SYMMETRY AND RIGIDITY RESULTS FOR COMPOSITE MEMBRANES AND PLATES
RISULTATI DI SIMMETRIA E RIGIDITÀ PER MEMBRANE E PIASTRE COMPOSITE

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Abstract. The *composite membrane problem* is an eigenvalue optimization problem deeply studied from the beginning of the '00's. In this note we survey most of the results proved by several authors over the last twenty years, up to the recent paper [14] written in collaboration with Giovanni Cupini.

We finally introduce an eigenvalue optimization problem for a fourth order operator, called *composite plate problem* and we present the symmetry and rigidity results obtained in this framework.

These last mentioned results are part of the papers [12, 13] written in collaboration with Francesca Colasuonno.

Sunto. Il problema della membrana composita è un problema di ottimizzazione di autovalori i cui primi contributi risalgono agli inizi degli anni '00. In questa nota presentiamo una sintesi dei principali risultati ottenuti negli ultimi venti anni, fino al recente contributo [14] scritto in collaborazione con Giovanni Cupini.

Introdurremo poi un problema di ottimizzazione di autovalori per un operatore del quart'ordine noto come problema della piastra composita, e presenteremo alcuni risultati di simmetria e rigidità in questo ambito. Questi ultimi risultati sono contenuti nei lavori [12, 13] scritti in collaborazione con Francesca Colasuonno.

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1. Introduction

Suppose you are given the prescribed shape and mass of a membrane, and several materials of varying densities: how can you build the membrane in such a way that the basic frequency is the smallest possible?
This is the physical content of the so called *composite membrane problem*. From the mathematical point of view, this turns out to be an eigenvalue optimization problem that has connections to several areas of mathematical analysis. Starting from the '00's, there has been a certain interest devoted to the above mentioned problem, as witnessed by the list of papers [8, 9, 10, 11, 23, 7]. Keeping in mind that the physically relevant case is provided by $n = 2$, we will introduce the problem in $\mathbb{R}^n$ with $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a non-empty open bounded domain with Lipschitz boundary $\partial\Omega$, let $0 \leq h < H$ be two positive constants, and $M \in [h|\Omega|, H|\Omega|]$. We define the set of admissible densities as

$$\mathrm{P} := \left\{ \rho : \Omega \to \mathbb{R} : \int_\Omega \rho(x)\,dx = M,\ h \leq \rho \leq H \text{ in } \Omega \right\}. \tag{1}$$

By *composite membrane problem* we mean the following minimization problem

$$\Theta(h, H, M) := \inf_{\rho \in \mathrm{P}} \inf_{u \in H^1_0(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2}{\displaystyle\int_\Omega \rho\, u^2}. \tag{2}$$

A couple (u, ρ) which realizes the double infimum in (2) is called (2)-optimal pair.
Once the problem is set, there are a bunch of questions that naturally arise. In this note we will focus on the following ones:

$$\begin{cases} \text{do optimal pair exist? Can they be characterized?} \\ \text{are there preservation or breaking of symmetry phenomena?} \\ \text{is the optimal pair unique?} \end{cases} \tag{3}$$

The (partial) answers to these questions are the content of section 2. In section 3 we then introduce the higher order version of Problem (2), namely the *composite plate problem*, and we report on the major results obtained so far (see also [2, 1] for similar results).

2. The composite membrane problem

In this section we collect most of results concerning the composite membrane problem. As already mentioned, Problem (2) as stated before has been considered for the first time in [8] and [9]. Among many other results, the authors proved there that Problem (2) can be seen as a particular case of a more general eigenvalue optimization problem, which can be stated as follows: let $\Omega \subset \mathbb{R}^n$ be a non-empty open and bounded set with Lipschitz boundary $\partial\Omega$. For every $A \in [0, |\Omega|]$, we denote by

$$\mathcal{D} := \{D \subset \Omega : D \text{ measurable set}, |D| = A\}, \tag{4}$$

the class of *admissible sets*. For any set $D \in \mathcal{D}$, let χ_D be its characteristic function. For every $\alpha > 0$ and $D \in \mathcal{D}$ we consider

$$\begin{cases} -\Delta u + \alpha\chi_D u = \lambda u & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Let us denote with $\lambda_\Omega(\alpha, D)$ the lowest eigenvalue of this boundary value problem. It is now possible to consider the following eigenvalue optimization problem:

$$\Lambda_\Omega(\alpha, A) := \inf_{D\in\mathcal{D}} \lambda_\Omega(\alpha, D), \tag{6}$$

whose variational characterization is given by

$$\Lambda_\Omega(\alpha, A) = \inf_{D\in\mathcal{D}} \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 + \alpha \int_\Omega \chi_D u^2}{\displaystyle\int_\Omega u^2}.$$

Any minimizer D in (6) is called an *optimal configuration* for the data (Ω, α, A). If moreover u satisfies (5) then (u, D) is called a (6)-optimal pair. The variational characterization of (6) clearly shows that changing D by a set of measure zero does not affect $\lambda_\Omega(\alpha, D)$ nor u, hence we are allowed to consider sets D that differs by a null-set as equal. Moreover, u can be chosen to be positive in Ω.

The existence and characterization of optimal pairs is pretty well-understood, as for their dependence on the initial data. In the next theorem we condensate several results which are parts of the content of [8, Theorem 1, Theorem 2, Proposition 10].

Theorem 2.1. *Let Ω be an open and bounded set of class $C^{1,1}$.* [1] *For any $\alpha > 0$ and $A \in [0, |\Omega|]$ there exists a (6)-optimal pair (u, D) with u positive. Moreover, every optimal pair satisfies the following properties:*

(i) $u \in H^2(\Omega) \cap C^{1,\delta}(\Omega) \cap C^{\gamma}(\overline{\Omega})$, *for some $\gamma > 0$ and every $\delta < 1$.*

(ii) *There exists a positive number $t = t(A, \Omega, u) > 0$ such that*

$$D = \{x \in \Omega : u(x) \leq t\}.$$

(iii) *D contains a tubular neighborhood of the boundary $\partial\Omega$ of Ω.*

(iv) *$\Lambda_\Omega(\alpha, \cdot)$ is strictly increasing for fixed $\alpha > 0$, and the function $\Lambda_\Omega(\cdot, A)$ is strictly increasing for fixed $A > 0$. Moreover, $\Lambda_\Omega(\alpha, A) - \alpha$ is strictly decreasing in α for any fixed $A \in (0, |\Omega|)$.*

(v) *Given $A \in [0, |\Omega|)$, there exists a unique positive number $\overline{\alpha}_\Omega(A) > 0$ such that*

$$\Lambda_\Omega(\overline{\alpha}_\Omega(A), A) = \overline{\alpha}_\Omega(A).$$

(vi) *If Ω is simply connected and $\alpha < \overline{\alpha}_\Omega(A)$, then D is connected.*

We refer to [8] for the proofs.

We want to stress that the positive number $t > 0$ appearing in (ii) is defined as

$$t = t(A, \Omega, u) := \sup\{s : |\{u < s\}| < A\},$$

see [8, Equation (8)]. We also note that, due to (iv), the positive number $\overline{\alpha}_\Omega(A)$ appearing in (v) is well defined. Unfortunately, as far as we know, the implicitly defined number $\overline{\alpha}_\Omega(A)$ has never been explicitly calculated nor estimated.

An immediate and important consequence of (iv) and (v) is that

$$\Lambda_\Omega(\alpha, A) - \alpha > 0, \quad \text{for every } \alpha < \overline{\alpha}_\Omega(A). \tag{7}$$

By means of Theorem 2.1, the connection between Problem (6) and Problem (2) can now be made truly explicit.

[1] We thank the referee for pointing out that the assumption on Ω in [8] is not correct in order to obtain the claimed regularity.

Theorem 2.2 ([8], Theorem 13)**.** (a) *Let (u,ρ) be a (2)-optimal pair, then ρ has the following form:*

$$\rho = h\,\chi_D + H\,\chi_{D^c\cap\Omega},$$

for a set of the form $D = \{u \le t\}$.

(b) *The pair (u,ρ) is a (2)-optimal pair with parameters (h,H,M) if and only if (u,D) is a (6)-optimal pair with parameters (α,A) given by*

$$\alpha = (H-h)\Theta, \quad A = \tfrac{H|\Omega|-M}{H-h}.$$

Moreover, the two minimal eigenvalues are related by

$$\Lambda = H\Theta.$$

We stress that the case $\alpha = \overline{\alpha}_\Omega(A)$ corresponds to $h = 0$, which actually reduces to the case of the classical first eigenvalue of the Dirichlet Laplacian. In particular, given A, when $\alpha \in (0, \overline{\alpha}_\Omega(A))$, the two variational problems are in one–to–one correspondence. Therefore, when dealing with the composite membrane problem, we are allowed to choose the formulation we like most.
Coming back to the list of questions, the first is now addressed. We want now to illustrate how the other two questions are correlated. First, the next result, once again proved in [8], shows that under appropriate assumptions on the set Ω preservation of symmetry occurs.

Theorem 2.3 ([8], Theorem 4)**.** *Assume that Ω is symmetric and convex with respect to the hyperplane $\{x_1 = 0\}$, i.e. for each $x' = (x_2,\dots,x_n)$ the set*

$$\{x_1 : (x_1,x') \in \Omega\}$$

is either empty or an interval of the form $(-c,c)$. Let (u,D) be a (6)-optimal pair. Then both u and D are symmetric with respect to $\{x_1 = 0\}$, $D^c \cap \Omega$ is convex with respect to $\{x_1 = 0\}$, and u is decreasing in x_1 for $x_1 \ge 0$.

An immediate consequence is the following

Corollary 2.1 ([8], Corollary 5). *Let $\Omega = B = \{\|x\| < 1\}$ be the unit ball. Then, for any $\alpha > 0$ and $A \in (0, |\Omega|)$ there is a unique optimal configuration D given by the shell region*

$$D = \{x \in \Omega : r(A) < \|x\| < 1\}.$$

We stress that Corollary 2.1 is a *uniqueness* result and, to the best of our knowledge, the case of the ball is the only one for which uniqueness is guaranteed so far. Concerning uniqueness, in [8] it is proved that even under the symmetry assumptions of Theorem 2.3, one can lose symmetry if the directional convexity is missing. The counterexample provided in [8] is the content of the following

Theorem 2.4 ([8], Theorem 7). *For $k \in (0, 1)$ define the dumbbell with handle width $2k$,*

$$\Omega_k := B_1(-2, 0) \cup ((-2, 2) \times (-k, k)) \cup B_1(2, 0),$$

where $B_r(p) := \{x \in \mathbb{R}^2 : \|x - p\|_{\mathbb{R}^2} < r\}$. Fix $\alpha > 0$ and $A \in (0, 2\pi)$. Then there is $k_0 = k_0(\alpha, A) > 0$ such that we have for $k < k_0$:

- *Any optimal pair (u, D) is not symmetric with respect to the x_2-axis.*
- *If $A > \pi$, then for any optimal pair (u, D) the complement $D^c \cap \Omega$ is contained in one of the balls $B_1(\pm 2, 0)$.*

This example shows the necessity of the directional convexity assumed in Theorem 2.3, and provides also an explicit example of breaking of symmetry phenomena.

Let us spend a few words concerning the proof of Theorem 2.3. The main technical tool used to prove Theorem 2.3 is Steiner symmetrization, combined with a result of Brothers and Ziemer [6]. We recall below the definition of Steiner rearrangement. Given a measurable function $u : \Omega \to \mathbb{R}$, we define the *distribution function* $\mu_u : \mathbb{R} \to \mathbb{R}$ of u as

$$\mu_u(\tau) := |\{x \in \Omega : u(x) > \tau\}|.$$

In order to simplify the notation, we will write $\{u > \tau\}$ in place of $\{x \in \Omega : u(x) > \tau\}$. An analogous notation will be adopted for sub-level and level sets as well. The *decreasing*

Steiner rearrangement $u^\sharp : [0, |\Omega|] \to \mathbb{R}$ of u is defined as

$$u^\sharp(s) := \begin{cases} \operatorname{ess\,sup} u, & s = 0, \\ \inf\{\tau : \mu_u(\tau) < s\}, & 0 < s \le |\Omega|. \end{cases}$$

The *increasing Steiner rearrangement* $u_\sharp : [0, |\Omega|] \to \mathbb{R}$ of u is defined as

$$u_\sharp(s) := \begin{cases} \operatorname{ess\,sup} u, & s = |\Omega|, \\ \inf\{\tau : |\{u < \tau\}| > s\}, & 0 \le s < |\Omega|. \end{cases}$$

The idea of the proof of Theorem 2.3 is the following: by Theorem 2.1, we know that there exists an optimal pair (u, D). Now, let $D_\sharp$ be the set defined through its characteristic function,

$$\chi_{D_\sharp}(x) := (\chi_D)_\sharp(x), \quad x \in \Omega. \tag{8}$$

One can now prove that $(u^\sharp, D_\sharp)$ is an admissible pair whose energy is less than the one of (u, D), thanks to the classical Pólya–Szegö principle.

This fact leads to a somehow natural question:

Is it possible to prove a Faber-Krahn-type inequality in this context?

We recall that the classical Faber-Krahn inequality for the Dirichlet Laplacian states that the balls minimize $\Omega \mapsto \lambda(\Omega)$ among all the sets of given measure. We refer to [21] for a proof and related results.

In order to state a precise result in this direction, let us recall the notion of Schwarz rearrangement. Given a bounded set Ω, we denote by $\Omega^\star$ the ball centered at the origin and volume $|\Omega^\star| = |\Omega|$. We also denote by ω_n the measure of the unit ball. Let now $u : \Omega \to \mathbb{R}$ be a measurable function. The *decreasing Schwarz symmetrization* $u^\star : \Omega^\star \to \mathbb{R}$ of u is defined as

$$u^\star(x) := u^\sharp(\omega_n \|x\|^n), \quad x \in \Omega^\star,$$

and the *increasing Schwarz symmetrization* $u_\star : \Omega^\star \to \mathbb{R}$ of u is defined as

$$u_\star(x) := u_\sharp(\omega_n \|x\|^n), \quad x \in \Omega^\star.$$

Once the above notation is fixed, we can state a result which answers positively to the above question. The precise statement is the content of the following

Theorem 2.5. *Given a ball $\Omega^\star$ centered at the origin, $A \in (0, |\Omega^\star|)$ and $\alpha \in (0, \overline{\alpha}_{\Omega^\star}(A))$, then*

$$\Lambda_{\Omega^\star}(\alpha, A) \leq \Lambda_\Omega(\alpha, A), \tag{9}$$

for every open and bounded connected set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, with $|\Omega| = |\Omega^\star|$. Moreover, the equality holds if and only if $\Omega = \Omega^\star$ up to translations.

The previous result can be proved along the same lines of the proof of [8, Theorem 4], that is by using Schwarz symmetrization and the Pólya-Szegö principle, in a standard way. Identification of equality cases could then be done by appealing to a celebrated result by Brothers and Ziemer (see [6, Theorem 1.1]). However, in [14] we gave a different proof of Theorem 2.5 for general sets, by adapting to our setting the proof of the classical Faber-Krahn inequality due to Kesavan [20], which in turn relies on a well-known result by Talenti [24] which we recall right now:

Theorem 2.6 ([24])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set of finite measure, and let $\Omega^\star$ be the ball centered at the origin and measure $|\Omega^\star| = |\Omega|$. Let $f \in L^2(\Omega)$ be nonnegative, and let $f^\star$ be its Schwarz symmetrization. If $u \in H^1_0(\Omega)$ is the weak solution of*

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and $v \in H^1_0(\Omega^\star)$ is the weak solution of

$$\begin{cases} -\Delta v = f^\star, & \text{in } \Omega^\star, \\ v = 0, & \text{on } \partial\Omega^\star, \end{cases}$$

then $v(x) \geq u^\star(x)$ for almost every $x \in \Omega^\star$.
Moreover, if $u^\star = v$ a.e. then Ω must be a ball.

Sketch of the proof of Theorem 2.5. We already recalled that the inequality (9) is proved in [8, Theorem 4].
We will now focus on the equality case, namely we start by assuming that that $\Lambda_\Omega = \Lambda_{\Omega^\star}$.

By Corollary 2.1, we know that $(u^\star, D_\star)$ is an optimal pair of (6) on $\Omega^\star$. We know that u solves

$$\begin{cases} -\Delta u = (\Lambda - \alpha\chi_D)u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Let v be the solution of

$$\begin{cases} -\Delta v = \left[(\Lambda - \alpha\chi_D)\, u\right]^\star, & \text{in } \Omega^\star, \\ v = 0, & \text{on } \partial\Omega^\star. \end{cases}$$

By Theorem 2.6

$$u^\star(x) \le v(x), \quad \text{for almost every } x \in \Omega^\star.$$

Since we can also prove, see [14, Proposition 2], that

$$\left[(\Lambda - \alpha\chi_D(x))\, u(x)\right]^\star = \left(\Lambda - \alpha\chi_{\Omega^\star\setminus\overline{(\Omega\setminus D)^\star}}\right) u^\star(x),$$

the function v actually solves

$$\begin{cases} -\Delta v = \left(\Lambda - \alpha\chi_{\Omega^\star\setminus\overline{(\Omega\setminus D)^\star}}\right) u^\star, & \text{in } \Omega^\star, \\ v = 0, & \text{on } \partial\Omega^\star. \end{cases} \tag{10}$$

Therefore

$$-\Delta v(x) = \left(\Lambda - \alpha\chi_{\Omega^\star\setminus\overline{(\Omega\setminus D)^\star}}(x)\right) u^\star(x) \le \left(\Lambda - \alpha\chi_{\Omega^\star\setminus\overline{(\Omega\setminus D)^\star}}(x)\right) v(x),$$

for almost every $x \in \Omega^\star$. Multiplying by v the former inequality and integrating by parts,

$$\frac{\displaystyle\int_{\Omega^\star} |\nabla v(x)|^2\, dx + \alpha \int_{\Omega^\star} \chi_{\Omega^\star\setminus\overline{(\Omega\setminus D)^\star}}(x) v(x)^2\, dx}{\displaystyle\int_{\Omega^\star} v(x)^2\, dx} \le \Lambda,$$

for $v \in H_0^1(\Omega^\star)$. Since $|D_\star| = |\Omega^\star \setminus \overline{(\Omega \setminus D)^\star}|$,

$$\frac{\displaystyle\int_{\Omega^\star} |\nabla v(x)|^2\, dx + \alpha \int_{\Omega^\star} \chi_{D_\star}(x) v(x)^2\, dx}{\displaystyle\int_{\Omega^\star} v(x)^2\, dx} \le \Lambda, \tag{11}$$

for $v \in H^1_0(\Omega^\star)$ and $D_\star \subset \Omega^\star$ with $|D_\star| = A$. But $(v, D_\star)$ is an admissible pair, so equality holds in (11) and $(v, D_\star)$ must be an optimal pair. Therefore v solves

$$\begin{cases} -\Delta v = (\Lambda - \alpha\chi_{D_\star})\, v, & \text{in } \Omega^\star, \\ v = 0, & \text{on } \partial\Omega^\star. \end{cases} \tag{12}$$

By (10) and (12), subtracting term by term we get

$$0 = (\Lambda - \alpha\chi_{\Omega^\star\setminus\overline{(\Omega\setminus D)^\star}})u^\star - (\Lambda - \alpha\chi_{D_\star})v = (\Lambda - \alpha\chi_{D_\star})(u^\star - v) \quad \text{a.e. in } \Omega^\star.$$

Recalling (7), the equality above directly implies

$$v = u^\star \quad \text{a.e. in } \Omega^\star,$$

and the conclusion now follows from Theorem 2.6. □

We close this section recalling that many other results concerning the regularity of the optimal configuration D and weaker notions of uniqueness of optimal pairs has been proved, see e.g. [9, 10, 11].

3. The composite plate problem

In the previous section we described a few results concerning the composite membrane problem. It is quite natural to ask up to what extent is it possible to generalize the same result to a *higher order eigenvalue optimization problem*, namely eigenvalues of the Navier Bilaplacian or the Dirichlet Bilaplacian. These operators find a natural application in the theory of plates, respectively hinged and clamped plates, see e.g. the monograph [18].
The composite hinged plate problem can be stated as follows: let $\Omega \subset \mathbb{R}^n$ be open and bounded set with C^4-smooth boundary $\partial\Omega$. As for Problem (2), let $0 \le h < H$ be two fixed constants and let $M \in [h\,|\Omega|, H\,|\Omega|]$. Then , we define the class of *admissible densities* as in (1). We can then consider the minimization problem

$$\Theta_N(h, H, M) := \inf_{\rho\in \mathrm{P}} \inf_{u\in H^2\cap H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2}{\int_\Omega \rho\, u^2}. \tag{13}$$

We say that any couple (u, ρ) which realizes the double infimum is a (13)-optimal pair.

Remark 3.1. *If in* (13) *we consider the Sobolev space $H^2_0(\Omega)$ instead of $H^2 \cap H^1_0(\Omega)$, we get the clamped case.*

It is now natural to ask the same questions as listed for the composite membrane problem. Nevertheless there is an immediate first striking difference: it is not more so evident that, given a (13)-optimal pair (u, ρ), u has a sign. Since this is quite a crucial property, we make use of a *strong minimum principle* proved in [17] which can be applied also in our case.

Lemma 3.1 ([17], Lemma 1)**.** *Let $\mathcal{C}^+ := \{w \in H^2 \cap H_0^1(\Omega) \,:\, w \geq 0$ a.e. in $\Omega\}$. Assume that $u \in H^2 \cap H_0^1(\Omega)$ is such that*

$$\int_\Omega \Delta u \Delta v \geq 0 \quad \textit{for every } v \in \mathcal{C}^+,$$

then $u \in \mathcal{C}^+$. Moreover, either $u \equiv 0$ or $u > 0$ a.e. in Ω.

We stress that we apply this minimum principle in order to get positivity of *minimizers* of our variational problem (see [12, Proposition 5.1]), and we are not able to get similar information for solutions of the Euler-Lagrange fourth order PDE associated to it which are not minimizers.

Once the positivity of u is established, we can also address the issue of existence of optimal pairs and of their characterization.

Theorem 3.1 ([12], Theorems 1.3 and 1.4)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^4-boundary $\partial\Omega$. For any $0 \leq h < H$ and every $M \in [h\,|\Omega|, H\,|\Omega|]$, there exists a* (13)*-optimal pair (u, ρ) with the following properties:*

(a) *$u \in C^{3,\gamma}(\overline{\Omega}) \cap W^{4,q}(\Omega)$, for every $\gamma \in (0,1)$ and $q \geq 1$;*

(b) *$\rho = h\,\chi_D + H\,\chi_{D^c}$, for a set of the form $D = \{u \leq t\}$, for a suitable $t = t(h, H, \Omega, u) > 0$.*

We stress that combining the regularity of u up to the boundary with the boundary condition $u = 0$ on $\partial\Omega$, we immediately get that the set $\{u \leq t\}$ contains a tubular neighborhood of $\partial\Omega$. We also point out that the existence part of the above theorem would hold even if the boundary $\partial\Omega$ is merely Lipschitz continuous. Nevertheless, we added it in order to get the sharpest possible regularity of u up to the boundary.

It is now quite natural to move to the symmetry properties seen in Section 2 in the case of the composite membrane problem. Before stating our result we must say that there is an immediately evident difficulty appearing. As briefly mentioned, the proof of Theorem 2.3 in [8] deeply relied on the Steiner symmetrization technique and this prevents us to follow the same scheme. Roughly speaking, this occurs because the higher order Sobolev space $H^2 \cap H_0^1(\Omega)$ *is not closed under symmetrization*. Therefore we need to find a new approach in order to prove any kind of symmetry. To this aim, we decide to work with the Euler-Lagrange equation rather than the functional appearing in the variational problem. Therefore, we have to deal with the following fourth order boundary value problem:

$$\begin{cases} \Delta^2 u = \Theta\rho u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

At this stage, we can also profit of the choice of the *Navier* boundary condition to write (14) as a second order cooperative elliptic system with Dirichlet boundary conditions: set $u_1 := u$, then

$$\begin{cases} -\Delta u_1 = u_2 & \text{in } \Omega, \\ -\Delta u_2 = \Theta\rho u_1 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

Symmetry of positive solutions of elliptic systems is a widely studied topic, whose main technical device is given by the *moving plane method*, originally introduced by Serrin [22] and then further developed and refined by many authors, see e.g. [19, 3] for scalar equations. The case of systems has also a quite long history, see e.g. [25, 16, 15] up to the more recent contributions [4, 5] that deals with positive sigular solutions.

In order to properly state our symmetry result, we need to introduce the classical formalism of the moving plane method. Given any $\lambda \in \mathbb{R}$, we introduce the (possibly empty) set

$$\Sigma_\lambda := \{x \in \Omega \,:\, x_1 > \lambda\}$$

and its reflection with respect to $T_\lambda = \{x \in \Omega \,:\, x_1 = \lambda\}$,

$$\Sigma'_\lambda := \{\varphi_\lambda(x) \in \mathbb{R}^n \,:\, x \in \Sigma_\lambda\},$$

where

$$\varphi_\lambda(x_1, x_2, \ldots, x_n) = (2\lambda - x_1, x_2, \ldots, x_n)$$

Since $\Omega \subset \mathbb{R}^n$ is bounded, T_λ does not intersect $\overline{\Omega}$ if λ is large enough. We define

$$\lambda_0 := \sup\{\lambda \in \mathbb{R} \,:\, T_\lambda \cap \overline{\Omega} \neq \emptyset\}.$$

Lowering the value of λ, the hyperplane T_λ cuts Σ_λ off from Ω. Clearly, at the beginning of the process, the reflection Σ'_λ of Σ_λ will be contained in Ω. Now, we define the value λ_1 as follows

$$\lambda_1 := \sup\{\lambda < \lambda_0 \,:\, \text{(i) or (ii) is verified}\}, \tag{16}$$

where

(i) Σ'_λ is internally tangent to the boundary $\partial\Omega$ at a certain point $P \notin T_\lambda$;
(ii) T_λ is orthogonal to the boundary $\partial\Omega$ at a certain point $Q \in T_\lambda \cap \partial\Omega$.

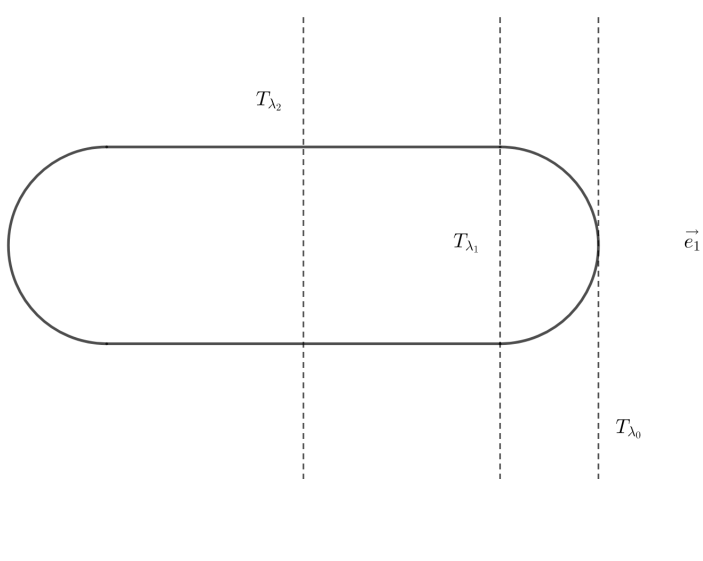
By construction,

$$\Sigma'_\lambda \subset \Omega \quad \text{for every } \lambda \in [\lambda_1, \lambda_0).$$

Nevertheless, by further decreasing the value of λ below λ_1, Σ'_λ might still be contained in Ω. Therefore we define the value

$$\lambda_2 := \inf\{\lambda < \lambda_0 \,:\, \Sigma'_\lambda \subset \Omega\}.$$

In order to clarify the previous definitions we include a picture where $0 = \lambda_2 < \lambda_1 < \lambda_0$.



Having fixed the standard notation related to the moving plane method, we can state our

Theorem 3.2 ([13], Theorem 1.2)**.** *Let Ω be symmetric and convex with respect to the hyperplane $\{x_1 = 0\}$, and with C^4-smooth boundary $\partial\Omega$. Let λ_1 be defined as in* (16)*, we assume that $\lambda_1 = 0$. If (u, ρ) is an optimal pair, then u is symmetric with respect to $\{x_1 = 0\}$ and strictly decreasing in x_1 for $x_1 > 0$. Consequently, ρ is symmetric with respect to the same hyperplane as well. Furthermore, the set $\{u > t\}$ is convex with respect to $\{x_1 = 0\}$.*

Let us now spend a few words concerning the proof of Theorem 3.2, which can be easily obtained combining the following technical lemmas. We stress that the upcoming lemmas hold in slightly more general cases, i.e. even when λ_1 and/or λ_2 are not equal to zero. We refer to [13] for all the proofs.

The first one guarantees that the moving plane method can effectively start.

Lemma 3.2 ([13], Lemma 3.1)**.** *Let $\partial\Omega \in C^4$, $\lambda \in (\lambda_1, \lambda_0]$, $x_0 \in T_\lambda \cap \partial\Omega$, and (u_1, u_2) be a weak solution of* (15)*. Fix $\epsilon > 0$ so small that the first component of the outer normal is strictly positive, i.e. $\nu_1(x) > 0$, for every $x \in \{x \in \partial\Omega : \|x - x_0\| < \epsilon\}$. Then, there exists a positive constant $\delta > 0$ such that*

$$\frac{\partial u_i}{\partial x_1} < 0 \quad \textit{in } \{x \in \Omega : \|x - x_0\| < \delta\} \quad \textit{for every } i = 1, 2.$$

The next two lemmas provide the desired monotonicity of both u_1 and u_2 up to Σ_{λ_1}.

Lemma 3.3 ([13], Lemma 3.2)**.** *Let $\lambda \in [\lambda_1, \lambda_0)$ and (u_1, u_2) be a weak solution of* (15)*. Suppose that*

$$u_i \leq u_i \circ \varphi_\lambda \quad \textit{but} \quad u_i \not\equiv u_i \circ \varphi_\lambda \quad \textit{in } \Sigma_\lambda \quad \textit{for some } i = 1, 2. \tag{17}$$

Then,

$$u_i < u_i \circ \varphi_\lambda \quad \textit{in } \Sigma_\lambda \quad \textit{for every } i = 1, 2$$

and

$$\frac{\partial u_i}{\partial x_1} < 0 \quad \textit{on } \Omega \cap T_\lambda \quad \textit{for every } i = 1, 2.$$

Lemma 3.4 ([13], Lemma 3.4). *Let $\partial\Omega \in C^4$, $\lambda \in (\lambda_1, \lambda_0)$, and (u_1, u_2) be a weak solution of* (15). *Then*

$$\frac{\partial u_i}{\partial x_1} < 0 \quad \text{and} \quad u_i < u_i \circ \varphi_\lambda \quad \text{in } \Sigma_\lambda \quad \text{for every } i = 1, 2.$$

We stress that the presence of the function ρ prevents from a direct use of the classical symmetry results for semilinear cooperative elliptic systems. Nevertheless, the explicit knowledge of it, see (b) of Theorem 3.1, makes it possible to prove the following

Lemma 3.5 ([13],Lemma 2.3). *Let $\lambda \in [\lambda_2, \lambda_0)$, (u, ρ) be an optimal pair, and $u \circ \varphi_\lambda - u \geq 0$ in Σ_λ. Then*

$$(\rho \circ \varphi_\lambda)(u \circ \varphi_\lambda) - \rho u \geq 0 \quad \text{in } \Sigma_\lambda.$$

This is crucial in order to prove Lemma 3.3, which in turn leads to Lemma 3.4. We refer to [13] for more details.

Remark 3.2. *The condition $\lambda_1 = 0$ seems to be a mere technical problem but at the moment it prevents us to consider sets which are flat in the x_1-direction, see* [13, Figure 1] *for a few examples of sets covered by the above Theorem. On the other hand, thanks to finer versions of the maximum principle in small domains, see* [16] *for the case of systems, we can also relax the regularity assumption on the boundary, at the price of a stronger assumption on the set $D^c \cap \Omega$, see* [13, Proposition 5.9].

Due to the nature of the moving plane technique, we can adopt it to prove rigidity results as well. The following theorem resembles the famous overdetermined problem of Serrin [22].

Theorem 3.3 ([13], Theorem 1.1). *Let $\partial\Omega$ be C^4-smooth and let (u, ρ) be a* (13)*-optimal pair. If u satisfies the additional condition*

$$\frac{\partial u}{\partial \nu} = c \quad \text{on } \partial\Omega \quad \text{for some } c < 0,$$

then Ω is a ball and u is radially symmetric and radially decreasing.

Remark 3.3. *As for the composite membrane problem, it is possible to consider a more general eigenvalue optimization problem, whose formulation resembles Problem* (6)*. Let* $\Omega \subset \mathbb{R}^n$ *be as in* (13)*, let* $\alpha > 0$ *and* $A \in [0, |\Omega|]$ *be fixed real numbers. Let* $\mathcal{D}$ *be the class of admissible sets as in* (4)*. Let* $\lambda_N = \lambda_N(\alpha, D)$ *be the lowest eigenvalue of the following boundary value problem with Navier boundary conditions:*

$$\begin{cases} \Delta^2 u + \alpha\chi_D u = \lambda\, u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \qquad \lambda \in \mathbb{R}.$$

We can then consider the variational problem

$$\Lambda_N(\alpha, A) = \inf_{D\in\mathcal{D}} \inf_{u\in H^2\cap H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2 dx + \alpha \int_\Omega \chi_D u^2 dx}{\int_\Omega u^2 dx}. \tag{18}$$

Once again it is possible to make the connection between Problem (18) *and Problem* (13) *explicit, see* [12, Theorem 1.4]*, but we can prove the symmetry and rigidity results only for Problem* (13)*. The main technical reason behind this fact is that when we consider an optimal pair* (u, D) *for Problem* (18)*, we are not able to prove the positivity of* u*. However, we expect them to be true even once Problem* (18) *is considered.*

We close this section with a brief account of the results holding in the clamped case. As far as we know, the only contributions in this direction have been obtained in [12]. The existence and characterization of optimal pairs is guaranteed by Theorem 3.1, which holds in this case as well, see [12, Theorems 1.3 and 1.4]. Positivity of the minimizers u and symmetry results have also been obtained in [12]: the biggest difference with respect to the hinged case concerns the fact that we are able to prove them only in the case of Ω being a ball. To be more precise, given a optimal pair (u, D), it is possible to prove that u has a sign exploiting once again Lemma 3.1. Unfortunately, in the clamped case this result is known to hold only in the case of the ball, see e.g. [17], because it requires to work with the explicit Green function, known only in the case of the ball. In particular, this forces us to study preservation of symmetry only in the case Ω being a ball. In this case, in [12, Theorem 1.5] is proved that there exists a unique optimal pair (u, D), where u is radial and D is an annulus of given volume. The technique used to prove [12, Theorem 1.5] is an adaptation of the one proposed in [17] in a different context.

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