Capacity Analysis of Interference Alignment With Bounded CSI Uncertainty

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***Abstract*—Interference alignment (IA) has been demonstrated to achieve the degree-of-freedom (DoF) of an interference channel given perfect global channel state information (CSI). In this letter, we consider the case of imperfect CSI with bounded errors and derive a capacity lower bound of the channel using IA. We show that this lower bound is within 1 bps/Hz of the capacity of the perfect CSI case up to a certain signal-to-noise ratio (SNR) which we refer to it as the *saturating* SNR. Further, we introduce a new metric called modified DoF (mDoF) in order to characterize the multiplexing performance of IA with imperfect CSI at *finite* SNR. Simulation results for the 3-user case are provided to illustrate the region within which the actual capacity of IA falls.**

***Index Terms*—Beamforming, interference alignment, interference channel, capacity bound, channel errors.**

I. Introduction

Wireless communications has revolutionized the way we live but continues to demand an ever higher spectral efficiency. One bottleneck of wireless communications is co-channel interference (CCI), which arises from frequency reuse in cellular networks or cognitive radio environments. In [1], the notion of interference alignment (IA) was introduced, which was further developed in [2], [3]. Remarkably, using IA, each user in the interference channel can achieve interference-free communication. A major result is that in a K-user interference channel, it can obtain a degree-of-freedom (DoF) of $(K/2)$ at high signal-to-noise ratio (SNR) in the case of infinite diversity [3] but a DoF of at most 2 with finite spatial-only diversity [4].

The reservation in IA is, however, the need of perfect global channel state information (CSI) at each transmitter.[1] Motivated by this, [5], [6] designed algorithms to perform IA given only local CSI while [7]–[10] took into account the errors due to channel estimation and feedback. For example, [8] presented the average achievable rate under a given measurement error power, and [9] established bounds on the average achievable rate with Gaussian CSI errors. Although the results in [8], [9] are indicative, the average rates are operationally unachievable.

In contrast to the previous work, this letter aims to derive an *achievable* capacity lower bound for IA with imperfect CSI under the model that the CSI errors are bounded. Our result reveals several properties that provide guidance in the design of interference networks, and is applicable to any perfect IA methods operating on the imperfect CSI [11], [12].

Notations: In this letter, uppercase bold letters denote matrices, while lowercase bold letters denote vectors. In addition, $(\cdot)^\dagger$ and $(\cdot)^T$ denote the conjugate transpose, and the transpose operations, respectively. $\texttt{span}(\mathbf{A})$ is the vector space generated by the columns of the matrix $\mathbf{A}$, $(\cdot)_k$ returns the kth row of an input matrix, and $\|\cdot\|_2$ is the square-norm.

II. The IA Model

Consider an interference channel with K pairs of transmitters and receivers. Each pair is regarded as a user. It is assumed that user i has m_i transmit antennas and n_i receive antennas. Every transmitter is assumed to possess the estimated channel matrices between transmitter j to receiver i, $\hat{\mathbf{H}}_{i,j}$, for all i, j. As in [11], [12], we consider that some perfect IA is adopted based on the estimated CSI, $\{\hat{\mathbf{H}}_{i,j}\}$, that permits the ith user to transmit d_i data streams. This means that for each user i, we are provided with the perfect precoder $\mathbf{V}_i$ and interference canceling matrix $\mathbf{U}_i$ that perform IA over all $\hat{\mathbf{H}}_{i,j}$.

In reality, the estimated channels are imperfect and the real channels, $\mathbf{H}_{i,j}$, can be written as

$$\mathbf{H}_{i,j} = \hat{\mathbf{H}}_{i,j} + \Delta\mathbf{H}_{i,j}, \tag{1}$$

where $\Delta\mathbf{H}_{i,j}$ denotes the channel measurement errors. In our model, we consider the errors bounded [13] such that

$$\delta_{i,j}^2 = \max_k \|(\Delta\mathbf{H}_{i,j})_k\|_2^2, \text{ for some given } \delta_{i,j} \geq 0, \tag{2}$$

where $(\Delta\mathbf{H}_{i,j})_k$ denotes the kth row of $\Delta\mathbf{H}_{i,j}$.

The received signals in vector form at user i are given by

$$\mathbf{y}_i = \mathbf{H}_{i,i}\mathbf{V}_i\mathbf{x}_i + \mathbf{H}_{-i}\mathbf{V}_{-i}\mathbf{x}_{-i} + \boldsymbol{\eta}_i, \tag{3}$$

where $\mathbf{H}_{-i} \triangleq [\mathbf{H}_{i,1}\cdots\mathbf{H}_{i,i-1}\mathbf{H}_{i,i+1}\cdots\mathbf{H}_{i,K}]$,

$$\mathbf{V}_{-i} \triangleq \begin{bmatrix} \mathbf{V}_1 & \mathbf{0} & \cdots & & & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & & & \vdots & & \vdots \\ & \mathbf{0} & \mathbf{V}_{i-1} & \mathbf{0} & & \cdots & \\ & \cdots & \mathbf{0} & & \mathbf{V}_{i+1} & \mathbf{0} & \\ \vdots & & & & \vdots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & & & \cdots & \mathbf{0} & \mathbf{V}_K \end{bmatrix}, \tag{4}$$

$\mathbf{x}_{-i} \triangleq [\mathbf{x}_1^T\cdots\mathbf{x}_{i-1}^T\mathbf{x}_{i+1}^T\cdots\mathbf{x}_K^T]^T$ in which $\mathbf{x}_j$ is the transmitted data stream vector by user j and $\boldsymbol{\eta}_i$ denotes the additive zero-mean N_0-variance Gaussian noise vector at user i.

For convenience, we assume that all users have the same average power constraint, $\mathbb{E}(\|\mathbf{x}_i\|_2^2) = \sum_{k=1}^{d_i}\mathbb{E}(|(\mathbf{x}_i)_k|^2) \leq \mathcal{E}$ where $\mathbb{E}(\cdot)$ returns the expectation of the input random entity.

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[1] Every transmitter needs to possess the CSI for every link in the entire interference network, including those not linking to the transmitter.

III. Capacity Lower Bound

In this section, we derive the capacity lower bound of any stream of a given user i in the IA model with imperfect CSI. We define, in the similar way as $\mathbf{H}_{-i}$, $\hat{\mathbf{H}}_{-i}$ for the estimated CSI matrix and $\Delta\mathbf{H}_{-i}$ for the CSI error matrix, excluding the direct channel for user i. Hence, $\mathbf{H}_{-i} = \hat{\mathbf{H}}_{-i} + \Delta\mathbf{H}_{-i}$.

Accordingly, the signal model (3) becomes

$$\mathbf{y}_i = \hat{\mathbf{H}}_{i,i}\mathbf{V}_i\mathbf{x}_i + \hat{\mathbf{H}}_{-i}\mathbf{V}_{-i}\mathbf{x}_{-i} + \Delta\mathbf{H}_{i,i}\mathbf{V}_i\mathbf{x}_i + \Delta\mathbf{H}_{-i}\mathbf{V}_{-i}\mathbf{x}_{-i} + \boldsymbol{\eta}_i. \quad (5)$$

Applying the interference canceling matrix on (5) gives

$$\mathbf{U}_i^\dagger\mathbf{y}_i = \underbrace{\mathbf{U}_i^\dagger\hat{\mathbf{H}}_{i,i}\mathbf{V}_i\mathbf{x}_i}_{\text{Desired Signal}} + \underbrace{\mathbf{U}_i^\dagger\hat{\mathbf{H}}_{-i}\mathbf{V}_{-i}\mathbf{x}_{-i}}_{=0} + \underbrace{\mathbf{U}_i^\dagger\Delta\mathbf{H}_{i,i}\mathbf{V}_i\mathbf{x}_i + \mathbf{U}_i^\dagger\Delta\mathbf{H}_{-i}\mathbf{V}_{-i}\mathbf{x}_{-i}}_{\text{Interference}} + \underbrace{\mathbf{U}_i^\dagger\boldsymbol{\eta}_i}_{\text{Noise}}. \quad (6)$$

We are going to bound the power of the different terms in the above expression to derive the capacity lower bound.

Let us first focus on the interference caused by other users at the ith receiver. The matrix $\Delta\mathbf{H}_{-i}\mathbf{V}_{-i}$ is responsible for that interference. In the general case, $\mathtt{span}(\Delta\mathbf{H}_{-i}\mathbf{V}_{-i})$ overlaps with the space designed for the desired signal and also that designed for the interference signal meaning that

$$\Delta\mathbf{H}_{-i}\mathbf{V}_{-i} = \underbrace{\mathbf{U}_i^\dagger\Delta\mathbf{H}_{-i}\mathbf{V}_{-i}}_{\text{span} \subset \text{ desired space}} + \underbrace{(\mathbf{I}-\mathbf{U}_i^\dagger)\Delta\mathbf{H}_{-i}\mathbf{V}_{-i}}_{\text{span} \subset \text{ interference space}}. \quad (7)$$

The worst case arises if all the interference goes to the signal space, i.e., $\Delta\mathbf{H}_{-i}\mathbf{V}_{-i} = \mathbf{U}_i^\dagger\Delta\mathbf{H}_{-i}\mathbf{V}_{-i}$. Thus, the interference power caused by other users can be upper bounded by

$$\mathcal{I}_i \le \mathbb{E}\left(\|\Delta\mathbf{H}_{-i}\mathbf{V}_{-i}\mathbf{x}_{-i}\|_2^2\right), \quad (8)$$

where the expectation is taken over the data stream $\mathbf{x}_{-i}$.

Proposition 1: We have the following upper bound for the received interference power caused by other users at user i:

$$\mathcal{I}_i \le n_i\delta_{\max}^2 D(K-1)\mathcal{E}, \quad (9)$$

where $D \triangleq \max_i \sum_{k=1, k\neq i}^{K} d_k$ and $\delta_{\max} \triangleq \max_{i,j} \delta_{i,j}$.

Proof: Let $\Delta\tilde{\mathbf{h}}_1$ denote the first column of $\Delta\mathbf{H}_{-i}\mathbf{V}_{-i}$ and $\boldsymbol{v}_1^{(1)}$ be the first column of $\mathbf{V}_1$. Then we have

$$\Delta\tilde{\mathbf{h}}_1 = \begin{bmatrix} (\Delta\mathbf{H}_{i,1})_1\boldsymbol{v}_1^{(1)} \\ (\Delta\mathbf{H}_{i,1})_2\boldsymbol{v}_1^{(1)} \\ \vdots \end{bmatrix}. \quad (10)$$

Clearly,

$$\left|(\Delta\mathbf{H}_{i,1})_k\boldsymbol{v}_1^{(1)}\right|^2 \le \underbrace{\|(\Delta\mathbf{H}_{i,1})_k\|_2^2}_{\le\delta_{i,1}^2}\underbrace{\left\|\boldsymbol{v}_1^{(1)}\right\|_2^2}_{\le 1} \le \delta_{i,1}^2 \le \delta_{\max}^2. \quad (11)$$

As a result, we get $\|\Delta\tilde{\mathbf{h}}_k\|_2^2 \le n_i\delta_{\max}^2$ because the same upper bound is valid for any column (say kth column) of $\Delta\mathbf{H}_{-i}\mathbf{V}_{-i}$.

Furthermore, we can write

$$\Delta\mathbf{H}_{-i}\mathbf{V}_{-i}\mathbf{x}_{-i} = \sum_j (\mathbf{x}_{-i})_j\Delta\tilde{\mathbf{h}}_j. \quad (12)$$

Now consider

$$\left\|\sum_j (\mathbf{x}_{-i})_j\Delta\tilde{\mathbf{h}}_j\right\|_2 \le \sum_j |(\mathbf{x}_{-i})_j|\,\|\Delta\tilde{\mathbf{h}}_j\|_2, \le \sqrt{n_i}\delta_{\max}\sum_j |(\mathbf{x}_{-i})_j|. \quad (13)$$

Note that $\sum_j |(\mathbf{x}_{-i})_j| = \|\mathbf{x}_{-i}\|_1$ and since $\|\mathbf{a}\|_1 \le \sqrt{N}\|\mathbf{a}\|_2$ (with N being the length of vector $\mathbf{a}$), we have

$$\mathbb{E}\left(\|\Delta\mathbf{H}_{-i}\mathbf{V}_{-i}\mathbf{x}_{-i}\|_2^2\right) \le n_i\delta_{\max}^2 D\mathbb{E}(\|\mathbf{x}_{-i}\|_2^2), \le n_i\delta_{\max}^2 D(K-1)\mathcal{E}, \quad (14)$$

which completes the proof. ■

Next, we consider the effects of the uncertainty on the kth stream of the ith user when the transmit power is $E_k^{(i)}$. Denote $\boldsymbol{v}_k^{(i)}$ and $\boldsymbol{u}_k^{(i)}$ as the kth column of $\mathbf{V}_i$ and $\mathbf{U}_i$, respectively. The signal component of the kth stream of user i is

$$\sqrt{E_k^{(i)}}\mathbf{H}_{i,i}\boldsymbol{v}_k^{(i)} = \sqrt{E_k^{(i)}}\hat{\mathbf{H}}_{i,i}\boldsymbol{v}_k^{(i)} + \sqrt{E_k^{(i)}}\Delta\mathbf{H}_{i,i}\boldsymbol{v}_k^{(i)}. \quad (15)$$

The worst case occurs if $\Delta\mathbf{H}_{i,i}\boldsymbol{v}_k^{(i)}$ is orthogonal to $\mathbf{H}_{i,i}\boldsymbol{v}_k^{(i)}$. In this case, the signal power in the kth stream at user i is

$$P_k^{(i)} = \left|\sqrt{E_k^{(i)}}\boldsymbol{u}_k^{(i)\dagger}\hat{\mathbf{H}}_{i,i}\boldsymbol{v}_k^{(i)}\right|^2 - \left|\sqrt{E_k^{(i)}}\boldsymbol{u}_k^{(i)\dagger}\Delta\mathbf{H}_{i,i}\boldsymbol{v}_k^{(i)}\right|^2. \quad (16)$$

Also, we have $|\sqrt{E_k^{(i)}}\boldsymbol{u}_k^{(i)\dagger}\Delta\mathbf{H}_{i,i}\boldsymbol{v}_k^{(i)}|^2 \le n_i\delta_{\max}^2 E_k^{(i)}$. Define $\sigma_k^{(i)} \triangleq \boldsymbol{u}_k^{(i)\dagger}\hat{\mathbf{H}}_{i,i}\boldsymbol{v}_k^{(i)}$. Therefore, we have

$$P_k^{(i)} \ge \left(\left(\sigma_k^{(i)}\right)^2 - n_i\delta_{\max}^2\right)E_k^{(i)}, \quad (17)$$

where we assume that we always have $(\sigma_k^{(i)})^2 \ge n_i\delta_{\max}^2$.

Proposition 2: The inter-stream interference power on the kth stream of user i is upper bounded by

$$\mathcal{S}_k^{(i)} \le n_i\delta_{\max}^2\left(\mathcal{E} - E_k^{(i)}\right). \quad (18)$$

Proof: Let $\mathbf{V}_i^{(-k)}$ be the precoding matrix $\mathbf{V}_i$ excluding the kth column and $\mathbf{x}_i^{(-k)}$ be the data vector $\mathbf{x}_i$ excluding the kth stream of the ith user. Then the worst case happens if all the power lost by other streams creates interference. That is,

$$\mathcal{S}_k^{(i)} = \left\|\Delta\mathbf{H}_{i,i}\mathbf{V}_i^{(-k)}\mathbf{x}_i^{(-k)}\right\|_2^2 = \sum_{l=1}^{n_i}\left|(\Delta\mathbf{H}_{i,i})_l\mathbf{V}_i^{(-k)}\mathbf{x}_i^{(-k)}\right|^2, \quad (19)$$

which can be upper bounded by

$$\mathcal{S}_k^{(i)} \le \sum_{l=1}^{n_i}\|(\Delta\mathbf{H}_{i,i})_l\|_2^2\left\|\mathbf{V}_i^{(-k)}\mathbf{x}_i^{(-k)}\right\|_2^2 \quad (20)$$

$$\le n_i\delta_{i,i}^2\left\|\mathbf{V}_i^{(-k)}\mathbf{x}_i^{(-k)}\right\|_2^2 \quad (21)$$

$$= n_i\delta_{i,i}^2\left\|\mathbf{x}_i^{(-k)}\right\|_2^2 \quad (22)$$

$$\le n_i\delta_{\max}^2\left\|\mathbf{x}_i^{(-k)}\right\|_2^2 = n_i\delta_{\max}^2\left(\mathcal{E} - E_k^{(i)}\right), \quad (23)$$

which is the desired result and the proof is completed. ■

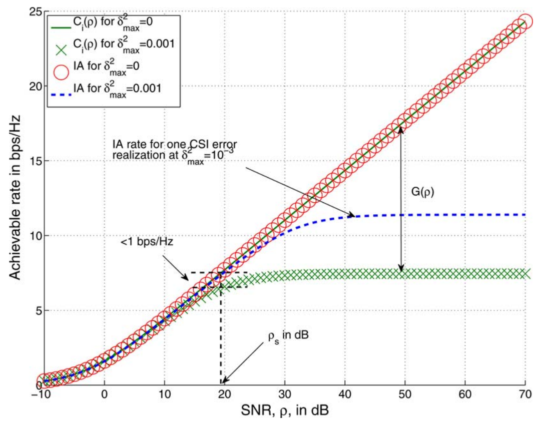

Fig. 1. The capacity lower bound in the single-stream case.

Theorem 1: A capacity lower bound for the kth stream of the ith user is given by

$$C_k^{(i)} \geq \log_2\left(1+\frac{\left(\left(\sigma_k^{(i)}\right)^2 - n_i\delta_{\max}^2\right)E_k^{(i)}}{N_0+n_i\delta_{\max}^2\left((D(K-1)+1)\,\mathcal{E}-E_k^{(i)}\right)}\right). \tag{24}$$

Proof: Using (9), (17), and (18) gives the result. ■

Corollary 1: If each user has only one stream, the capacity lower bound in (24) becomes

$$\underline{C}_i(\rho) = \log_2\left(1+\frac{\left(\sigma_i^2 - n_i\delta_{\max}^2\right)\rho}{1+n_i\delta_{\max}^2(K-1)^2\rho}\right), \tag{25}$$

where $\rho \triangleq \mathcal{E}/N_0$ and $\sigma_k^{(i)}$ in (24) is replaced by σ_i.

Proof: In this case, (18) is not used and $D = K-1$. ■

IV. Saturating SNR and mDoF

In Fig. 1, we plot the capacity lower bound $\underline{C}_i$ for the cases $\delta_{\max}^2 = 0$ and $\delta_{\max}^2 = 0.001$ assuming that $\sigma_i = 1$. We observe that there is a saturation point in SNR where after this point any further increase in SNR will not lead to a useful increase in the achievable rate due to the CSI errors. We refer to this point as the saturating SNR and we present it below.

Theorem 2: The saturating SNR, ρ_s, is given by

$$\rho_s = \frac{1}{\sigma_i^2} + \frac{1-\frac{n_i\delta_{\max}^2}{\sigma_i^2}}{n_i\delta_{\max}^2(K-1)^2} \overset{(a)}{\simeq} \frac{1}{\sigma_i^2} + \frac{1}{n_i\delta_{\max}^2(K-1)^2}, \tag{26}$$

where (a) is due to the fact that typically $n_i\delta_{\max}^2 \ll \sigma_i^2$.

Proof: Let $A = \sigma_i^2 - n_i\delta_{\max}^2$ and $B = n_i\delta_{\max}^2(K-1)^2$. Hence, we have $\underline{C}_i(\rho) = \log_2(1+(A\rho/(1+B\rho)))$. Further, define the SNR in dB as $\rho_{\text{dB}} = 10\log_{10}\rho$. When $\delta_{\max} = 0$ and at high SNR, the capacity lower bound becomes

$$\begin{aligned}\underline{C}_i(\rho_{\text{dB}})|_{\delta_{\max}=0} &\simeq \log_2\left(\sigma_i^2 10^{\frac{\rho_{\text{dB}}}{10}}\right)\\ &= \log_2\sigma_i^2 + \frac{\log_2 10}{10}\rho_{\text{dB}}.\end{aligned} \tag{27}$$

The saturating SNR occurs when

$$\underline{C}_i(\rho_{s,\text{dB}})|_{\delta_{\max}=0} = \underline{C}_i(\infty)|_{\delta_{\max}\neq 0}, \tag{28}$$

which implies that

$$\begin{aligned}\log_2\sigma_i^2 + \frac{\log_2 10}{10}\rho_{s,\text{dB}} &\overset{(a)}{=} \log_2\left(1+\frac{A}{B}\right)\\ \rho_{s,\text{dB}} &= 10\log_{10}\left(\frac{1}{\sigma_i^2}+\frac{A}{B\sigma_i^2}\right),\end{aligned}$$

where (a) is due to high SNR approximation and (27). The desired result in the linear scale is immediately obtained. ■

Corollary 2: At $\rho = \rho_s$, if $n_i\delta_{\max}^2 \ll \sigma_i^2$, the capacity lower bound is within 1 bps/Hz of the rate without CSI errors, i.e.,

$$G(\rho_s) = \underline{C}_i(\rho_s)|_{\delta_{\max}=0} - \underline{C}_i(\rho_s)|_{\delta_{\max}\neq 0} \leq 1 \text{ bps/Hz}. \tag{29}$$

Proof: At the saturating SNR, we have

$$\begin{aligned}G(\rho_s) &\overset{(a)}{=} \log_2\left(1+\sigma_i^2\rho_s\right) - \log_2\left(1+\frac{A\rho_s}{1+B\rho_s}\right),\\ &\overset{(b)}{=} \log_2\left(2+\frac{A}{B}\right) - \log_2\left(1+\frac{\frac{A}{\sigma_i^2}\left(1+\frac{A}{B}\right)}{1+\frac{B}{\sigma_i^2}+\frac{A}{\sigma_i^2}}\right),\\ &= \log_2\left(2+\frac{A}{B}\right) - \log_2\left(1+\frac{1+\frac{A}{B}}{\frac{\sigma_i^2}{A}+\frac{B}{A}+1}\right),\end{aligned} \tag{30}$$

where (b) uses $\rho_s = (1/\sigma_i^2)(1+(A/B))$ due to (26). Now, consider

$$\frac{A}{\sigma_i^2} \simeq 1, \text{ as } n_i\delta_{\max}^2 \ll \sigma_i^2. \tag{31}$$

Substituting this result back into (30) gives

$$G(\rho_s) \simeq \underbrace{\log_2\left(2+\frac{A}{B}\right) - \log_2\left(1+\frac{1+\frac{A}{B}}{2+\frac{B}{A}}\right)}_{\in\left[\log_2\left(\frac{9}{5}\right),1\right] \text{ for } \frac{A}{B}\in[0,+\infty)} \tag{32}$$

Therefore, $G(\rho_s) \leq 1$ and we complete the proof. ■

In Fig. 1, we also see that IA with no CSI errors achieves the same rate of the capacity lower bound if $\delta_{\max} = 0$. In other words, the capacity lower bound is tight and that the saturating SNR, ρ_s, tells exactly where the capacity of IA can be achieved within one bit in the presence of CSI errors.

Corollary 3: The rate ceiling for $\underline{C}_i$ is given by

$$\lim_{\rho\to\infty} \underline{C}_i(\rho) = \log_2\left(\sigma_i^2\rho_s\right). \tag{33}$$

Proof: Taking the limit for $\underline{C}_i(\rho)$ gives the result. ■

Corollary 4: At high SNR ($\geq \rho_s$), the gap between the rate achievable by IA with no CSI errors and the capacity lower bound can be approximated by

$$G(\rho) \approx \log_2\rho - \log_2\rho_s. \tag{34}$$

Proof: This result can be shown by

$$\begin{aligned}G(\rho) &\overset{(a)}{=} \underline{C}_i(\rho)|_{\delta_{\max}=0} - \lim_{\rho\to\infty}\underline{C}_i(\rho)|_{\delta_{\max}\neq 0},\\ &\overset{(b)}{=} \log_2\left(\sigma_i^2\rho\right) - \log_2\left(\sigma_i^2\rho_s\right),\end{aligned} \tag{35}$$

where (b) uses the high SNR approximation and the result in *Corollary 3* to reach the desired result. ■

Conventionally, the DoF is defined as [14]

$$\text{DoF} = \lim_{\rho\to\infty}\frac{C_\Sigma(\rho)}{\log_{10}\rho}. \tag{36}$$

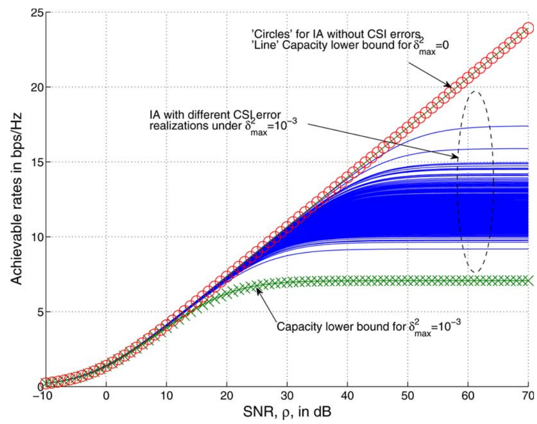

Fig. 2. Achievable rates for $\delta_{\max}^2 = 0, 10^{-3}$.

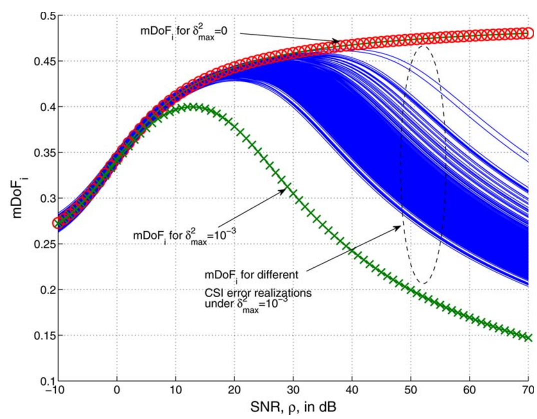

Fig. 3. mDoF for $\delta_{\max}^2 = 0, 10^{-3}$.

This metric represents the total number of streams achievable by the network but it is only defined at infinite SNR and will be zero with CSI errors. We thus define here a different metric called the modified DoF (mDoF) which is more meaningful in the case of CSI errors and is defined as a function of SNR. The mDoF for user i is defined as

$$\mathrm{mDoF}_i(\rho) = \frac{\underline{C}_i(\rho)}{\min\{m_i, n_i\} \times \log_2\left(1 + \alpha_i^2 \rho\right)}, \quad (37)$$

where α_i is the maximum singular value of $\mathbf{H}_{i,i}$.

This quantity is the ratio of our capacity lower bound $\underline{C}_i$ and the capacity upper bound for a single-user multiple-input multiple-output (MIMO) channel with the same direct channel $\mathbf{H}_{i,i}$ with no CSI errors. It is defined for any SNR value and characterizes the performance of a given user in comparison to the case where this user sees no incoming interference. Based on this definition, clearly, if $\delta_{\max} \neq 0$, we have

$$\lim_{\rho \to \infty} \mathrm{mDoF}_i(\rho) = 0. \quad (38)$$

That is, with CSI errors, we lose all the DoF at high SNR, as is expected because of the inevitable interference. We also have $\mathrm{mDoF}(\rho) = \sum_{\forall i} \mathrm{mDoF}_i(\rho)$ as the network mDoF.

V. Simulation Results

In this section, we compare the capacity lower bound (25) to the capacity IA achieves without CSI errors in the 3-user case with $m_i = 3$, $n_i = 2$, $d_i = 1$, and $\delta_i = \delta_{\max} \forall i$. In the simulations, all the matrices (including the CSI errors) have random entries drawn from a complex Gaussian distribution with zero mean and unit variance, but the error matrices are normalized to fulfill the δ_i^2 constraint on their norm.

In Fig. 2, we provide the achievable rate results including the capacity lower bounds with $\delta_{\max} = 0$ and $\delta_{\max}^2 = 10^{-3}$, the rates achievable by IA with perfect CSI and that by IA with 500 different CSI error realizations. As pointed out earlier, the capacity lower bound stays very close to the rate of IA with perfect CSI initially but they depart as SNR keeps increasing. Also, the actual achievable rate of IA with CSI errors can go anywhere between the bound and the perfect CSI case.

Fig. 3 shows the results for the mDoF for a given user of the channel under the cases $\delta_{\max}^2 = 0$ and $\delta_{\max}^2 = 10^{-3}$. Again, the mDoF of the perfect CSI case and that based on our bound provide a region within which the actual IA with CSI errors achieve. Also, we see that mDoF approaches 0 at high SNR.

VI. Conclusion

This letter presented a capacity lower bound for the MIMO interference channel using IA in the presence of bounded CSI errors. In the single-stream case, we illustrated that there is a saturating SNR, after which considerable loss in the achievable rates from the perfect CSI case is anticipated. The saturating SNR can therefore be viewed as the effective operating point before which the benefit of IA is fully delivered in the practical case of imperfect CSI. After that SNR, the rates saturate and the transmitters generate more interference to the channel.

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