ON s-HARMONIC FUNCTIONS ON CONES
FUNZIONI s-ARMONICHE SU CONI

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Abstract. We deal with non negative functions which are s-harmonic on a given cone of the n-dimensional Euclidean space with vertex at zero, vanishing on the complementary. We consider the case when the parameter s approaches 1, wondering whether solutions of the problem do converge to harmonic functions in the same cone or not. Surprisingly, the answer will depend on the opening of the cone through an auxiliary eigenvalue problem on the upper half sphere. These conic functions are involved in the study of the nodal regions in the case of optimal partitions and other free boundary problems and play a crucial role in the extension of the Alt-Caffarelli-Friedman monotonicity formula to the case of fractional diffusions.

Sunto. Ci occupiamo di funzioni non negative che sono s-armoniche su un dato cono dello spazio euclideo n-dimensionale con vertice in zero, e che si annullano sul complementare. Consideriamo il caso in cui il parametro s converge a 1, chiedendoci se le soluzioni del problema convergano o meno a funzioni armoniche nello stesso cono. Sorprendentemente, la risposta dipenderá dall'apertura del cono attraverso un problema agli autovalori ausiliario sulla semisfera superiore. Queste funzioni coniche si incontrano nello studio delle regioni nodali nel caso di partizioni ottimali e altri problemi di frontiera libera e svolgono un ruolo cruciale nell'estensione della formula di monotonia di Alt-Caffarelli-Friedman al caso di diffusioni frazionarie.

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1. Introduction

This is a note based on the work [22], written in collaboration with S. Terracini and G. Tortone. The aim is to give an introduction to the problem, trying to explain the reasons behind this study and the interest of the results obtained. Any further detail can be found in [22].

Let $n \geq 2$ and let C be an open cone in $\mathbb{R}^n$ with vertex at 0; for a given $s \in (0,1)$, we consider the problem of the classification of nontrivial functions which are s-harmonic inside the cone and vanish identically outside, that is:

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in} \quad C, \\ u_s \geq 0 & \text{in} \quad \mathbb{R}^n \\ u_s \equiv 0 & \text{in} \quad \mathbb{R}^n \setminus C. \end{cases} \tag{1}$$

Here we define the fractional Laplacian

$$(-\Delta)^s u(x) = C(n,s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(\eta)}{|x-\eta|^{n+2s}} \mathrm{d}\eta \,,$$

where u is a sufficiently smooth function and

$$C(n,s) = \frac{2^{2s} s \Gamma(\frac{n}{2} + s)}{\pi^{n/2} \Gamma(1-s)} > 0, \tag{2}$$

where

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t.$$

The principal value is taken at $\eta = x$: hence, though u needs not to decay at infinity, it has to keep an algebraic growth with a power strictly smaller than $2s$ in order to make the above expression meaningful. We consider the following definition of distributional solutions to (1) (see [6])

Definition 1.1. *Let $s \in (0,1)$, $n \geq 2$ and C be an open cone in $\mathbb{R}^n$ with vertex at 0. We say that the function u_s is solution to* (1) *if belongs to $\mathcal{L}_s$; that is,*

$$\int_{\mathbb{R}^n} \frac{|u_s(x)|}{(1+|x|)^{n+2s}} < +\infty,$$

it is continuous in C, and it is s-harmonic in C in the sense of distributions; that is, for any $\phi \in C_c^\infty(C)$

$$\int_{\mathbb{R}^n} u_s(-\Delta)^s\phi = 0.$$

By Theorem 3.2 in [4], it is known that there exists a homogeneous, nonnegative and nontrivial solution to (1) of the form

$$u_s(x) = |x|^{\gamma_s} u_s\left(\frac{x}{|x|}\right),$$

where $\gamma_s := \gamma_s(C)$ is a definite homogeneity degree (characteristic exponent), which depends on the cone. Moreover, such a solution is continuous in $\mathbb{R}^n$ and unique, up to multiplicative constants. We can normalize it in such a way that $\|u_s\|_{L^\infty(S^{n-1})} = 1$. We consider the case when s approaches 1, wondering whether solutions of the problem do converge to a harmonic function in the same cone and, in case, which are the suitable spaces for convergence.

Such conic s-harmonic functions appear as limiting blow-up profiles and play a major role in many free boundary problems with fractional diffusions and in the study of the geometry of nodal sets, also in the case of partition problems (see, e.g. [1, 5, 11, 14, 16]). Moreover, as we shall see later, they are strongly involved with the possible extensions of the Alt-Caffarelli-Friedman monotonicity formula to the case of fractional diffusion. The study of their properties and, ultimately, their classification is therefore a major achievement in this setting. The problem of homogeneous s-harmonic functions on cones has been deeply studied in [4, 6, 7, 17]. The present note mainly focuses on the limiting behaviour as $s \nearrow 1$.

Our problem (1) can be linked to a specific spectral problem of local nature in the upper half sphere; indeed let us look at the extension technique popularized by Caffarelli and Silvestre (see [10]), characterizing the fractional Laplacian in $\mathbb{R}^n$ as the Dirichlet-to-Neumann map for a function v depending on one more space dimension and satisfying:

$$\begin{cases} L_s v = \mathrm{div}(y^{1-2s}\nabla v) = 0 & \text{in} \quad \mathbb{R}^{n+1}_+, \\ v(x,0) = u(x) & \text{on} \quad \mathbb{R}^n . \end{cases} \tag{3}$$

Such an extension exists unique (for a suitable class of functions u) in the sense of convolution with the Poisson kernel of the half space and it is given by the formula:

$$v(x,y) = d(n,s)\int_{\mathbb{R}^n} \frac{y^{2s}u(\eta)}{(|x-\eta|^2+y^2)^{n/2+s}}\mathrm{d}\eta \qquad \text{where } d(n,s)^{-1} := \int_{\mathbb{R}^n} \frac{1}{(|\eta|^2+1)^{n/2+s}}\mathrm{d}\eta \,.$$

Then, the nonlocal original operator translates into a boundary derivative operator of Neumann type:

$$-\frac{C(n,s)}{d(n,s)}\lim_{y\to 0} y^{1-2s}\partial_y v(x,y) = (-\Delta)^s u(x).$$

Now, let us consider an open region $\omega \subseteq S^{n-1} = \partial S^n_+$, with $S^n_+ = S^n \cap \{y>0\}$, and define the eigenvalue

$$\lambda_1^s(\omega) = \inf\left\{ \frac{\displaystyle\int_{S^n_+} y^{1-2s}|\nabla_{S^n} w|^2 \mathrm{d}\sigma}{\displaystyle\int_{S^n_+} y^{1-2s} w^2 \mathrm{d}\sigma} \ : \ w \in H^1(S^n_+; y^{1-2s}\mathrm{d}\sigma)\setminus\{0\} \text{ and } w\equiv 0 \text{ in } S^{n-1}\setminus\omega \right\}.$$

Next, define the *characteristic exponent* of the cone C_ω spanned by ω as

$$\gamma_s(C_\omega) = \gamma_s(\lambda_1^s(\omega))\,, \tag{4}$$

where the function $\gamma_s(t)$ is defined by

$$\gamma_s(t) := \sqrt{\left(\frac{n-2s}{2}\right)^2 + t} - \frac{n-2s}{2}\,.$$

Remark 1.1. *There is a remarkable link between the nonnegative $\lambda_1^s(\omega)$-eigenfunctions and the $\gamma_s(\lambda_1^s(\omega))$-homogeneous L_s-harmonic functions: let consider the spherical coordinates (r,θ) with $r>0$ and $\theta\in S^n$. Let φ_s be the first nonnegative eigenfunction associated to $\lambda_1^s(\omega)$ and let v_s be its $\gamma_s(\lambda_1^s(\omega))$-homogeneous extension to $\mathbb{R}^{n+1}_+$, i.e.*

$$v_s(r,\theta) = r^{\gamma_s(\lambda_1^s(\omega))}\varphi_s(\theta),$$

which is well defined as soon as $\gamma_s(\lambda_1^s(\omega)) < 2s$ (this fact is always granted by an easy inclusion of spaces which says that the maximal homogeneity degree is given by the function with zero trace in $\mathbb{R}^n$; that is, y^{2s}). By [19], the operator L_s can be decomposed as

$$L_s w = \sin^{1-2s}(\theta_n)\frac{1}{r^n}\partial_r\left(r^{n+1+2s}\partial_r w\right) + \frac{1}{r^{1+2s}}L_s^{S^n} w$$

where $y = r\sin(\theta_n)$ and the Laplace-Beltrami type operator is defined as

$$L_s^{S^n} w = \mathrm{div}_{S^n}(\sin^{1-2s}(\theta_n)\nabla_{S^n} w)$$

with ∇_{S^n} denoting the tangential gradient on S^n. Then, we easily get that v_s is L_s-harmonic in the upper half-space; moreover its trace $u_s(x) = v_s(x,0)$ is s-harmonic in the cone C_ω spanned by ω, vanishing identically outside: in other words u_s is a solution of our problem (1)*.*

In a symmetric way, for the standard Laplacian, we consider the problem of γ-homogeneous functions which are harmonic inside the cone spanned by ω and vanish outside:

$$\begin{cases} -\Delta u_1 = 0 & \text{in} \quad C_\omega, \\ u_1 \geq 0 & \text{in} \quad \mathbb{R}^n \\ u_1 = 0 & \text{in} \quad \mathbb{R}^n \setminus C_\omega. \end{cases} \tag{5}$$

Is is well known that the associated eigenvalue problem on the sphere is that of the Laplace-Beltrami operator with Dirichlet boundary conditions:

$$\lambda_1(\omega) = \inf\left\{ \frac{\displaystyle\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 \mathrm{d}\sigma}{\displaystyle\int_{S^{n-1}} u^2 \mathrm{d}\sigma} \ : \ u \in H^1(S^{n-1}) \setminus \{0\} \text{ and } u = 0 \text{ in } S^{n-1} \setminus \omega \right\},$$

and the *characteristic exponent* of the cone C_ω is

$$\gamma(C_\omega) = \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1(\omega)} - \frac{n-2}{2} = \gamma_{s|s=1}(\lambda_1(\omega)) \ . \tag{6}$$

In the classical case, the characteristic exponent enjoys a number of nice properties: it is minimal on spherical caps among sets having a given measure. Moreover for the spherical caps, the eigenvalues enjoy a fundamental convexity property with respect to the colatitude θ ([3, 15]). The convexity plays a major role in the proof of the Alt-Caffarelli-Friedman monotonicity formula, a key tool in the Free Boundary Theory ([9]).

Since the standard Laplacian can be viewed as the limiting operator of the family $(-\Delta)^s$ as $s \nearrow 1$, some questions naturally arise:

Problem 1.1. *Is it true that*

(a) $\lim_{s\to 1}\gamma_s(C)=\gamma(C)$*?*

(b) $\lim_{s\to 1} u_s = u_1$ *uniformly on compact sets, or better, in Hölder local norms?*

(c) *for spherical caps of opening* θ *is there any convexity of the map* $\theta \mapsto \lambda_1^s(\theta)$ *at least, for* s *near* 1*?*

We therefore addressed the problem of the asymptotic behavior of the solutions of problem (1) for $s \nearrow 1$, obtaining a rather unexpected result: our analysis shows high sensitivity to the opening solid angle ω of the cone C_ω, as evaluated by the value of $\gamma(C)$. In the case of wide cones, when $\gamma(C) < 2$ (that is, $\theta \in (\pi/4, \pi)$ for spherical caps of colatitude θ), our solutions do converge to the harmonic homogeneous function of the cone; instead, in the case of narrow cones, when $\gamma(C) \geq 2$ (that is, $\theta \in (0, \pi/4]$ for spherical caps), then the limit of the homogeneity degree will be always two and the limiting profile will be something different, though related, of course, through a correction term. Similar transition phenomena have been detected in other contexts for some types of free boundary problems on cones ([2, 20]). As a consequence of our main result, we will see a lack of convexity of the eigenvalue as a function of the colatitude. Our main result is the following Theorem (corresponding to Theorem 1.3 in [22]).

Theorem 1.1. *Let* C *be an open cone with vertex at the origin. There exist finite the following limits:*

$$\overline{\gamma}(C) := \lim_{s\to 1^-}\gamma_s(C) = \min\{\gamma(C), 2\}$$

and

$$\mu(C) := \lim_{s\to 1^-}\frac{C(n,s)}{2s-\gamma_s(C)} = \begin{cases} 0 & \text{if } \gamma(C)\leq 2, \\ \mu_0(C) & \text{if } \gamma(C)\geq 2, \end{cases}$$

where $C(n,s)$ *is defined in* (2) *and*

$$\mu_0(C) := \inf\left\{ \frac{\displaystyle\int_{S^{n-1}} |\nabla_{S^{n-1}}u|^2 - 2nu^2 \mathrm{d}\sigma}{\left(\displaystyle\int_{S^{n-1}}|u|\mathrm{d}\sigma\right)^2} \;:\; u\in H^1(S^{n-1})\setminus\{0\} \text{ and } u = 0 \text{ in } S^{n-1}\setminus C \right\}.$$

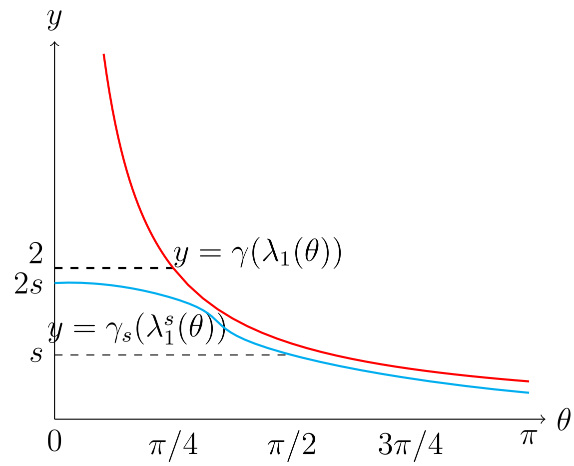

FIGURE 1. Characteristic exponents of spherical caps of aperture 2θ for $s<1$ and $s=1$.

Let us consider the family (u_s) of nonnegative solutions to (1) *such that $\|u_s\|_{L^\infty(S^{n-1})}=1$. Then, as $s\nearrow 1$, up to a subsequence, we have*

1. *$u_s\to\overline{u}$ in $L^2_{\rm loc}(\mathbb{R}^n)$ to some $\overline{u}\in H^1_{\rm loc}(\mathbb{R}^n)\cap L^\infty(S^{n-1})$.*
2. *The convergence is uniform on compact subsets of C, $\overline{u}$ is nontrivial with $\|\overline{u}\|_{L^\infty(S^{n-1})}=1$ and is $\overline{\gamma}(C)$-homogeneous.*
3. *The limit $\overline{u}$ solves*

$$\begin{cases}-\Delta\overline{u}=\mu(C)\displaystyle\int_{S^{n-1}}\overline{u}\,\mathrm{d}\sigma & \text{in } C,\\ \overline{u}=0 & \text{in } \mathbb{R}^n\setminus C\,.\end{cases}\tag{7}$$

Remark 1.2. *Uniqueness of the limit $\overline{u}$ and therefore existence of the limit of u_s as $s\nearrow 1$ holds in the case of connected cones and, in any case, whenever $\gamma(C)>2$. One can see that under symmetry assumptions on the cone C, the limit function $\overline{u}$ is unique and hence it does not depend on the choice of the subsequence (we refer to Proposition 4.3 in* [22]*).*

A nontrivial improvement of the main Theorem concerns uniform bounds in Hölder spaces holding uniformly for $s\to 1$ (corresponding to Theorem 1.5 in [22]).

Theorem 1.2. *Assume the cone is spanned by a cap $\omega\subset S^{n-1}$ which is a finite union of caps of class $\mathcal{C}^{1,1}$. Let $\alpha\in(0,1)$, $s_0\in(\max\{1/2,\alpha\},1)$ and A an annulus centered* *at zero. Then the family of solutions u_s to* (1) *is uniformly bounded in $C^{0,\alpha}(A)$ for any $s \in [s_0, 1)$.*

1.1. **On the fractional Alt-Caffarelli-Friedman monotonicity formula.** In the case of reaction-diffusion systems with strong competition between a number of densities which spread in space, one can observe a segregation phenomenon: as the interspecific competition rate grows, the populations tend to separate their supports in nodal sets, separated by a free boundary. For the case of standard diffusion, both the asymptotic analysis and the properties of the segregated limiting profiles are fairly well understood, we refer to [8, 12, 13, 18, 21] and references therein. Instead, when the diffusion is nonlocal and modeled by the fractional Laplacian, the only known results are contained in [23, 24, 25, 26]. As shown in [23, 24], estimates in Hölder spaces can be obtained by the use of fractional versions of the Alt-Caffarelli-Friedman (ACF) and Almgren monotonicity formulæ. For the statement, proof and applications of the original ACF monotonicity formula we refer to the book by Caffarelli and Salsa [9] on free boundary problems. Moreover, the nonlocal ACF monotonicity formula has the following statement (see Proposition 4 in [23])

Proposition 1.1. *Let $v_1, v_2 \in H^{1,1-2s}(B_R^+(x_0,0)) = H^1(B_R^+(x_0,0), y^{1-2s}\mathrm{d}z)$ which corresponds to $\overline{C^\infty(B_R^+(x_0,0))}^{\|\cdot\|_{H^{1,1-2s}}}$ with*

$$\|\cdot\|^2_{H^{1,1-2s}(B_R^+(x_0,0))} = \int_{B_R^+(x_0,0)} y^{1-2s}\left(|\cdot|^2 + |\nabla \cdot|^2\right).$$

Let moreover v_1, v_2 be continuous functions such that

- $v_1 v_2|_{\{y=0\}} = 0$, $v_i(x_0,0) = 0$;
- *for every non negative $\phi \in C_c^\infty(B_R(x_0,0))$,*

$$\int_{\mathbb{R}^{n+1}_+} (L_s v_i) v_i \phi \;+\; \int_{\mathbb{R}^n} (\partial_y^{1-2s} v_i) v_i \phi = \int_{\mathbb{R}^{n+1}_+} y^{1-2s} \nabla v_i \cdot \nabla(v_i \phi) \le 0.$$

Then the function

$$\Phi(r) = \frac{1}{r^{4\nu_s^{ACF}}} \left(\int_{B_r^+(x_0,0)} y^{1-2s} \frac{|\nabla v_1|^2}{|z-(x_0,0)|^{n-2s}} \right) \left(\int_{B_r^+(x_0,0)} y^{1-2s} \frac{|\nabla v_2|^2}{|z-(x_0,0)|^{n-2s}} \right)$$

is monotone non decreasing in r for $r \in (0,R)$.

As for the local case, the usefulness of the ACF monotonicity formula in free boundary problems lies in the fact that it gives control over the local behaviour of the solution on both sides of the free boundary. Let us state here the fractional version of the spectral problem beyond the ACF formula used in [23, 24]: consider the set of 2-partitions of S^{n-1} as

$$\mathcal{P}^2 := \left\{(\omega_1,\omega_2) : \omega_i \subseteq S^{n-1} \text{ open}, \ \omega_1 \cap \omega_2 = \emptyset, \ \overline{\omega_1} \cup \overline{\omega_2} = S^{n-1}\right\}$$

and define the optimal partition value as:

$$\nu_s^{ACF} := \frac{1}{2} \inf_{(\omega_1,\omega_2)\in\mathcal{P}^2} \sum_{i=1}^{2} \gamma_s(\lambda_1^s(\omega_i)). \tag{8}$$

It is easy to see, by a Schwarz symmetrization argument, that ν_s^{ACF} is achieved by a pair of complementary spherical caps $(\omega_\theta, \omega_{\pi-\theta}) \in \mathcal{P}^2$ with aperture 2θ and $\theta \in (0,\pi)$ (for a detailed proof of this kind of symmetrization we refer to [25]), that is:

$$\nu_s^{ACF} = \min_{\theta\in[0,\pi]} \Gamma^s(\theta) = \min_{\theta\in[0,\pi]} \frac{\gamma_s(\theta)+\gamma_s(\pi-\theta)}{2}.$$

This gives a further motivation to our study of (1) for spherical caps. A classical result by Friedland and Hayman, [15], yields $\nu^{ACF} = 1$ (case $s = 1$), and the minimal value is achieved for two half spheres; this equality is the core of the proof of the classical Alt-Caffarelli-Friedman monotonicity formula.

It was proved in [23] that ν_s^{ACF} is linked to the threshold for uniform bounds in Hölder norms for competition-diffusion systems, as the interspecific competition rate diverges to infinity, as well as the exponent of the optimal Hölder regularity for their limiting profiles. It was also conjectured that $\nu_s^{ACF} = s$ for every $s \in (0,1)$. Unfortunately, the exact value of ν_s^{ACF} is still unknown, and we only know that $0 < \nu_s^{ACF} \leq s$ (see [23, 24]). Actually one can easily give a better lower bound given by $\nu_s^{ACF} \geq \max\{s/2, s-1/4\}$ when $n=2$ and $\nu_s^{ACF} \geq s/2$ otherwise, which however it is not satisfactory. As already remarked in [1], this lack of information implies also the lack of an exact Alt-Caffarelli-Friedman monotonicity formula for the case of fractional Laplacians. Our contribution to this open problem is a byproduct of the main Theorem 1.1.

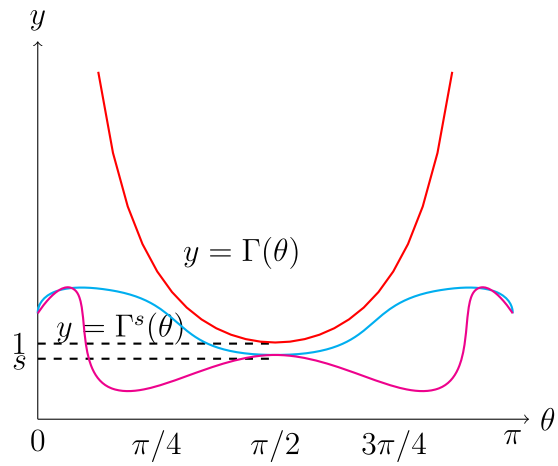

FIGURE 2. Possible values of $\Gamma^s(\theta) = \Gamma^s(\omega_\theta, \omega_{\pi-\theta})$ for $s < 1$ and $s = 1$ and $n = 2$.

Corollary 1.1. *In any space dimension we have*

$$\lim_{s \to 1} \nu_s^{ACF} = 1 \ .$$

1.2. **Ingredients for the main Theorem 1.1.** The following is the idea of the proof of the main Theorem. First, one has to obtain local $C^{0,\alpha}$-estimates in compact subsets of C and local H^s-estimates in compact subsets of $\mathbb{R}^n$ for solutions u_s of (1). These bounds are crucial in order to give sense to the limit for $s \to 1$. One can prove the following facts:

1) Local Hölder bounds: fix $K \subset C$ and $s_0 \in (0, 1)$, there exists $\overline{\alpha} \in (0, 1)$ such that $\|u_s\|_{C^{0,\overline{\alpha}}(K)} \leq c\left(1 + \frac{C(n,s)}{2s-\gamma_s(C)}\right)$ for any $s \in [s_0, 1)$.
2) Local H^s bounds: fix $\eta \in C_c^\infty(\mathbb{R}^n)$ and $s_0 \in (0, 1)$, $[\eta u_s]_{H^s(\mathbb{R}^n)} \leq c\left(1 + \frac{C(n,s)}{2s-\gamma_s(C)}\right)$ for any $s \in [s_0, 1)$.

An important quantity which appears in this estimates and plays a fundamental role is

$$\frac{C(n, s)}{2s - \gamma_s(C)},$$

where $C(n, s) > 0$ is the normalization constant given in (2). It will be therefore very important to bound this quantity uniformly in s. In this way our estimates are uniform in s as $s \to 1$, and we can ensure some regularity to the limit function and the belonging to the right Sobolev space.

Then, one has to analyze the asymptotic behaviour of $\gamma_s(C)$ as s converges to 1, in order to understand the quantities $\overline{\gamma}(C)$ and $\mu(C)$ which appear in Theorem 1.1 and which correspond respectively to the homogeneity and the possible deviation from harmonicity referred to the limit function. Hence, one can prove the following facts:

3) The function $s \mapsto \gamma_s(C)$ is non decreasing and there holds the upper bound $\gamma_s(C) \leq \gamma(C)$ (where $\gamma(C)$ is the homogeneity of the harmonic function of the same cone).

These results hold for regular cones (spanned by caps which are of class $C^{1,1}$ and connected). The boundary regularity is required in order to apply a fundamental tool in our analysis; that is, the inequality in [17]

$$\frac{1}{c}|x|^{\gamma_s - s}\mathrm{dist}(x, \partial C)^s \leq u_s(x) \leq c|x|^{\gamma_s - s}\mathrm{dist}(x, \partial C)^s.$$

The validity of such inequality relies on a boundary Harnack principle and on sharp estimates for the Green function for bounded $C^{1,1}$ domains.

In order to prove point 3), a distributional semigroup property for the fractional Laplacian for functions which grow at infinity is required. That is, the fact that if $\delta \in (0, 1-s)$, for any $\phi \in C_c^\infty(C)$

$$((-\Delta)^{s+\delta}u_s, \phi) = ((-\Delta)^\delta[(-\Delta)^s u_s], \phi).$$

This result allows to give a distributional sense to $(-\Delta)^{s+\delta}u_s$ and most of all, to give a sign to it: more precisely, one can show that $(-\Delta)^{s+\delta}u_s \geq 0$ in C for any fixed $\delta \in (0, 1-s]$, and this allows to order $\gamma_s(C) \leq \gamma_{s+\delta}(C)$ by a comparison between u_s and $u_{s+\delta}$ (and consequently the universal bound $\gamma_s(C) \leq \gamma(C)$).

Then, as we said, the next step is the study of

4) The limit $\mu(C) = \lim_{s\to 1} \frac{C(n,s)}{2s - \gamma_s(C)}$ for regular cones.

We remark the fact that the normalization constant $C(n,s) \to 0$ as $s \to 1$. By the universal bound $\gamma_s(C) \le \gamma(C)$, one easily obtain that $\mu(C) = 0$ for wide cones (by definition $\gamma(C) < 2$). Nevertheless, in case of narrow cones, it is possible to construct for any $s \in [s_0, 1)$ a function v_s which is $(-\Delta)^s v_s \le 0$ in a spherical narrow enough cone $C_\theta \subset C$. Such a function is $\gamma_s^*(\theta)$-homogeneous with $\gamma_s(C) \le \gamma_s(\theta) \le \gamma_s^*(\theta) := 2s - s\frac{C(n,s)}{\mu_0(\theta)}$ and

$$\mu_0(\theta) := \min_{\substack{u \in H_0^1(S^{n-1} \cap C_\theta) \\ u \neq 0}} \frac{\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 - 2nu^2 \mathrm{d}\sigma}{\left(\int_{S^{n-1}} |u| \mathrm{d}\sigma\right)^2}.$$

So

$$\mu(C) = \lim_{s \to 1} \frac{C(n,s)}{2s - \gamma_s(C)} \le \lim_{s \to 1} \frac{C(n,s)}{2s - \gamma_s^*(\theta)} = \mu_0(\theta) < +\infty.$$

Thanks to these facts, we have eventually the uniformity of bounds 1) and 2), and we can give sense to the limit $u_s \to \overline{u}$ in Theorem 1.1. Then, the limit equation satisfied by $\overline{u}$

$$\begin{cases} -\Delta \overline{u} = \mu(C) \displaystyle\int_{S^{n-1}} \overline{u} \mathrm{d}\sigma & \text{in } C, \\ \overline{u} = 0 & \text{in } \mathbb{R}^n \setminus C \ , \end{cases}$$

and the fact that the limit function $\overline{u}$ is homogeneous of degree $\overline{\gamma}(C) = \min\{\gamma(C), 2\}$ follow from direct computations.

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