$W^{2,p}$ A PRIORI ESTIMATES FOR NONVARIATIONAL OPERATORS: THE SHARP MAXIMAL FUNCTION TECHNIQUE
LA TECNICA DELLA FUNZIONE MASSIMALE SHARP NELLE STIME A PRIORI $W^{2,p}$ PER OPERATORI NON VARIAZIONALI

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ABSTRACT. We consider a nonvariational degenerate elliptic operator, structured on a system of left invariant, 1-homogeneous, Hörmander vector fields on a Carnot group, where the coefficient matrix is symmetric, uniformly positive on a bounded domain and the coefficients are locally VMO. We discuss a new proof (given in [7] and also based on results in [6]) of the interior estimates in Sobolev spaces, first proved in [2]. The present proof extends to this context Krylov' technique, introduced in [26], consisting in estimating the sharp maximal function of second order derivatives.

SUNTO. Si considera un operatore nonvariazionale ellittico degenere, strutturato su un sistema di campi vettoriali di Hörmander, invarianti a sinistra e 1-omogenei su un gruppo di Carnot, dove la matrice dei coefficienti è simmetrica, uniformemente positiva su un dominio limitato e i coefficienti sono localmente VMO. Si discute una nuova dimostrazione (contenuta in [7] e basata anche su risultati in [6]) delle stime all'interno in spazi di Sobolev, provate in [2]. La presente dimostrazione estende a questo contesto la tecnica di Krylov, introdotta in [26], che consiste nello stimare la funzione massimale sharp delle derivate del second'ordine.

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1. A priori estimates for nonvariational operators: a brief historical survey

This paper is based on a talk given by the author in Bologna at the "Bruno Pini Mathematical Analysis Seminar", which in turn was based on the papers [7] and [6], and aims to keep the informal, but, I hope, informative, style of the talk.

Let us consider linear uniformly elliptic operators in nondivergence form

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j}$$

over a bounded and sufficiently smooth domain $\Omega \subset \mathbb{R}^n$, where

$$\mu |\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \leqslant \mu^{-1} |\xi|^2$$

for a.e. $x \in \Omega$, every $\xi \in \mathbb{R}^n$ and some positive constant μ. We look for strong solutions to the Dirichlet problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \\ Lu = g \text{ a.e. in } \Omega \end{cases}$$

for an assigned $g \in L^p(\Omega)$, $1 < p < \infty$. The classical theory by Agmon-Douglis-Nirenberg [1], 1959, ensures that if the coefficients a_{ij} are uniformly continuous then the problem is well-posed in the above functional framework. In the special case $n = 2$ (and $p = 2$) it is enough to assume $a_{ij} \in L^\infty(\Omega)$ (Talenti, [34]), while for $n > 2$ classical examples by Pucci (quoted by Talenti in [33], see also the book [35] by Maugeri, Palagachev, Softova) show that as soon as the coefficients a_{ij} possess a discontinuity point, existence or uniqueness of the solution may fail to be true. So, one can ask whether the continuity assumption on the coefficients can be weakened, in dimension $n > 2$, keeping the well-posedness of the problem. A first answer was given by Miranda [31], 1963, who assuming $a_{ij} \in W^{1,n} \cap L^\infty$ proved the solvability in $W^{2,2}(\Omega)$ for $g \in L^q(\Omega), q > 2$. Almost 30 years later, Chiarenza-Frasca-Longo [11], 1991, [12], 1993, assuming $a_{ij} \in VMO \cap L^\infty$ proved the well-posedness of the problem in $W^{2,p}(\Omega)$ for every $p \in (1, \infty)$. Recall that VMO, the space of functions having vanishing mean oscillation, introduced by Sarason [32], is defined as

$$VMO = \left\{ a \in L^1_{loc} : \sup_{\rho \leqslant r} \sup_{x} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |a(y) - a_{B_r}|\, dy \to 0 \text{ for } r \to 0 \right\}$$

(with $a_B = \frac{1}{|B|}\int_B a(x)\,dx$). Since $W^{1,n} \subset VMO$, this theory significantly extends Miranda's result. Functions in VMO can present some kind of discontinuity, although they cannot be, for instance, homogeneous of degree zere as the functions considered in Pucci's counterexamples are.

For future comparison, let us briefly sketch the technique that Chiarenza-Frasca-Longo use in order to prove *local* estimates in $W^{2,p}$ (estimates near the boundary require some further ideas, that we are not going to discuss here). They start writing a representation formula for second order derivatives of functions $u \in C_0^\infty(\Omega)$, namely

$$u_{x_i x_j}(x) = c_{ij}(x)\, Lu(x) + \sum_{h,k=1}^{n} \mathrm{PV} \int [a_{hk}(x) - a_{hk}(y)]\, \Gamma_{ij}(x, x-y)\, u_{y_h y_k}(y)\, dy,$$

where $\Gamma(x,\cdot)$ is the fundamental solution of the constant coefficient elliptic operator obtained freezing the coefficients of L at the point x, $\Gamma_{ij}(x,y) = \frac{\partial^2}{\partial y_i \partial y_j}\Gamma(x,y)$, and $c_{ij}(x)$ are suitable bounded functions. In compact form, the above formula can be rewritten as

$$u_{x_i x_j} = c_{ij} \cdot Lu + \sum_{h,k=1}^{n} [a_{hk}, T_{ij}]\, u_{x_h x_k},$$

where every T_{ij} is a Calderón-Zygmund operator "with variable kernel" and $[a_{hk}, T_{ij}]$ is its commutator with the multiplication operator for a_{hk}. Expanding $\Gamma_{ij}(x,\cdot)$ in series of spherical harmonics (according to a classical procedure introduced by Calderón-Zygmund, [10]) every T_{ij} and every commutator can be represented in terms of singular integrals of convolution type. Then, Chiarenza-Frasca-Longo invoke a deep real analysis theorem proved by Coifman-Rochberg-Weiss [14] ensuring that, for a classical Calderón-Zygmund operator T of convolution type, and any function $a \in BMO(\mathbb{R}^n)$,

$$\|[a,T]\, f\|_p \leqslant c\, \|a\|_*\, \|f\|_p$$

for every $p \in (1,\infty)$, where $\|\cdot\|_*$ is the BMO seminorm:

$$\|a\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |a(y) - a_{B_r}|\, dy.$$

This fact, together with L^p continuity of singular integrals and a suitable uniform control over the coefficients in the expansion in series of spherical harmonics, gives an estimate of the kind

$$\left\|u_{x_i x_j}\right\|_p \leqslant c\left\|Lu\right\|_p + c\sum_{h,k=1}^{n}\left\|a_{hk}\right\|_*\left\|u_{x_h x_k}\right\|_p.$$

Exploiting the fact that VMO functions can be approximated in BMO seminorm with uniformly continuous functions, the last estimate can be refined proving that, for $a_{hk}\in VMO$ and every fixed $\varepsilon>0$, there exists $r>0$ such that for $u\in C_0^\infty\left(B_r\right)$,

$$\left\|u_{x_i x_j}\right\|_p \leqslant c\left\|Lu\right\|_p + \varepsilon\sum_{h,k=1}^{n}\left\|u_{x_h x_k}\right\|_p$$

which finally gives

$$\left\|u_{x_h x_k}\right\|_p \leqslant c\left\|Lu\right\|_p$$

for every $u\in C_0^\infty\left(B_r\right)$ with r small enough, and every $p\in(1,\infty)$. From this result, the proof of interior $W^{2,p}$ estimates of the kind

$$\|u\|_{W^{2,p}(\Omega')} \leqslant c\left\{\|Lu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right\} \text{ for } \Omega'\Subset\Omega$$

is then a routine.

Note that if we knew the *uniform continuity* of the coefficients a_{ij}, as in the classical theory, then a much easier procedure would give the same result just exploiting L^p continuity of singular integrals of convolution type.

The technique devised by Chiarenza-Frasca-Longo has been subsequently extended to other nonvariational operators with VMO coefficients, for instance:

Parabolic equations: Bramanti-Cerutti, [4];

Systems of elliptic equations: Chiarenza-Franciosi-Frasca, [13];

Ultraparabolic equations of Kolmogorov-Fokker-Planck type: Bramanti-Cerutti-Manfredini, [5];

Nonvariational equations structured on Hörmander vector fields: Bramanti-Brandolini [2], [3], Bramanti-Zhu, [8].

While each of the above extensions poses some new problems, a common feature in all these works is the heavy use of singular integral theory. In particular, one needs to assure the L^p continuity of singular integrals and their commutators with BMO functions. Moreover, the involved singular integrals are not of convolution type: instead, they need to be expanded in an infinite series of convolution-type singular integrals, which poses the problem of getting some control on the coefficients in this expansion.

In 2007, Krylov [26] (see also [27] and the papers [22] by Kim-Krylov and [24] by Kim) introduced a different technique to prove $W^{2,p}$ a priori estimates for nonvariational operators with VMO coefficients, getting similar results also under weaker assumptions. Instead of expressing the second order derivatives $u_{x_ix_j}$ in terms of singular integrals and their commutators, a *pointwise* estimate is proved for the *sharp maximal function* of $u_{x_ix_j}$. Recall that, for a locally integrable function f, the Hardy-Littlewood maximal function Mf and the Fefferman-Stein sharp maximal function $f^{\#}$ are defined as:

$$Mf(x) = \sup_{x \ni B_r} \frac{1}{|B_r|} \int_{B_r} |f(y)| \, dy; \tag{1}$$

$$f^{\#}(x) = \sup_{x \ni B_r} \frac{1}{|B_r|} \int_{B_r} |f(y) - f_{B_r}| \, dy$$

and satisfy the well-known estimates:

$$c_1 \|Mf\|_{L^p(\mathbb{R}^n)} \leqslant \|f\|_{L^p(\mathbb{R}^n)} \leqslant c_2 \left\|f^{\#}\right\|_{L^p(\mathbb{R}^n)} \text{ for } f \in L^p, 1 < p < \infty. \tag{2}$$

Postponing for the moment a precise statement, let us just say that the key estimate in [26] looks as follows:

$$\left(u_{x_ix_j}\right)^{\#}(x) \leqslant c\left(M\left(|Lu|^{\alpha}\right)(x)\right)^{1/\alpha} + (...)$$

for $\alpha \in (1, \infty)$, where (...) contains some terms that in the subsequent L^p estimates can be made small and taken to the left-hand side, under the assumption of VMO coefficients and functions u supported in small balls. Actually, taking L^p norms in the last inequality (after choosing $1 < \alpha < p$) and applying (2), we can get

$$\left\|u_{x_ix_j}\right\|_p \leqslant c \|Lu\|_p + \|(...)\|_p$$

and finally

$$\left\|u_{x_ix_j}\right\|_p \leqslant c \|Lu\|_p \,,$$

as desired. Exploiting this technique, in [26] and several subsequent papers, the Authors prove $W^{2,p}$ estimates for elliptic or parabolic operators having coefficients $a_{ij}(x,t)$ which belong to VMO with respect to the x-variable, and are merely L^{∞} in t, or VMO in space with respect to all but one variable, or other extensions of these results (a few more details about these results will be given in the final section of this paper). See the papers [27], [22], [24], [23], [29], [16], [17], [18] and the book [28].

The research program that we want to describe here can be stated as follows. We want to investigate whether it is possible to adapt the above technique to *nonvariational operators structured on Hörmander vector fields*, getting new results, or getting known result with a shorter proof. So far, we have been able to achieve the second goal: in the paper [7] by Bramanti-Toschi, we get $W_X^{2,p}$ local estimates for operators

$$Lu = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j u$$

structured on Hörmander vector fields on Carnot groups, with VMO coefficients a_{ij}; this result was firstly established in [2]. In the present approach we do not directly exploit L^p estimates on singular integrals or commutators of singular integrals, nor expansions in series of spherical harmonics, but we use a local sharp maximal function, over local homogeneous spaces, and the corresponding L^p estimate of Fefferman-Stein type, obtained in [6] by Bramanti-Fanciullo.

The plan of the remaining parts of this paper is the following. After introducing, in section 2, the necessary background on Hörmander vector fields and real analysis, in section 3 we will describe the main result and the basic ideas of its proof. Finally, in section 4 we will describe some of the existing results proved by this technique and present some open problems, in the same line of the one discussed in the present paper.

2. The context. Hörmander vector fields on Carnot groups

Let us consider, in $\mathbb{R}^n$, a family of q real and smooth vector fields

$$X_i = \sum_{j=1}^{n} b_{ij}(x) \partial_{x_j},\ i = 1, 2, ..., q < n$$

and let us assume that they satisfy Hörmander's condition in $\mathbb{R}^n$, which means the following. Define the commutator of two vector fields X, Y as

$$[X, Y] = XY - YX.$$

We assume that the system consisting in the vector fields X_i and their iterated commutators

$$\left[X_{i_1}, \left[X_{i_2}, \left[..., \left[X_{i_{k-1}}, X_{i_k}\right]\right]\right]\right] \text{ for } k = 2, 3, ..., r$$

up to some step r, spans $\mathbb{R}^n$ at every point. Under these assumptions, Hörmander's theorem (see [20]) states that the operator

$$L = \sum_{i=1}^{q} X_i^2$$

is hypoelliptic, that is for every distribution u in Ω and open set $A \subset \Omega$, we have $Lu \in C^\infty(A) \Longrightarrow u \in C^\infty(A)$. Note that the second order operator L has nonnegative characteristic form but, as soon as $q < n$, is degenerate elliptic. This means that its hypoellipticity is a nontrivial, highly informative regularizing property.

Example 2.1. *In* $\mathbb{R}^3 \ni (x, y, t)$*, let*

$$X = \partial_x + 2y\partial_t$$
$$Y = \partial_y - 2x\partial_t.$$

Then

$$[X, Y] = -4\partial_t$$

and since the three vector fields $X, Y, [X, Y]$ *are linearly independent at every point, the second order differential operator*

$$\begin{aligned} L = X^2 + Y^2 &= (\partial_x + 2y\partial_t)^2 + (\partial_y - 2x\partial_t)^2 \\ &= \partial_x^2 + \partial_y^2 + 4\left(x^2 + y^2\right)\partial_t^2 + 4\left(y\partial_{xt}^2 - x\partial_{yt}^2\right) \end{aligned}$$

is hypoelliptic.

A *homogeneous group* (in $\mathbb{R}^n$) is a Lie group $(\mathbb{R}^n, \circ)$ (where we think $\circ$ as a "translation"), having 0 as its neutral element, endowed with a family $\{D_\lambda\}_{\lambda>0}$ of group automorphisms ("dilations") acting as follows:

$$D_\lambda(x_1, x_2, ..., x_n) = (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, ..., \lambda^{\alpha_n} x_n)$$

for assigned integers $1 = \alpha_1 \leqslant \alpha_2 \leqslant ... \leqslant \alpha_n$.

We will denote this structure by $\mathbb{G} = (\mathbb{R}^n, \circ, D_\lambda)$. The number

$$Q = \sum_{i=1}^{n} \alpha_i$$

is called *homogeneous dimension* of $\mathbb{G}$.

Example 2.2 (The Heisenberg group $\mathbb{H}^n$)**.** *In $\mathbb{R}^{n+n+1} \ni (x, y, t)$, let*

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' - 2 (x \cdot y' - x' \cdot y))$$
$$D (\lambda) (x, y, t) = \left(\lambda x, \lambda y, \lambda^2 t\right).$$

With this structure, $\mathbb{R}^N$ (with $N = 2n+1$) becomes a homogeneous group, of homogeneous dimension $Q = 2n + 2$. This group is denoted by $\mathbb{H}^n$.

A differential operator P on a homogeneous group $\mathbb{G}$ is called:

left invariant if

$$P (L_y f) (x) = L_y (P f (x)) \;\; \forall x, y \in \mathbb{R}^n$$

for every smooth function f, where

$$L_y f (x) = f (y \circ x);$$

homogeneous of degree $\alpha \in \mathbb{R}$ (or α-homogeneous) if

$$P (f (D (\lambda) x)) = \lambda^\alpha (P f) (D (\lambda) x)$$

for every $x \in \mathbb{G}$, $\lambda > 0$ and every smooth function f.

Let us denote by X_i $(i = 1, 2, ..., n)$ the only left invariant vector field which agrees at the origin with ∂_{x_i}.

We will assume that, for some integer $q < n$, the vector fields $X_1, X_2, ..., X_q$ satisfy Hörmander's condition and are 1-homogeneous. Under these assumptions, $\mathbb{G}$ is called a *Carnot group* and

$$L = \sum_{i=1}^{q} X_i^2$$

is called the *canonical sublaplacian* over $\mathbb{G}$. The operator L is hypoelliptic (by Hörmander's theorem), 2-homogeneous and left invariant.

Example 2.3 (The Kohn Laplacian on the Heisenberg group $\mathbb{H}^1$)**.** *The operator L described in Example 2.1 is the canonical sublaplacian on the Heisenberg group $\mathbb{H}^1$ defined in Example 2.2, since X, Y are 1-homogeneous, left invariant, agree at the origin with ∂_x, ∂_y respectively, and satisfy Hörmander's condition.*

In every Carnot group we can define (in several ways) a homogeneous norm $\|\cdot\|$ satisfying the properties

$$\|D(\lambda)x\| = \lambda\|x\| \;\; \forall x \in \mathbb{G}, \lambda > 0$$
$$\|x \circ y\| \leqslant c(\|x\| + \|y\|) ; \left\|x^{-1}\right\| = \|x\| .$$

Then we can also define the (quasi)distance

$$d(x, y) = \left\|y^{-1} \circ x\right\| ,$$

and the corresponding balls $B_r(x)$, whose volume satisfy the simple condition $|B_r(x)| = cr^Q$. The space $(\mathbb{R}^n, d, dx)$ is a *space of homogeneous type* in the sense of Coifman-Weiss [15]: it is a quasimetric measure space, satisfying the doubling condition for the volume of balls. Therefore, the Hardy-Littlewood maximal function can be defined at the usual way (as in (1)), and the maximal inequality holds:

$$\|Mf\|_p \leqslant c\|f\|_p \text{ for } 1 < p \leqslant \infty.$$

Recall that, in the applications to PDEs that we have in mind, we will work in a bounded domain Ω. If we want to regard Ω as a metric (or quasimetric) space, its balls are actually the sets $\Omega \cap B_r(x)$, so the application of the standard theory of spaces of homogeneous type to bounded domains often requires to compute integrals over sets of the kind $\Omega \cap B_r(x)$ or compare the volumes of these sets for different radii r. This can be troublesome in the context of general Hörmander vector fields, therefore we prefer to develop some real analysis machinery in the context of *locally homogeneous spaces*, in the sense of Bramanti-Zhu [9].

In particular, as to the *sharp maximal function*, a local version of this concept and the Fefferman-Stein-type L^p inequality have been given by Bramanti-Fanciullo in [6].

Let us consider three bounded domains $\Omega_0 \Subset \Omega' \Subset \Omega'' \Subset \mathbb{R}^n$ and, for every function $f \in L^1_{loc}(\Omega'')$ and $x \in \Omega'$, let

$$f^{\#}_{\Omega',\Omega''}(x) = \sup_{\substack{B(\overline{x},r)\ni x \\ \overline{x}\in\Omega', r\leqslant\varepsilon}} \frac{1}{|B(\overline{x},r)|} \int_{B(\overline{x},r)} |f(y) - f_B|\, dy.$$

Here the number $\varepsilon > 0$ is chosen small enough so that if $\overline{x} \in \Omega'$ and $r \leqslant \varepsilon$, then $B(\overline{x},r) \subset \Omega''$. The result proved in [6] then reads as follows:

Theorem 2.1. *For every $R > 0$ small enough, the domain Ω_0 can be covered by balls $B_R = B(x_i, R)$ such that for every f supported in B_R such that $f \in L^1(B_R)$ and $\int_{B_R} f = 0$,*

$$\|f\|_{L^p(B_R)} \leqslant c \left\| f^{\#}_{\Omega',\Omega''} \right\|_{L^p(B_{\gamma R})}$$

(for some $\gamma > 1$, such that $B_{\gamma R} \subset \Omega'$).

3. Main result and line of the proof

After these preliminaries we are now in position to state the main result proved in [7].

Let $X_1, ..., X_q$ be vector fields such that $\sum_{i=1}^q X_i^2$ is the canonical sublaplacian on a Carnot group $\mathbb{G}$ in $\mathbb{R}^n$, and for a bounded domain $\Omega \subset \mathbb{R}^n$ let us consider the operator

$$Lu = \sum_{i,j=1}^q a_{ij}(x) X_i X_j u,$$

where a_{ij} is a symmetric matrix of measurable functions satisfying

$$\mu |\xi|^2 \leqslant \sum_{i,j=1}^q a_{ij}(x) \xi_i \xi_j \leqslant \mu^{-1} |\xi|^2$$

for some $\mu > 0$, every $\xi \in \mathbb{R}^q$ and $x \in \Omega$. Assume that $a_{ij} \in VMO_{loc}(\Omega)$, that is: for every $\Omega_0 \Subset \Omega' \Subset \Omega$, letting

$$\eta_{a_{ij}}(r) = \sup_{\overline{x}\in\Omega_0, \rho\leqslant r} \frac{1}{|B(\overline{x},\rho)|} \int_{B(\overline{x},\rho)} \left| a_{ij}(y) - (a_{ij})_B \right| dy$$

(where r is chosen small enough so that $\overline{x} \in \Omega_0, \rho \leqslant r \Longrightarrow B(\overline{x},\rho) \subset \Omega'$) we have

$$\eta_{a_{ij}}(r) \to 0 \text{ as } r \to 0.$$

Then our main result is the following:

Theorem 3.1. *For every $p\in(1,\infty)$, the domain Ω_0 can be covered by a finite number of balls $B_R(x_i)$ such that for every $u\in C_0^\infty(B_R(x_i))$ we have:*

$$\|X_iX_ju\|_{L^p}\leqslant c\,\|Lu\|_{L^p}\,.$$

The above result is the key step in the proof of interior a priori estimates for functions in $W_X^{2,p}(\Omega)$. As already said, this result was firstly proved in [2]; here we are presenting the proof given in [7], making use of the sharp maximal function approach.

In the rest of this section we will illustrate the main steps of the proof of Theorem 3.1. The reader is referred to [7] for details. In order to make more transparent the idea of the proof, we will start illustrating the last step of the proof, and then we will proceed backwards.

The key point is the proof of the following

Theorem 3.2. *Let $p,\alpha,\beta\in(1,\infty)$ with $\alpha^{-1}+\beta^{-1}=1$ and $R\in(0,\varepsilon)$ (for a suitable ε small enough). There exists $c>0$, depending on $p,\alpha,\mathbb{G},\mu$, but not on R, such that for every $u\in C_0^\infty(B_R)$, $k\geq 4\Lambda^3$ we have*

$$\begin{aligned}(X_hX_lu)^{\#}_{\Omega',\Omega''}(x)\leqslant{}&\frac{c}{k}\sum_{i,j=1}^{q}M(X_iX_ju)(x)+ck^{2+Q/p}\left(M\left(|Lu|^p\right)(x)\right)^{1/p}\\&+ck^{2+Q/p}\left(a_R^{\sharp}\right)^{1/\beta p}\sum_{i,j=1}^{q}\left(M(|X_iX_ju|^{p\alpha})(x)\right)^{1/\alpha p}\end{aligned}$$

for $h,l=1,2,...,q$, every $x\in B_R$, where $a_R^{\sharp}=\max_{i,j=1,\dots,q}\eta_{a_{ij}}(R)$. The number Λ above is a suitable constant depending on $\mathbb{G}$, whose meaning will be explained later.

Let us show how the previous theorem implies the desired result (Theorem 3.1).

Applying Theorem 3.2 for suitable p,p_1,α,β and the maximal inequality on $M\left(|Lu|^p\right)$ we get:

$$\begin{aligned}\sum_{i,j=1}^{q}\left\|(X_iX_ju)^{\#}_{\Omega',\Omega''}\right\|_{L^p\left(B_{\gamma R}\right)}\leqslant{}&\frac{c}{k}\sum_{i,j=1}^{q}\|X_iX_ju\|_{L^p(B_R)}\\&+ck^{2+Q/p_1}\left\{\|Lu\|_{L^p(B_R)}+\left(a^{\sharp}_{\gamma R}\right)^{1/\beta p_1}\sum_{i,j=1}^{q}\|X_iX_ju\|_{L^p(B_R)}\right\}.\end{aligned}\tag{3}$$

Since $u \in C_0^\infty(B_R)$, the derivatives X_iX_ju have vanishing avarage, which enables us to apply the sharp maximal inequality of Theorem 2.1, getting

$$\sum_{i,j=1}^{q} \|X_iX_ju\|_{L^p(B_R)} \leqslant c \sum_{i,j=1}^{q} \left\| (X_iX_ju)^{\#}_{\Omega',\Omega''} \right\|_{L^p\left(B_{\gamma R}\right)}$$

which combined with (3) for k large enough gives

$$\sum_{i,j=1}^{q} \|X_iX_ju\|_{L^p(B_R)} \leqslant c \left\{ \|Lu\|_{L^p(B_R)} + \left(a^{\sharp}_{\gamma R}\right)^{1/\beta p_1} \sum_{i,j=1}^{q} \|X_iX_ju\|_{L^p(B_R)} \right\}.$$

If R is small enough, and therefore $a^{\sharp}_{\gamma R}$ is small enough, the quantity $\sum_{i,j=1}^{q} \|X_iX_ju\|_{L^p(B_R)}$ on the right hand side can be taken to the left hand side and we get the desired bound.

Next, let us discuss how Theorem 3.2 can be actually proved. In order to show how $(X_iX_ju)^{\#}$ can be bound in terms of $M(X_iX_ju)$ and $M(|Lu|^p)$, one proves an estimate on the corresponding *constant coefficient operator*:

$$\overline{L}u = \sum_{i,j=1}^{q} \overline{a}_{ij} X_iX_ju$$

where the constant matrix belongs to the same "ellipticity class" of $\{a_{ij}(x)\}_{i,j}$:

$$\mu |\xi|^2 \leqslant \sum_{i,j=1}^{q} \overline{a}_{ij}\xi_i\xi_j \leqslant \mu^{-1} |\xi|^2 .$$

In other words, the constant in the following bound will depend on the constant coefficients $\overline{a}_{ij}$ only through the number μ. The result is the following:

Theorem 3.3. *For every $p \in (1,\infty)$ there exists $c > 0$, depending on $p, \mathbb{G}, \mu$ such that for every $k \geq 4\Lambda^3$, $r > 0$, $u \in C^\infty(\mathbb{R}^n)$*

$$\frac{1}{|B_r|}\int_{B_r} |X_iX_ju(x) - (X_iX_ju)_{B_r}| dx$$
$$\leqslant \frac{c}{k} \sum_{i,j=1}^{q} \frac{1}{|B_{kr}|}\int_{B_{kr}} |X_iX_ju(x)| dx + ck^{2+Q/p} \left(\frac{1}{|B_{kr}|}\int_{B_{kr}} |\overline{L}u(x)|^p dx \right)^{1/p}.$$

($\Lambda > 1$ is the same constant appearing in Theorem 3.2).

Let us show how Theorem 3.3 implies Theorem 3.2. This is a key point in the general strategy, where the role of constant coefficient operators in this technique appears.

We want to show that for every $u \in C_0^\infty(B_R), x \in B_{\gamma R}$ and k large enough,

$$(X_iX_ju)^{\#}_{\Omega',\Omega''}(x) \leqslant \frac{c}{k}\sum_{i,j=1}^{q} M(X_iX_ju)(x) + ck^{2+Q/p}\left(M(|Lu|^p)(x)\right)^{1/p}$$
$$+ ck^{2+Q/p}\left(a_R^{\sharp}\right)^{1/\beta p}\sum_{i,j=1}^{q}\left(M(|X_iX_ju|^{p\alpha})(x)\right)^{1/\alpha p}.$$

To do this, pick $x \in B_{\gamma R}$ and a ball $B_r(\overline{x})$, admissible for the evalutation of $(X_iX_ju)^{\#}_{\Omega',\Omega''}(x)$, that is: $x \in B_r(\overline{x}) \subset \Omega''$, $\overline{x} \in \Omega'$. By Theorem 3.3 we can bound

$$\frac{1}{|B_r|}\int_{B_r}|X_iX_ju(y) - (X_iX_ju)_{B_r}|dy$$
$$\leqslant \frac{c}{k}\sum_{i,j=1}^{q}\frac{1}{|B_{kr}|}\int_{B_{kr}}|X_iX_ju(y)|dy + ck^{2+Q/p}\left(\frac{1}{|B_{kr}|}\int_{B_{kr}}|\overline{L}u|^p dy\right)^{1/p}. \tag{4}$$

To bound the right hand side, we can write

$$\frac{1}{|B_{kr}|}\int_{B_{kr}}|X_iX_ju(y)|dy \leqslant M(X_iX_ju)(x)$$
$$\left\|\overline{L}u\right\|_{L^p(B_{kr})} \leqslant \|Lu\|_{L^p(B_{kr})} + \left\|\overline{L}u - Lu\right\|_{L^p(B_{kr})}$$

and exploit the fact that

$$\left(\frac{1}{|B_{kr}|}\int_{B_{kr}}|Lu(y)|^p dy\right)^{1/p} \leqslant \left(M(|Lu|^p)(x)\right)^{1/p}$$

so that

$$\frac{1}{|B_r|}\int_{B_r}|X_iX_ju(y) - (X_iX_ju)_{B_r}|dy \leqslant \frac{c}{k}\sum_{i,j=1}^{q}M(X_iX_ju)(x)$$
$$+ ck^{2+Q/p}\left\{\left(M(|Lu|^p)(x)\right)^{1/p} + \frac{1}{|B_{kr}|^{1/p}}\left\|\overline{L}u - Lu\right\|_{L^p(B_{kr})}\right\}. \tag{5}$$

Next, let us write

$$\int_{B_{kr}}|\overline{L}u(x) - Lu(x)|^p dx$$
$$\leqslant c\sum_{i,j=1}^{q}\left(\int_{B_{kr}\cap B_R}|\overline{a}_{ij} - a_{ij}(x)|^{p\beta}dx\right)^{1/\beta}\left(\int_{B_{kr}\cap B_R}|X_iX_ju|^{p\alpha}dx\right)^{1/\alpha}. \tag{6}$$

Since the coefficients $\overline{a}_{ij}, a_{ij}$ are bounded by $1/\mu$ we have

$$\int_{B_{kr}\cap B_R} |\overline{a}_{ij} - a_{ij}(x)|^{p\beta} dx \leqslant \mu^{-\beta p+1} \int_{B_{kr}\cap B_R} |a_{ij}(x) - \overline{a}_{ij}|\, dx. \tag{7}$$

We now choose a particular constant matrix $\{\overline{a}_{ij}\}$, depending on r, k:

$$\overline{a}_{ij} = \begin{cases} (a_{ij})_{B_R} & \text{if } kr \geq R \\ (a_{ij})_{B_{kr}} & \text{if } kr \leqslant R \end{cases}$$

and with this definition we easily get

$$\int_{B_{kr}\cap B_R} |a_{ij}(x) - \overline{a}_{ij}|\, dx \leqslant c(kr)^Q a_R^\sharp,$$

which together with (5), (6), (7), gives

$$\frac{1}{|B_r|}\int_{B_r} |X_iX_ju(y) - (X_iX_ju)_{B_r}| dy \leqslant \frac{c}{k}\sum_{i,j=1}^q M(X_iX_ju)(x) + ck^{2+Q/p}\left(M(|Lu|^p)(x)\right)^{1/p}$$

$$+ ck^{2+Q/p}\frac{1}{|B_{kr}|^{1/p}}\sum_{i,j=1}^q \left(\mu^{-\beta p+1}c(kr)^Q a_R^\sharp\right)^{1/p\beta}\left(\int_{B_{kr}\cap B_R}|X_iX_ju|^{p\alpha}dx\right)^{1/p\alpha} \tag{8}$$

and the last line is bounded by

$$ck^{2+Q/p}\left(a_R^\sharp\right)^{1/p\beta}\sum_{i,j=1}^q \frac{1}{(kr)^{Q/p\alpha}}\left(\int_{B_{kr}}|X_iX_ju|^{p\alpha}dx\right)^{1/p\alpha}$$
$$\leqslant ck^{2+Q/p}\left(a_R^\sharp\right)^{1/p\beta}\sum_{i,j=1}^q \left(M(|X_iX_ju|^{p\alpha})(x)\right)^{1/p\alpha}$$

with c also depending on μ. Inserting this bound into (8) and finally taking the supremum over all the admissible balls B_r, we get the statement of Theorem 3.2.

So, we are reduced to the proof of Theorem 3.3, which deals with constant coefficient operators. Note how in the previous reasoning the "right" choice of the constant matrix does not consist in *freezing* the coefficients at some point, but in *averaging* them over suitable balls. This is an ingenious technique, firstly devised in [26] to reduce the analysis of a variable coefficient operator to that of a constant coefficient operator, and brings a relevant simplification to the general strategy.

Let us note, however, that the proof of Theorem 3.3 is not easy, especially in the present context where the model operator $\overline{L}$ is not a constant coefficient elliptic or parabolic operator, but a sublaplacian on a Carnot group.

We will not describe in detail this proof. What follows is just a list of the tools which enter the proof of Theorem 3.3 given in [7]:

Poincaré's inequality on Carnot groups (Jerison, [21]). For every $p \in [1,\infty)$ there exist constants $c > 0, \Lambda > 1$ such that for every ball $B = B(x_0, r)$ and function $u \in C^1\left(\overline{\Lambda B}\right)$,

$$\left(\frac{1}{|B|}\int_B |u(x) - u_B|^p \, dx\right)^{1/p} \leqslant cr\left(\frac{1}{|\Lambda B|}\int_{\Lambda B} |Xu(x)|^p \, dx\right)^{1/p}.$$

Note that the constant Λ appearing in this theorem is the one which enters the statements of Theorems 3.2 and 3.3.

This result, applying the standard Lax-Milgram approach, also gives the next tool:

Solvability in weak sense of the Dirichlet problem

$$\begin{cases} \overline{L}u = f \in L^2(B_R) & \text{in } B_R \\ u = g \in W_X^{1,2}(B_R) & \text{on } \partial B_R \end{cases}$$

where we look for $u \in W_X^{1,2}(B_R)$ such that $u - g \in W_{0,X}^{1,2}(B_R)$. A maximum principle for this problem is also established and used.

Subelliptic estimates (Kohn, [25]**)** for the model operator $\overline{L}$, with a uniform control on the constant as the matrix $\overline{a}_{ij}$ ranges in the ellipticity class (the uniformity of the bound can be checked just going through the proof). There exists $\varepsilon > 0$ and, for every pair of cutoff functions $\eta, \eta_1 \in C_0^\infty(\mathbb{R}^n)$ with $\eta_1 = 1$ on $\operatorname{sprt}\eta$, and for every $\sigma, \tau > 0$, there exists $c > 0$ such that

$$\|\eta u\|_{H^{\sigma+\varepsilon}} \leqslant c\left(\left\|\eta_1 \overline{L}u\right\|_{H^\sigma} + \|\eta_1 u\|_{H^{-\tau}}\right)$$

where H^σ denotes the standard fractional Sobolev space over $\mathbb{R}^n$, defined via Fourier transform, and c depends on $\overline{a}_{ij}$ only through the number μ.

Existence of a $(2-Q)$**-homogeneous global fundamental solution for** $\overline{L}$ (Folland, [19]), with a uniform estimate as the matrix $\overline{a}_{ij}$ ranges in the ellipticity class (the uniformity of this bound is proved in [2]):

$$|\Gamma_{\overline{a}}(x)| \leqslant \frac{c}{\|x\|^{Q-2}} \ \forall x \in \mathbb{R}^n \setminus \{0\},$$

where c depends on $\overline{a}_{ij}$ only through the number μ.

Other tools entering the proof of Theorem 3.3 are an essential use of *dilations* on the group, and the use of *interpolation inequalities* to bound norms of $X_i u$ by norms of $X_i^2 u$ and u.

4. Generalizations and open problems

As the previous discussion shows, an interesting feature of the sharp function technique is the way it reduces the study of operators with variable coefficients to that of operators with constant coefficients. On the other hand, the proof of the required estimate for these model operators is far from being trivial. As we shift our interest from elliptic equations with VMO coefficients to more and more general classes of operators, the study of the class of "model operators" which play the role of "constant coefficient operators" becomes more challenging.

Let us end with a brief survey of some classes of operators which have been successfully studied with this technique, and some open problems which are naturally suggested concerning operators structured on Hörmander vector fields.

In [26], Krylov considers uniformly parabolic operators with coefficients $a_{ij}(t,x)$ which are VMO with respect to x and just L^∞ in t. Namely, the assumption is:

$$\sup_{(t,x)} \sup_{r<R} \frac{1}{r^2 |B_r(x)|} \int_t^{t+r^2} \int_{y,z \in B_r(x)} |a_{ij}(s,y) - a_{ij}(s,z)| \, dydzds \to 0 \text{ as } R \to 0.$$

Under this assumption, well-posedness of the Cauchy problem in $W_p^{1,2}$ on the whole space for $p \in (1,\infty)$ is established. In this situation, the model operators are:

$$\overline{L}u = u_t - \sum_{i,j=1}^n a_{ij}(t) \, u_{x_i x_j}.$$

Analogously, Kim and Krylov in [22] consider elliptic operators with coefficients $a_{ij}(x', x_n)$ which are VMO with respect to x' and just L^∞ in t, and they prove well-posedness of the problem $Lu = f$ on the whole space in $W^{2,p}$ for $p \in (2,\infty)$.

So, a class of natural open problems is the following: can one prove $W_X^{1,2,p}$ estimates for evolution operators structured on Hörmander vector fields

$$Lu = u_t - \sum_{i,j=1}^{q} a_{ij}\left(t,x\right) X_i X_j u,$$

assuming a_{ij} to be VMO in space and just L^∞ in time? This should require a deep analysis of the model operators

$$\overline{L}u = u_t - \sum_{i,j=1}^{q} a_{ij}\left(t\right) X_i X_j u.$$

An analogous problem could be posed for Kolmogorov-Fokker-Planck operators, modeled on the class introduced by Lanconelli-Polidoro in [30], of the kind

$$Lu = \sum_{i,j=1}^{q} a_{ij}\left(t,x\right) u_{x_i x_j} + \langle x, BDu\rangle - u_t$$

under similar assumptions on a_{ij}. In this case the class of model operators is:

$$\overline{L}u = \sum_{i,j=1}^{q} a_{ij}\left(t\right) u_{x_i x_j} + \langle x, BDu\rangle - u_t.$$

Both the problems seem to be challenging.

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