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From the Mathematics Olympiad to the Maximum Principle

The following problem is from the Math Olympiad collection [1].

Problem A In a chess board with 64 squares, if a set of 64 numbers are placed one on each square satisfying the condition that the value on any square equals the average of its neighbours, then show that all the numbers are the same.

The following arises in many areas of science and engineering.

Maximum Principle Let $f(x, y)$ be a continuous function on the unit disc $D \equiv \{(x, y) : x^2 + y^2 \leq 1\}$ on R^2 that satisfies the *mean-value property* (MVP): for all (x_0, y_0) in the interior D^0 of D, i.e. $x_0^2 + y_0^2 < 1$,

$$f(x_0, y_0) = \frac{1}{\pi r^2} \int_{B((x_0,y_0),r)} f(x, y)\, dx\, dy$$

where $B((x_0, y_0), r) \equiv \{(x, y) : (x - x_0)^2 + (y - y_0)^2 \leq r\}$ is a disc of radius r with center at (x_0, y_0) and $r > 0$ is such that $B((x_0, y_0), r) \subset D^0$. Then f achieves its maximum only on $\{(x, y) : x^2 + y^2 = 1\}$, the boundary of D, unless it is a constant.

It will be shown in this note that *Problem A* leads to the *Maximum principle* in a natural way.

Suppose you have solved this problem A. What are some generalisations of this problem? To find generalisations of a problem or a result it is good to have a solution to the original problem. Then one can see what parts of the hypotheses are being used, how they can be relaxed or changed a little such that the original method still works. One can also see if the original method can be modified to allow for different sets of hypotheses.

Keywords
Maximum principle, mean-value property, stochastic matrix.

Here are a few generalisations of Problem A that leads to the Maximum Principle. Hints for solutions are given at the end. You are encouraged to try to solve the problems and consult the hints only as a last resort.

Problem 2 has a socioeconomic interpretation. In a community of connected households if the income of every household is no larger than the average of its neighbors then all households must have the same income. That is, local socialism implies global socialism in a connected world.

Problem 1. Suppose in Problem A the value at each square is less than or equal to the average of the values of its neighbours. Then all the values are the same.

Problem 2. Let S be a finite set and $f : S \to R$ be a real valued function. Suppose for each s in S there is a neighbourhood N_s such that $f(s)$ is less than or equal to the average of the values of f over N_s (it is assumed that s is not included in N_s). Suppose further that for any s_1 and s_2 in S there exist $t_1, t_2, \ldots, t_k$ in S such that $t_1 \in N_{s_1}, t_2 \in N_{t_1}, \ldots, t_k \in N_{t_{k-1}}, s_2 \in N_{t_k}$. Then show that f is constant over S.

Problem 3. Let S be a finite or countably infinite set identified by $S \equiv \{1, 2, \ldots, k\}, k \leq \infty$. Let $P \equiv ((p_{ij}))$ be a stochastic matrix, i.e.,

i) for each $i, j, p_{ij} \geq 0$;

ii) for each i, $\sum_{j=1}^{k} p_{ij} = 1$.

Let $f : S \to R^+$ be such that

i) for each i

$$f(i) \leq \sum_{j=1}^{k} p_{ij} f(j);$$

ii) for some j_0 in S

$$f(j_0) \geq f(i) \text{ for all } i \text{ in } S.$$

Let P satisfy the 'irreducibility' condition:

For any i, j *in* S, $i \neq j$, *there exists a finite set* $\{i_1, i_2, \ldots i_n\} \subset S$ *such that*

$$p_{ii_1} p_{i_1 i_2} \ldots p_{i_{n-1} i_n} > 0.$$

Then show that $f(j) = f(j_0)$ for all j in S. Deduce that any $f : S \to R^+$ satisfying (i) attains its maximum in S iff it is a constant (provided the irreducibility condition holds).

Problem 4. Let $S = (a, b)$ be an open interval in R. Let $f : S \to R$ be a continuous function. Let $p : S \times S \to R^+ = (0, \infty)$ be a continuous function such that for each x in S, $\int p(x, y)dy = 1$. Suppose that

i) for each x in (a, b), $f(x) \leq \int_a^b f(y)\ p(x, y)dy$;

ii) for some x_0 in S, $f(x_0) \geq f(y)$ for all y in S. Then show that $f(x) = f(x_0)$ for all x in S.

Problem 5. Let S be an open connected set in some Euclidean space R^k. Let $f : S \to R$ be a continuous function. Suppose f has the *super mean-value property*; for each x in S, $f(x) \leq \frac{1}{V_r} \int_{B(x,r)} f(y)\ dy$ where $B(x, r) \equiv \{y : d(x, y) \leq r\}$ is the ball of radius r with center at $x, d(x, y)$ being the Euclidean distance between x and y and $r > 0$ is such that $B(x, r) \subset S$, and V_r is the k-dimensional volume of $B(x, r)$. Suppose, further, that for some x_0 in S, $f(x_0) \geq f(y)$ for all y in S. Then show that $f(x) = f(x_0)$ for all x in S.

Problem 6. Establish the Maximum Principle, stated earlier.

Problem 7. Let f have the MVP on all of R^2. Suppose f is bounded above and attains the upper-bound at some θ in R^2. Show that f is a constant.

Problem 8. Let S be an open set in some Euclidean space R^k. Let $p : S \times S \to [0, \infty]$ be such that i) it is continuous and ii) for each x in S, $\int p(x, y)dy = 1$. Let $f : S \to R$ be a continuous function such that a) for each x in $S, \int |f(y)|\ p(x, y)dy < \infty$ and $\int f(x) \leq f(y)\ p(x, y)dy$ and b) there is an x_0 such that $f(x_0) \geq$

$f(x)$ for all x in S. Suppose for any two x and y in S, $x \neq y$, there $z_1, z_2, \ldots z_k$ in S such that $p(x,z) > 0, p(z_1,z_2) > 0$, $p(z_2,z_3) > 0, \ldots, p(z_k,y) > 0$. Then show that $f(x) = f(x_0)$ for all x in S.

Hints for Solutions to the Above Problems

As mentioned earlier consult these hints only after you have tried hard to solve the above problems. It is likely that your solutions could be simpler and nicer than these hints. So why not give yourself that opportunity!

Problem 1. Consider a square B where the value is the largest, say, M, among all the sixty four numbers. Show that the values on all the squares neighbouring B should equal M. Now move to the squares neighbouring each of these and so on.

Problem 2. Since S is finite there is an s_0 in S such that $f(s_0) \geq f(s)$ for all s in S. Now argue as in Problem 1 starting with a neighbourhood of s_0 and then going from there to other points s in S.

Problem 3. Show first that

$$\sum_j p_{j_0 j}(f(j) - f(j_0)) = 0$$

and hence that for any j, $p_{j_0 j} > 0 \Rightarrow f(j) = f(j_0)$. Next, iterate the given inequality to show that for all $i, f(i) \leq \sum_k \left(\sum_j p_{ij} p_{jk} \right) f(k)$ and conclude that if for any k, $\sum_j p_{j_o j} p_{jk} > 0$ then $f(k) = f(j_0)$ and so on.

Problem 4. Note that if for some y in S, $f(y) < f(x_0)$, then by continuity of f, $f(y') < f(x_0)$ for all y' in a small interval around y. Next show that this coupled with inequality (i) and the strict positivity of p will imply that $f(x_0) < f(x_0)$ a contradicicition.

Problem 5. First consider a ball $B(x_0, r)$ of radius r small enough so that $B(x_0, r) \subset S$. Use the continuity of

f at x_0 and the conditions $f(x_0) \geq f(y)$ for all y in S and $f(x_0) \leq \frac{1}{V_r} \int\limits_{B(x_0,r)} f(y)\,dy$ to conclude that $f(y) = f(x_0)$ for all y in $B(x_0, r)$. Now use the connectedness of S to conclude that for any y_0 in $S, y_0 \neq x_0$ there is a finite number of open balls $B(x_0, r_0), B(x_1, r_1), \ldots, B(x_k, r_k)$, such that $B(x_i, r_i) \cap B(x_{i+1}, r_{i+1}) \neq \emptyset$ for $i = 0, 1, \ldots, k-1$ and $y_0 \in B(x_k, r_k)$ and conclude that $f(y_0) = f(x_0)$.

Problem 6. Show this is a special case of Problem 5.

Problem 7. Take any large disk of radius R centered at the origin and containing θ and apply problem 6.

Problem 8. By now you should be able to do this.

Suggested Reading

[1] V Krishnamurthy, C R Pranesachar, K N Ranganathan and B J Venkatachala, *Mathematics Olympiad Primer*, Interline Publishing Pvt. Ltd., 1996.

"... And my own view is that, at bottom and to a first approximation, Ramanujan was intellectually as sound an infidel as Bertrand Russell or Littlewood ... One thing I am sure. Ramanujan was not in the least the 'inspired idiot' that some people seem to have thought him. On the contrary, he was (except for a brief period when his mental equilibrium was definitely upset by illness) a very shrewd and sensible person; very individual, of course, and with a reasonable allowance of the minor eccentricities of genius, but fundamentally normal and sane"

– From
letter from G H Hardy to S Chandrasekhar,
Feb.19, 1936.