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The determinant bundle on the moduli space of stable triples over a curve

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Abstract. We construct a holomorphic Hermitian line bundle over the moduli space of stable triples of the form (E_1, E_2, ϕ), where E_1 and E_2 are holomorphic vector bundles over a fixed compact Riemann surface X, and $\phi : E_2 \longrightarrow E_1$ is a holomorphic vector bundle homomorphism. The curvature of the Chern connection of this holomorphic Hermitian line bundle is computed. The curvature is shown to coincide with a constant scalar multiple of the natural Kähler form on the moduli space. The construction is based on a result of Quillen on the determinant line bundle over the space of Dolbeault operators on a fixed C^∞ Hermitian vector bundle over a compact Riemann surface.

Keywords. Moduli space; stable triples; determinant bundle; Quillen metric.

1. Introduction

Moduli spaces of stable vector bundles over a compact Riemann surface are now very extensively studied objects. We recall that a fundamental work of Quillen [Q], implies that the determinant line bundle over such a moduli space has a natural Hermitian metric, such that the curvature of its Chern connection coincides with a constant scalar multiple of the natural Kähler form on the moduli space.

In recent years there has been a lot of interest in vector bundles with some additional structure, generally called augmented bundles. Examples of augmented bundles are k-pairs (i.e., bundles with k distinguished holomorphic sections), Higgs bundles, coherent systems etc. Holomorphic triples are other such augmented objects which have been considered recently, [G] and [BG]. A *holomorphic triple* consists of a pair of holomorphic vector bundles E_1 and E_2, together with a holomorphic vector bundle homomorphism $\phi : E_2 \longrightarrow E_1$. Such triples form a fairly general class of augmented bundles, in the sense that Higgs bundles and coherent systems can be thought of as specialized triples. See [BDGW] for a general survey of augmented bundles over a Riemann surface.

There is a notion of stability for holomorphic triples, which arises from the study of certain differential equations called the *coupled vortex equations*. These equations were studied by García-Prada in [G], and they were related to the stability of triples by a work of Bradlow and García-Prada in [BG]. The stability of a holomorphic triple on X depends on

Dedicated to the memory of Subhashis Nag.

a parameter α. Bradlow and García-Prada construct a moduli space N_α of α-stable triples over a compact Riemann surface X, which turns out to be a quasi-projective variety. They also construct a natural Kähler metric on this moduli space N_α. It should be remarked that although Higgs bundles are specialized triples, the stability of a Higgs bundle is not the same as that of the associated triple, see ([BDGW], §1.5). (A similar remark is also valid for coherent systems.)

In this paper, we construct a natural holomorphic line bundle $\mathcal{L}$, equipped with a Hermitian structure, over the above mentioned moduli space N_α of triples. We compute the curvature of the Chern connection of this holomorphic Hermitian line bundle. It is proved here that this curvature coincides with a constant scalar multiple of the natural Kähler form on N_α. In particular, this proves that the Kähler class on N_α is rational. Our construction is based on a result of Quillen [Q] on determinants of $\overline{\partial}$-operators over vector bundles over a Riemann surface and also on the existence of a special metric on a stable triple established in [BG].

Here is a brief outline of the contents of the paper. In §2, we recall the basic notions about holomorphic triples, their stability, and their moduli spaces. In §3, we construct a natural holomorphic line bundle $\mathcal{L}$ on the moduli space of stable triples, and show that it has a certain universal property for families of triples. Section 4 deals with the construction of a natural Hermitian metric on the above line bundle $\mathcal{L}$, and in this section we prove the main result of the paper (Theorem 4.1), which asserts that the first Chern form of the Hermitian holomorphic line bundle $\mathcal{L}$ is a constant scalar multiple of the natural Kähler form on the moduli space.

2. Triples and their moduli

In this section we review the basic notions about holomorphic triples, their stability and their moduli spaces. We also recall the construction of the Kähler structure on these moduli spaces. The proofs of the statements in this section can be found in [G], [BG] and [BDGW].

2a *Stable triples*

Let X be a compact Riemann surface of genus g. If E is a vector bundle over X, we denote the rank of E by $\mathrm{rank}(E)$, its degree by $\deg(E)$, and its slope by $\mu(E)$; by definition, $\mu(E) = \deg(E)/\mathrm{rank}(E)$.

DEFINITION 2.1

A *triple* $T = (E_1, E_2, \phi)$ consists of two holomorphic vector bundles E_1 and E_2 over X, together with a homomorphism $\phi : E_2 \longrightarrow E_1$ of holomorphic vector bundles. A *morphism* $f : T' \longrightarrow T$ from a triple $T' = (E'_1, E'_2, \phi')$ to another triple $T = (E_1, E_2, \phi)$ is a pair $f = (f_1, f_2)$ consisting of vector bundle homomorphisms $f_1 : E'_1 \longrightarrow E_1$ and $f_2 : E'_2 \longrightarrow E_2$ such that the following diagram commutes:

$$\begin{array}{ccc} E'_1 & \xrightarrow{f_1} & E_1 \\ \phi' \uparrow & & \uparrow \phi \\ E'_2 & \xrightarrow[f_2]{} & E_2 \end{array}$$

If both f_1 and f_2 are vector bundle isomorphisms, we say that $f = (f_1, f_2)$ is an *isomorphism* of triples.

DEFINITION 2.2

A triple $T = (E_1, E_2, \phi)$ is said to be the *zero triple*, denoted by $T = 0$, if $E_1 = E_2 = 0$. We say that $T' = (E_1', E_2', \phi')$ is a *subtriple* of $T = (E_1, E_2, \phi)$ if each E_i', $i = 1, 2$, is a sub-bundle of E_i such that $\phi(E_2') \subseteq E_1'$, with ϕ' being the restriction of ϕ to E_2'. We say that a sub-triple $T' = (E_1', E_2', \phi')$ of $T = (E_1, E_2, \phi)$ is *proper* unless $T' = 0$, or $T' = T$ (i.e., $E_1' = E_1$, $E_2' = E_2$ and $\phi' = \phi$).

DEFINITION 2.3

Let α be a real number, and let $T = (E_1, E_2, \phi)$ be a triple. We define the *α-degree* and the *α-slope* of T by

$$\deg_\alpha(T) := \deg(E_1 \oplus E_2) + \alpha \cdot \operatorname{rank}(E_2)$$

and

$$\mu_\alpha(T) := \frac{\deg_\alpha(T)}{\operatorname{rank}(E_1 \oplus E_2)}$$

respectively. We say that a triple T is *α-stable* (respectively, *α-semistable*) if

$$\mu_\alpha(T') < \mu_\alpha(T) \quad (\text{respectively,} \quad \mu_\alpha(T') \leq \mu_\alpha(T))$$

for every proper subtriple T' of T.

Remark 2.4. We note that if $T = (E_1, E_2, \phi)$ is α-stable, and both E_1 and E_2 are non-zero, then the homomorphism ϕ must be non-zero.

2b *Moduli spaces*

DEFINITION 2.5

If $T = (E_1, E_2, \phi)$ is a triple, we call the sequence

$$(\operatorname{rank}(E_1), \operatorname{rank}(E_2), \deg(E_1), \deg(E_2))$$

as the *type* of T.

Fix $(r_1, r_2, d_1, d_2) \in \mathbb{N}^2 \times \mathbb{Z}^2$. Let α be a positive rational number. Define τ to be equal to $\mu_\alpha(T)$ for any triple T of type (r_1, r_2, d_1, d_2), i.e.,

$$\tau = \frac{d_1 + d_2 + \alpha r_2}{r_1 + r_2},$$

and let $\tau' = \tau - \alpha$. Fix two C^∞ complex vector bundles E_1 and E_2 on X with $\operatorname{rank}(E_i) = r_i$ and $\deg(E_i) = d_i$, where $i = 1, 2$. Fix Hermitian metrics h_i on E_i $(i = 1, 2)$ and a Hermitian metric μ on X. Let ω be the Kähler form of (X, μ), and assume, without any loss of generality, that the volume of X with respect to the metric μ is 1, i.e., $\int_X \omega = 1$.

Let $\mathcal{A}_i$ be the space of holomorphic structures on E_i $(i = 1, 2)$, which is an affine space for $A^{0,1}(\mathcal{E}nd(E_i))$. Consider the Cartesian product

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathrm{C}^{\infty}(\mathcal{H}om(E_2, E_1)).$$

Then $\mathcal{A}$ is a complex affine space modelled after the direct sum

$$A^{0,1}(\mathcal{E}nd(E_1)) \oplus A^{0,1}(\mathcal{E}nd(E_2)) \oplus \mathrm{C}^{\infty}(\mathcal{H}om(E_2, E_1)).$$

Let $\mathcal{B}$ be the subset of $\mathcal{A}$ given by

$$\mathcal{B} = \{(D_1, D_2, \phi) \in \mathcal{A} \quad | \quad D_1 \circ \phi = \phi \circ D_2\}.$$

It is evidently a closed analytic subset of $\mathcal{A}$. For each point $D = (D_1, D_2, \phi) \in \mathcal{B}$, we have a holomorphic triple $T_D = (E_{1,D_1}, E_{2,D_2}, \phi)$ of type (r_1, r_2, d_1, d_2), where E_{i,D_i} is the holomorphic vector bundle defined by the holomorphic structure D_i on E_i. Let $\mathcal{B}_{\alpha}^{\mathrm{s}}$ denote the subset of $\mathcal{B}$ consisting of all D such that the associated holomorphic triple T_D is α-stable. From the openness of the stability condition it follows that $\mathcal{B}_{\alpha}^{\mathrm{s}}$ is an open subset of $\mathcal{B}$.

Let $\mathcal{G}_i$ and $\mathcal{G}_i^{\mathrm{C}}$ denote the unitary and complex gauge groups, respectively, of E_i $(i = 1, 2)$, and let $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ and $\mathcal{G}^{\mathrm{C}} = \mathcal{G}_1^{\mathrm{C}} \times \mathcal{G}_2^{\mathrm{C}}$. Then there is a holomorphic right action of $\mathcal{G}^{\mathrm{C}}$ on $\mathcal{A}$ given by

$$(D_1, D_2, \phi) \cdot (g_1, g_2) = (g_1^{-1} \circ D_1 \circ g_1, g_2^{-1} \circ D_2 \circ g_2, g_1^{-1} \circ \phi \circ g_2).$$

This action leaves $\mathcal{B}_{\alpha}^{\mathrm{s}}$ invariant. Embed $\mathbb{C}^*$ in $\mathcal{G}^{\mathrm{C}}$ using the homomorphism defined by $\lambda \mapsto (\lambda \cdot \mathrm{Id}_{E_1}, \lambda \cdot \mathrm{Id}_{E_2})$. Then $\mathbb{C}^*$ acts trivially on $\mathcal{A}$, and the induced action of $\overline{\mathcal{G}}^{\mathrm{C}} = \mathcal{G}^{\mathrm{C}}/\mathbb{C}^*$ on $\mathcal{B}_{\alpha}^{\mathrm{s}}$ is free ([BG], Corollary 3.12). The quotient $M_{\alpha}^{\mathrm{s}} = \mathcal{B}_{\alpha}^{\mathrm{s}}/\overline{\mathcal{G}}^{\mathrm{C}}$ is the moduli space of α-stable triples of type (r_1, r_2, d_1, d_2). It is known that M_{α}^{s} has a natural structure of a quasi-projective variety ([BG], Theorem 6.1). Let N_{α} be the set of non-singular points of M_{α}^{s}, and let $\mathcal{C}_{\alpha} := \pi^{-1}(N_{\alpha})$, where $\pi : \mathcal{B}_{\alpha}^{\mathrm{s}} \longrightarrow M_{\alpha}^{\mathrm{s}}$ is the canonical projection. Then N_{α} is a non-singular quasi-projective variety of dimension $1 + r_2 d_1 - r_1 d_2 + (r_1^2 + r_2^2 - r_1 r_2)(g - 1)$, provided it is non-empty, where g is the genus of X ([BG], Theorem 6.1). Moreover, any triple $D = (D_1, D_2, \phi) \in \mathcal{B}_{\alpha}^{\mathrm{s}}$ with ϕ either surjective or injective, actually lies inside $\mathcal{C}_{\alpha}$ ([BG], Proposition 6.3). It is easy to see that $\mathcal{C}_{\alpha}$ is a locally closed $\overline{\mathcal{G}}^{\mathrm{C}}$-invariant complex analytic subset of $\mathcal{A}$. Indeed, it can be shown that the canonical projection $\pi : \mathcal{C}_{\alpha} \longrightarrow N_{\alpha}$ is a holomorphic principal bundle with structure group $\overline{\mathcal{G}}^{\mathrm{C}}$, and consequently $\mathcal{C}_{\alpha}$ is a locally closed complex submanifold of $\mathcal{A}$.

2c *Kähler metric on the moduli space*

Recall that we have fixed a Hermitian metric h_i in E_i $(i = 1, 2)$ and a Hermitian metric μ of unit volume on the Riemann surface X, whose Kähler form is denoted by ω. Further, we had defined

$$\tau = \frac{d_1 + d_2 + \alpha r_2}{r_1 + r_2}$$

and $\tau' = \tau - \alpha$.

There is a natural Kähler metric on $\mathcal{A}$ which can be described as follows. Recall that $\mathcal{A}$ is the Cartesian product

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathrm{C}^\infty(\mathcal{H}om(E_2, E_1)).$$

The holomorphic tangent bundle of $\mathcal{A}_i$ is canonically trivial with $A^{0,1}(\mathcal{E}nd(E_i))$ as the fiber. Thus, to construct a Kähler metric on $\mathcal{A}_i$, it is enough to give a Hermitian inner product on the vector space $A^{0,1}(\mathcal{E}nd(E_i))$. Since both X and E_i carry Hermitian metrics, there is a natural choice for the inner product on $A^{0,1}(\mathcal{E}nd(E_i))$, namely the L^2 inner product. In other words, $\mathcal{A}_i$ becomes a Kähler manifold in a natural way. Similarly, the L^2 inner product on $C^\infty(\mathcal{H}om(E_2, E_1))$ makes it a Kähler manifold. Thus each factor of $\mathcal{A}$ carries a natural Kähler metric. The product of these Kähler metrics makes $\mathcal{A}$ a Kähler manifold.

Recall that $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ is the product of the unitary gauge groups of E_1 and E_2. Embed $U(1)$ in $\mathcal{G}$ via the homomorphism $\lambda \mapsto (\lambda \cdot \mathrm{Id}_{E_1}, \lambda \cdot \mathrm{Id}_{E_2})$. Then $U(1)$ acts trivially on $\mathcal{A}$, and hence we get an induced action of $\overline{\mathcal{G}} := \mathcal{G}/U(1)$ on $\mathcal{A}$. This action leaves invariant the complex submanifold $\mathcal{C}_\alpha$ of $\mathcal{A}$. We thus get an induced action of $\overline{\mathcal{G}}$ on $\mathcal{C}_\alpha$. The restriction of the Kähler metric on $\mathcal{A}$ to $\mathcal{C}_\alpha$ makes $\mathcal{C}_\alpha$ a Kähler manifold. The action of $\overline{\mathcal{G}}$ on $\mathcal{C}_\alpha$ clearly preserves the Kähler metric. Hence it preserves the associated symplectic structure on $\mathcal{C}_\alpha$. In other words, the action of $\overline{\mathcal{G}}$ on $\mathcal{C}_\alpha$ is a symplectic action. It is in fact a Hamiltonian action; we will explicitly describe a moment map for it.

Let $\mathcal{E}nd(E_i, h_i)$ be the real vector bundle over X given by the skew-Hermitian endomorphisms of the Hermitian vector bundle E_i. Then, the Lie algebra $\overline{\mathfrak{g}}$ of $\overline{\mathcal{G}}$ is canonically isomorphic to the Lie subalgebra of the direct sum $C^\infty(\mathcal{E}nd(E_1, h_1)) \bigoplus C^\infty(\mathcal{E}nd(E_2, h_2))$ consisting of pairs of the form (f_1, f_2), where $f_i \in C^\infty(\mathcal{E}nd(E_i, h_i))$ $(i = 1, 2)$, satisfying the condition

$$\int_X (\mathrm{tr}(f_1) + \mathrm{tr}(f_2))\, \omega = 0\,.$$

Let

$$\Lambda : A^p(\mathcal{E}nd(E_i, h_i)) \longrightarrow A^{p-2}(\mathcal{E}nd(E_i, h_i))$$

denote the adjoint of the operator

$$L = \mathrm{e}(\omega) : A^{p-2}(\mathcal{E}nd(E_i, h_i)) \longrightarrow A^p(\mathcal{E}nd(E_i, h_i)),$$

which is the exterior multiplication by the Kähler form ω of (X, μ). Define

$$\Phi : \mathcal{C}_\alpha \longrightarrow \overline{\mathfrak{g}}$$

by

$$\begin{aligned}\Phi(D_1, D_2, \phi) \;&= (\Lambda R(\nabla_{D_1}) - \sqrt{-1}\phi\phi^* + 2\pi\sqrt{-1}\tau, \Lambda R(\nabla_{D_2}) \\ &\quad + \sqrt{-1}\phi^*\phi + 2\pi\sqrt{-1}\tau').\end{aligned}$$

Then Φ is a moment map for the action of $\overline{\mathcal{G}}$ on $\mathcal{C}_\alpha$ ([BG], §6.3). (The moment map is a differentiable map from $\mathcal{C}_\alpha$ to the dual vector space $\overline{\mathfrak{g}}^*$ of $\overline{\mathfrak{g}}$; here we are identifying $\overline{\mathfrak{g}}$ with its dual using the L^2 inner product.) Thus the action of $\overline{\mathcal{G}}$ on $\mathcal{C}_\alpha$ is Hamiltonian. It can be checked that the origin $0 \in \bar{\mathfrak{g}}$ is a regular value of Φ, and hence $\Phi^{-1}(0)$ is a closed C^∞ submanifold of $\mathcal{C}_\alpha$. Moreover, the action of $\overline{\mathcal{G}}$ on $\Phi^{-1}(0)$ is proper and free. Finally, the inclusion $\Phi^{-1}(0) \hookrightarrow \mathcal{C}_\alpha$ induces a diffeomorphism

$$\Phi^{-1}(0)/\overline{\mathcal{G}} \cong \mathcal{C}_\alpha/\overline{\mathcal{G}}^{\mathsf{C}} = N_\alpha.$$

Now, the standard procedure of symplectic reduction ([K], p. 273, Theorem 5.11) provides a Kähler metric on $\Phi^{-1}(0)/\overline{\mathcal{G}}$ and hence on N_α. The Kähler metric on N_α obtained this way will be called *the natural Kähler metric*.

3. The determinant bundle

In this section, we will define the determinant line bundle for a family of triples, and construct a natural Hermitian holomorphic line bundle over the moduli space of α-stable triples. We will continue with the notation of the preceding section.

3a *Families of triples*

Fix a positive rational number α.

DEFINITION 3.1

Let S be a complex manifold. A *family* T_S of triples over X, parametrized by S, consists of two holomorphic vector bundles $E_{1,S}$ and $E_{2,S}$ over $S \times X$, and a homomorphism $\phi_S : E_{2,S} \longrightarrow E_{1,S}$ of holomorphic vector bundles over $S \times X$. We say that two families $T_S = (E_{1,S}, E_{2,S}, \phi_S)$ and $T'_S = (E'_{1,S}, E'_{2,S}, \phi'_S)$ are *equivalent* if there exist a holomorphic line bundle L over S and isomorphisms $f_i : E_{i,S} \otimes p_S^* L \longrightarrow E'_{i,S}$ $(i = 1, 2)$, where $p_S : S \times X \longrightarrow S$ is the obvious projection, such that the diagram

$$\begin{array}{ccc} E_{1,S} \bigotimes p_S^* L & \xrightarrow{f_1} & E'_{1,S} \\ \phi_S \times \mathrm{Id} \uparrow & & \uparrow \phi'_S \\ E_{2,S} \bigotimes p_S^* L & \xrightarrow[f_2]{} & E'_{2,S} \end{array}$$

commutes.

PROPOSITION 3.2

Let $T_S = (E_{1,S}, E_{2,S}, \phi_S)$ and $T'_S = (E'_{1,S}, E'_{2,S}, \phi'_S)$ be two families of α-stable triples on X parametrized by a complex manifold S. Suppose that for each point s in S, the triples T_s and T'_s are isomorphic, where T_s (respectively, T'_s) is the restriction of T_S (respectively, T'_S) to $X_s = \{s\} \times X$. Then T_S and T'_S are equivalent.

Proof. Let $\mathcal{F}$ denote the sheaf of homomorphisms from T_S to T'_S, i.e., for any open subset U of $S \times X$, the space of sections $\Gamma(U, \mathcal{F})$ consists of pairs of the form $f = (f_1, f_2)$, where for each $i = 1, 2$, the map $f_i : E_{i,S}|_U \longrightarrow E'_{i,S}|_U$ is a vector bundle homomorphism satisfying the condition $f_1 \circ \phi_S = \phi'_S \circ f_2$. Then $\mathcal{F}$ is a coherent $\mathcal{O}_{S\times X}$-module, and $H^0(X_s, \mathcal{F}_s)$ is a one-dimensional vector space for all $s \in S$ ([BG], Corollary 3.12).

Therefore, by a standard result on direct images, $L = (p_S)_* \mathcal{F}$ is a locally free sheaf over S of rank one, and the fibre L_s of the line bundle L at any point $s \in S$ is canonically isomorphic to $\mathrm{Hom}(T_s, T'_s)$, the space of global homomorphisms from the triple T_s over X_s to T'_s. From the definition of the line bundle L, we have an obvious homomorphism $f_i : E_{i,S} \otimes p_S^* L \longrightarrow E_{i,S}$ for each $i = 1, 2$. These homomorphisms f_i are actually isomorphisms, for if T and T' are two α-stable triples of the same type (r_1, r_2, d_1, d_2), then every non-zero homomorphism from T to T' is in fact an isomorphism ([BG], Proposition 3.10). Clearly, the diagram

$$\begin{array}{ccc} E_{1,S} \bigotimes p_S^* L & \xrightarrow{\ f_1\ } & E'_{1,S} \\ {\scriptstyle \phi_S \bigotimes \mathrm{Id}} \uparrow & & \uparrow {\scriptstyle \phi'_S} \\ E_{2,S} \bigotimes p_S^* L & \xrightarrow[\ f_2\ ]{} & E'_{2,S} \end{array}$$

commutes. Therefore, T_S and T'_S are equivalent. This completes the proof of the proposition.

3b *Classifying map of a family*

Let E_1 and E_2 be fixed C^∞ complex vector bundles over X, as in §2. For $i = 1, 2$, let $\mathcal{E}_i = \mathcal{A} \times E_i = p_X^* E_i$ be the vector bundle over $\mathcal{A} \times X$, obtained by pulling back E_i using the natural projection $p_X : \mathcal{A} \times X \longrightarrow X$; the space $\mathcal{A}$ was defined in §2b. Then there is a natural holomorphic structure on $\mathcal{E}_i$. This holomorphic structure is determined by the following two conditions: (a) for any point $D = (D_1, D_2, \phi) \in \mathcal{A}$, the holomorphic structure on the restriction of $\mathcal{E}_i$ to the submanifold $\{D\} \times X \subset \mathcal{A} \times X$ is given by D_i; and (b) for any point $x \in X$, the holomorphic structure on the restriction of $\mathcal{E}_i$ to the subset $\mathcal{A} \times \{x\} \subset \mathcal{A} \times X$ coincides with the natural trivialization of this vector bundle over $\mathcal{A} \times x$.

Consider the family $\mathcal{T} = (\mathcal{E}_1, \mathcal{E}_2, \Psi)$ of C^∞ triples parametrized by $\mathcal{A}$, where $\Psi : \mathcal{E}_2 \longrightarrow \mathcal{E}_1$ is the obvious C^∞ homomorphism, i.e.,

$$\Psi(D, e_2) = (D, \phi(e_2)) \quad \text{for} \quad D = (D_1, D_2, \phi) \in \mathcal{A} \quad \text{and} \quad e_2 \in E_2.$$

Although Ψ is only C^∞ on $\mathcal{A}$, it follows from the definition of $\mathcal{B}$ that the restriction of Ψ to $\mathcal{B}$ is in fact holomorphic. Restrict $\mathcal{T}$ to $\mathcal{B}_\alpha^{\mathrm{s}}$, and denote this restriction also by $\mathcal{T}$. It is a holomorphic family of α-stable holomorphic triples over X parametrized by $\mathcal{B}_\alpha^{\mathrm{s}}$. We wish to show that this family $\mathcal{T}$ is a locally universal family on $\mathcal{B}_\alpha^{\mathrm{s}} \times X$.

Indeed, let $T_S = (E_{1,S}, E_{2,S}, \phi_S)$ be a family of α-stable triples of type (r_1, r_2, d_1, d_2), parametrized by a complex manifold S. Then S can be covered by open subsets U such that $E_{i,S}|_{U \times X}$ is isomorphic, as a C^∞ vector bundle, to $p_X^* E_i$ on $U \times X$ for each $i = 1, 2$, where $p_X : U \times X \longrightarrow X$ is the canonical projection. For each point $s \in U$, the $\overline{\partial}$-operator acting on the holomorphic bundle $E_{i,s}$ defines a holomorphic structure $D_{i,s}$ in E_i. Further, the homomorphism $\phi_s : E_{2,s} \longrightarrow E_{1,s}$, which is the restriction of ϕ_S to the subset $X_s = \{s\} \times X \subset S \times X$, defines a homomorphism $E_2 \longrightarrow E_1$, which we again denote by ϕ_s itself. In other words, for each point $s \in U$, we get a point $D_s = (D_{1,s}, D_{2,s}, \phi_s) \in \mathcal{A}$. The fact that the given homomorphism $\phi_s : E_{2,s} \longrightarrow E_{1,s}$ is holomorphic, ensures that

$D_s \in \mathcal{B}$ whenever $s \in U$. Since each triple T_s is α-stable, we in fact get $D_s \in \mathcal{B}_\alpha^{\mathrm{s}}$ for all $s \in U$. Clearly the resulting map $f_U : U \longrightarrow \mathcal{B}_\alpha^{\mathrm{s}}$ is a holomorphic map. Moreover, for each $s \in U$, we have $T_s \cong ((f_U \times \mathrm{Id})^*(\mathcal{T}))_s$. Thus, by Proposition 3.2, the two families of triples, namely $(f_U \times \mathrm{Id})^*(\mathcal{T})$ and $T_S|_U$, are equivalent. This shows that the family $\mathcal{T}$ on $\mathcal{B}_\alpha^{\mathrm{s}} \times X$ has the local universal property.

In the setup of the above paragraph, it is clear that if U' is another open subset of S such that $E_{i,S}|_{U' \times X}$ is C^∞ isomorphic to $p_X^* E_i$ $(i = 1, 2)$ on $U' \times X$, where $p_X : U' \times X \longrightarrow X$ is the obvious projection, then the corresponding map $f_{U'} : U' \longrightarrow \mathcal{B}_\alpha^{\mathrm{s}}$ has the property that $\pi \circ f_U = \pi \circ f_{U'}$ on $U \cap U'$, where $\pi : \mathcal{B}_\alpha^{\mathrm{s}} \longrightarrow M_\alpha^{\mathrm{s}}$ is the canonical projection. We thus get a well-defined holomorphic map $h_S : S \longrightarrow M_\alpha^{\mathrm{s}}$ such that the restriction of h_S to any open set $U \subset S$, as above, equals $\pi \circ f_U$. If we view M_α^{s} as the space of isomorphism classes of α-stable triples of type (r_1, r_2, d_1, d_2), then the image of $s \in S$ under h_S is precisely the isomorphism class of the triple $T_s = T_S|_{X_s}$. We call $h_S : S \longrightarrow M_\alpha^{\mathrm{s}}$ the *classifying map* for the family T_S.

3c *Determinant line bundle for a family*

To begin with, let us recall the determinant line bundle for a family of usual vector bundles. Fix a point x_0 on the Riemann surface X. Let S be a connected complex manifold, and let E_S be a family of vector bundles over X parametrized by S, i.e., E_S is a holomorphic vector bundle over $S \times X$. Let $\mathrm{Det}(E_S) \longrightarrow S$ be the determinant of the cohomology of E_S, i.e.,

$$\mathrm{Det}(E_S) = \det\, ((p_S)_* E_S)^{-1} \otimes \det\, (R^1\, (p_S)_*\, E_S),$$

where $p_S : S \times X \longrightarrow S$ is the projection onto the first factor, and det denotes the determinant line bundle for a coherent analytic sheaf; the construction of det can be found in Chapter V, §6 of [K]. We will use the notation L^{-1} for the dual of a line bundle L. The fiber of the holomorphic line bundle $\mathrm{Det}(E_S)$ over any point $s \in S$ is canonically isomorphic to

$$\bigwedge^{\mathrm{top}} (H^0(X_s, E_s))^* \otimes \bigwedge^{\mathrm{top}} (H^1(X_s, E_s)).$$

Define a line bundle $\Theta(E_S)$ over S by

$$\Theta\, (E_S) = \mathrm{Det}(E_S)^r \otimes \det(E^{x_0})^{\chi(E_s)},$$

where E^{x_0} is the restriction of E_S to the slice $S^{x_0} = S \times \{x_0\} \subset S \times X$, $r = \mathrm{rank}(E_S)$, and $\chi(E_s) = \dim H^0(X_s, E_s) - \dim H^1(X_s, E_s)$ is the Euler characteristic of the vector bundle $E_s \longrightarrow X_s$ for some $s \in S$. Since S is connected, the number $\chi(E_s)$ is independent of $s \in S$. We call $\Theta\, (E_S)$ the *determinant line bundle* for the family of vector bundles E_S.

DEFINITION 3.3

Let $T_S = (E_{1,S}, E_{2,S}, \phi_S)$ be a holomorphic family of triples on X parametrized by S. We define the *determinant line bundle* $\mathcal{D}(T_S)$ of T_S to be

$$\mathcal{D}(T_S) = \Theta\left(E_{1,S}\right)^{r_2} \otimes \Theta\left(E_{2,S}\right)^{r_1},$$

where $r_i = \mathrm{rank}(E_{i,S})$, $i = 1, 2$.

Remark 3.4. The projection formula implies that

$$\mathrm{Det}(E_S \otimes p_S^* L) = \mathrm{Det}(E_S) \otimes L^{-\chi(E_s)},$$

where L is any line bundle over S and p_S is the natural projection from $S \times X$ to S. From this it follows immediately that if $E_S \longrightarrow S \times X$ is a family of vector bundles, then $\Theta\left(E_S \otimes p_S^* L\right)$ is canonically isomorphic to $\Theta\left(E_S\right)$ for every holomorphic line bundle L over S. Moreover, using the standard properties of the determinant of the cohomology, we conclude that if $f : S' \longrightarrow S$ is a holomorphic map, then $\Theta\left((f \times \mathrm{Id})^* E_S\right)$ is isomorphic to $f^* \Theta\left(E_S\right)$. From these facts it follows that the same properties continue to hold for the determinant of the triples. In other words, if T_S and T'_S are equivalent families of triples, then $\mathcal{D}(T_S) \cong \mathcal{D}(T'_S)$. Similarly, if $f : S' \longrightarrow S$ is a morphism, then $\mathcal{D}((f \times \mathrm{Id})^* T_S)$ is isomorphic to $f^* \mathcal{D}(T_S)$.

3d *Determinant bundle on the moduli space*

Let $\mathcal{T} = (\mathcal{E}_1, \mathcal{E}_2, \Psi)$ be the family of canonical triples on $\mathcal{B} \times X$, as defined in §3b. Let $\mathcal{D} = \mathcal{D}(\mathcal{T})$ denote the determinant line bundle of this family as in Definition 3.3. Then $\mathcal{D}$ is a holomorphic line bundle on $\mathcal{B}$. Recall that, by definition,

$$\mathcal{D} = \Theta\left(\mathcal{E}_1\right)^{r_2} \otimes \Theta\left(\mathcal{E}_2\right)^{r_1},$$

where

$$\Theta\left(\mathcal{E}_i\right) = \mathrm{Det}(\mathcal{E}_i)^{r_i} \otimes \det(E_{i,x_0})^{\chi(E_i)}.$$

The action of the complex gauge group $\mathcal{G}_i^{\mathbf{C}}$ on $\mathcal{B}$ lifts to a right action on $\mathrm{Det}(\mathcal{E}_i)$ as follows. If $D = (D_1, D_2, \phi)$ is a point in $\mathcal{B}$, then the fibre of $\mathrm{Det}(\mathcal{E}_i)$ at D is canonically isomorphic to the one-dimensional complex vector space

$$V_{i,D} = \bigwedge^{\mathrm{top}}(K_{i,D}^*) \bigotimes \bigwedge^{\mathrm{top}}(C_{i,D}),$$

where $K_{i,D}$ is the kernel and $C_{i,D}$ the cokernel of the Dolbeault operator $D_i : A^0(E_i) \longrightarrow A^{0,1}(E_i)$. If $g \in \mathcal{G}_i^{\mathbf{C}}$, then the induced $C^\infty(X)$-module isomorphism $g : A^0(E_i) \longrightarrow A^0(E_i)$ carries $K_{i,D\cdot g}$ onto $K_{i,D}$. Therefore, we get a dual isomorphism $g^* : K_{i,D}^* \longrightarrow K_{i,D\cdot g}^*$. This, in turn, induces an isomorphism

$$\bigwedge^{\mathrm{top}} g^* : \bigwedge^{\mathrm{top}}(K_{i,D}^*) \longrightarrow \bigwedge^{\mathrm{top}}(K_{i,D\cdot g}^*)$$

of the top exterior powers. Similarly, the isomorphism $g^{-1} : A^{0,1}(E_i) \longrightarrow A^{0,1}(E_i)$ maps the subspace $\mathrm{Im}(D_i)$ onto $\mathrm{Im}(D_i \cdot g)$, and hence it induces an isomorphism $g^{-1} : C_{i,D} \longrightarrow C_{i,D\cdot g}$. Let

$$\bigwedge^{\mathrm{top}} g^{-1} : \bigwedge^{\mathrm{top}}(C_{i,D}) \longrightarrow \bigwedge^{\mathrm{top}}(C_{i,D\cdot g})$$

be the isomorphism of top exterior powers defined by the isomorphism g^{-1}. Now, the right action

$$\mathrm{Det}(\mathcal{E}_i) \times \mathcal{G}_i^{\mathbf{C}} \longrightarrow \mathrm{Det}(\mathcal{E}_i)$$

of $\mathcal{G}_i^{\mathbb{C}}$ on $\mathrm{Det}(\mathcal{E}_i)$ is defined by

$$(\alpha \otimes v, g) \longmapsto \left(\bigwedge^{\mathrm{top}} g^*\right)(\alpha) \bigotimes \left(\bigwedge^{\mathrm{top}} g^{-1}\right)(v)\,,$$

where α (respectively, v) is a vector in the one-dimensional complex vector space $\bigwedge^{\mathrm{top}}(K_{i,D}^*)$ (respectively, $\bigwedge^{\mathrm{top}}(C_{i,D})$) for some $D \in \mathcal{B}$. Similarly there is a natural right action

$$E_{i,x_0} \times \mathcal{G}_i^{\mathbb{C}} \longrightarrow E_{i,x_0}$$

of the group $\mathcal{G}_i^{\mathbb{C}}$ on the vector space E_{i,x_0} defined by

$$(e, g) \longmapsto g^{-1}(e)\,.$$

Consequently, we get an induced action of $\mathcal{G}_i^{\mathbb{C}}$ on $\det(E_{i,x_0})$. Therefore, $\mathcal{G}_i^{\mathbb{C}}$ acts on the line bundle $\Theta\,(\mathcal{E}_i)$ in a natural fashion. Under this action, the subgroup $\mathbb{C}^* \subset \mathcal{G}_i^{\mathbb{C}}$ acts on $\mathrm{Det}(\mathcal{E}_i)$ via the character $\lambda \mapsto \lambda^{\chi(E_i)}$, and it acts on $\det(E_{i,x_0})$ via the character $\lambda \mapsto \lambda^{-r_i}$. As a result, the action of $\mathbb{C}^*$ on $\Theta\,(\mathcal{E}_i)$ is trivial. We thus get an induced action of $\overline{\mathcal{G}}_i^{\mathbb{C}} = \mathcal{G}_i^{\mathbb{C}}/\mathbb{C}^*$ on $\Theta\,(\mathcal{E}_i)$, and hence an action of $\overline{\mathcal{G}}_1^{\mathbb{C}} \times \overline{\mathcal{G}}_2^{\mathbb{C}}$ on $\mathcal{D}$. Now the canonical homomorphism $\overline{\mathcal{G}}^{\mathbb{C}} \longrightarrow \overline{\mathcal{G}}_1^{\mathbb{C}} \times \overline{\mathcal{G}}_2^{\mathbb{C}}$ gives an action of $\overline{\mathcal{G}}^{\mathbb{C}}$ on $\mathcal{D}$. In other words, $\mathcal{D}$ is a $\overline{\mathcal{G}}^{\mathbb{C}}$-linearized line bundle over $\mathcal{B}$.

Restrict the line bundle $\mathcal{D}$ to $\mathcal{B}_\alpha^{\mathrm{s}}$. Since the canonical projection $\pi : \mathcal{B}_\alpha^{\mathrm{s}} \longrightarrow M_\alpha^{\mathrm{s}}$ is a holomorphic principal bundle with structure group $\overline{\mathcal{G}}^{\mathbb{C}}$, it follows immediately that the line bundle $\mathcal{D}$ descends to a holomorphic line bundle $\mathcal{L}$ over M_α^{s}, i.e., there is given a $\overline{\mathcal{G}}^{\mathbb{C}}$-equivariant isomorphism of $\pi^*\mathcal{L}$ with $\mathcal{D}$. We have thus constructed a natural line bundle $\mathcal{L}$ over M_α^{s} that we are seeking.

DEFINITION 3.5

We will call the above holomorphic line bundle $\mathcal{L}$, on M_α^{s}, the *determinant line bundle* over M_α^{s}.

The determinant line bundle on M_α^{s} has the following universal property.

PROPOSITION 3.6

If $T_S = (E_{1,S}, E_{2,S}, \phi_S)$ is a family of α-stable pairs of type (r_1, r_2, d_1, d_2) parametrized by a complex manifold S, then there is a canonical isomorphism $\mathcal{D}(T_S) \cong h_S^*\mathcal{L}$, where $h_S : S \longrightarrow M_\alpha^{\mathrm{s}}$ is the classifying map for the family T_S, and $\mathcal{L}$ is the determinant bundle over M_α^{s}.

Proof. Let $\mathcal{U}$ be an open cover of S such that if U is a member of the collection $\mathcal{U}$, then the restriction $E_{i,S}|_{U\times X}$ of $E_{i,S}$ to $U \times X$ is isomorphic, as a C^∞ vector bundle, to the restriction of $p_X^* E_i$ to the open subset $U \times X$, for each $i = 1, 2$. We have seen in §3b that for every $U \in \mathcal{U}$, there exists a holomorphic map $f_U : U \longrightarrow \mathcal{B}_\alpha^{\mathrm{s}}$ such that the restricted family $T_S|_U$ is equivalent to $(f_U \times \mathrm{Id})^*\mathcal{T}$ on $U \times X$, where $\mathcal{T}$ is the above family of triples on $\mathcal{A} \times X$. The map f_U has the property that $h_S|_U = \pi \circ f_U$, where $\pi : \mathcal{B}_\alpha^{\mathrm{s}} \longrightarrow M_\alpha^{\mathrm{s}}$, as before, is the canonical projection. Thus on each member U of $\mathcal{U}$, we have a chain of isomorphisms

$$\mathcal{D}(T_S) \cong \mathcal{D}((f_U \times \mathrm{Id})^* \mathcal{T}) \cong f_U^* \mathcal{D}(\mathcal{T}) \cong f_U^*(\pi^* \mathcal{L}) \cong h_S^* \mathcal{L}\,,$$

which is obtained using Remark 3.4 and the fact that $\mathcal{L}$ is the descent of $\mathcal{D}(\mathcal{T})$ from $\mathcal{B}_\alpha^{\mathrm{s}}$. Therefore, for each member $U \in \mathcal{U}$, we get a canonical isomorphism γ_U on U from $\mathcal{D}(T_S)$ to $h_S^*\mathcal{L}$. The canonical nature of γ_U ensures that if U and U' are two members of $\mathcal{U}$, then γ_U and $\gamma_{U'}$ have to coincide on the overlap $U \cap U'$. Consequently, the various local isomorphisms γ_U glue together to give a canonical global isomorphism of $\mathcal{D}(T_S)$ with $h_S^*\mathcal{L}$ over S. This completes the proof of the proposition. □

The above proposition shows that to establish a local property of $\mathcal{D}(T_S)$ for an arbitrary family of triples T_S, it is enough to prove for $\mathcal{L}$.

4. Metric on the determinant bundle

In this section, we will construct a natural Hermitian metric on the determinant line bundle $\mathcal{L}$ over the moduli space N_α. The variety N_α, as before, is the smooth locus of the quasi-projective variety M_α^{s}. Since $\mathcal{L}$ is a holomorphic line bundle, this Hermitian metric determines a canonical connection on $\mathcal{L}$, which is usually known as the Chern connection. We will also compute the curvature of this canonical connection.

4a *Quillen metrics*

Recall that we have a natural family of triples $\mathcal{T} = (\mathcal{E}_1, \mathcal{E}_2, \Psi)$ over $\mathcal{A} \times X$. We had defined the holomorphic $\overline{\mathcal{G}}^{\mathbb{C}}$-line bundle $\mathcal{D}$ to be the determinant bundle for this family (see Definition 3.3), i.e.,

$$\mathcal{D} = \Theta\,(\mathcal{E}_1)^{r_2} \otimes \Theta\,(\mathcal{E}_2)^{r_1}\ ,$$

where

$$\Theta\,(\mathcal{E}_i) = \mathrm{Det}(\mathcal{E}_i)^{r_i} \otimes_{\mathbb{C}} \det(E_{i,x_0})^{\chi(E_i)}\,.$$

Here E_{i,x_0} is the fibre of E_i at the point $x_0 \in X$, and $r_i = \mathrm{rank}(E_i)$.

For each $i = 1, 2$, the determinant line bundle $\mathrm{Det}(\mathcal{E}_i) \longrightarrow \mathcal{A}$ carries a natural Hermitian metric, called the *Quillen metric*, which we recall briefly. A detailed description can be found [Q]. The fibre of $\mathrm{Det}(\mathcal{E}_i)$ at any point $D = (D_1, D_2, \phi)$ is canonically isomorphic to the one-dimensional complex vector space

$$V_{i,D} = \bigwedge^{\mathrm{top}}(K_{i,D}^*) \otimes \bigwedge^{\mathrm{top}}(C_{i,D})\,,$$

where $K_{i,D}$ (respectively, $C_{i,D}$), as before, is the kernel (respectively, cokernel) of the Dolbeault operator $D_i : A^0(E_i) \longrightarrow A^{0,1}(E_i)$. Now the Hermitian metrics on X and E_i induce an L^2 inner product on $K_{i,D}$ and $C_{i,D}$, and hence an inner product on $V_{i,D}$. We denote this L^2 inner product on $V_{i,D}$ by $\lambda_{i,D}$. On the other hand, the Laplacian $D_i^* D_i$: $A^0(E_i) \longrightarrow A^0(E_i)$ defines a zeta function $\zeta_{i,D}(s) = \mathrm{trace}(D_i^* D_i)^{-s}$. The Quillen inner product $\rho_{i,D}$ on $V_{i,D}$ is defined to be $\rho_{i,D} = \exp(-\zeta'_{i,D}(0)) \cdot \lambda_{i,D}$. It follows immediately from the results of [Q] that the inner product $\rho_{i,D}$ varies smoothly as D varies in $\mathcal{A}$, and hence defines a C^∞ Hermitian metric ρ_i in $\mathrm{Det}(\mathcal{E}_i)$. Since $\mathrm{Det}(\mathcal{E}_i)$ is a holomorphic line bundle, the metric ρ_i determines a unique connection on $\mathrm{Det}(\mathcal{E}_i)$ compatible with both the holomorphic and metric structures. Let $c_1(\mathrm{Det}(\mathcal{E}_i), \rho_i)$ denote the first Chern form of this canonical connection. Then, the main result of [Q] implies that

$$c_1(\mathrm{Det}(\mathcal{E}_i), \rho_i) = p_i^* \Omega_i \, , \tag{4a.1}$$

where for each $i = 1, 2$, $p_i : \mathcal{A} \longrightarrow \mathcal{A}_i$ is the projection defined by $(D_1, D_2, \phi) \mapsto D_i$, and Ω_i is the Kähler form on $\mathcal{A}_i$.

The Hermitian metric in E_i defines a flat Hermitian metric in the trivial line bundle $\det(E_{i,x_0}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{A}}$. This flat metric in $\det(E_{i,x_0}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{A}}$, together with the Quillen metric ρ_i in $\mathrm{Det}(\mathcal{E}_i)$ induces a Hermitian metric θ_i in $\Theta\,(\mathcal{E}_i)$. It follows immediately from eq. (4a.1) that

$$c_1(\Theta\,(\mathcal{E}_i)\,, \theta_i) = r_i \cdot p_i^* \Omega_i \, , \tag{4a.2}$$

where $r_i = \mathrm{rank}(E_i)$.

4b *A property of the inner product*

Let V be a complex Hilbert space with inner product $\langle \cdot\,, \cdot \rangle : V \times V \longrightarrow \mathbb{C}$, and let $N : V \longrightarrow \mathbb{R}$ denote the norm square function $x \mapsto \langle x, x \rangle$. Let $\Omega \in A^{1,1}(V)$ denote the Kähler form of V with respect to the Kähler metric defined by the inner product $\langle \cdot\,, \cdot \rangle$. Then one observes that

$$\Omega = \sqrt{-1}\partial\bar{\partial} N \, .$$

Let us check this fact with a simple computation.

Indeed, a differential form ω on V is zero if and only if it vanishes on every finite-dimensional subspace of V. For, if ω is a p-form on V which vanishes on every finite-dimensional subspace of V, then for every point $x \in V$ and for all tangent vectors $v_1, \ldots, v_p$ in $V \cong \mathrm{T}_x V$, we have

$$\omega_x(v_1, \ldots, v_p) = (i_W^* \omega)_x (v_1, \ldots, v_p) \, ,$$

where W is the subspace of V generated by $\{x, v_1, \ldots, v_p\}$, and $i_W : W \hookrightarrow V$ is the inclusion map. The hypothesis on ω implies that $i_W^* \omega = \omega|_W = 0$, hence $\omega_x(v_1, \ldots, v_p) = 0$. Therefore ω is the zero form on V.

In view of the above fact, we can assume without loss of generality that V is finite-dimensional. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V, and $\{z_1, \ldots, z_n\}$ be the dual basis of V^*. Then $\Omega = \sqrt{-1}\sum_{i=1}^n \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$, whereas $N = \sum_{i=1}^n z_i \bar{z}_i$. Therefore, we have

$$\sqrt{-1}\partial\bar{\partial} N = \sqrt{-1}\sum_i \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i = \Omega \, ,$$

and this proves the assertion made at the beginning of this subsection.

4c *Metric on $\mathcal{L}$*

Recall that $\mathcal{D}$ is the holomorphic line bundle over $\mathcal{A}$ defined by

$$\mathcal{D} = \Theta\,(\mathcal{E}_1)^{r_2} \otimes \Theta\,(\mathcal{E}_2)^{r_1}\ .$$

In §4a, we had constructed a natural metric θ_i in $\Theta\,(\mathcal{E}_i)$. Let $\|\cdot\|$ denote the L^2 norm on $C^\infty(\mathcal{H}om(E_2, E_1))$, and define a C^∞ function $N : \mathcal{A} \longrightarrow \mathbb{R}$ by

$$N(D_1, D_2, \phi) := \|\phi\|^2, \quad (D_1, D_2, \phi) \in \mathcal{A}\,.$$

Now define a Hermitian metric δ in $\mathcal{D}$ by

$$\delta = \exp(-2\pi r_1 r_2 N) \cdot \theta_1^{r_2} \otimes \theta_2^{r_1}\,, \tag{4c.1}$$

where θ_i is the Hermitian metric in $\Theta\,(\mathcal{E}_i)$ constructed in §4a.

Recall that in subsection 2b, we defined the subset $\mathcal{B}_\alpha^{\mathrm{s}} \subset \mathcal{A}$ consisting of all holomorphic structures giving a stable triple. Now restrict the Hermitian holomorphic line bundle $(\mathcal{D}, \delta)$ to the locally closed complex submanifold $\mathcal{B}_\alpha^{\mathrm{s}}$ of $\mathcal{A}$. From §2c, we have a commutative diagram

$$\begin{array}{ccc} \Phi^{-1}(0) & \xrightarrow{\ \ i\ \ } & \mathcal{C}_\alpha \\ \pi_0 \Big\uparrow & & \Big\uparrow \pi \\ N_\alpha & \xrightarrow[\ \ \mathrm{Id}\ \ ]{} & N_\alpha \end{array}\quad ,$$

where the top arrow is the inclusion map i of $\Phi^{-1}(0)$ in $\mathcal{B}_\alpha^{\mathrm{s}}$, π the canonical projection from $\mathcal{B}_\alpha^{\mathrm{s}}$ onto N_α, and π_0 the restriction of π to $\Phi^{-1}(0)$. Here $\Phi : \mathcal{B}_\alpha^{\mathrm{s}} \longrightarrow \overline{\mathfrak{g}}$ is the moment map for the action of $\overline{\mathcal{G}} = (\mathcal{G}_1 \times \mathcal{G}_2)/U(1)$, where $\mathcal{G}_i$ is the unitary gauge group of E_i, $i = 1, 2$. The map π is a principal $\overline{\mathcal{G}}^{\mathbb{C}}$-bundle, and π_0 is a principal $\overline{\mathcal{G}}$-bundle.

It is clear that the metric δ in $\mathcal{D}$ is preserved under the action of the group $\overline{\mathcal{G}}$. Since the determinant bundle $\mathcal{L} \longrightarrow N_\alpha$ is the descent of $\mathcal{D}$ from $\mathcal{B}_\alpha^{\mathrm{s}}$, the $\overline{\mathcal{G}}$-invariance of the metric δ in $\mathcal{D}$ implies that it descends to a C^∞ Hermitian metric λ in $\mathcal{L}$, i.e., there exists a unique metric λ in $\mathcal{L}$ such that $\pi_0^*\lambda = i^*\delta$. We call λ the *natural metric* in $\mathcal{L}$.

Given that the line bundle $\mathcal{D}$ has a natural holomorphic structure, the Hermitian metric δ on it defines a canonical (Chern) connection $\nabla_{\mathcal{D}}$ on $\mathcal{D}$. It is straight-forward to check that this connection $\nabla_{\mathcal{D}}$ descends from $\Phi^{-1}(0)$ to a connection $\nabla_{\mathcal{L}}$ on $\mathcal{L}$, i.e., $\pi_0^*\nabla_{\mathcal{L}} = i^*\nabla_{\mathcal{D}}$. A similar argument can be found in Theorem 3.2 of [GS]. We claim that this descended connection $\nabla_{\mathcal{L}}$ on $\mathcal{L}$ is the canonical connection of the holomorphic line bundle $\mathcal{L}$ with respect to the Hermitian metric λ on $\mathcal{L}$. To see this, we need to check that $\nabla_{\mathcal{L}}$ is

(a) Hermitian, i.e., it is compatible with the Hermitian metric λ on $\mathcal{L}$; and
(b) holomorphic, i.e., its connection form with respect to any local holomorphic frame of $\mathcal{L}$ is of bidegree $(1, 0)$.

We get (a) easily from the fact that $\nabla\mathcal{D}$ is Hermitian with respect to the metric δ on $\mathcal{D}$. To check (b), let s be a holomorphic frame of $\mathcal{L}$ on an open subset U of N_α, i.e., s is a nowhere zero holomorphic section of $\mathcal{L}$ on U. Let ω be the connection form of $\nabla_{\mathcal{L}}$ with respect to the frame s. We are required to show that ω is of type $(1, 0)$, i.e.,

$$\omega_p(J_p v) = \sqrt{-1} \cdot \omega_p(v) \tag{4c.2}$$

for all $p \in U$ and $v \in \mathrm{T}_p N_\alpha$, where $\mathrm{T}_p N_\alpha$ is the real tangent space to N_α at p, and $J_p : \mathrm{T}_p N_\alpha \longrightarrow \mathrm{T}_p N_\alpha$ is the almost complex structure on $\mathrm{T}_p N_\alpha$. Let $D \in \Phi^{-1}(0)$ be a point lying over p, i.e., $\pi_0(D) = p$. One easily checks that

$$\mathrm{T}_D \Phi^{-1}(0) = \mathrm{T}_D \left(\overline{\mathcal{G}} \cdot D\right) \oplus V\,,$$

where $\overline{\mathcal{G}} \cdot D$ is the $\overline{\mathcal{G}}$-orbit through D, and V is the orthogonal complement of $\mathrm{T}_D \left(\overline{\mathcal{G}} \cdot D\right)$ with respect to the Riemannian metric on $\Phi^{-1}(0)$. Now, let $t = \pi^* s$ be the $\overline{\mathcal{G}}^{\mathsf{C}}$-invariant holomorphic frame of $\mathcal{D}$ on $\pi^{-1}(U)$ obtained by pulling back s via π. Let $\tilde{\omega}$ be the connection form of $\nabla_{\mathcal{D}}$ with respect to the holomorphic frame t of $\mathcal{D}$. Then, the fact that $\nabla_{\mathcal{L}}$ is the descent of $\nabla_{\mathcal{D}}$ from $\Phi^{-1}(0)$ immediately gives that $\pi_0^* \omega = i^* \tilde{\omega}$. From this observation, and the fact that π_0 is a submersion, to establish the relation (4c.2), it suffices to check that the subspace V of $\mathrm{T}_D \Phi^{-1}(0)$ is invariant under the almost complex structure $J_D : \mathrm{T}_D \mathcal{C}_\alpha \longrightarrow \mathrm{T}_D \mathcal{C}_\alpha$. Let us verify this. Take $w \in V$ and $\xi \in \overline{\mathfrak{g}}$. We need to check that

$$\langle J_D w\,,\ X(\xi)_D \rangle = 0\,,$$

where $X(\xi)$ is the fundamental vector field on $\mathcal{C}_\alpha$ defined by the Lie algebra element ξ. But, from the definitions of the symplectic structure on a Kähler manifold and moment map, we have

$$\langle J_D w\,,\ X(\xi)_D \rangle = \Omega_{\mathcal{A}}(w, X(\xi)_D) = \mathrm{d}(\Phi^\xi)_D = 0\,,$$

where $\Omega_{\mathcal{A}}$ is the Kähler form on $\mathcal{A}$, and $\Phi^\xi : \mathcal{C}_\alpha \longrightarrow \mathbb{R}$ is the function defined by $\Phi^\xi(x) = B(\Phi(x), \xi)$. Here $B : \overline{\mathfrak{g}}^* \times \overline{\mathfrak{g}} \longrightarrow \mathbb{R}$ is the canonical duality pairing. This proves the assertion that $\nabla_{\mathcal{L}}$ is the canonical connection on the holomorphic Hermitian line bundle $\mathcal{L}$.

We are now ready to compute the curvature of the determinant line bundle $\mathcal{L}$ equipped with the natural Hermitian metric λ.

Theorem 4.1. *The first Chern form $c_1(\mathcal{L}, \lambda)$ of the determinant line bundle $\mathcal{L}$ on the moduli space N_α, with respect to the natural Hermitian metric λ in $\mathcal{L}$ is given by*

$$c_1(\mathcal{L}, \lambda) = r_1 r_2 \cdot \Omega_{N_\alpha}\,,$$

where Ω_{N_α} is the natural Kähler form of N_α.

Proof. Since $\pi_0 : \Phi^{-1}(0) \longrightarrow N_\alpha$ is a submersion, to prove the theorem it suffices to verify that

$$\pi_0^*(c_1(\mathcal{L}, \lambda)) = r_1 r_2 \cdot \pi_0^* \Omega_{N_\alpha}.$$

But, as we have observed above, the canonical connection $\nabla_{\mathcal{L}}$ is the descent of $\nabla_{\mathcal{D}}$ via π_0, hence we get

$$\pi_0^* c_1(\mathcal{L}, \lambda) = i^* c_1(\mathcal{D}, \delta)\,.$$

From the defining property of symplectic reduction, we get

$$\pi_0^* \Omega_{N_\alpha} = i^* \Omega_{\mathcal{A}} \,,$$

where $\Omega_{\mathcal{A}}$ is the Kähler form on $\mathcal{A}$. Therefore to prove the theorem, it is enough to show that

$$c_1(\mathcal{D}, \delta) = r_1 r_2 \cdot \Omega_{N_\alpha} \quad \text{on } \mathcal{A} \,.$$

So let $s = s_1^{r_2} \otimes s_2^{r_1}$ be a local holomorphic section of $\mathcal{D}$ on an open set $U \subset \mathcal{A}$, where s_i is a holomorphic section of $\Theta\,(\mathcal{E}_i)$ on U $(i = 1, 2)$; then we have

$$c_1(\mathcal{D}, \delta) = \frac{-1}{2\pi\sqrt{-1}} \bar{\partial}\partial \log \delta(s, s)$$

on U. On the other hand, from eq. (4c.1), we immediately get

$$\begin{aligned} \frac{-1}{2\pi\sqrt{-1}} \log \delta(s, s) \; &= \frac{-1}{2\pi\sqrt{-1}} [\, -2\pi r_1 r_2 N + r_2 \log \theta_1(s_1, s_1) \\ &\qquad + r_1 \log \theta_2(s_2, s_2)] \,. \end{aligned}$$

Therefore, using a combination of the observation in §4b that the norm square is a potential for the Kähler form, and eq. (4a.2), we obtain

$$\begin{aligned} c_1(\mathcal{D}, \delta) &= -r_1 r_2 \cdot \sqrt{-1}\bar{\partial}\partial N + r_2 \cdot c_1(\Theta\,(\mathcal{E}_1)\,, \theta_1) + r_1 \cdot c_1(\Theta\,(\mathcal{E}_2)\,, \theta_2) \\ &= r_1 r_2 \cdot p_3^* \Omega_3 + r_2 r_1 \cdot p_1^* \Omega_1 + r_1 r_2 \cdot p_2^* \Omega_2 \,, \end{aligned}$$

where Ω_i, $i = 1, 2$, is the Kähler form on $\mathcal{A}_i$, the form Ω_3 is the Kähler form on $C^\infty(\mathcal{H}om(E_2, E_1))$, and p_j, $j = 1, 2, 3$, are the projections from $\mathcal{A}$ to each of its three factors. On the other hand, $\mathcal{A}$ carries the product Kähler structure coming from $\mathcal{A}_1$, $\mathcal{A}_2$ and $C^\infty(\mathcal{H}om(E_2, E_1))$, hence the Kähler form on $\mathcal{A}$ is precisely

$$\Omega_{\mathcal{A}} = p_1^* \Omega_1 + p_2^* \Omega_2 + p_3^* \Omega_3 \,.$$

Thus the Chern form $c_1(\mathcal{D}, \delta)$ coincides with $\Omega_{\mathcal{A}}$ on U. This completes the proof of the theorem.

4d *Concluding remarks*

Let $\mathcal{N}$ be a moduli space of stable vector bundles over a compact connected hyperbolic Riemann surface X. For any positive integer k, consider the space of holomorphic sections of the kth tensor power of the determinant line bundle over $\mathcal{N}$. As the Riemann surface moves in a family, these vector spaces fit together to give a holomorphic vector bundle over the Teichmüller space. Hitchin proved that this vector bundle has a natural projectively flat connection [H]. One of the reasons for the existence of this connection is that the curvature of the Chern connection on the determinant line bundle is a constant scalar multiple of the natural Kähler form on $\mathcal{N}$. Therefore, in view of Theorem 4.1, it would be interesting to know whether the construction of Hitchin can be generalized to get a projectively flat connection on the vector bundle constructed using the holomorphic sections of a tensor power of the line bundle $\mathcal{L}$ over N_α.

Triples can defined in the more general context of parabolic vector bundles. A *parabolic triple* consisting of (E_{2*}, E_{1*}, ϕ_*) where E_{1*} and E_{2*} are parabolic vector bundles, and $\phi_* : E_{2*} \longrightarrow E_{1*}$ is a homomorphism preserving the parabolic structures. The notion of stability can be generalized analogously. Using the method of [BR], Theorem 4.1 can be extended to the more general situation of parabolic stable triples with parabolic structure on a n-pointed Riemann surface. However, in view of [BR], such a generalization is now quite straight-forward, and we leave the details.

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