Infinite number of exponents for a spin-glass transition

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We consider the behavior of the overlap of $m(\geqslant 2)$ paths at the spin-glass transition for a directed polymer in a random medium. We show that an infinite number of exponents is required to describe these overlaps. This is done in an $\epsilon = d-2$ expansion without using the replica trick.

When disorder induces a new thermodynamic phase not found in the pure system, the description of the transition itself poses new challenges. This is exemplified by the efforts made to understand the spin-glass transition.[1,2] Other examples are magnets in random fields,[3] polymers in disordered media,[4–7] or with random interactions,[8] etc. Several concepts emerged from the solution of the infinite range ''mean-field'' spin-glass problem using the replica trick, but their validity for finite dimensional systems, remains to be established.[9,10]

The most important concept in the replica approach is that of the overlap which plays the role of the order parameter for the transition.[1] The overlap purports to characterize the rugged free-energy landscape in the spin-glass phase through the statistics of the pairwise common configurations in the various minima. In principle, a many valley free-energy surface would require higher order overlaps for a more detailed description.[11,12] These are the overlap of say, three or more (m) minima, to be called the ''*m overlap.*'' All of these overlaps vanish at the transition point as $q_m \sim |T_c - T|^{\beta_m}$ for $T \to T_c -$, because the free-energy surface goes over to a smooth one in the high-temperature phase. Now, how many exponents are needed to describe these overlaps? The answer is one for *infinite range or infinite dimensional* models,[1,11] justifying the use of the pair overlap, q_2, as the sole order parameter in the replica approach (with β_2 as the order parameter exponent).[13,14] What about finite dimensional systems? The question assumes importance because the number of independent exponents tells us the number of quantities one requires to characterize the transition. Alas, so little is known of the spin-glass problem.[2]

In this paper, we develop a method to calculate these m overlaps for arbitrary m in an $\epsilon = d-2$ expansion for the spin-glass transition of a directed polymer in a random medium (DPRM) *without using the replica trick.* A $d+1$ dimensional directed polymer (DP) is a string stretched in a preferred direction and with free fluctuations in the transverse d dimensions. In a random medium, the gain in the potential energy from randomness can win over the ''random walk'' entropy, producing a disorder dominated ''super diffusive'' phase.[5–7,15,16] For $d>2$, there is a transition from a low-temperature, strong disorder, spin-glass-type phase to a pure-type phase.[6,17,18] The transition is described by an unstable fixed point $[\sim O(\epsilon)]$ in a renormalization group (RG) approach via the mapping[19] to a nonlinear noisy stochastic equation for the free energy (Kardar-Parisi-Zhang equation).[5,6] The simplicity of the model, nontrivial solutions in many cases,[20,11,12] and the possibility of studying various fundamental questions[21,22] related to disorder systems in general, make the strong disorder phase and the transition a topic of paramount importance.[7] In fact, various techniques have been used for this purpose, as, e.g., Bethe ansatz for $d=1$,[20] dynamic renormalization group,[23,5,6,24] scaling theory,[18] numerical simulations,[25] mode coupling,[26] $1/d$ expansion,[27] on a Cayley tree,[11] on hierarchical lattices,[12] etc. Replica symmetry breaking[28] has been tried by some[15,16,29] and vehemently opposed by a few others.[30,31]

For DP, the overlap (m overlap) describes the fraction of paths common to two (m) minimum (free) energy paths, and is therefore equal to the fraction of the paths two (m) polymers go together when placed in the same random medium.[15,16,11,12,24,32] The two (m) polymers act as the replicas of the one chain system. In the spin-glass phase, if there is only one minimum free energy path, then both the chains would follow the same path, an attraction induced by the disorder. In case there are many valleys, then the chains can get separated by hopping to a neighboring valley. Thus overlaps contain information about the valley structure.

The m overlaps, q_m, have been calculated for a DPRM on a Cayley tree and on hierarchical lattices.[12,11] The Cayley tree problem can be thought of as an infinite dimensional case while the hierarchical lattices are definitely finite dimensional with tunable dimensionality. For the Cayley tree problem, closed form expressions for q_m show that $\beta_m = 1$ for all m.[11] For hierarchical lattices, there is a critical dimension above which a transition takes place.[12] Numerically the transition temperature has been obtained by locating the temperature where q_2 and q_3 vanish. Nothing, unfortunately, is known for β_m. A linear dependence of β_m on m, in a multifractal analysis, would also mean that only one exponent is needed, as, e.g., for pure noninteracting Gaussian chains, (see below). The crucial question is, therefore, whether such linearity is maintained for finite dimensional systems. The answer we find is no.

In this paper, we use the continuum approach. The polymer is described by the Hamiltonian

$$H = \int_0^t d\tau \left[\frac{\gamma}{2} \dot{\mathbf{x}}^2(\tau) + \frac{\lambda}{2\gamma} V[\mathbf{x}(\tau), \tau] \right] \tag{1}$$

where $\mathbf{x}(t)$ is the d-dimensional transverse spatial coordinate of the DP at the contour length t, and $\dot{\mathbf{x}}(\tau) = d\mathbf{x}(\tau)/d\tau$. The

first term on the right hand side represents the entropic fluctuations of a free Gaussian chain with γ as the bare line tension. V corresponds to a space- and time-dependent random potential seen by the chain, and the amplitude $\lambda/2\gamma$ is chosen for convenience. The random potential is taken to be uncorrelated, normally distributed[5] with zero mean, and $\overline{V(\mathbf{x},\tau)V(\mathbf{y},\tau')}=2\Delta\,\delta(\mathbf{x}-\mathbf{y})\delta(\tau-\tau')$, where the overbar stands for disorder averaging.

A formal way to define the m overlap is to put m chains in the system and take

$$q_m=-\frac{1}{t}\int_0^t d\tau\overline{\left\langle\prod_{i=1}^{m-1}\delta[\mathbf{x}_{i,i+1}(\tau)]\right\rangle}, \tag{2}$$

where $\mathbf{x}_{i,i+1}(\tau)=\mathbf{x}_i(\tau)-\mathbf{x}_{i+1}(\tau)$, and $\langle\cdots\rangle$ stands for thermal average for a realization. One way of computing q_m is to couple the m chains or replicas with a weak m-body interaction. Then overlap would follow from an appropriate derivative of the total free energy with respect to the hypothetical coupling constant (see below). This procedure was adopted for the overlap (q_2) in the numerical work of Mezard in 1+1 dimensions[15] and by one of us in a one-loop RG approach[24] (see also Ref. 32). We generalize the method of Ref. 24 for q_m. The RG analysis is geared towards calculating the scaling exponent of the coupling constant. A judicious use of finite-size scaling, as explained below, then gives us the exponent β_m.

With the definition of the m overlap in Eq. 2, we consider an m chain interacting Hamiltonian

$$\mathscr{H}_m=\sum_{i=1}^{m}H_i+(\lambda/2\gamma)v_m\int_0^t d\tau\prod_{i=1}^{m-1}\delta[\mathbf{x}_{i,i+1}(\tau)], \tag{3}$$

where H_i is the Hamiltonian of Eq. 1 for the ith polymer. Defining the quenched free energy $f_m(v_m,t)=\overline{\ln Z_m}$ where Z_m is the partition function for $\mathscr{H}_m$, the m overlap is obtained as $q_m=-t^{-1}\,df_m(v_m,t)/dv_m|_{v_m=0}$.

Our interest is in the scaling part of the free energy, $f_m\approx t^{\chi/z}f(v_m t^{-\phi_m/z})$, where ϕ_m is the scaling exponent for v_m, and χ and z are the single-chain-free energy fluctuation ($\Delta f\sim t^{\chi/z}$) and dynamic (size $x\sim t^{1/z}$) exponents.[24,5] We have verified that, as expected, there is no change in the single-chain exponent. Taking derivative then gives $q_m\sim t^{\Sigma_m}$, with $\Sigma_m=(\chi-\phi_m-z)/z$. Incidentally, the system is taken to be infinite in extent in all the d transverse directions and is of length t in the preferred direction. This form of q_m can, therefore, be treated as a finite-size scaling form.[33] Now, the transition takes place only in the thermodynamic limit of $t\to\infty$. In that limit, for $d>2$, there is a diverging length scale with exponent ν, $\xi_\parallel\sim|T-T_c|^{-\nu}$, parallel to the special t-like direction.[5,17] Finite-size scaling suggests a scaling form $q_m=t^{-\beta_m/\nu}g(t/\xi_\parallel)$. Therefore, right at the critical point (i.e., the unstable fixed point in RG), $q_m\sim t^{-\beta_m/\nu}$. A comparison then yields $\beta_m=-\nu\Sigma_m$. Remember, that ν is strictly m independent. Our strategy is therefore to calculate ϕ_m.

Define $h(\{\mathbf{x}_j\},t)=(2\gamma/\lambda)\ln Z(\{\mathbf{x}_j\},t)$, where $Z(\{\mathbf{x}_j\},t)$ is the partition function for chains with end points at $\{\mathbf{x}_j\}$, all starting at the origin. This h satisfies the equation,[24]

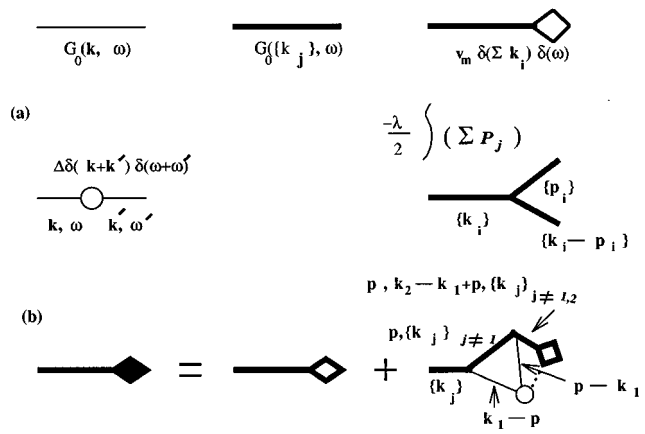

FIG. 1. Diagrammatic representation of the parameters (a) and the solution (b). [See also Figs. 1 and 2 of Ref. 5(b).] The solid square represents the vertex function which for zero external momenta gives v_{mR}. The dotted line in the series for v_m is a dummy line signifying loop closing. This line indicates that two different indices are coupled by the dummy momentum p. There are two factors in (b). (i) A combinatorial factor 8 from the insertions of λ vertices and the subsequent noise contraction and (ii) $\binom{m}{2}$ for choosing the wave vectors.

$$\frac{\partial}{\partial t}h=\sum_{j=1}^{m}\left[\gamma\nabla_j^2 h+\frac{\lambda}{2}(\nabla_j h)^2\right]+g_0, \tag{4}$$

where $g_0=\Sigma_{j=1}^{m}V(\mathbf{x}_j,t)+v_m\Pi_{j=1}^{m-1}\delta[\mathbf{x}_{j,j+1}(\tau)]$. This equation, though resembling a higher (md) dimensional KPZ equation,[5] is actually not so for the peculiar noise term. We prefer this equation to Eqs. (1) or (3) because an RG can be implemented with the nonlinear term λ as perturbation. This is different from a perturbation in the random potential around the Gaussian chains. Moreover, the equation describes the free energy and averaging h will naturally give the quenched average free energy without any recourse to the replica trick.

A scaling of $\mathbf{x}\to b\mathbf{x}, t\to b^z t$, then shows that, in the absence of nonlinearity λ, $v_m\to b^{z-(m-1)d-\chi}v_m$. An anomalous part η_m would creep in through renormalization when the nonlinearity λ is present. The crossover exponent is therefore $\phi_m=-[z-(m-1)d-\chi+\eta_m]$. This gives

$$\beta_m=\nu[d(m-1)+\eta_m]/z. \tag{5}$$

The m dependence, apart from the Gaussian one, therefore comes from η_m. At the Gaussian level the exponents depend linearly on m.

The formal solution of Eq. (4), in the Fourier space $(\mathbf{k}_j,\omega)$ conjugate to $(\mathbf{x}_j,t)$, can be written as

$$h(\{\mathbf{k}_j\},\omega)=G_0(\{\mathbf{k}_j\},\omega)\tilde{g}_0-(\lambda/2)G_0(\{\mathbf{k}_j\},\omega)\times\int_{\{\cdot\}}\left(\sum_{j=1}^{m}\mathscr{P}_j\right)h(\{\mathbf{p}_j\},\Omega)h(\{\mathbf{k}_j-\mathbf{p}_j\},\omega-\Omega), \tag{6}$$

where $G_0(\{\mathbf{k}_j\},\omega)=[\gamma\Sigma_j k_j^2-i\omega]^{-1}$ represents the bare m particle propagator (Green's function) (see Fig. 1), and $\tilde{g}_0$ is the Fourier transform of g_0. A shorthand notation is used,

viz., $\mathscr{P}_j = \mathbf{p}_j \cdot (\mathbf{k}_j - \mathbf{p}_j)$, $\int_{\{\cdot\}} = (2\pi)^{-md-1} \int d\Omega \prod_{i=1}^{m} d\mathbf{p}_i$, and $\{\mathbf{k}_j\}$ to represent all the m $\mathbf{k}$ vectors. To tame possible divergences, we put an upper cutoff Λ $(=1)$ for all p integrals. This cutoff actually comes from a short distance cutoff in real space.

We now use a dynamic renormalization group approach to determine the behavior of v_m for large length scales by integrating out fluctuations on smaller scales. This is based on Eq. (6). The procedure is well documented, especially for the KPZ equation[5] and for the $m=2$ case of Eq. (4).[24] The basic idea is to (i) integrate out the fluctuations at the shortest scale by taking a slice $\Lambda e^{-\delta l} \leqslant p \leqslant \Lambda$ from the momentum integral, (ii) absorb it in the coupling constant, and (iii) rescale all the momenta, etc., to get back the original cutoff Λ. The resulting changes are then absorbed by renormalizing the parameters of the problem. In the limit $\delta l \to 0$, these changes are expressed in terms of differential equations (recursion relations) that tell us the flow of the parameters as we go to longer length scales. Special care is needed, for the problem at hand, to keep track of the momenta indices that get coupled by the noise. These interchain connections produce the necessary anomalous part in the RG equation. We skip the algebraic details.

The flow of the disorder is described in terms of the dimensionless coupling $U = K_d \lambda^2 \Delta / (2\gamma^3)$, where $K_d = (2\pi)^{-d} S_d$, S_d being the surface area of the unit d-dimensional sphere, and can be found in Ref. 5. We have verified that this single chain equation is recovered from Eq. (6), and is independent of m. Let us recapitulate that at $d=2$, U is marginally relevant. This leads to a new critical point for $d>2$. For $d=2+\epsilon$, the unstable fixed point $U^* = 2\epsilon$ corresponds to the spin-glass transition point, with $\nu = 2/(d-2)$ as the length scale exponent.[34]

We concentrate on the renormalization of the coupling v_m. For the long wavelength, long time limit, the external wavevectors and frequency are small or zero. In this limit, for arbitrary m, the effective coupling constant to one-loop order (see Fig. 1) is given by (suppressing the zero wave vectors)

$$v_{mR} = v_m + 8 \binom{m}{2} \left(-\frac{\lambda}{2} \right)^2 (2 v_m \Delta) \times \int_{p,\Omega} p^4 G_0(\mathbf{p},\Omega) [G_0(-\mathbf{p},-\Omega)]^2 G_0(\mathbf{p},-\mathbf{p},0). \tag{7}$$

The recursion relation for v_m follows from Eq. (7) as

$$\frac{dv_m}{dl} = \left[z - \chi - (m-1)d + \binom{m}{2} U \right] v_m . \tag{8}$$

Since our interest is in the crossover exponent for v_m at $v_m = 0$, this first order (in v_m), one-loop equation is sufficient for us. Higher loops will generate higher order terms in U (and hence ϵ).

Using the one-loop fixed point value $U^* = 2\epsilon$ for the transition point, we find $\eta_m = -m(m-1)\epsilon$, which from Eq. (5) gives

$$\beta_m = \nu \zeta_c [2(m-1) - (m-1)^2 \epsilon + O(\epsilon^2)], \tag{9}$$

where $\zeta_c = 1/z_c$ is the size exponent at the transition point. We now see that the linear relation between β_m and m at the Gaussian level is not respected in the first order. In other words the exponents β_m are not linearly dependent on each other, and higher order terms are expected to make the interdependence more complicated. Hence the need for an infinite number of exponents at the transition point.

From the nature of the perturbation series, we see that the change in exponent to first order in ϵ is due to the effective two-body interaction induced by the disorder. The loop in Fig. 1 and in Eq. (7) comes when disorder couples two different chains. With δ-correlated noise, this means that the two chains are going through the same point in space. The purpose of overlap is to count these. It is well known that two-body interaction changes the reunion behavior of m chains, and each m requires a new exponent for reunion.[35,36] This is the situation here.

Let us now try to connect this result to a replica analysis. We take n chains in the random medium and average the resulting partition function, or equivalently, get the effective Hamiltonian for the nth moment of the partition function. The effect of the disorder is to couple these chains through a two-body interaction.[37] The m overlap, then corresponds to the reunion of m chains out of these m, in the limit $n \to 0$ (the replica trick). At the critical point, taking the chains to be Gaussian (since $\zeta_c = 1/z_c = 1/2$),[18,6] the reunion of a subset of interacting random walkers can be studied following Ref. 36. The only difference with Ref. 36 is that the relevant fixed point is the unstable one, and the $n \to 0$ limit can be taken [see, e.g., Ref. 36(b)] to get the anomalous part of Eq. (9). This is essentially correct but it still needs to be established that the chains are actually Gaussian (not just $\zeta_c = 1/2$). These problems are not present in the differential equation approach used in this paper.

What do all of these mean for the spin-glass transition in finite dimensions? One expects to write down (in the $n \to 0$ limit) a Landau-Ginzburg type free-energy functional with the overlaps as the order parameters. There can be two possibilities. (i) One is that the simple minded description through the overlap of two copies is not sufficient, and one has to worry about the higher overlaps, and in fact an infinite number of them. Even if one starts with the Gaussian distribution, renormalization effects will generate the higher overlaps (arbitrary distributions generate then in any case). In this situation, the conventional replica approach may not be useful. The problem here again may be the interchange of the two limits, viz, $n \to 0$ and the thermodynamic limit. (ii) The other option is that the two copy overlap is good enough in the sense that higher m overlaps are irrelevant. Naively speaking, at the transition point with $z=2$, $\chi=0$, v_m is irrelevant at $d=2$ for $m>2$. However, this does not necessarily imply irrelevance of the m overlaps in the single-chain problem. Remember, that v_m is a coupling introduced by hand in the many chain Hamiltonian in Eq. (3) to calculate the overlaps and *does not* appear in the description of the single-chain problem. This opens up the possibility where under special conditions higher order overlaps can become important as in multicritical cases or polymers with higher order composite operators becoming relevant. Proper choice of parameters can then lead to multicritical analogs of spin glasses. These are not the random version of pure multicriti-

cal models but are of inherently different type. In both cases, if the m overlaps are important at the transition, they are expected to be so in the spin-glass phase also. We hope that this will motivate further detailed numerical work to settle this issue. We conclude that, like multifractals, a spin-glass transition in finite dimensions subsumes an infinite number of independent exponents. This indicates either a failure of the simple replica picture in finite dimensions or the possibility of highly complex spin-glass phases.

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