Computation of Kolmogorov's Constant in Magnetohydrodynamic Turbulence

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Abstract

In this paper we calculate Kolmogorov's constant for magnetohydrodynamic turbulence to one loop order in perturbation theory using the direct interaction approximation technique of Kraichnan. We have computed the constants for various $E^u(k)/E^b(k)$, i.e., fluid to magnetic energy ratios when the normalized cross helicity is zero. We find that K increases from 1.47 to 4.12 as we go from fully fluid case ($E^b = 0$) to a situation when $E^u/E^b = 0.5$, then it decreases to 3.55 in a fully magnetic limit ($E^u = 0$). When $E^u/E^b = 1$, we find that $K = 3.43$.

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Kolmogorov [1] hypothesized that the energy spectrum $E(k)$ of fluid turbulence in the inertial range (i.e., length scale intermediate between the energy feeding and energy damping ranges) is a power law with a spectral index of $-5/3$, i.e.,

$$E(k) = K_{Ko}\Pi^{2/3}k^{-5/3}, \tag{1}$$

where K_{Ko} is an universal constant called Kolmogorov's constant, k is the wavenumber, and Π is the nonlinear cascade of energy which is also equal to the dissipation rate of the fluid.

The calculations based on Direct interaction approximations [2, 3], renormalization group technique [4], self-consistent mode coupling [5] etc. show that $K_{Ko} \approx 1.5$ in three dimensions (3D). However, in two dimensions (2D) Kraichnan [2], Olla [6], and Nandy and Bhattacharjee [7] show that $K_{Ko} \approx 6.4$ in the region where inverse cascade of energy occurs. Experiments and numerical simulations [8] yield constants approximately same as those predicted by the above calculations.

Magnetohydrodynamic (MHD) turbulence is more complex than the fluid turbulence. In MHD there are velocity field $\mathbf{u}$ and magnetic field $\mathbf{B} = \mathbf{B_0} + \mathbf{b}$, where $\mathbf{B_0}$ is the mean magnetic field and $\mathbf{b}$ is the fluctuation in the magnetic field. It is customary to use Elsässer variables $\mathbf{z}^{\pm} = \mathbf{u} \pm \mathbf{b}/\sqrt{\mathbf{4}\pi\rho}$ that can be interpreted as Alfvén waves having positive and negative velocity and magnetic field correlations respectively. Here ρ is the density of the magnetofluid. In this paper we assume incompressible approximation which in most of the situations correspond to a constant density of the fluid. Matthaeus and Zhou, Zhou and Matthaeus, and Marsch [9] constructed a phenomenology when $z^{\pm} \gg B_0/\sqrt{4\pi\rho}$, where B_0 is the mean magnetic field or the magnetic field of the largest eddies. In this phenomenology, the energy spectra $E^{\pm}(k)$ of $z^{\pm}$ fluctuations are proportional to $k^{-5/3}$ and

$$E^{\pm}(k) = K^{\pm}(\Pi^{\pm})^{4/3}(\Pi^{\mp})^{-2/3}k^{-5/3}, \tag{2}$$

where $K^{\pm}$ are constants, which we will refer to as Kolmogorov's constants for MHD turbulence, and $\Pi^{\pm}(k)$ are the nonlinear cascade rates of $z^{\pm}$ fluctuations respectively. The normalized cross helicity σ_c, (strictly speaking spectral normalized cross helicity) defined as $(E^+(k) - E^-(k))/(E^+(k) + E^-(k))$, play an important role in MHD turbulence. In this paper, however, we restrict ourselves to zero normalized cross helicity where, by symmetry, $\Pi^+ = \Pi^-$ and $K^+ = K^- = K$. This choice of σ_c is motivated by the simplicity of the calculation and by the fact that in the solar wind most of the fluctuations in the outer heliosphere (beyond earth) and some of them in the inner heliosphere (from sun to earth) have negligible σ_c. There is another important energy called residual energy E^R defined as the difference between the kinetic (E^u) and magnetic (E^b) energies. Note, however, that the residual energy is not conserved while both E^+ and E^- are conserved.

In the other limit when $z^{\pm} \ll B_0/\sqrt{4\pi\rho}$, Kraichnan and Dobrowolny et al. [10] argued that

$$\Pi^+ = \Pi^- = \frac{1}{A^2} E^+(k) E^-(k) k^3, \tag{3}$$

where A is a constant called Kraichnan's constant.

The solar wind observations indicate that the it exhibits MHD turbulence. The energy spectra of the solar wind has been found to be closer to $k^{-5/3}$ than to $k^{-3/2}$ [11]. However, it should be noted that these exponents are difficult to distinguish. Recently we have performed direct numerical simulations of MHD turbulence [12] in that we could not conclude whether the spectral index was 5/3 or 3/2. However, the nonlinear energy cascade rates $\Pi^{\pm}$ appear to follow Eq. (2) rather than Eq. (3). Motivated by the above mentioned solar wind observations, in this paper we assume that the energy spectra $E^{\pm}(k)$ obey Kolmogorov-like phenomenology [Eq. (2)] for $z^{\pm} > B_0$.

We determine for the first time the constant K associated with the Kolmogorov spectrum in MHD turbulence. To this end we attempt to find K of Eq. (2) using perturbative technique similar to DIA (see Fournier et al. and Camargo and Tasso [13] for application of Renormalization Group techniques to MHD). In this paper we only discuss the situations when $E^+(k) = E^-(k)$ and $B_0 = 0$. However, we vary E^u/E^b from 0 to ∞, i.e., from magnetic case to fully fluid case. The computation of $K^{\pm}$ when $E^-(k)/E^+(k)$ is different from unity and when B_0 is nonzero is under progress; forthcoming results will be presented elsewhere.

The MHD equations in terms of $z^{\pm}$, in absence of B_0, are given by [10]

$$\begin{array}{c} \frac{d}{dt} z_i^{\pm}(\mathbf{k},t) + \nu_+ k^2 z_i^{\pm}(\mathbf{k},t) + \nu_- k^2 z_i^{\mp}(\mathbf{k},t) = f_i^{\pm} \\ +\epsilon M_{ijm}(\mathbf{k}) \int d\mathbf{p} z_j^{\mp}(\mathbf{p},t) z_m^{\pm}(\mathbf{k}-\mathbf{p},t) \end{array} \tag{4}$$

where

$$M_{ijm}(\mathbf{k}) = k_j P_{im}(\mathbf{k});\ P_{im}(\mathbf{k}) = \delta_{im} - \frac{k_i k_m}{k^2};\ \nu_{\pm} = \frac{\nu \pm c^2/(4\pi\sigma)}{2}, \tag{5}$$

ν is kinematic viscosity, c is the speed of light, σ is conductivity, $f^{\pm}$ are forcing, and ϵ is the expansion parameter which is set to one at the end. We assume independent forcing of $z^{\pm}$, i.e., $\partial f^{s_1}/\partial f^{s_2} = \delta^{s_1 s_2}$, where $s_1, s_2 = \pm$. We solve for self-energy $\hat{\Sigma}$ (a 2×2 matrix), which is defined as

$$\hat{G} = \hat{G}_0 - \hat{G}_0 \hat{\Sigma} \hat{G}, \tag{6}$$

to first order using the technique of Wyld [14] and Leslie [3] (the details of the calculation are be presented in [15]). In our calculation we postulate the frequency dependence of the $z - z$ correlations functions as [3]:

$$C^{\pm\pm}(k,\omega) = \frac{1}{2\pi} \frac{C^{\pm}(k)}{(i\omega - \eta_k^{\pm})}; C^{\pm\mp}(k,\omega) = \frac{1}{2\pi} \frac{C^{R}(k)}{(i\omega - \frac{\eta_k^{+}+\eta_k^{-}}{2})}, \tag{7}$$

where η_k^+ and η_k^- are the reciprocal of the mean response time of z_k^+ and z_k^- fluctuations respectively. The self-energy can be written as a following 2×2 matrix:

$$\hat{\Sigma} = \begin{pmatrix} \Sigma^{++} & \Sigma^{+-} \\ \Sigma^{-+} & \Sigma^{--} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \eta^{+} & \eta^{-} \\ \eta^{+} & \eta^{-} \end{pmatrix}, \tag{8}$$

$$\begin{aligned} \eta^{\pm}(k) = & \lim_{\omega \to 0} \frac{1}{2} k^2 \int \int d\mathbf{p} d\omega' \times \\ & [b_1(k,p,q)(G^{\mp\pm}(\mathbf{p},\omega') + G^{\mp\mp}(\mathbf{p},\omega'))C^R(\mathbf{k}-\mathbf{p},\omega-\omega') \\ & + b_2(k,p,q)(G^{\pm\mp}(\mathbf{p},\omega') + G^{\pm\pm}(\mathbf{p},\omega'))C^{\mp}(\mathbf{k}-\mathbf{p},\omega-\omega') \\ & + b_3(k,p,q)(G^{\pm\pm}(\mathbf{p},\omega') + G^{\pm\mp}(\mathbf{p},\omega'))C^{\mp}(\mathbf{k}-\mathbf{p},\omega-\omega') \\ & + b_4(k,p,q)(G^{\mp\pm}(\mathbf{p},\omega') + G^{\mp\mp}(\mathbf{p},\omega'))C^R(\mathbf{k}-\mathbf{p},\omega-\omega')] \end{aligned} \tag{9}$$

where G is the Green's function, $b_i(k,p,q) = k^{-2} B_i(k,p,q)$, and B_i's are the functions of (x,y,z), the cosines of the angles between (**p**,**q**), (**q**,**k**), and (**k**,**p**) respectively; B_i's given in the appendix of Leslie [3]. Substitution of $\hat{\Sigma}$ of (8) in Eq. (6) and ignoring kinematic viscosity and resistivity as compared to $\hat{\Sigma}/k^2$ yields $\hat{G}$

$$\hat{G}(k,\omega) = \frac{1}{-\omega\left(\omega + i\frac{\eta^{+}+\eta^{-}}{2}\right)} \begin{pmatrix} -i\omega + \frac{\eta^{+}}{2} & \frac{\eta^{-}}{2} \\ \frac{\eta^{-}}{2} & -i\omega + \frac{\eta^{+}}{2} \end{pmatrix}$$

Note that when $C^{+}(k) = C^{-}(k) = C^{R}(k)$, i.e., when the magnetic energy is zero, the above equations yield $\eta^{+} = \eta^{-} = \eta_{fluid}$.

As mentioned earlier, in this paper we restrict ourselves to situations when $C^+(k) = C^-(k) = C(k)$ and assume the Kolmogorov scaling law, i.e., $E(k)$ are proportional to $k^{-5/3}$. In this situation $\eta_k^+ = \eta_k^-$ by symmetry. Since the energy spectra $E(k)$, which is equal to $C(k)/(4\pi k^2)$, is proportional to $k^{-5/3}$, the correlation functions $C^\pm(k)$ and self-energies will be [3]

$$C(k) = \frac{1}{4\pi} K \Pi^{2/3} k^{-11/3}; \;\; \eta^\pm(k) = \Lambda \Pi^{1/3} k^{2/3}, \tag{10}$$

where Λ is a constant. We denote the ratio C^R/C by α. Using the above quantities and with the change of variables $p = \zeta k$ and $q = \kappa k$ we obtain

$$\begin{aligned} \frac{\Lambda^2}{K} &= \frac{1}{4} \int\int d\zeta d\kappa \zeta \frac{\kappa^{-11/3}}{(\zeta^{2/3} + \kappa^{2/3})} k^{-2} \\ &\quad \times \left[\alpha(b_1(k,p,q) + b_4(k,p,q)) + b_2(k,p,q) + b_3(k,p,q)\right]. \end{aligned} \tag{11}$$

The integrals of Eq. (12) suffer from the well known "infrared problem" which comes from the strong dynamic coupling of fluctuations with widely differing wavenumbers. Over the years, various techniques have been developed to tackle the difficulty in the context of pure fluid turbulence. The earlier methods (essentially introduction of a cut off) are discussed in Leslie[3]. Later work involving either Renormalization group technique [4], or the Lagrangian or semi-Lagrangian pictures [16], or self-consistent screening [5] shows how the theory can be made naturally finite and cut off independent. The infrared difficulties associated with Eq. (12) can be similarly avoided. Knowing that the full theory is constrained to be finite, we adopt the practical procedure of evaluating the integral in Eq. (12) with a cut off $k_0 = \lambda k$ and choosing $\lambda = 1$ so that the pure fluid value of Λ^2/K are obtained correctly when $\alpha = 1$. Thereafter λ is not varied. With this procedure we obtain Λ^2/K as a function of α. For some of the characteristic values of α, Λ^2/K is listed in Table 1.

To obtain the numerical value of K we need another equation involving Λ and K. To this end, we derive an expression for energy cascade rate Π in terms of correlations $C(k)$ and Green's functions $G(k)$. Following an approach similar to that of Leslie [3], when $C^+(k) = C^-(k) = C(k)$, we obtain

$$\left(\frac{d}{dt} + 2\nu_+ k^2\right) E(k) + 2\nu_- k^2 E^R(k) = T(k,t). \tag{12}$$

Using $T(k,t)$ we can write an expression for energy cascade rate, which is $\Pi = -\int_0^k dk' T(k',t)$. Substitution of C's and η's, and change of variables $k' = k/u, p = v(k/u)$, and $q = w(k/u)$ yield an expression for Λ/K^2, that is,

$$\frac{\Lambda}{K^2} = \int_0^1 dv \ln\frac{1}{v} \int_{v^*}^{1+v} dw[\Psi(1,v,w) + \Psi(1,w,v)] \tag{13}$$

where

$$\begin{aligned} \Psi(1,v,w) &= \frac{1}{4} vw(1 + v^{2/3} + w^{2/3})^{-1} \\ &\times [b_1(1,v,w)w^{-11/3}((1+\alpha^2)v^{-11/3} - 2\alpha) \\ &+ (1+\alpha^2)b_2(1,w,v)v^{-11/3}(w^{-11/3} - 1) \\ &- 2\alpha b_3(1,v,w)w^{-11/3} - (1+\alpha^2)b_4(1,v,w)v^{-11/3}], \end{aligned} \tag{14}$$

and $v^* = \max(v, |1-v|)$. See Table 1 for Λ/K^2 for various values of α.

We calculate K from Λ^2/K and Λ/K^2. The values of K for various values of α are listed in Table 1. We have also calculated the constant K in 2D. In 2D when $E^u = E^b$, we choose $\lambda = 1$, same as λ of 3D, but not 0.065 which yields $K = 6.6$ in fluid limit. The choice of same λ in 2D MHD was motivated by fact that in MHD turbulence, forward cascade of energy and inverse cascade of cross helicity occur in both 2D and 3D[17]. Note that the behaviour of fluids turbulence in 2D and in 3D are dramatically different.

The results presented in this paper are in good agreement with the simulation results of Verma et al. [12] ($K = 3.7$ in three-dimensions for $E^-/E^+ = 0.6$, and average $K = 6.6$ in two-dimensions for $E^-/E^+ = 1, 0.6$). In 3D simulation, however, only a single run with a relatively lower resolution (128^3) was performed. We need to perform more runs to come to a definite conclusion.

We find that K monotonically increases from 1.47 to 4.12 as we go from fully fluid case ($E^R = E$) to a situation when $E^R/E = -1/3$, i.e., $E^u/E^b = 0.5$, then it decreases and finally reaches 3.55 in fully magnetic limit ($E^R = -E$). The maxima of K occurring for $E^u/E^b = 0.5$ is a curious result because average E^u/E^b for the solar wind in the outer heliosphere, where $E^+ \approx E^-$ is a good approximation, is 0.5. Note that maximum K corresponds to minimum energy dissipation, hence it appears as if the solar wind is settling to a minimum dissipation state. Further investigation in this direction could possibly yield interesting results regarding minimum dissipation states.

Recent temperature evolution studies of solar wind plasma show that turbulent heating in the solar wind can account for the observed temperature evolution when $K = 1.0$ for nonAlfvénic streams (streams for which $E^+ \approx E^-$) [18]. In the inner heliosphere, where the solar wind is fluid dominated, the turbulent heating is possibly the major heating mechanism. However, in the outer heliosphere, since K is in the range of 3 to 4, turbulent heating could account for only 20-25 % of the total dissipation; rest of the dissipation should be provided by other sources, e.g., shock heating, stream-stream interactions etc.

To conclude, we have calculated the Kolmogorov's constant for MHD turbulence for zero σ_c to one loop order using perturbative technique similar to the direct interaction approximation of Kraichnan. We find that in fluid dominated case, $K \approx 1.5-2$, but as magnetic energy increases, K increases as well and reaches a maximum value of 4.12 when $E^b = 2E^u$, then it decreases and reaches 3.55 for a fully magnetic case. These results are consistent with the recently performed simulation results and also shed light on the evolution of E^u/E^b ratio and temperature in the solar wind.

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References

[1] A. N. Kolmogorov, C. R. Acad. Sci. URSS, **30**, 301, 1941.

[2] R. H. Kraichnan, J. Fluid Mech., **47**, 525, 1971.

[3] D. C. Leslie, *Developments in the theory of Turbulence*, Claredon, Oxford, 1973.

[4] V. Yakhot and S. A. Orszag, J. Sci. Comput., **1**, 3, 1986; D. Ronis, Phys. Rev. A, **36**, 3322, 1987; D. Forster, D. Nelson, and M. Stephen, Phys. Rev. A, **16**, 732, 1977; For recent calculations and reviews refer to articles in *Transition, Turbulence, and Combustion*, ed. M. Y. Hussaini et al., Kluwer, 1994.

[5] J. K. Bhattacharjee, Phys. Fluids, **A 3**, 879, 1991; V. I. Belinicher and V. S. L'vov, Sov. Phys. JETP, **66**, 303, 1987.

[6] P. Olla, Phys. Rev. Lett., **67**, 2465, 1991.

[7] M. K. Nandy and J. K. Bhattacharjee, in press, Int. J. Mod. Phys., 1995.

[8] H. L. Grant, R. W. Stewart, and A. Molliet, J. fluid Mech., **12**, 241, 1962; U. Frisch and P. Sulem, Phys. Fluids, **27**, 1921, 1984.

[9] W. H. Matthaeus and Y. Zhou, Phys. Fluids, **B1**, 1929, 1989; Y. Zhou and W. H. Matthaeus, J. Geophys. Res., **95**, 14881, 1990; E. Marsch, in *Reviews of Modern Astronomy, Vol. 4*, ed. G. Klare, Springer-Verlog, 1990.

[10] R. H. Kraichnan, Phys. Fluids, **8**, 1385, 1965; M. Dobrowolny, A. Mangeney, and P. Veltri, Phys. Rev. Lett., **45**, 144, (1980).

[11] W. H. Matthaeus and M. L. Goldstein, J. Geophys. Res., **87**, 6011, 1982; E. Marsch and C.-Y. Tu, J. Geophys. Res., **95**, 8211, 1990.

[12] M. K. Verma, Ph. D. thesis, University of Maryland, College Park, 1994; M. K. Verma, D. A. Roberts, M. L. Goldstein, S. Ghosh, and W. T. Stribling, preprint, 1995.

[13] J. D. Fourrnier, P. L. Sulem, and A. Pouquet, J. Phys. A, **15**, 1393, 1982; S. J. Camargo and H. Tasso, Phys. Fluids B, **4**(5), 1199, 1992.

[14] H. W. Wyld, Ann. Phys. (NY), **14**, 143, 1961.

[15] M. K. Verma and J. K. Bhattacharjee, to be sent to Phys. Rev. E, 1995

[16] R. H. Kraichnan, Phys. Fluids, **8**, 575, 1965.

[17] A. C. Ting, W. H. Matthaeus, and D. Montgomery, Phys. Fluids, **29**, 9695, 1986.

[18] M. K. Verma, in press, J. Geophys. Res., 1995.

E^R/E	E^u/E^b	Λ^2/K	Λ/K^2	K
1	∞	0.114	0.190	1.47
0	1	0.176	0.066	3.43
$-1/3$	$1/2$	0.197	0.053	4.12
$-2/3$	$1/5$	0.218	0.056	4.11
-1	0	0.239	0.073	3.55
0^{2D}	1	0.070	0.021	5.41

Table 1: Kolmogorov's constant K of MHD turbulence for various values of $\alpha = E^R/E$ (assuming $E^+ = E^-$). The symbols E, E^R, E^u, E^b denote the total energy, residual energy, kinetic and magnetic energy respectively.