by

$$\mathbb{E}\big(\mathrm{e}^{-tS_F}\big) = \exp\left(\int_0^\infty \frac{\mathrm{e}^{-tu} - 1}{F(u)} F(du)\right)$$

for $t > 0$. It is clear that the random variable S_F is infinitely divisible with its Lévy measure L_F given by the relation

$$L_F(dx) = \frac{F(dx)}{F(x)}.$$

Further, it is the case that, all infinitely divisible distributions supported on $[0, \infty)$ with Lévy measure L, such that $h_L(x) = L([x, \infty))$ is a continuous function of x on $(0, \infty)$, must arise in this way. For more detailed discussions on infinitely divisible distributions supported on $[0, \infty)$, we refer the reader to Sato (1999) or Bondesson (1992).

The above result is useful in simulation of infinitely differentiable laws (see Bose et al., 2002, for a discussion on this topic.) Indeed, given the Lévy measure, we can easily construct the corresponding distribution function F by the formula $F(x) = \exp(-L([x, \infty)))$ for $x \geq 0$. If the Lévy measure has finite total mass, i.e., $L([0, \infty)) < \infty$, then F has a jump at 0 and after a finite random time, say N, $R_N^{(L)}$ becomes 0, and then records after that remains at 0. Therefore, we have that $\sum_{n=1}^\infty R_n^{(L)} = \sum_{n=1}^N R_n^{(L)}$ and it can be exactly evaluated.

In case, $L([0, \infty)) = \infty$, we have that $F(0) = 0$. As a consequence, $R_n^{(L)} > 0$ for all $n \geq 1$. In such a case, we can simulate the random variables $\{R_n^{(L)} : 1 \leq n \leq N\}$ where N is a large constant (fixed or random), using the fact that $\{R_n^{(L)} : n \geq 1\}$ is a Markov chain with initial distribution F and transition kernel $P(x, dy)$ given by $P(x, dy) = 1\{y \leq x\}F(dy)/F(x)$ and approximate the infinite sum S_F by the finite sum $\sum_{n=1}^N R_n^{(L)}$. Hence, in this case, it is important to estimate how much error is made in this approximation. In other words, we would like to investigate the behaviour of $\sum_{n=N+1}^\infty R_n^{(L)}$ as $N \to \infty$.

A crucial observation in the study of record values is the following: Let G be a given distribution function and $\{Y_i : i \geq 1\}$ be a sequence of i.i.d. random variables with exponential distribution having mean 1. For a nondecreasing function ϕ, define the (left continuous) inverse of ϕ as

$$\phi^{\leftarrow}(y) = \inf\{s : \phi(s) \geq y\}.$$

Also define

$$\psi_G(x) = G^{\leftarrow}\big(1 - \mathrm{e}^{-x}\big), \tag{2}$$

and

$$H_G(x) = 1 - \exp\Big[-\big[-\log\big(1 - G(x)\big)\big]^{1/2}\Big]. \tag{3}$$

Then, the joint distribution of $\{\psi_G(Y_1), \psi_G(Y_1 + Y_2), \dots, \psi_G(\sum_{i=1}^n Y_i)\}$ is the same as that of the first n upper records from the distribution G, namely, $\big\{R_1^{(U)}(G),\, R_2^{(U)}(G), \dots, R_n^{(U)}(G)\big\}$, (see, for example, Resnick, 1973). Using this representation, Resnick (1973) has shown that if the limiting distribution function $\tilde{G}$, of the properly centered and scaled sequence of records $(R_n^{(U)}(G) - a_n(G))/b_n(G)$ (where $a_n(G)$ and $b_n(G)$ are suitable sequences of constants) exists, then it must be one of the three distributions, given in Cases (i)–(iii) below.

In the following $N(x)$ denotes the standard normal distribution function. For a distribution function J, we say that J belongs to max-domain of attraction of K and denote $J \in D(K)$ if for some normalizing constants a_n and b_n,

$$J^n(b_n x + a_n) \Rightarrow K(x)$$

at all continuity points x of K. Resnick (1973) showed that only one of the following is possible:

- **Case (i)** The limiting distribution is $\tilde{G}(x) = N(x)$. This happens if and only if $H_G \in D(\Lambda)$ where $\Lambda(x) = \exp\big(-\mathrm{e}^{-x}\big)$. In this case, it turns out that

$$a_n^{(1)}(G) = \psi_G(n) \text{ and } b_n^{(1)}(G) = \psi_G(n + \sqrt{n}) - \psi_G(n). \tag{4}$$

- **Case (ii)** The limiting distribution is

$$\tilde{G}(x) = N_{1,\alpha}(x) = \begin{cases} 0 & \text{if } x < 0, \\ N(\alpha \log x) & \text{if } x \geq 0 \end{cases} \tag{5}$$

where $\alpha > 0$. This happens if and only if $H_G(x) \in D(\Phi_{\alpha/2})$ where

$$\Phi_{\alpha/2}(x) = \begin{cases} 0 & \text{if } x < 0 \\ \exp(-x^{-\alpha/2}) & \text{if } x \geq 0. \end{cases}$$

In this case, we have

$$a_n^{(2)}(G) = 0 \text{ and } b_n^{(2)}(G) = \psi_G(n). \tag{6}$$

- **Case (iii)** The limiting distribution is

$$\tilde{G}(x) = N_{2,\alpha}(x) = \begin{cases} N\big(-\alpha \log(-x)\big) & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \tag{7}$$

where $\alpha > 0$. This happens if and only if $H_G \in D(\Psi_{\alpha/2})$ where

$$\Psi_{\alpha/2}(x) = \begin{cases} \exp\big(-(-x)^{\alpha/2}\big) & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

In this case,

$$a_n^{(3)}(G) = x_0 \text{ and } b_n^{(3)}(G) = x_0 - \psi_G(n)$$

where x_0 is the (necessarily finite) supremum of support of G, i.e., $G(x_0) = 1$ and $G(x_0 - \epsilon) < 1$ for any $\epsilon > 0$. We shall drop G from the above notations whenever there is no chance of confusion.

From now on we will assume that $F(0) = 0$. Then, F is continuous everywhere. Further, we assume that $\int_0^1 xF(dx)/F(x) < \infty$. Hence, the tail part of the sum $T_n(F)$ converges to 0. In this article, we study conditions under which $\big(\log T_n(F) - \alpha_n\big)/\beta_n$ converges to a proper random variable in distribution for suitable choices of constants α_n and β_n.

We define $W_k = -\log X_k$, for $k \geq 1$. Then, $\{W_k : k \geq 1\}$ is a sequence of i.i.d. random variables. Let G be the distribution function of W_1. Then, G is given by

$$G(x) := 1 - F\big(\mathrm{e}^{-x}\big). \tag{8}$$

The support of G has upper endpoint ∞. Since $x \mapsto -\log x$ is a monotone decreasing function, the k-th upper record generated from the sequence $\{W_k : k \geq 1\}$ is given by

$$R_k^{(U)}(G) = -\log R_k^{(L)}(F). \tag{9}$$

Corresponding to this distribution function G, there will be the associated distribution H_G (see (3)). We will drop the subscript G from H_G. Let us set

$$H(x) := 1 - \exp\big[-\big(-\log(1-G(x))\big)^{1/2}\big] = 1 - \exp\Big[-\big[-\log F\big(\exp(-x)\big)\big]^{1/2}\Big]. \tag{10}$$

The following function which is important for the characterization of domain of attraction will also be useful for our purpose. Let us define

$$U(x) := \Big(\frac{1}{1-H}\Big)^{\leftarrow}(x) = -\log F^{\leftarrow}\big(\exp(-\log^2 x)\big). \tag{11}$$

The relation between $U(\cdot)$ and $\psi_G(\cdot)$, defined in (2) is given by

$$\psi_G(x) = U\big(\mathrm{e}^{\sqrt{x}}\big). \tag{12}$$

As $F\big(\mathrm{e}^{-x}\big) > 0$ for all $x \in \mathbb{R}$, we have $G(x) = 1 - F\big(\mathrm{e}^{-x}\big) < 1$. Therefore, using Resnick's result, only two cases, viz., case (i) that is,

$H \in D(\Lambda)$ and case (ii) that is, $H \in D(\Phi_{\alpha/2})$ for some $\alpha > 0$, are possible here.

The next section contains the two main theorems along with two results (Propositions 2 and 3) on Π-varying and regularly varying functions. Proofs of the main theorems are given in Section 3 and the proofs of the propositions are given in Section 4.

2. Main Results

We study each of the above mentioned cases separately.

Case (i) $H \in D(\Lambda)$: It is well known (see Resnick, 1987, page 28, Proposition 0.10) that, equivalently $U(\cdot)$ is Π-varying, that is, there exists an auxiliary function $a(\cdot)$ such that

$$\frac{U(tx) - U(x)}{a(x)} \to \log t \text{ as } x \to \infty.$$

A choice of the function $a(\cdot)$ is given by $a(x) = U(\mathrm{e}\, x) - U(x)$. In this case, we have the following result:

Theorem 1. *Assume that F is continuous. If $U(\cdot)$ is Π-varying with auxiliary function $a(\cdot)$ satisfying $(\log\log t)/a(t) \to 0$ as $t \to \infty$, then*

$$\frac{\log T_n(F) + a_n^{(1)}}{b_n^{(1)}} \Rightarrow \xi_1$$

where ξ_1 follows a standard normal distribution, i.e., $P(\xi_1 \le x) = N(x)$. Moreover, we may choose $a_n^{(1)}$ and $b_n^{(1)}$ as in (4).

The relation between the conditions imposed on $U(\cdot)$ and its auxiliary function $a(\cdot)$ in the above theorem and the finiteness of S_F is given below.

Proposition 1. *If $U(x) = -\log F^{\leftarrow}(\exp(-\log^2 x))$ is Π varying with an auxiliary function $a(\cdot)$ satisfying $(\log\log x)/a(x) \to 0$, then $F(0) = 0$ and $\int_0^1 xF(dx)/F(x) < \infty$.*

Proof. First note that for any $u > 0, F(F^{\leftarrow}(u)) = u$ since F is continuous. Thus, from the definition of $U(\cdot)$, we have that, for all $x \ge 0$, $F\left(\mathrm{e}^{-U(x)}\right) = \exp(-\log^2 x)$. Observe that since $U(\cdot)$ is Π-varying, $U(x)/a(x) \to \infty$ (see Resnick, 1987, Page 35, Exercise 0.4.3.1) and $a(x)/\log\log x \to \infty$ as $x \to \infty$. Thus, we have that, $U(x) \to \infty$ as $x \to \infty$. Therefore, letting $x \to \infty$ in the formula for F above and using the right continuity of F at 0, it follows that $F(0) = 0$.

Now define $\theta_0 = 1$ and for $n \geq 1$, define $\theta_n = F^{\leftarrow}(F(\theta_{n-1})/\,\mathrm{e})$. Then, we have

$$F(\theta_n) = F(F^{\leftarrow}(F(\theta_{n-1})/\,\mathrm{e})) = F(\theta_{n-1})/\,\mathrm{e} = \cdots = F(1)/\,\mathrm{e}^n\,. \qquad (13)$$

Since $F(u) > 0$ for all $u > 0$, we get that $\theta_n \downarrow 0$. This implies that

$$\begin{aligned} \int_0^1 \frac{xF(dx)}{F(x)} &= \sum_{n=0}^{\infty} \int_{\theta_{n+1}}^{\theta_n} \frac{xF(dx)}{F(x)} \leq \sum_{n=0}^{\infty} \theta_n \int_{\theta_{n+1}}^{\theta_n} \frac{F(dx)}{F(x)} \\ &= \sum_{n=0}^{\infty} \theta_n \big[\log F(\theta_n) - \log F(\theta_{n+1})\big] = \sum_{n=0}^{\infty} \theta_n, \end{aligned}$$

since, from (13), we have $\log F(\theta_n) - \log F(\theta_{n+1}) = 1$.

Now, from the definition of $U(\cdot)$ and (13), we have $\theta_n = F^{\leftarrow}(F(\theta_{n-1})/\,\mathrm{e}) = F^{\leftarrow}(F(1)/\,\mathrm{e}^n) = \exp\Big(-U\big(\exp(\sqrt{n - \log F(1)})\big)\Big)$.

Let $\delta > 0$. Since $U(x)/a(x) \to \infty$ and $a(x)/\log\log x \to \infty$, as $x \to \infty$, we get

$$\frac{U\big(\exp(\sqrt{n - \log F(1)})\big)}{\log\log\big(\exp(\sqrt{n - \log F(1)})\big)} = \frac{2U\big(\exp(\sqrt{n - \log F(1)})\big)}{\log(n - \log F(1))} \geq 4$$

for all n sufficiently large. Thus, we have that

$$U\big(\exp(\sqrt{n - \log F(1)})\big) \geq 2\log(n - \log F(1))$$

for all n sufficiently large. Therefore, we have, for all n sufficiently large,

$$\theta_n \leq \exp\Big(-2\log\big(n - \log F(1)\big)\Big) = \big(n - \log F(1)\big)^{-2}.$$

Hence, $\sum_{n=0}^{\infty} \theta_n < \infty$. □

Remark 1. In Theorem 1, we can replace $b_n^{(1)}(G)$ by $a\big(\exp(\sqrt{n})\big)/2$. To see this, let $t_n = \exp(\sqrt{n + \sqrt{n}})/\exp(\sqrt{n}) = \exp(\sqrt{n + \sqrt{n}} - \sqrt{n}) \to \exp(1/2)$ as $n \to \infty$. Therefore,

$$\begin{aligned} \frac{b_n^{(1)}(G)}{a\big(\exp(\sqrt{n})\big)} &= \frac{U\big(\exp(\sqrt{n + \sqrt{n}})\big) - U\big(\exp(\sqrt{n})\big)}{a\big(\exp(\sqrt{n})\big)} \\ &= \frac{U\big(t_n \exp(\sqrt{n})\big) - U\big(\exp(\sqrt{n})\big)}{a\big(\exp(\sqrt{n})\big)} \to 1/2 \end{aligned}$$

as $\big[U(tx) - U(x)\big]/a(x) \to \log t$ as $x \to \infty$ uniformly on compact sets of $(0, \infty)$.

Further observe that, $a(\cdot)$ being an auxiliary function of a Π-varying function, is slowly varying (see Proposition 0.12 of Resnick, 1987).

Also, $\exp(\sqrt{n}-\sqrt{n-1}) \to 1$. Hence, $a(\mathrm{e}^{\sqrt{n}}) \sim a(\mathrm{e}^{\sqrt{n-1}})$ and $b_{n-1}^{(1)}(G) \sim b_n^{(1)}(G)$.

Example 1. The powers of uniform distribution satisfy the conditions of the theorem. Let F be a distribution function defined by

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x^\alpha & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1. \end{cases}$$

Then, we have, for $0 < x < 1$, $F^{\leftarrow}(x) = x^{1/\alpha}$. Therefore, we have $U(x) = \log^2 x/\alpha$. In this case, $a(x) = 2\log x/\alpha$. Clearly, we have $\log\log x/a(x) \to 0$ as $x \to \infty$.

It may be mentioned that if F is the uniform distribution then the n-th lower record has the representation $R_n^L = \prod_{i=1}^n U_i$ where U_i are iid uniform random variables. Hence the partial sum of R_n is a sum of product of iid random variables. More general quantities have been considered in the literature, for example by taking other random variables instead of uniform U_i and replacing the product operation by other suitable operations. For the interested reader, some relevant references are Rachev and Samorodnitsky (1995), Goldie and Maller (2000).

Case (ii) $H \in D(\Phi_{\alpha/2})$: Again it is well known that an equivalent condition in this case is that $U(\cdot)$ is regularly varying with index $2/\alpha$ (follows from Propositions 1.11 and 0.8 of Resnick, 1987). We denote this by $U(\cdot) \in RV_{2/\alpha}$. We then have the following result:

Theorem 2. *If $U(\cdot)$ is regularly varying with index $2/\alpha > 0$, then*

$$\frac{\log T_n(F)}{b_n^{(2)}(G)} \Rightarrow \xi_2$$

where ξ_2 has distribution $N_{2,\alpha}$ given by (7), same as that of the negative of the α-th root of lognormal distribution and $b_n^{(2)}(G) = \psi_G(n) = U\big(\mathrm{e}^{\sqrt{n}}\big)$ is as in (6).

Remark 2. As in Proposition 1, the condition that $U(\cdot)$ is regularly varying with an index $2/\alpha > 0$, implies that the basic conditions on F for finiteness of S_F are automatically satisfied. To see this, note that $F\big(\mathrm{e}^{-U(x)}\big) = \exp(-\log^2 x)$. Since $U(\cdot) \in RV_{2/\alpha}$, we have $U(x) \to \infty$ as $x \to \infty$. Hence, we conclude that $F(0) = 0$.

Further, using the same notations and following the same computations as in Proposition 1, we can prove that $\int_0^1 xF(dx)/F(x) \le \sum_{n=0}^\infty \theta_n$.

Since $U(\cdot) \in RV_{2/\alpha}$, we have for all n sufficiently large,

$$U\big(\exp(\sqrt{n - \log F(1)})\big) \geq \exp\big(\sqrt{n - \log F(1)}/\alpha\big) \geq n.$$

Therefore, we have here also, $\sum_{n=0}^{\infty} \theta_n < \infty$.

Remark 3. As in Remark 1, since $U(\cdot)$ is regularly varying and $\exp(\sqrt{n} - \sqrt{n-1}) \to 1$, we have $b_{n-1}^{(2)}(G) \sim b_n^{(2)}(G)$.

We end this section with an example of Theorem 2.

Example 2. As an example of an F which satisfies the conditions of Theorem 2, take $U(x) = x^{2/\alpha}$. Easy computations show that we can define F as follows:

$$F(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ \exp\Big(-\frac{\alpha^2 \log^2(-\log x)}{4}\Big) & \text{if } 0 < x < \mathrm{e}^{-1} \\ 1 & \text{if } x \geq \mathrm{e}^{-1}\,. \end{cases}$$

The proofs of the main theorems will require the following two results on Π-varying and regularly varying functions. These are of independent interest.

Proposition 2. *Let $U(\cdot)$ be an eventually non-decreasing Π-varying function with $a(\cdot)$ as the auxiliary function such that $(\log\log x)/a(x) \to 0$ as $x \to \infty$. Then, for any $\kappa > 0$,*

$$\frac{1}{a(x)} \log\left[\mathrm{e}^{U(x)} \int_x^\infty \frac{\mathrm{e}^{-U(u)} \log^\kappa u}{u} du\right] \to 0$$

and hence

$$U(\mathrm{e}\, x) + \log \int_x^\infty \frac{\mathrm{e}^{-U(u)} \log^\kappa u}{u} du \sim a(x).$$

The analogue of the above proposition for regularly varying functions is given below.

Proposition 3. *Let $U(\cdot) \in RV_\beta$ for $\beta > 0$. Then, for any $\kappa > 0$,*

$$\frac{1}{U(x)} \log\left[\mathrm{e}^{U(x)} \int_x^\infty \frac{\mathrm{e}^{-U(u)} \log^\kappa u}{u} du\right] \to 0$$

and hence

$$-\log \int_x^\infty \frac{\mathrm{e}^{-U(u)} \log^\kappa u}{u} du \sim U(x).$$

The proofs of the above propositions are given in Section 4.

3. Proof of the Main Theorems

In order to prove the theorems, we note that, for any two sequences $\{\alpha_n\}$ and $\{\beta_n\}$, we have, using (9),

$$\begin{aligned} \frac{\log T_n(F) + \alpha_n}{\beta_n} &= \frac{\log R_n^{(L)}(F) + \alpha_n}{\beta_n} + \frac{\log T_n(F) - \log R_n^{(L)}(F)}{\beta_n} \\ &= -\frac{R_n^{(U)}(G) - \alpha_n}{\beta_n} + \frac{\log\bigl[T_n(F)/R_n^{(L)}(F)\bigr]}{\beta_n}. \end{aligned}$$

Using conditions of Theorem 1, if $\alpha_n = a_n^{(1)}(G)$ and $\beta_n = b_n^{(1)}(G)$, then the first term converges to ξ_1. Similarly, from the condition of Theorem 2, with $\alpha_n = 0$ and $\beta_n = b_n^{(2)}$, the first term converges to ξ_2.

Hence, in order to prove Theorems 1 and 2, it is enough to show that the second term in the above expression converges to 0 in probability in either case. The rest of this section is devoted towards proving this.

Denote $\Sigma_i = \sum_{j=1}^{i} Y_j$ and $Z_i := \mathrm{e}^{\sqrt{\Sigma_i}}$. Also we can write $R_i^{(L)}(F) = \exp\Bigl(-\bigl(-\log R_i^{(L)}(F)\bigr)\Bigr) = \exp\bigl(-R_i^{(U)}(G)\bigr)$. Using the representation of the upper records, for $i \geq 1$, we have

$$R_i^{(U)}(G) = \psi_G\bigl(\textstyle\sum_{j=1}^{i} Y_j\bigr) = U(Z_i).$$

This implies that

$$\frac{T_n(F)}{R_n^{(L)}(F)} = \sum_{i=n}^{\infty} \frac{R_i^{(L)}(F)}{R_n^{(L)}(F)} = \sum_{i=n}^{\infty} \mathrm{e}^{-U(Z_i)+U(Z_n)} = \mathrm{e}^{U(Z_n)} \sum_{i=n}^{\infty} \mathrm{e}^{-U(Z_i)}.$$

Now we define a random function V, which is a linear interpolation of the Σ_i's. Let

$$V(u) = \sum_{j=1}^{[u]} Y_j + (u - [u])Y_{[u]+1} = \Sigma_{[u]} + (u - [u])\bigl(\Sigma_{[u]+1} - \Sigma_{[u]}\bigr). \quad (14)$$

Since almost surely each $Y_i > 0$, V is almost surely strictly increasing. Further, since F is continuous, from the definition of $U(\cdot)$, see (11), it follows that $U(\cdot)$ is also strictly increasing. Hence,

$$\mathrm{e}^{-U(Z_i)} \leq \int_{i-1}^{i} \exp\Bigl(-U\bigl(\mathrm{e}^{\sqrt{V(z)}}\bigr)\Bigr) dz.$$

Thus,

$$\frac{T_n(F)}{R_n^{(L)}(F)} \leq \mathrm{e}^{U(Z_n)} \int_{n-1}^{\infty} \exp\Bigl(-U\bigl(\mathrm{e}^{\sqrt{V(z)}}\bigr)\Bigr) dz$$

$$= \mathrm{e}^{U(Z_{n-1})} \int_{n-1}^{\infty} \exp\Big(-U\big(\mathrm{e}^{\sqrt{V(z)}}\big)\Big) dz \times \frac{\mathrm{e}^{U(Z_n)}}{\mathrm{e}^{U(Z_{n-1})}}.$$

Now, taking logarithm and dividing by β_n, we conclude that it is enough to show that

$$\frac{\log\Big[\mathrm{e}^{U(Z_n)} \int_n^\infty \exp\Big(-U\big(\mathrm{e}^{\sqrt{V(z)}}\big)\Big) dz\Big]}{\beta_n} \xrightarrow{P} 0 \tag{15}$$

and

$$\frac{U(Z_n) - U(Z_{n-1})}{\beta_n} \xrightarrow{P} 0, \tag{16}$$

since $\beta_n/\beta_{n-1} \to 1$ by Remarks 1 and 3.

We will first estimate the above integral. The following lemma shows that the above integral could be bounded by deterministic integrals.

Lemma 1. *Given any $\delta > 0$, on a set of probability 1, $\int_n^\infty \exp\Big(-U\big(\mathrm{e}^{\sqrt{V(z)}}\big)\Big) dz$ is bounded from both above and below, eventually (in n), by expressions of the form $2\int_{\mathrm{e}^{\sqrt{\Sigma_n}}}^\infty \frac{1}{u}\, \mathrm{e}^{-U(u)} \log^\kappa u\, du$, where κ is $3+\delta$ and $1-\delta$ for the upper and the lower bound respectively.*

Proof. On the interval $(i, i+1)$, the function V is linear and strictly increasing. So on each of the pieces (Σ_i, Σ_{i+1}), the inverse function is well defined and is actually linear. Suppose g is the inverse function of V. The exact expression of g is also easy to obtain:

$$g(u) = i + \frac{u - \Sigma_i}{\Sigma_{i+1} - \Sigma_i} = i + \frac{u - \Sigma_i}{Y_{i+1}} \quad \text{if } \Sigma_i < u < \Sigma_{i+1}.$$

Thus, substituting, $u = \mathrm{e}^{\sqrt{V(z)}}$ or $z = g(\log^2 u)$, we obtain that

$$\begin{aligned} \int_i^{i+1} \exp\Big(-U\big(\mathrm{e}^{\sqrt{V(z)}}\big)\Big) dz &= 2\int_{\mathrm{e}^{\sqrt{\Sigma_i}}}^{\mathrm{e}^{\sqrt{\Sigma_{i+1}}}} \frac{\mathrm{e}^{-U(u)}\, g'(\log^2 u) \log u}{u} du \\ &= 2\int_{\mathrm{e}^{\sqrt{\Sigma_i}}}^{\mathrm{e}^{\sqrt{\Sigma_{i+1}}}} \frac{\mathrm{e}^{-U(u)} \log u}{u\, Y_{i+1}} du \end{aligned} \tag{17}$$

since $g'(u) = 1/Y_{i+1}$ for $\Sigma_i < u < \Sigma_{i+1}$.

Now let us fix $\delta_1 > 0$ such that $2(1+\delta_1)/(1-\delta_1) < 2+\delta$ and $2\delta_1/(1-\delta_1) < \delta$. Consider the event $E_n^{(1)} = \{Y_{n+1} < n^{-1-\delta_1}\} \cup \{Y_{n+1} > n^{\delta_1}\}$. Then, $P\left(E_n^{(1)}\right) \leq 1 - \exp(-n^{-1-\delta_1}) + \exp(-n^{\delta_1}) \leq n^{-1-\delta_1} + \exp(-n^{\delta_1})$, which is summable. Hence $P\left(\limsup_{n\to\infty} E_n^{(1)}\right) = 0$. Let

$E_n^{(2)} = \{\Sigma_n < n^{1-\delta_1}\}$. Since $\Sigma_n/n \to 1$ with probability 1, we have, $P\left(\limsup_{n\to\infty} E_n^{(2)}\right) = 0$.

Define E through $E^c = \limsup_{n\to\infty} E_n^{(1)} \bigcup \limsup_{n\to\infty} E_n^{(2)}$. Then clearly $P(E) = 1$. From now on, we will concentrate only on the set E. For any sample point $\omega \in E$, there exists $N \equiv N(\omega)$, such that for all $n \geq N$, we have $\Sigma_n \geq n^{1-\delta_1}$ and $n^{-1-\delta_1} \leq Y_n \leq n^{\delta_1}$. Now, let $n \geq N$ be given. Then for $i \geq n \geq N$ and $u \in (Z_i, Z_{i+1})$, we have $\log u \geq \sqrt{\Sigma_i} \geq i^{(1-\delta_1)/2}$ and hence, by choice of δ_1, we have

$$\frac{1}{Y_{i+1}} \leq i^{1+\delta_1} \leq \left(\log u\right)^{2(1+\delta_1)/(1-\delta_1)} \leq \log^{2+\delta} u$$

and

$$\frac{1}{Y_{i+1}} \geq i^{-\delta_1} \geq \left(\log u\right)^{-2\delta_1/(1-\delta_1)} \geq \log^{-\delta} u.$$

The bounds are then obtained by putting these estimates in (17) and summing over $i \geq n$. □

The following Lemma about regularly varying functions will be useful in the proof of the Theorems.

Lemma 2. *Let $f \in RV_\beta$ for $\beta \in \mathbb{R}$. Then, $\frac{f(Z_n)}{f\left(e^{\sqrt{n}}\right)}$ converges weakly to a random variable with distribution $N_{1,(2/|\beta|)}$, given by* (5).

Note that, when $\beta = 0$, i.e., the function is slowly varying, the limiting random variable with distribution $N_{1,\infty}$ is interpreted to be degenerate at 1 and hence $f(Z_n)/f\left(e^{\sqrt{n}}\right)$ converges to 1 in probability.

Proof. Observe that the functions $h_n(x) = f(e^{\sqrt{n}}\, x)/f(e^{\sqrt{n}})$, for $n \in \mathbb{N}$ converge to the function $h(x) = x^\beta$ uniformly on compact sets of $(0, \infty)$. The result then follows easily from Theorem 5.5 of Billingsley (1968) and the fact that

$$e^{-\sqrt{n}}\, Z_n = \exp\left(\frac{\Sigma_n - n}{\sqrt{n}} \cdot \frac{1}{1 + \sqrt{\Sigma_n/n}}\right) \Rightarrow e^{\frac{1}{2}\zeta}$$

where ζ is a standard normal random variable. □

Now we are ready to prove the theorems and we begin with the first one.

Proof of Theorem 1. The expression on the left side of (15) can be written as

$$\frac{\log\Big[\mathrm{e}^{U(Z_n)}\int_n^\infty \exp\Big(-U\big(\mathrm{e}^{\sqrt{V(z)}}\big)\Big)dz\Big]}{a(Z_n)} \times \frac{a(Z_n)}{a\big(\mathrm{e}^{\sqrt{n}}\big)} \times \frac{a\big(\mathrm{e}^{\sqrt{n}}\big)}{b_n^{(1)}(G)}.$$

We have already shown that $b_n^{(1)}(G)/a\big(\exp(\sqrt{n})\big) \to 1/2$ as $n \to \infty$ (see Remark 1). The second factor converges in probability to 1 by Lemma 2, since $a(\cdot)$, being the auxiliary function of a Π-varying function, is slowly varying (see Proposition 0.12 of Resnick, 1987). We shall now show that the first factor converges to 0 with probability 1. Consider any ω such that $Z_n \to \infty$ and the bounds for $\int_n^\infty \exp\Big(-U\big(\exp(\sqrt{V(z)})\big)\Big)dz$ in Lemma 1 holds eventually in n. Since $Z_n \to \infty$, by Proposition 2, each bound for the first factor converges to 0. So the first factor converges to 0 on the set of all such ω, which has probability 1, by Lemma 1 and the fact $Z_n \to \infty$ almost everywhere. This shows (15).

Now we concentrate on the left side of (16). Replacing $b_n^{(1)}(G)$ by $a\big(\mathrm{e}^{\sqrt{n}}\big)$, we have,

$$\frac{U(Z_n) - U(Z_{n-1})}{a\big(\mathrm{e}^{\sqrt{n}}\big)} = \frac{U(Z_n) - U(Z_{n-1})}{a(Z_{n-1})} \times \frac{a(Z_{n-1})}{a\big(\mathrm{e}^{\sqrt{n-1}}\big)} \times \frac{a\big(\mathrm{e}^{\sqrt{n-1}}\big)}{a\big(\mathrm{e}^{\sqrt{n}}\big)}.$$

We have already shown that the second factor above converges to 1 in probability by Lemma 2. As shown in Remark 1, the last factor also converges to 1. We now show that the first factor converges to zero almost surely. Given $\eta > 0$, we can choose $0 < \epsilon < 1$ and M so large that for all $x \geq M$,

$$\sup\Big\{\Big|\frac{U(tx) - U(x)}{a(x)}\Big| : t \in [1-\epsilon,\ 1+\epsilon]\Big\} < \eta.$$

Let $\theta_n = Z_n/Z_{n-1} = \exp(\sqrt{\Sigma_n} - \sqrt{\Sigma_{n-1}})$. Then, $\theta_n \to 1$ almost surely as $\log\theta_n = \sqrt{\Sigma_n} - \sqrt{\Sigma_{n-1}} = Y_n/(\sqrt{\Sigma_n} + \sqrt{\Sigma_{n-1}}) \to 0$ almost surely. Also, $Z_n \to \infty$ almost surely. Then for almost all ω, we can choose $N \equiv N(\omega)$, such that for $n \geq N$, we have $\theta_n \in [1-\epsilon,\ 1+\epsilon]$ and $Z_{n-1} > M$. Then, for all $n \geq N$, we have

$$\Big|\frac{U(Z_n) - U(Z_{n-1})}{a(Z_{n-1})}\Big| = \Big|\frac{U(\theta_n Z_{n-1}) - U(Z_{n-1})}{a(Z_{n-1})}\Big| < \eta.$$

So, $\big|\big(U(Z_n) - U(Z_{n-1})\big)\big|/a(Z_{n-1}) \to 0$ almost everywhere. This completes the proof. □

Next, we prove Theorem 2.

Proof of Theorem 2. The expression on the left side of (15) can be written as

$$\frac{\log\Big[\mathrm{e}^{U(Z_n)}\int_n^\infty \exp\Big(-U\big(\mathrm{e}^{\sqrt{V(z)}}\big)\Big)dz\Big]}{U(Z_n)} \times \frac{U(Z_n)}{U\big(\mathrm{e}^{\sqrt{n}}\big)}.$$

It can be shown that the first factor converges to 0 with probability 1, using Proposition 3 and Lemma 1, in exactly the same way as in the proof of Theorem 1. We omit the details. The second factor converges weakly to $N_{1,\alpha}$ by Lemma 2, since $U(\cdot) \in RV_{2/\alpha}$. This proves (15).

Next, we concentrate on the left side of (16). We have,

$$\frac{U(Z_n)) - U(Z_{n-1})}{U\big(\exp(\sqrt{n})\big)} = \frac{U(Z_n) - U(Z_{n-1})}{U(Z_{n-1})} \times \frac{U(Z_{n-1})}{U\big(\mathrm{e}^{\sqrt{n-1}}\big)} \times \frac{U\big(\mathrm{e}^{\sqrt{n-1}}\big)}{U\big(\mathrm{e}^{\sqrt{n}}\big)}.$$

By Lemma 2, the second factor above converges in distribution to $N_{1,\alpha}$. By Remark 3, the third factor converges to 1. We now show that the first factor converges to zero almost everywhere. Given $\eta > 0$, we can choose $0 < \epsilon < 1$ and M so large that for all $x \geq M$,

$$\sup\Big\{\Big|\frac{U(tx) - U(x)}{U(x)}\Big| : t \in [1-\epsilon,\ 1+\epsilon]\Big\} < \eta$$

for all $x \geq N$. Then arguing as before that $\theta_n = Z_n/Z_{n-1} \to 1$ and $Z_n \to \infty$ almost surely, for almost all ω, we can choose $N \equiv N(\omega)$, such that for all $n \geq N$, we have $\theta_n \in [1-\epsilon,\ 1+\epsilon]$ and $Z_n > M$, so that

$$\Big|\frac{U(Z_n) - U(Z_{n-1})}{U(Z_{n-1})}\Big| = \Big|\frac{U(\theta_n Z_{n-1}) - U(Z_{n-1})}{U(Z_{n-1})}\Big| < \eta.$$

So, $\big|\big(U(Z_n) - U(Z_{n-1})\big)\big|/U(Z_{n-1}) \to 0$ almost everywhere. This completes the proof. □

4. Proof of Propositions 2 and 3

Finally we give the proofs of Propositions 2 and 3.

Proof of Proposition 2. Let $\eta > 0$ be given. Then choose K so large that $U(\cdot)$ is non-decreasing beyond e^K and

$$2^\kappa \sum_{i=K}^{\infty} \big[(i-2)!\big]^{-\kappa} < 1. \tag{18}$$

Next, using the facts that $U(\cdot)$ is Π-varying, powers of logarithm are slowly varying and $(\log\log x)/a(x) \to 0$, choose large enough N such that the following holds for all $x \geq N$:

$$\sup\left\{\left|\frac{U(tx)-U(x)}{a(x)} - \log t\right| : t \in \left[1, \mathrm{e}^K\right]\right\} \leq \eta, \tag{19}$$

$$1-\eta \leq \sup\left\{\frac{\log^\kappa(tx)}{\log^\kappa x} : t \in \left[1, \mathrm{e}^K\right]\right\} \leq 1+\eta, \tag{20}$$

$$a(x) \geq \kappa \log\log x, \tag{21}$$

$$\sup\left\{\frac{j+1+\log x}{j-1+\log x} : j \geq 0\right\} = \frac{\log x + 1}{\log x - 1} \leq 2 \tag{22}$$

and

$$\mathrm{e}^{-Ka(x)} \leq \frac{1}{2}. \tag{23}$$

Now

$$\begin{aligned} \mathrm{e}^{U(x)} \int_x^\infty \frac{\mathrm{e}^{-U(t)} \log^\kappa t}{t} dt &= \mathrm{e}^{U(x)} \int_1^\infty \frac{\mathrm{e}^{-U(tx)} \log^\kappa (tx)}{t} dt \\ &= \int_1^{\mathrm{e}^K} \frac{\mathrm{e}^{-[U(tx)-U(x)]} \log^\kappa(tx)}{t} dt + \int_{\mathrm{e}^K}^\infty \frac{\mathrm{e}^{-[U(tx)-U(x)]} \log^\kappa(tx)}{t} dt, \end{aligned} \tag{24}$$

where K is the constant chosen above in (18). We will estimate each of the above integrals separately.

For the first integral, for $t \in \left[1, \mathrm{e}^K\right]$, we have from (19), for all $x \geq N$,

$$U(tx)-U(x) = \left[\frac{U(tx)-U(x)}{a(x)} - \log t\right] a(x) + a(x)\log t \geq -\eta a(x) + a(x)\log t.$$

Thus, we have, also using (20), for all $x \geq N$,

$$\begin{aligned} &\int_1^{\mathrm{e}^K} \frac{\mathrm{e}^{-[U(tx)-U(x)]} \log^\kappa(tx)}{t} dt \\ &\qquad \leq (1+\eta)\log^\kappa x \int_1^{\mathrm{e}^K} \mathrm{e}^{\eta a(x)}\, t^{-a(x)-1}\, dt \leq \frac{(1+\eta)\,\mathrm{e}^{\eta a(x)} \log^\kappa x}{a(x)}. \end{aligned}$$

For the second integral, we have, using the fact that $U(\cdot)$ is non-decreasing beyond e^K,

$$\int_{\mathrm{e}^K}^\infty \frac{\mathrm{e}^{-[U(tx)-U(x)]} \log^\kappa(tx)}{t} dt$$

$$
\begin{aligned}
&= \sum_{j=K}^{\infty} \int_{\mathrm{e}^j}^{\mathrm{e}^{j+1}} \frac{\mathrm{e}^{-[U(tx)-U(x)]} \log^{\kappa}(tx)}{t} dt \\
&\le \sum_{j=K}^{\infty} \exp\bigl(-\bigl[U(\mathrm{e}^j\, x) - U(x)\bigr]\bigr) \log^{\kappa}(\mathrm{e}^{j+1}\, x) \\
&= \sum_{j=K}^{\infty} \exp\bigl(-\bigl[U(\mathrm{e}^j\, x) - U(x)\bigr] + \kappa \log\log(\mathrm{e}^{j+1}\, x)\bigr). \qquad (25)
\end{aligned}
$$

Using the fact that $a(\cdot)$ can be taken to be $a(x) = U(\mathrm{e}\, x) - U(x)$, we see that $U(\mathrm{e}^j\, x) - U(x) = \sum_{l=0}^{j-1} U(\mathrm{e}^{l+1}\, x) - U(\mathrm{e}^l\, x) = \sum_{l=0}^{j-1} a(\mathrm{e}^l\, x)$. Now, for $x \ge N$, we have, using (21) and (22),

$$
\begin{aligned}
\sum_{l=0}^{j-1} a(\mathrm{e}^l\, x) - \kappa \log\log(\mathrm{e}^{j+1}\, x) &\ge \kappa \sum_{l=0}^{j-1} \log\log(\mathrm{e}^l\, x) - \kappa \log\log(\mathrm{e}^{j+1}\, x) \\
&\ge \kappa \sum_{l=1}^{j-2} \log l - \kappa \Bigl[\log\log(\mathrm{e}^{j+1}\, x) - \log\log(\mathrm{e}^{j-1}\, x)\Bigr] \\
&= \kappa \sum_{l=1}^{j-2} \log l - \kappa \Bigl[\log \frac{(j+1+\log x)}{(j-1+\log x)}\Bigr] \ge \kappa \sum_{l=1}^{j-2} \log l - \kappa \log 2.
\end{aligned}
$$

Therefore, we obtain, using (25),

$$
\begin{aligned}
&\int_{\mathrm{e}^K}^{\infty} \frac{\mathrm{e}^{-[U(tx)-U(x)]} \log^{\kappa}(tx)}{t} dt \\
&\qquad \le 2^{\kappa} \sum_{j=K}^{\infty} \exp\bigl(-\kappa \textstyle\sum_{l=1}^{j-2} \log l\bigr) = 2^{\kappa} \sum_{j=K}^{\infty} \bigl[(j-2)!\bigr]^{-\kappa} \le 1
\end{aligned}
$$

from the choice of K in (18).

Putting the estimates in (24) together, taking logarithm and dividing by $a(x)$, we have

$$
\begin{aligned}
&\frac{1}{a(x)} \log\Bigl[\mathrm{e}^{U(x)} \int_x^{\infty} \frac{\mathrm{e}^{-U(t)} \log^{\kappa} t}{t} dt\Bigr] \\
&\le \frac{1}{a(x)} \log\Bigl[1 + \frac{(1+\eta)\, \mathrm{e}^{\eta a(x)} \log^{\kappa} x}{a(x)}\Bigr] \\
&\sim \eta + \frac{\log(1+\eta)}{a(x)} + \kappa \frac{\log\log x}{a(x)} - \frac{\log a(x)}{a(x)},
\end{aligned}
$$

since $(1+\eta)\,\mathrm{e}^{\eta a(x)}\log^\kappa x/a(x) \to \infty$. Then using $\log\log x/a(x) \to 0$, we have,

$$\limsup_{x\to\infty} \frac{1}{a(x)} \log\left[\mathrm{e}^{U(x)} \int_x^\infty \frac{\mathrm{e}^{-U(t)}\log^\kappa t}{t} dt\right] \le \eta.$$

On the other hand, using (24) and then (19), (20) and (23), we have

$$\begin{aligned}
\mathrm{e}^{U(x)} \int_x^\infty \frac{\mathrm{e}^{-U(t)}\log^\kappa t}{t} dt &\ge \int_1^{\mathrm{e}^K} \frac{\mathrm{e}^{-[U(tx)-U(x)]}\log^\kappa (tx)}{t} dt \\
&\ge (1-\eta)\log^\kappa x\,\mathrm{e}^{-\eta a(x)} \int_1^{\mathrm{e}^K} t^{-a(x)-1} dt \\
&= \frac{(1-\eta)\log^\kappa x\,\mathrm{e}^{-\eta a(x)}\big(1-\mathrm{e}^{-Ka(x)}\big)}{a(x)} \\
&\ge \frac{(1-\eta)\log^\kappa x\,\mathrm{e}^{-\eta a(x)}}{2a(x)}.
\end{aligned}$$

Then taking logarithm, dividing by $a(x)$ and taking limit as $x \to \infty$, we have arguing as before

$$\liminf_{x\to\infty} \frac{1}{a(x)} \log\left[\mathrm{e}^{U(x)} \int_x^\infty \frac{\mathrm{e}^{-U(t)}\log^\kappa t}{t} dt\right] \ge -\eta.$$

Since $\eta > 0$ is arbitrary, we have

$$\frac{1}{a(x)} \log\left[\mathrm{e}^{U(x)} \int_x^\infty \frac{\mathrm{e}^{-U(u)}\log^\kappa u}{u} du\right] \to 0.$$

Further conclusion follows since $U(\mathrm{e}\,x) - U(x) \sim a(x)$. □

Proof of Proposition 3. Let $\eta \in (0,1)$ be given. First we may choose $N > \mathrm{e}$ so that for all $x \ge N$, we have, $\log^\kappa u \le u$ and

$$(1-\eta)t^{\beta-\eta}U(x) \le U(tx) \le (1+\eta)t^{\beta+\eta}U(x) \text{ for all } t \ge 1. \tag{26}$$

The set of inequalities (26) is Potter's bound for regularly varying functions (see Resnick, 1987, page 22, Proposition 0.8). Thus, we have, for $x \ge N$,

$$\begin{aligned}
I_x = \int_x^\infty \frac{\mathrm{e}^{-U(u)}\log^\kappa u}{u} du &\le \int_x^\infty \mathrm{e}^{-U(u)}\,du = x\int_1^\infty \mathrm{e}^{-U(tx)}\,dt \\
&\le x\int_1^\infty \exp\big(-(1-\eta)t^{\beta-\eta}U(x)\big)dt \\
&= \frac{x}{(\beta-\eta)(1-\eta)U(x)} \frac{\int_{(1-\eta)U(x)}^\infty \mathrm{e}^{-t}\,t^{1/(\beta-\eta)-1}dt}{\big[(1-\eta)U(x)\big]^{1/(\beta-\eta)-1}}.
\end{aligned}$$

Now, using L'Hôpital's rule, it is easy to check that

$$\int_x^\infty \mathrm{e}^{-t}\, t^\theta dt \sim \mathrm{e}^{-x}\, x^\theta \tag{27}$$

as $x \to \infty$. Thus, we can choose $K \geq N$ so that for all $x \geq K$,

$$\int_x^\infty \frac{\mathrm{e}^{-U(u)} \log^\kappa u}{u} du \leq \frac{(1+\eta)x}{(\beta-\eta)(1-\eta)U(x)}\, \mathrm{e}^{-(1-\eta)U(x)}\,.$$

Therefore, multiplying I_x by $\mathrm{e}^{U(x)}$, taking logarithm and dividing by $U(x)$, we have,

$$\limsup_{x\to\infty} \frac{1}{U(x)} \log\left[\mathrm{e}^{U(x)} \int_x^\infty \frac{\mathrm{e}^{-U(u)} \log^\kappa u}{u} du\right]$$
$$\leq \limsup_{x\to\infty} \left(\eta + \frac{1}{U(x)} \log\Big[\frac{(1+\eta)x}{(\beta-\eta)(1-\eta)U(x)}\Big]\right) = \eta$$

using the fact that $U(\cdot) \in RV_\beta$ for $\beta > 0$.

On the other hand, using (26), for all $x \geq N$, we have,

$$\begin{aligned}
\int_x^\infty \frac{\mathrm{e}^{-U(u)} \log^\kappa u}{u} du &\geq \int_x^\infty \frac{\mathrm{e}^{-U(u)}}{u} du = \int_1^\infty \frac{\mathrm{e}^{-U(tx)}}{t} dt \\
&\geq \int_1^\infty \frac{\exp\big(-(1+\eta)t^{\beta+\eta}U(x)\big)}{t} dt \\
&= \frac{1}{\beta+\eta} \int_{(1+\eta)U(x)}^\infty \mathrm{e}^{-s}\, s^{-1} ds.
\end{aligned}$$

Then, again using (27), we can choose $K' \geq N$ so that for all $x \geq K'$, we have

$$\int_x^\infty \frac{\mathrm{e}^{-U(u)} \log^\kappa u}{u} du \geq \frac{(1-\eta)x\, \mathrm{e}^{-(1+\eta)U(x)}}{(\beta+\eta)(1+\eta)U(x)}.$$

Then, arguing as before,

$$\liminf_{x\to\infty} \frac{1}{U(x)} \log\left[\mathrm{e}^{U(x)} \int_x^\infty \frac{\mathrm{e}^{-U(u)} \log^\kappa u}{u} du\right] \geq -\eta.$$

Since $\eta \in (0,1)$ is arbitrary, the results follow. □

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