MINIMAL CONNECTIONS: THE CLASSICAL STEINER PROBLEM AND GENERALIZATIONS

CONNESSIONI MINIME: IL PROBLEMA CLASSICO DI STEINER E GENERALIZZAZIONI

EMANUELE PAOLINI

Abstract. The classical Steiner problem is the problem of finding the shortest graph connecting a given finite set of points. In this seminar we review the classical problem and introduce a new, generalized formulation, which extends the original one to infinite sets in metric spaces.

Sunto. Il problema classico di Steiner richiede di determinare il grafo di lunghezza minimica che connette un dato insieme finito di punti. In questo seminario rivedremo il problema classico e introdurremo una nuova formulazione piuú generale che estende il problema originario a insiemi anche infiniti in spazi metrici.

2010 MSC. 30L99; 05C63.

Keywords. Steiner problem, minimal connections

1. Introduction

Generally speaking, the *Steiner Problem* can be thought as the problem of finding the set S which connects a given set of points A and which has minimal length. The solutions to the Steiner Problem are called *Steiner trees.*

For example if A is the set of the vertices of a square $A = \{a_1, a_2, a_3, a_4\}$, a Steiner tree is given by the set Σ depicted in Figure 1 (left). In this example, the Steiner tree is actually an embedded finite dimensional acyclic graph. Notice however that this graph has additional vertices other than the given points of A. These additional vertices will be called *Steiner points.*

Bruno Pini Mathematical Analysis Seminar, Vol. 1 (2012) pp. 72–87

Department of Mathematics, University of Bologna.

ISSN 2240-2829.

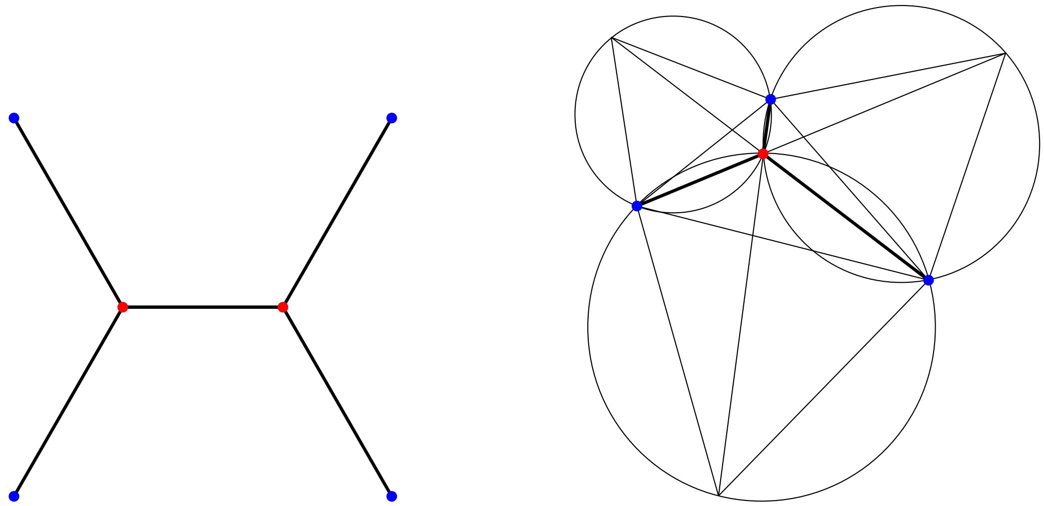

Figure 1. *On the left:* a solution of the Steiner problem for 4 points placed in the vertices (blue points) of a square. This solution has two Steiner points (in red). Another solution is obtained by a rotation of 90 degrees. *On the right:* the Fermat point of a triangle corresponds to the Steiner point (red) for three vertices (blue).

This example suggests that the Steiner problem could be stated in the setting of embedded weighted graphs. In fact the statement of the Steiner problem is usually reduced by adding the following restrictions: the given set A must be finite, the set S must be a connected finite (immersed) graph, the ambient space is the Euclidean plane. In our opinion these restriction hide some of the beauties of the Steiner problem while they are not really needed. In fact we will see that the restriction on competitors is not necessary: if the set A is finite, the minimal set S *must* be a finite graph even if we do not impose it. Moreover we can drop the requirement of A to be finite and we can work in a very general class of ambient spaces. What we found is that the problem has anyway a solution which is a topological tree (maybe not finite). We will present these results which are proven with details in [PS12].

We notice that the Steiner problem has a relevant geometrical component. As opposite to other apparently similar problems such as: the *minimal spanning tree* problem or the *lazy salesman* problem, the Steiner tree problem cannot be reduced to a problem on abstract weighted graphs. This is due to the fact that solving the Steiner problem requires to find the location of an undetermined number of additional vertices (the Steiner points) and in this *location* problem the geometry of the ambient space plays a fundamental role.

2. History of the problem

The simplest interesting case of the Steiner problem is the case when A is composed by three points in the plane[1]. This case was first studied by Torricelli, Cavalieri and Fermat in the 17th century. They formulated the problem as follows.

Problem 2.1 (Fermat problem)**.** *Given a triangle with vertices a_1, a_2, a_3 find the point p such that the sum of the distances of p from the three vertices is minimal.*

The point p is usually called the *Fermat point* of the triangle. If the triangle has an angle greater or equal to 120 degrees, the point p coincides with the vertex of that angle. Otherwise the point p is internal to the triangle and has very nice geometrical properties (see Figure 1):

(1) it is the intersection of the three circles passing through the vertices of the three equilateral triangles constructed on the sides of the triangle $a_1a_2a_3$;
(2) it is the intersection of the three lines passing by a vertex of the triangle and the third vertex of the equilateral triangles constructed on the opposite side;
(3) it is the point which *sees* the three sides of the triangle with an equal angle of 120 degrees.

The Fermat problem can be generalized by considering a finite set of points $a_1, \dots, a_N$ and looking for a point p such that the sum of the distances $\sum_i d(p, a_i)$ is minimal. This problem was introduced by Steiner and was called the *generalized Fermat problem*. However this is not what we call the Steiner problem since, as we have seen, the solution to the Steiner problem can have more than a single Steiner point if we want to connect a set with more than three points.

In 1934 Jarnìck and Kössler [aMK34] introduced for the first time the problem with finite graphs: given a set of points $a_1, \dots, a_N$ find the shortest graph among all connected

[1] As a matter of fact the case when A is composed by two points is also important: in that case the problem becomes the problem of finding geodesics.

graphs containing the given points. They proved the existence of a minimizer. Also they found the minimum in some special cases, namely in the cases when the given set A is composed by the vertices of a regular polygon with $N = 3, 4, 5, 6$ and $N \geq 13$ vertices. The gap $N = 6, \dots, 12$ was closed in 1987 by Du, Hwang and Weng [DHW87]. In 1941 Courant and Robbins published the famous book "what is mathematics" [CR41] where they include a presentation of Fermat problem and consider the generalization of Steiner (with a single additional point) which they call "Steiner Problem". Then they introduce the same idea of Jarnìck and Kössler and call it "generalized Steiner problem". The success of the book gave a wide spread to this problem which hence became widely known as the "Steiner problem".

We will not try to change the historical name of the Steiner problem even if we recognize that it would be more appropriate to name it after Jarnìck and Kössler.

More information on the Steiner problem can be found in [FKH92, IT94].

3. Classical statement and existence result

The Steiner problem, as introduced in the 20th century, has the following statement.

Problem 3.1 (classical Steiner problem)**.** *Given the set of points $A = \{a_1, \dots, a_N\}$ in the euclidean plane, find the shortest graph Σ among all (finite) connected graphs containing all the given points.*

Even if the problem looks quite simple, the existence of a solution is not straightforward. In fact the space of all finite graphs is not finite dimensional since we do not have an upper limit on the *order* of the graphs (i.e. the number of vertices) since the number of Steiner points might, in principle, become arbitrarily large. To prove the existence of solutions it turns out that it is first necessary to find some variational properties of minimal graphs which will enable us to modify the minimizing sequences in order to make them converge to a finite graph.

The first property is that minimal graphs cannot contain loops (in fact we always call them minimal *trees*). This is trivially true because if the graph contained a loop, the

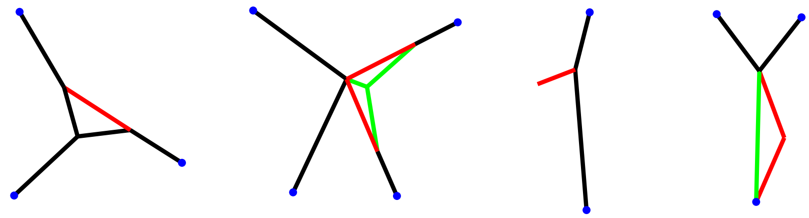

FIGURE 2. The four variations which are used to decrease the length of a competitor graph. The blue points are the points to be connected. When the red lines are removed and replaced by green lines the total length decreases.

graph would be doubly-connected hence we could remove an edge of the loop without disconnecting the graph.

The second property is that every Steiner point (i.e. every vertex of the graph which is not one of the given points $a_1, \dots, a_n$) has order not larger than 3. This is due to the property of Fermat points. In fact if we had 4 or more segments meeting at a point, at least two of them must define an angle which is less than 120 degrees[2]. Then by applying Fermat construction in a small triangle with a vertex in the Steiner point and two sides on the two segments, we are able to decrease the total length of the given graph, without disconnecting it (see Figure 2). Applying this construction we conclude that every Steiner point of a minimal Steiner tree must have order 3 (vertices with order 2 are clearly not optimal) and moreover the three edges joining in the Steiner point must define equal angles of 120 degrees[3].

The third property is that every vertex of order 1 is a point of A. In fact if the graph has a vertex V of order 1 and if it is not a point of A, it is possible to remove the (unique) edge of the graph which has V as endpoint (if some points of A lie on that edge, we remove only a part of the edge, until we reach the first point of A). We can repeat this procedure until all vertex of order one are points of A.

[2]Notice that this is true even in higher dimensions, not only in the plane.

[3]Again this is true also in higher dimension (i.e. in the space) where we conclude that Steiner Trees must be locally planar around every Steiner point.

The fourth property is that every vertex of order 2 is also a point of A. In this case we notice that if V is a vertex of order 2 there are two edges joining it. We can hence remove the two edges (and the vertex) and replace them with a single edge with the same end-points. Of course if points of A lie on one of the two edges, we remove only a part of that edge, until we reach a point of A.

These four properties, taken together, state that in a minimal graph (if it exists!) all the vertices which are not in A are of order 3. This enables us to obtain an estimate on the total number of vertices. In fact let v_1, v_2, v_3 be, respectively, the number of vertices of order 1, 2 and 3. We claim that if the graph has at least two vertices then $v_3 \leq v_1 - 2$. This is true for all graphs which are connected, contain no loop, and have vertices of order not greater than 3. The claim can be proven by induction removing, one after the other, the terminal points. In our case we know that $v_1 + v_2 \leq |A|$ and hence the total number of vertices $v = v_1 + v_2 + v_3 \leq |A| + v_1 - 2 \leq 2|A| - 2$.

Notice that up to now we have stated some nice property of a minimal graph even if we do not yet know that it exists... To prove the actual existence of solutions we will apply these properties to a minimizing sequence. If S_k is a minimizing sequence (i.e. a sequence of connected graph containing a given finite set A, such that the length of S_k converges to the infimum length of all possible connected graphs containing A) then we apply to each S_k all the local modification we have mentioned so that we obtain a sequence S'_k of connected graphs which again is minimizing but now we can assume that S'_k satisfies the estimate on the number of vertices: $v \leq 2|A| - 2$. The number of possible different topologies of the graphs S'_k is hence finite and we are able to find a subsequence S'_{k_j} which all have the same number of vertices and the same topology. This means that every such graph S'_{k_j} can be represented by a point $p_j \in (\mathbb{R}^2)^n$ where n is the number of vertices of the graph, and the length of the graph is a (fixed) continuous function $f\colon (\mathbb{R}^2)^n \to \mathbb{R}$. It is not difficult to prove that the points p_j must be equibounded otherwise the length of the corresponding graphs would go to infinity. So, by Weierstraß Theorem we known that, up to a subsequence, the points p_j converge to some point p. This point represents a graph S which, by continuity, must be a connected graph containing all points of A and whose length is minimal. Hence we have found the existence of a solution to the classical Steiner problem.

4. Generalized Steiner Problem: existence

We notice that the statement of the Steiner problem only need two basic concepts: length and connectedness. In any metric space X we have both: the connectedness is defined by topology and the length can be defined by the Haudorff measure. It is not necessary to restrict ourselves to the class of finite graphs, we can deal with the larger class of compact sets $S \subset X$.

The family of non-empty compact sets can be endowed with the Hausdorff uniform distance, defined by:

$$d(S_1, S_2) = \inf\{\varepsilon > 0 \colon S_1 \subset (S_2)_\varepsilon \text{ and } S_2 \subset (S_1)_\varepsilon\}$$

where $(S)_\varepsilon = \{x \in X \colon d(x, S) < \varepsilon\}$ is the ε-neighbourhood of a set S.

We know that the space of compact subsets of a compact metric space is compact with respect to this metric (Blaschke Theorem). Moreover the Gołąb Theorem asserts that the $\mathcal{H}^1$-measure is lower semicontinuous on connected sets. Taken together these tools give the following result:

Theorem 4.1. *Let X be a compact, connected metric space and let $A \subset X$ be a compact set. Then there exists a set S which has minimal $\mathcal{H}^1$ measure among all compact connected sets containing A.*

The proof is simple. Consider the family $\mathcal{F}$ of all compact, connected sets which contain A. By hypothesys $X \in \mathcal{F}$, hence $\mathcal{F}$ is not empty. Let $m = \inf\{\mathcal{H}^1(S) \colon S \in \mathcal{F}\}$ and let S_k be a minimizing sequence, i.e. $S_k \in \mathcal{F}$ such that $\mathcal{H}^1(S_k) \to m$. By Blaschke Theorem, up to a subsequence, we might suppose that S_k converge uniformly to a compact set S. By continuity $S \in \mathcal{F}$. Moreover by Gołąb theorem we conclude that $\mathcal{H}^1(S) \leq \liminf_k \mathcal{H}^1(S_k) = m$ and hence S is a minimal set.

It is not really necessary to require that X be compact. In fact if S_k is a sequence of sets all containing A and with a uniform bound on the length, then all these sets must

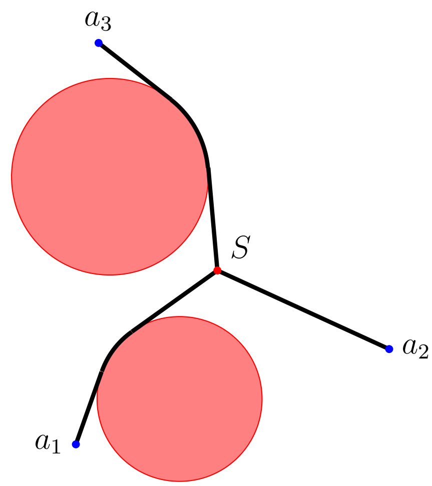

FIGURE 3. An example of Steiner problem with obstacles. The ambient space X is the plane with the two red circle removed. The set A is composed by the three blue points. The set S is the resulting minimizer.

be contained in a sufficiently large ball. Hence the previous theorem holds true in the following more general form:

Theorem 4.2. *Let X be a metric space with the Heine-Borel property (that is: every closed ball is compact) and let A be a compact subset of X. If there exists a compact connected set containing A, then there exists a set S which has minimal length among all compact connected sets containing A.*

The previous theorem applies, for example, when X is any closed subset of $\mathbb{R}^N$. Even this simple generalization is quite useful even in the case when A is finite. For example we can model the concept of *obstacle* in the Steiner problem (see for example Figure 3).

We have added the hypothesys that a compact connected set containing A exists. This is needed because otherwise the family of competitors would be empty and hence it would be impossible to find a minimizer.

However notice that we do not exclude the case $\mathcal{H}^1(S) = +\infty$. In fact in this general setting it may happen that all connected sets containing A have infinite length.

Here are two simple examples. In the first example we have an infinite set A in the plane.

Example 4.1. *Let $X = [0,1]^2 \subset \mathbb{R}^2$ and consider the compact set*

$$A = \left\{ \left(\frac{1}{n}, \frac{1}{m} \right) : n, m \in 1, 2, \dots, \infty \right\}$$

where we let $1/\infty = 0$. If S is a connected set such that $S \supset A$, then $\mathcal{H}^1(S) = +\infty$.

To prove the assertion in the previous example it is enough to consider a disjoint family of balls centered in the points of A. We know that S being connected and passing trought the center of every ball, its length must be larger than the sum of all the radii of the balls. By direct computation we can prove that such sum is infinite.

In the second example we have a finite set in a connected but not arcwise connected metric space.

Example 4.2. *Let $X \subset \mathbb{R}^2$ be defined by*

$$X = \{(x, y) \in \mathbb{R}^2 \colon x \in [-1, 1] \setminus \{0\}, y = \sin(\pi/x)\} \cup \{(0, y) \colon y \in [-1, 1]\}$$

and consider the set $A = \{(-1, 0), (1, 0)\} \subset X$. X is compact and connected. If S is any connected subset of X containing A, then $\mathcal{H}^1(S) = +\infty$.

To prove the assertion in the previous example just notice that every such set S must necessarily contain the graph of the function $y = \sin(1/x)$ which has infinite length.

On the other hand it is easy to prove that if X is compact and endowed with a geodesic distance (i.e. the distance between two points is equal to the infimum of the lengths of the curves joining them) and if A is finite, then there exists a compact and connected set containing A with finite length. In fact one can consider the union of a finite number of curves with finite length which connect every point of A with some fixed point.

Theorem 4.2 is quite more general than the classical Steiner problem. However this formulation makes no sense when the given set A has infinite $\mathcal{H}^1$ measure. A possible formulation which can be easily handled with the same tools seen so far is the following:

Theorem 4.3. *Let X be a metric space with the Heine-Borel property. Let A be a compact subset of X. If there exists a compact connected set containing A, then there exists a compact connected set S with minimal length among all compact connected sets such that $S \cup A$ is connected.*

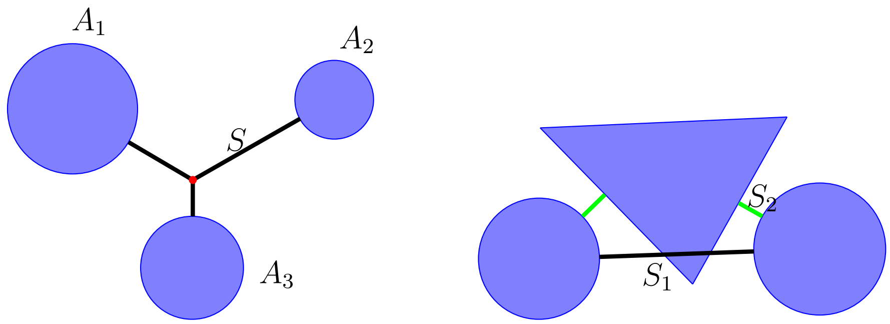

FIGURE 4. *On the left:* the set $A = A_1 \cup A_2 \cup A_3$ has infinite $\mathcal{H}^1$-measure, nevertheless it is natural to look for a minimizing set S which connects the three components. *On the right:* The black segment represents the minimal set S_1 which is connected and such that $S_1 \cup A$ is connected (the set A has three components represented in blue). The green set composed by two segments is the minimal set S_2 such that $S_2 \cup A$ is connected without requiring S_2 itself being connected.

With this formulation we don't require S to contain A but only to let $S \cup A$ become connected. This is a nice formulation of the concept of *connecting a set*. The fact that S is also required to be connected is needed to adapt the proof already exposed for Theorem 4.1. In fact the Gołąb Theorem need the sequence of sets S_k to be connected, otherwise it is not guaranteed that the $\mathcal{H}^1$ measure is lower semicontinuous[4].

Unfortunately the requirement of S to be connected leads to an undesiderable behaviour: if we require $S \cup A$ to be connected, it is not natural to require also S to be connected. See Figure 4 as an example.

It is possible to drop the requirement of S being connected by considering a slight generalization of the Gołąb Theorem. What we need to do is considering the sets $A \cup S_k$ which are connected, but to prove that the semicontinuity result holds for $\mathcal{H}^1(S_k \setminus A)$. This requires a *localized* version of Gołąb Theorem which is proved in [PS12]. With these considerations one can get the following result:

[4]for example a sequence of finite sets (each of which has zero $\mathcal{H}^1$ measure) can converge to a segment with positive length

Theorem 4.4. *Let X be a metric space satisfying the Heine-Borel property. Let A be a compact subset of X. If there exists a compact connected set containing A then there exists a compact set S such that $S \cup A$ is connected and $\mathcal{H}^1(S)$ is minimal among all compact sets S' such that $S' \cup A$ is connected.*

The previous statement seems very close to the *intuitive* statement: *find the shortest set connecting a given set A.* The only somewhat innatural requirements are about the compactness of S, even if, in view of Blaschke Theorem, one might believe that the compactness of the competitors is required. It was a surprise to discover that this is not the case. In fact we can state the following:

Theorem 4.5. *Let X be a connected metric space with the Heine-Borel property. Let A be a compact subset of X. We say that a set S* connects *A if $S \cup A$ is connected. Then there exists a set S such that $\mathcal{H}^1(S)$ is minimal among all sets which connect A.*

In this statement we require that the ambient space X is connected because this is a nice way to be sure that the family of competitors is not empty, since X is now itself a competitor (we don't require competitors to be compact anymore).

The proof of Theorem 4.5 is quite involved. The key point is the fact that if S is a connected set with finite length then $\bar{S}$ is also connected and has the same length. Moreover a set with finite length is also bounded and hence $\bar{S}$ turns out to be compact. So it is possible to recover the setting of Theorem 4.4.

5. Generalized Steiner Problem: properties of minimizers

Once we have stated the existence of solutions to the Steiner problem in the quite general setting of Theorem 4.5 it is natural to ask whether such minimizers enjoy some nice property as the classical minimizers do.

In this section we will state that, in fact, all the properties of classical minimizers can be recovered when they make sense.

First of all we want to highlight a property of classical minimizers which plays an important role also in the generalized case. We say that a Steiner tree S over a set A is *full* or *complete* or *indecomoposable* if the set $S \setminus A$ is connected. Looking at Figure 5 we

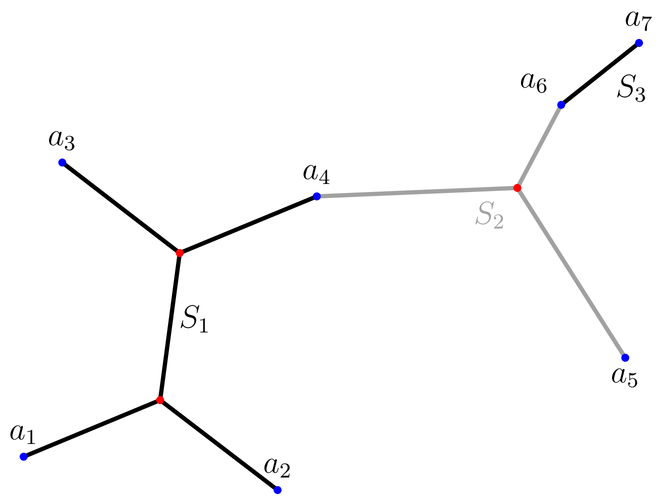

FIGURE 5. An example of decomposable Steiner Tree. The minimizer S over the set $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ is the union $S = S_1 \cup S_2 \cup S_3$ where S_1 is the minimizer over $A_1 = \{a_1, a_2, a_3, a_4\}$, S_2 is the minimizer over $A_2 = \{a_4, a_5, a_6\}$ and S_3 is the minimizer over $A_3 = \{a_6, a_7\}$. The points a_4 and a_6, if removed, cause a disconnection of S.

understand how any classical Steiner Tree can be decomposed in the union of smaller full Steiner trees.

A complete Steiner trees is characterized by not having any vertex of order two. In fact such vertices must be points of A and if removed cause a disconnection of the tree. So every point of A is an *end-point* of the graph (i.e. a vertex of order one). If a is the number of points of A (vertices of order one), s is the number of steiner points (vertices of order three) and e is the number of edges of a complete Steiner tree, we notice that $s = a - 2$ and $e = 2a - 3$. These equalities can be found by induction, as we have done in Section 2.

In the following statement we collect the properties which are valid in the generalized Steiner problem (see [PS12] for details).

Theorem 5.1. *Let X be a metric space and $A \subset X$ a compact set. Let S be a minimizer of $\mathcal{H}^1(S)$ among all subsets of X such that $S \cup A$ is connected. Suppose moreover that $\mathcal{H}^1(S) < +\infty$. Then $S \setminus A$ has at most a countable number of connected components.*

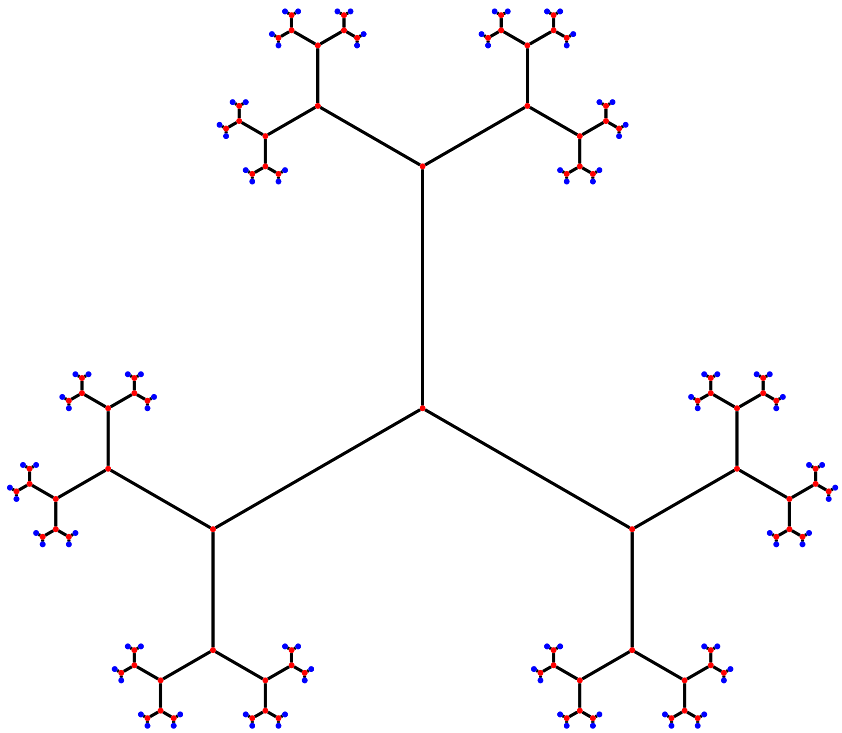

FIGURE 6. The sixth iteration of a splitting tree S_6 ($n = 6$). The boundary set A_6 is composed by $|A_6| = 3 \cdot 2^{n-1} = 96$ points (in blue). The tree S has $3 \cdot 2^{n-1} - 2 = 94$ Steiner points (in red) and $3 \cdot (2^n - 1) = 93$ edges.

Moreover each component S_0 of $S \setminus A$ is a topological tree composed by at most a countable number of vertices and edges and which is locally finite in $X \setminus A$.

The results stated in the previous theorem are optimal. Here are some examples.

Example 5.1. *Let $X = [0,1]$ and let $A \subset X$ be the Cantor set $A = A/3 \cup (1 - A/3)$. It is easy to see that $S = X \setminus A$ is a minimizer, in fact every point of S is necessary to connect A while $S \cup A = [0,1]$ is connected. Notice that $S = S \setminus A$ is a countable union of disjoint intervals. Notice that even if A is not countable, S is composed by a countable number of disjoint intervals. Every such interval connects only two points of A. As a consequence we might observe that there are points of A which are not in the closure of any connected component but, on the contrary, are accumulation points of a sequence of connected components.*

Example 5.2. *Another example is depicted in Figure 6. We consider a set S_1 which is composed by three edges which join on a triple point. Once we have constructed S_n we construct S_{n+1} by adding a triple point on every terminal vertex using edges with sufficiently small length t_n. We let A_n be the set of terminal points of S_n. It is possible to prove that when t_n goes to zero fast enough, the limit set $S = \bigcup_n S_n$ is a minimizer over the set A which is the limit, in the Hausdorff metric, of the terminal points A_n.*

We notice that the set A is homeomorphic to a Cantor set. In particular A is more than countable. On the other hand the set S is complete ($S \setminus A$ is connected) and it is an infinite tree composed by a countable number of edges and a countable number of Steiner points. However the set S is locally finite outside A, in fact given any $\varepsilon > 0$ we notice that only a finite number of Steiner points and edges of S have distance greater than ε from A.

As in the last example, we notice that since the minimizer S is locally finite, we can apply the small variations considered in Section 2 and recover all the properties of classical Steiner trees:

Theorem 5.2. *Let X be a complete connected Riemann manifold and let $A \subset X$ be compact. Then among all sets $S \subset X$ such that $S \cup A$ is connected there exists one set S which minimizes $\mathcal{H}^1(S)$. Moreover any minimizer with finite length is composed by the union of countable many geodesics which meet in triple points with angles of 120 degrees. The set of triple points is locally finite in $X \setminus A$.*

6. Open problems

In this section we address some of the questions which, in our knowledge, remain open.

Question 6.1. *Is it possible to find a bounded set A in $X = \mathbb{R}^N$ such that the problem: "find $S \subset X$ such that $\mathcal{H}^1(S)$ is minimal among all sets for which $S \cup A$ is connected" does not have a solution?*

In this question the set A must not be closed, otherwise we know that a solution exists. Also remember that we do not require the minimal set to have finite length (see Example 4.1).

An example of non-existence when A is not bounded is easy to find, just consider $X = \mathbb{R}^2$ and $A = \{(x,y) \in \mathbb{R}^2 \colon |y| = e^x\}$. For each $\varepsilon > 0$ the set $S_\varepsilon = \{(\log\varepsilon, y)\colon |y| \leq \varepsilon\}$ has length $2\varepsilon > 0$ and $S_\varepsilon \cup A$ is connected. However A cannot be connected with a set of zero length.

Here are some examples where the minimizer exists even if A is not closed.

Example 6.1. *Let $X = \mathbb{R}$, $A = [0,1] \cap \mathbb{Q}$. The set $S = [0,1]$ is a minimal set connecting A with length 1. If instead we choose $A = [0,1] \setminus \mathbb{Q}$ the set $S = [0,1] \cap \mathbb{Q}$ is a minimal set connecting A and has length zero.*

Example 6.2. *Let $X = \mathbb{R}$, $A = [-1,1] \setminus \{0\}$. The set $S = \{0\}$ is a minimal set connecting A.*

Example 6.3. *Let $X = \mathbb{R}^2$, $A = \{(x,y) \in \mathbb{R}^2 \colon x \neq 0, y = \sin(1/x), |x| < 1\}$. Then the set $S = \{(0,0)\}$ is a minimal set connecting A. Notice that in this case $S \cup A$ is connected but not arcwise connected!*

Question 6.2. *Consider the set constructed in Example 5.2. We know that choosing the length t_n in some appropriate way, such a set is indeed minimal. However if we choose $t_n = \lambda^n$ for some $\lambda > 0$, the resulting set S becomes a* self-similar *fractal tree. Does exist some $\lambda > 0$ such that this fractal tree is a minimizer (with respect to its terminal points)?*

About the previous question we notice that for some λ the resulting fractal tree develops some self-touching parts, hence it turns out not actually being a topological tree. Hence if λ is not small enough, the set cannot be minimal.

References

[aMK34] V. Jarník and M. Kössler. O minimálních grafech, obsahujících n daných bodů. *Časopis pro pěstovánímatematiky a fysiky*, 063(8):223–235, 1934.

[CR41] R. Courant and Robbins. *What Is Mathematics?* Oxford Univ. Press, 1941.

[DHW87] Ding-Zhu Du, Frank K. Hwang, and J. F. Weng. Steiner minimal trees for regular polygons. *Discrete & Computational Geometry*, 2:65–84, 1987.

[FKH92] P. Winter F. K. Hwang, D. S. Richards. *The Steiner tree problem.* Elsevier Science Publisher B.V., 1992.

[IT94] A. O. Ivanov and A. A. Tuzhilin. *Minimal networks: the Steiner problem and its generalizations.* CRC Press, 1994.

[PS12] Emanuele Paolini and Eugene Stepanov. Existence and regularity results for the steiner problem. *Calc. Var. Partial Diff. Equations*, 2012.

Dipartimento di Matematica "U. Dini", Universit di Firenze

E-mail address: paolini@unifi.it