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# Tracking and Judging Debates using Argumentation: The Roles of Confirmation, Preclusion and Reflection

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**Signed Declaration**

I, Robert Godfrey Bowerman, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Signature and Date:

# Abstract

Using argumentation to debate and reach conclusions is a particularly human activity relevant to many professions and applications. Debates exist not only in the Houses of Parliament, but also in such disciplines as medicine and law. In this theoretical thesis I explore three new logical constructs for realistic debate modelling, namely: confirmation, preclusion and reflection. Confirmation is two or more arguments for a claim, used to provide corroboration of evidence. Preclusion is an attacking argument that says 'one or other of your arguments is wrong'; an argumentation technique used adeptly by Sherlock Holmes and many politicians. Reflection is a way of identifying logical redundancies (i.e. predictable patterns) in the argument data structure of a debate. A reflection originates from an unpredictable 'reflector' argument and gives rise to the predictable or 'reflected' argument. One type of reflection can be said to 'flow down' a tree of arguments, where the reflector is nearer the root and the reflected arguments further from the root, while another kind 'flows up' the tree in the reverse direction. Incorporating preclusion into the model of reflection increases this to four distinct types of reflection, two up-tree and two down-tree. The value of identifying and removing reflections is to ensure intuitive, or arguably 'correct', results when judging debates, be that judgement based on the existence or number of arguments. Removing reflection also aids human comprehension of the debate as it reduces the number of arguments involved. This logical analysis of reflection and preclusion leads to the definition of a reflection-free, preclusion-aware, debate-tracking tree. Finally, the framework addresses judging the tree to determine who won the debate, with a proposal that takes confirmation into account when reaching conclusions. Confirmation assessment is helpful in resolving inconsistencies. Out of scope are notions of alternating moves by competing players and computational complexity.

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I wish to thank my tutor and supervisor Dr Anthony Hunter and my second supervisor Dr Robin Hirsh for their support and teaching. I also wish to acknowledge that I first observed the logical approach of confirmation and judgement presented herein (including the approaches of confirmation thresholds for acceptable levels of evidence, judges that resolve inconsistency by counting arguments and by the notion of decomposition) in the ancient Indian astrology system known as Jyotish. The Sanskrit texts, with English translations and commentaries, by Parasara, Vrahamira and others, provided me with the first inspiration for this approach to debate tracking and judgement. I am grateful to Pundit Vishwanath Sharma, Sri Sri Ravi Shankar and Maharishi Mahesh Yogi for explaining this ancient system of logic to me. I am also grateful to the software engineer who sought me out me, because of my book *Putting Expert Systems into Practice*, to ask me how a Jyotish expert system should handle inconsistency and redundancy in its processing. I did not know the answer at that time, however, this thesis is now an answer to that question. I would also like to thank Dr John Fox and his team at Cancer Research UK for explaining to me how this approach to debate tracking and judgement would be helpful in modern medicine. I am grateful to my wife Liz and my children Nadine, Gaia, Julia and Niel for their patience over the last six years of my work on this thesis.

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# Chapter 1

# Introduction

In common usage the word 'argument' tends to have two meanings; one is a single argument where a set of premises are used to infer a claim; the second meaning is a set of many of these single arguments that interact with each other in various ways allowing a final or overall conclusion about a particular claim to be drawn. I call the second usage a 'debate'. Debates occur in many professional walks of life, not just in the formal debates in the Houses of Parliament or the Oxford Union. Professional debates occur, for example, in medicine, journalism, economics, business management, systems engineering, law, politics, forensic science, detective work and investment banking.

The underlying goal of this thesis is to allow a computer to have a deep and rich understanding of a debate, but not to actively participate. Although I hold up 'professional debate' as the guide and motivation of the formalisms in this thesis I can offer no gold standard that defines professional debate in its own right. By professional debate I am implying one conducted by courteous individuals in an orderly and precise way with no tolerance for ambiguity or carelessness. I assume an emphasis on avoiding errors and thereby on reaching as logically correct a conclusion as possible. I would suggest that in such debates there is rarely a single absolute truth to be located, however, application of these principles will provide outcomes that are more helpful than if such an approach were not used. This thesis examines professional debate in two steps; one is tracking or recording the form of the debate in some kind of data structure; the other is judging the outcome or conclusion of the debate, given that data structure. My approach is to analyse the completed debate data structure as a whole and not to consider the sequence or temporal build up of moves by individual players. Hence my approach is monological in nature. Dialectical considerations, of alternating moves by competing players, are outside the scope of this thesis. There is a developing literature on how to track and judge debates in the field of artificial intelligence (AI), with some overlap into at least business administration, politics, jurisprudence, medicine, mathematics and philosophy. However, I present evidence and reasoning in the thesis that there is no published debate formalism that is definitely necessary and sufficient, i.e. that covers the whole debate and nothing but the debate. As I use the phrase 'whole debate' later I should be clear that I am meaning all of the unique arguments that can be constructed with a) a minimal consistent set of premises taken from the overall set of available premises and b) which use classical logic to deduce their claim. I do not refer to informal aspects such as a video tape of the proceedings or the body language of speakers.

Consequently, this thesis proposes a new formalism to track and judge professional debates in a way that is necessary and sufficient. So when I propose that a debate should be conducted in a certain way, that is at the same time a) my claiming to know something of how professional debate is practised in the world (based primarily on my 30 years of information technology experience in industry analysing and designing systems for these professions), b) my own views on how a debate should be conducted to yield an accurate result and last but not least c) the formal framework defined in this thesis.

To track a debate is to populate a data structure or record of the evidence, arguments and interactions between those arguments that comprise the debate. In the body of this thesis I assume that all arguments are of equal weight and come from reputable sources so that only logic can be used to distinguish between them. The challenge here is to record each relevant argument once and only once, and also not to include any superfluous arguments. Hence the data structure should only be as large and as complex as necessary to keep track of the debate and no more. Any missing or excessive argumentation in the tracking, which would or could lead to errors or changes in the judgement outcome, is of importance. The data structure does not have to be fully populated, for example at some interim point in the progress of the debate, for the judge function to provide an outcome. Only when the debate is complete, however, will the judgement be conclusive. Forming this data structure of related arguments is commonly referred to as 'argument aggregation'; some authors include judgement within argument aggregation, but I would recommend making that step distinct.

To judge a debate is to decide the overall outcome to all of the arguments presented in the debate. A debate will have a motion (regardless of whether that is formally declared or not), which could be phrased as 'this house believes such and such'. A motion can be represented as a logical proposition, $\alpha$. At the outset of the debate the motion is an open question that could be true or false. Once the debate has run its course, judgement occurs and the question of the motion is answered. The outcome will either be $\alpha$, $\neg\alpha$ or possibly the empty set $\emptyset$. If the outcome is $\neg\alpha$ it is commonly said that the motion was defeated. If $\alpha$, then the motion carried. The empty set may arise for a variety of reasons including either i) there was a stalemate with equal evidence, support or weight for both $\alpha$ or $\neg\alpha$, or ii) there could be not enough compelling evidence to say definitively whether $\alpha$ or $\neg\alpha$ is the case, or iii) some technical or procedural problem occurred.

A debate may be judged by a vote or poll of some kind (e.g. in Parliament, Oxford Union or a jury) or it may be judged by an individual (e.g. a judge, a senior doctor or a newspaper editor). The aim of this thesis, and of related work in the AI literature, is to empower a computer to track and judge a debate and hence determine the outcome. Most authors believe that such an automated judge should be logical and methodical, free from arbitrary or random judgements and able to explain why it reached a particular conclusion. Note that not all AI judges use argumentation or even logic, for example neural networks can be said to judge outcomes and their inner workings are typically free from logic. They usually also lack any form of explanation facility to show why a particular conclusion was reached. Judges based on argumentation are formally introduced in Chapter 2 and revisited in Chapter 7.

Any debate is driven by its knowledgebase, which is a single repository of all of the assumptions available to be reasoned with. Within this knowledgebase may be i) domain-specific knowledge (for example the contents of medical text books), ii) facts or evidence specific to the current case being considered and iii) world knowledge, which is generally known as common sense information, but which is often not written down. In addition to the purely logical knowledgebase, a debate may also have access to metaknowledge, meaning any additional knowledge available about the knowledge in the knowledgebase. Metaknowledge can include such things as priorities between assumptions, thresholds for evidence and domain specific ontologies.

## 1.1 Problem Statement

The problem taken up by this thesis, of debate tracking and judging, was well summarised as long ago as 1937 by the legal expert Professor John H. Wigmore in the following paragraph.

> *'What is wanted is simple enough in purpose, namely, some method which will enable us to lift into consciousness and state in words the reasons why a total mass of evidence does or does not persuade us to a given conclusion and why our conclusion would or should have been different or identical if some part of that total mass of evidence had been different.'*
> (Wigmore, 1937; Twining, 1985)

This quotation draws out two facets of the problem, firstly to track the debate, i.e. *'to lift into consciousness or state in words'*, and secondly to judge the outcome of the debate, i.e. *'persuade us'* or *'our conclusion'*. In tracking a debate, my view is that the whole debate and nothing but the debate should be captured in some formal and visible structure. If the whole debate is not tracked, tracking is incomplete. If something other than the debate is captured, tracking is excessive or inordinate. This problem thus has five parts or steps, i) the *'total mass of evidence'* or knowledgebase, which is the input ii) the *'reasons'* or individual arguments that follow from this evidence, iii) capturing the relevant set of arguments and their interactions in some formal structure, i.e. *'some method'* or tracking the debate, iv) the process of judgement of the motion under debate and v) the output which is *'our conclusion'*.

Inherent in this problem are six key challenges, namely:

1. **Inconsistency**. The evidence is usually inconsistent. Classical logic collapses in the face of inconsistency, a phenomena known as 'ex falso quodlibet'. Anything can be inferred from an inconsistency (including erroneous conclusions). Inconsistency is also *useful* as it captures the two sides of the debate around the motion and also around subservient points argued over in the debate.

2. **Repetition**. Repetition in the evidence and reasons is also common. Such repetition clutters presentation thereby hampering judgement. More problematic, however, is that any judgement scheme which analyses the existence of arguments (i.e. existential judges) or counts arguments (i.e. quantitative judges) can be skewed by repetition to reach a different answer. On the other hand, repetition can be useful as it provides *confirmation* which is helpful for resolving inconsistency, filtering out noise and establishing judge outcomes.

3. **Vital arguments excluded**. Sufficient or complete versus incomplete – it is necessary to capture the whole debate including all of the legitimate ways that arguments interact, with the main two kinds of interaction known as undercut (where one argument attacks or negates the premises of another argument) and rebuttal (where one argument attacks the claim of another argument).

4. **Superfluous arguments included**. Necessary or ordinate versus inordinate – nothing but the debate must be tracked. Anything that is utterly predictable or redundant adds nothing and must be excluded.

5. **Non-monotonic**. The knowledgebase can be subject to revision and a different or revised judgement must be possible, as pointed out by Wigmore. I take this challenge as related to the notion of appeal in courts of law, whereby the total mass of evidence may be augmented or reduced, even in minor ways. Certainly the legal notion of appeal is broader than just revision of the knowledgebase as it can also cover errors in due process, which I touch on in Section 7.2.

6. **Computational complexity**. The issue of deducing an inference from a set of assumptions is a computer science problem known to be at least as hard as non-deterministic polynomial time (NP) problems (Cook, 1971). The computation complexity of argument aggregation is an active area of research (Eiter & Gottlob, 1995; Dunne & Bench-Capon, 2002; Parsons et al., 2003; Wooldridge et al., 2006) with a growing consensus that it is even harder than the problems described by Cook. Thus practical applications may require heuristics to reach approximate conclusions in a reasonable time. The computational complexity of debate tracking and judging, together with algorithms and heuristics, are outside the scope of this thesis and left as an area for future research.

These six challenges are not unique to this thesis, but rather are well known in the fields of logic, mathematics and computer science. This problem of tracking and judging debates is found in a large number of application areas. These applications fall broadly into the two areas of i) professional debate and ii) intra-system decision making. Professional debate needs this problem solved particularly in the areas of law, medicine, journalism, politics and systems engineering. More advanced instances of the problem occur within systems, such as AI agents, that are involved in perception, learning, planning or taking action. Given a mass of inconsistent evidence these system may be presented with questions such as: What do I perceive? Given what I know and what I perceive what do I believe? Given what I believe what should I do now? What should my plan be? What do we believe? What is our plan? Addressing this problem of how to correctly track and judge debates is essential so that correct, rather than erroneous, conclusions can be drawn for these questions. An automated solution is especially needed in areas where the mass of evidence is large and complex enough to make manual processing (i.e. generating the complete set of arguments from the given premises, tracking the debate interactions as a whole and judging the outcome) time consuming and error prone. Consequently, in these situations, automation ought to save time, reduce errors and render the outcome more plausible and thus dependable.

## 1.2 My Solution in the Context of the Literature

This thesis employs a theoretical formal approach and falls within that part of AI known as argumentation. Such an approach ensures that different implementations will obtain the same results; something not established by the informal and applied branches of argumentation. Argumentation is one approach to inconsistency and is found not just in AI, but being logical is also in philosophy and mathematics, dating back at least to Aristotle. Modern interest in argumentation arises with (Wigmore, 1937; Toulmin, 1958).

In this field there are four main formal approaches to defining an individual argument, namely i) minimal consistent subset (mincons) arguments following classical propositional logic (Doyle, 1979), ii) defeasible logic argumentation schemes (Nute, 1988), iii) semi-abstract (Bondarenko et al., 1997) and iv) fully abstract arguments from (Dung, 1993). I employ mincons (comprising a minimal consistent set of assumptions, classical logic and an inference or claim), and augment them with a labelled deduction system (Gabbay, 1991) so as to track the contributions of multiple experts or agents.

Arguments in the literature interact by attacking each other, either with an abstract (Dung, 1993), semi-abstract (Bondarenko et al., 1997) or a concrete approach. A concrete attack is either a rebuttal or an undercut (Pollock, 1970). I extend argument interaction by including *confirmation*, also known as corroboration, (Wigmore, 1937)), as a core relationship. I extend undercut by generalising the work of (Pollock, 1970) and (Besnard & Hunter, 2001) to bring undercut closer to professional debate. A minor, but useful, construct I introduce is the *contradiction* which is an aggregation of arguments for and against a claim.

The variety of approaches in the literature to tracking a debate almost all use some form of graph or tree to show how the arguments in the debate interact. These graphs and trees are directed as attacks are drawn as edges with an arrow head on them indicating the direction of attack. In trees the direction of attack is always from leaf towards root. Proposers of formal argument aggregation trees include (Pollock, 1992; Simari et al., 1994; Prakken & Sartor, 1997; Besnard & Hunter, 2001; Amgoud & Cayrol, 2002; García & Simari, 2004; Dung et al., 2006). In contrast, the more abstract approaches of (Dung, 1993; Dung, 1995; Bondarenko et al., 1997) can be represented as graphs, but those representations are not explicitly relied upon in the way that trees are in the other schemes. My analysis suggests that none of these trees or graphs captures 'the whole debate and nothing but the debate'. I prove that finding for one leading tree and show an indication for other trees in the literature. My research finding is that a category of redundant arguments, which I call reflected arguments, are present in these trees from the literature, making them *inordinate*. Some of the trees appear to aim to exclude reflected arguments, but with excessive success, making them *incomplete*. The representative tree from the literature I have picked for detailed analysis is in (Besnard & Hunter, 2001), allowing me to establish several ways that these argument trees are incomplete and inordinate. I then respond to these issues by proposing a tree that is both complete and ordinate – the matt opaque contradiction tree or debate tree for short.

Finally, in the literature and this thesis, the outcome or conclusion of a debate is drawn through the act of judgement. There have been a number of proposals for judge functions including (Pollock, 1992;

Benferhat et al., 1993), with useful elaborations including those from (Elvang-Gøransson & Hunter, 1995; Prakken & Sartor, 1997; Bondarenko et al., 1997; Besnard & Hunter, 2001; Amgoud & Cayrol, 2002; García & Simari, 2004; Dung et al., 2006). Conceptually, the input to a judge is the debate tracking argument aggregation, and the output is the debate conclusion. In practice, the input to my judge functions is i) the motion, ii) the knowledgebase and iii) optional judge-specific metaknowledge. In this way each unique judge can compute the debate tracking data structure it requires, be that a single contradiction, a tree, or some other form, prior to then performing its deliberations and outputting the conclusion. I hold that judges in the literature, whether based on existential or quantitative methods are likely to give skewed results when applied to inordinate or incomplete trees, demonstrating my point by a set of examples.

The broad approach of tracking and judging debates found in the literature is applicable to many areas of professional debate. Of particular potential utility are applications in medicine (Fox & Das, 2000) and law (Prakken, 1997). This thesis suggests the systems engineering applications of design rationale and design trade-off as particular fruitful ones – building on the work of (Buckingham-Shum, 1994).

## 1.3 My Contributions: Confirmation, Preclusion and Reflection

In this section I touch on a) the structure of this thesis and b) its novel concepts not covered elsewhere in the literature. Clearly a central theme running though the majority of the literature on argumentation is that of *attack* - where one argument attacks another. The distinction that an attack is either abstract or concrete is also widespread in the literature, where the abstract approach builds on (Dung, 1993) and the concrete approach covers most of the rest of the literature. My focus is on the particular form of the concrete approach that uses minimal consistent subsets and classical propositional logic together with rebuttal and undercut, as is summarised in, for example, (Besnard & Hunter, 2001) and detailed in my Chapter 2.

The main point of my thesis is that this form of argumentative attack exhibits a phenomena I call *reflection*. I establish, given a knowledgebase of assumptions, plus the attacks and arguments derivable from it, that the existence of some arguments is utterly predictable from the existence of other arguments. So, given their prerequisites, reflected arguments always exist. Likewise, I establish that the existence of some attacks is *predictable* from the existence of other attacks. I call the predictable arguments *reflected arguments* and the predictable attacks *reflected attacks*, while the unpredictable arguments and attacks I call *direct*. I present the case that the direct arguments and attacks are the useful ones and reflected ones are akin to noise (e.g. in an analogue hi-fi music system) or waste (e.g. in the crafts of woodwork or sculpture) which should be eliminated or dismissed.

Included within this thesis is an analysis showing that reflection is not unique to this mincon approach, but rather also exists within other argumentation frameworks and is therefore a more general problem in the argumentation literature. This analysis looks at the assumption-based argumentation approach of (Dung et al., 2006). While the semi-abstract approach began with (Bondarenko et al., 1997) and is often referred to as the BDKT (Bondarenko, Dung, Kowalski and Toni) approach (see for example the review of (Prakken & Vreeswijk, 2002)), I have chosen to base my analysis on the subsequent paper (Dung et al., 2006) as it has slightly evolved its basic definitions to provide a somewhat clearer description of what they call assumption-based argumentation or ABA for short. I show ABA as being concrete enough to establish that it allows reflection. I examine one instance of ABA showing that it can contain reflection; what happens in other instances is an open question. With the fully abstract approach of (Dung, 1993), however, it is not possible to say whether it does or does not contain reflection. Any implementation in software of the fully abstract approach that determines what arguments exist and what arguments attack each other could contain reflection and consequently consideration of the possibility of reflection in such an implementation would be prudent.

In summary so far, an attack is either concrete or abstract and concrete attacks are either reflected or direct. This thesis then makes further distinctions as follows. In many places in the literature there is clarity that a concrete attack is either a rebuttal or an undercut. I am consequently able to show that there exist reflected rebuttals, reflected undercuts, direct rebuttals and direct undercuts. I analyse the properties of these reflected and direct functions in Chapter 4. Undercuts can be reflected off other undercuts or off rebuttals. Likewise rebuttals can be reflected off other rebuttals or off undercuts.

A second theme, in addition to reflection, running through this thesis is that of *confirmation*; that arguments can be logically aggregated into sets of arguments all with the same claim. A simple enough idea in essence, confirmation is introduced, motivated and illustrated in Chapter 2. While confirmation is definitely not a novel introduction to the literature, see for example (Wigmore, 1937; Twining, 1985; Amgoud et al., 2004; Reed & Rowe, 2006), I would argue that my presentation and analysis provides a convenient set of basic argument aggregation tools not found elsewhere. The heart of the novelty of this thesis is the use of confirmation to deepen the analysis of reflection; but first one other distinction.

Just as arguments can attack each other, I establish that confirmations can attack each other. Confirmation is employed in professional debate and is useful for reaching sound debate outcomes. Attack between confirmations can take two forms: undercut or rebuttal. An undercutting attack between confirmations I call a *preclusive undercut*, or *preclusion* for short. Similarly rebuttal attacks exist between confirmations, which I call confirmation rebuttals. Chapter 3, introduces the concept of preclusive undercut. When a confirmation is subject to a preclusive undercut, the attack is effectively saying *'one or other of your arguments is wrong'*, in contrast to the canonical undercut (Besnard & Hunter, 2001) between arguments which says *'one or other of your premises is wrong'*. Consequently a use of preclusive undercuts is for attacking sets of arguments with mutually inconsistent supports. A preclusive undercut can attack many arguments or it can just attack one. Likewise the attacking confirmation can contain one or many arguments. Given preclusive undercut and confirmation rebuttal it is also possible to define confirmation attack with reflected and direct forms.

Combining the notion of attacks between confirmations with that of reflection provides access to a number of interesting behaviours. I discuss this area of reflected and direct confirmations in Chapter 5. My analysis follows the same pattern as the previous chapter, but on the level of confirmations rather than on the level of individual arguments. Thus there exist reflected preclusive undercuts, reflected confirmation rebuttals, direct preclusive undercuts, direct confirmation rebuttals, reflected confirmation attacks and direct confirmation attacks. A feature of these reflections is that the actual number of reflections often does not match what might intuitively be expected. There are two mechanisms involved that can make the number of reflections more than might be anticipated and a further mechanism that can lead to the number of reflections being fewer than might be anticipated. Consequently reflections can be enlarged, reduced or exact. I call the ratio between the anticipated number of arguments and the actual number of reflected arguments the *reflection scaling factor*. Distorted reflections may occur with different parts of a reflection having different reflection scaling factors. It is also the case that different kinds of reflection have different possible ranges for their reflection scaling factors.

A growing part of the literature that is not just on attack addresses the question of *bipolar* frameworks where arguments can have an assisting or positive relationship in addition to the more common attacking or negative relationship. My confirmation functions venture into this bipolar territory. I show that reflections occur with these positive relationships in a similar, but different way from attack. The remainder of this paragraph introduces reflections of a positive or assisting nature. All of the reflections described so far have in common that an attack is engendering an attack, causing what is commonly called *reinstatement*. I define a nomenclature of a chain of arguments with a head (attacked but not attacking) and a tail (attacking but not attacked), showing that these reinstatement type reflections have the reflector at the head of the chain and the reflected argument at the tail. Other kinds of reflection exist within sets of interacting arguments. I have identified another form where the reflector is at the tail of chain and the reflected argument is at the head - these I call *reflected confirming arguments*. As above the symmetry continues with these reflections existing not only at the level of individual arguments but also at the level of confirmations. Hence, I also introduce the *reflected confirming confirmation*. Likewise these tail-to-head forms show that there exist direct confirming arguments and direct confirming confirmations which are free from reflection. These tail-to-head reflections are subject to scaled reflections and have their own pattern of scaling factors. The final chapter, Chapter 8, touches on a further form of assisting arguments and shows that they too can contain reflections.

The last two chapters of the thesis build on the understanding of confirmation, preclusion and reflection to tackle the questions of how best to track and judge debates. My analysis leads to the conclusion that reflected arguments are not good candidates or proxies to represent the arguments they are reflected off. Hence I advocate using only direct arguments within the tracking of debates. Building on my analysis of trees in the literature, showing them to be incomplete, inordinate and inchoate, I propose the debate tree to track *'the debate, the whole debate and nothing but the debate'*. The vertices of a debate tree map to direct contradictions, while the edges map to direct preclusions. Debate trees are finite. In the penultimate chapter, Chapter 7, I define a judge function that assesses debate trees using quantitative methods. This judge is the only one I know of that takes reflection into account and is thus, I believe, likely to give more intuitive results for professional debates than judges skewed by reflection.

The novelty of contributions is high in the middle of the thesis (particularly Chapters 4 and 5) and lower in the early (Chapters 1 to 3) and late material (Chapters 7 and 8). The utility of the material, however, rises steadily as the thesis progresses. The key formal definitions are in Chapters 2 through 6. There is no separate literature review chapter as strong links to the literature exist in almost most chapters, except for the more novel Chapters 4 and 5 where such links are rare.

An outline system design for a tool to help facilitate professional debates is described in Chapter 8. This design covers virtually all of the theory introduced in this thesis. The tool differs from existing tools in that it automates three things i) construction of arguments from premises, ii) building of the debate tracking structure (which is a single tree) and iii) judging the outcome of the debate. Layered on top of this automation is argumentation visualisation, allowing users to understand more clearly the state of play in the debate tracking and judging and thus participate more actively in progressing the debate.

## Chapter 2

# Basic Argumentation Framework

## 2.1 Overview of the Chapter

This chapter formally introduces my basic argumentation framework. The chapter proceeds in six steps that build on one another, here is a quick overview:

### 2.1.1 Six Steps to the Basic Framework

**Step 1 - Knowledgebase** Section 2.2, the initial knowledgebase of labelled assumption formulae, this finite set is denoted $\Delta$. It may come from multiple experts or agents.

**Step 2 - Deductions** Section 2.3, the set of everything that can be deduced from the knowledgebase, denoted $\mathsf{deductions}(\Delta)$, contains labelled deduction formulae.

**Step 3 - Arguments** Section 2.4, those deductions with consistent and minimal premises, are called arguments and denoted $\mathsf{arguments}(\Delta)$. The premises must be consistent to counter the problem of ex falso quodlibet.

**Step 4 - Confirmations** Section 2.5, I then perform a further aggregation of all arguments for each claim. I call these sets of arguments *confirmations* and denote the set of all confirmations, denoted $\Diamond(\Delta)$. I contribute the *inconsistent confirmation* to the literature.

**Step 5 - Contradictions** Section 2.6, a further aggregation that I call *contradiction* identifies the sets of arguments for and against each claim and is denoted $\blacklozenge(\Delta)$.

**Step 6 - Judgement** Section 2.7, the judge function provides a 'useful' subset of the claims. The choice of what is useful depends upon the application, but must include resolving inconsistencies. The $\mathsf{judge}()$ function has many possible forms, a number of which are explored in subsequent chapters.

At the end of the chapter is a comprehensive worked example, Section 2.8, retracing the same six steps as above, this time with specific formulae. Finally comes the conclusion, Section 2.9.

### 2.1.2 Context Within the Literature

Steps one, two and three, from the above list of six steps, are broadly shared by many authors, e.g. (Doyle, 1979; Pollock, 1992; Benferhat et al., 1993; Elvang-Gøransson & Hunter, 1995; Bondarenko et al., 1997; Pollock, 2000; Amgoud & Cayrol, 2002). The use of labels to formally track and distinguish the contributions of multiple experts or agents during argument aggregation is a relatively underdeveloped part of the literature, e.g. (Walton, 1999; Chesñevar & Simari, 2005). The basic ideas of argument aggregation steps four and five are tangentially touched on in the literature, e.g. (Fox et al., 1993a; Krause et al., 1995; Besnard & Hunter, 2001; Amgoud et al., 2004; Walton & Reed, 2006), but not in the form stated here. They are sometimes less formally stated, sometimes more formally, however there is no consensus between argumentation authors on representation, nor on a way to handle multiple experts. With step six I return to common ground with many authors, e.g. (Manor & Rescher, 1970; Cayrol et al., 1992; Pollock, 1992; Benferhat et al., 1993; Elvang-Gøransson & Hunter, 1995; Bondarenko et al., 1997; Prakken & Sartor, 1997; Krause et al., 1998; Bourne & Parsons, 1999; Fox & McBurney, 2002; Konieczny & Pérez, 2002; Dung et al., 2006). While there is consensus on the conceptual goal there is little consensus on nomenclature nor on how to get there. I will show that my framework provides a) a formal notation for subsuming the past work of many authors and b) a convenient way of exploring fresh and useful ground for future AI research and applications in the area of inconsistency.

Readers may feel that the framework of this chapter is overly complex, however it helps to simplify the work of subsequent chapters. This chapter, in isolation, adds nothing significant to the literature; its value will become clear later in the thesis.

## 2.2 Knowledgebase of Labelled Assumptions

My approach is to use classical deduction and augment it with the labelled deductive system (LDS) approach of (Gabbay, 1991; Gabbay et al., 2000). My framework is limited to propositional calculus, leaving the complications of first order predicate calculus, higher order and other logics as areas for future research.

Ordinary set theory and notation are also employed, in particular: if $\mathcal{X}$ and $\mathcal{Y}$ are arbitrary sets then $\wp(\mathcal{X})$ is the power set of $\mathcal{X}$; $\mathcal{X} \times \mathcal{Y}$ is the Cartesian product of $\mathcal{X}$ and $\mathcal{Y}$; $|\mathcal{X}|$ is the cardinality of $\mathcal{X}$;$\mathcal{X} \setminus \mathcal{Y}$ is set difference giving elements that belong to $\mathcal{X}$ but not to $\mathcal{Y}$; and $\emptyset$ is the empty set. $\mathbb{N}$ is the set of natural numbers (the non-negative integers), $\mathbb{R}$ is the set of real numbers and $\mathbb{N}^+$ is the set of natural numbers greater than zero. $\perp$ is inconsistency and $\top$ is tautology. I now use a recursive definition to define my starting point, the language $\mathcal{L}$.

**Definition 2.2.1.** *I employ classical propositional logic in defining a* **language** $\mathcal{L}$. *Let lower case Greek letters be formulae* $\alpha, \beta, \gamma, \ldots$ *in* $\mathcal{L}$. *I assume* $\mathcal{L}$ *contains a finite set of atoms (or equivalently atomic formulae). Furthermore let* $\mathcal{L}$ *use the standard classical connectives,* $\neg$ *negation,* $\wedge$ *conjunction,* $\vee$ *disjunction and* $\rightarrow$ *implication. If* $\alpha$ *is a formula then* $\neg\alpha$ *is also a formula. If* $\alpha$ *and* $\beta$ *are formulae then* $\alpha \wedge \beta$, $\alpha \vee \beta$ *and* $\alpha \rightarrow \beta$ *are also formulae.*

Although the number of atomic formulae is finite, the classical propositional proof rules, such as $\vee$I and $\wedge$I, ensure that any number of compound formulae can be built up and thus that the language $\mathcal{L}$ is infinite. Consequently, $\mathcal{L}$ allows formulae of arbitrary complexity thereby providing the set of all possible formulae. The biconditional symbol $\leftrightarrow$ can also be constructed following the standard definition

$\alpha \leftrightarrow \beta$ iff $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$. I use $\vdash$ as classical deduction with the standard classical propositional proof rules.

**Example 2.2.1. Formulae.** $\alpha, \beta, \gamma, \alpha \vee \delta, \pi \wedge \neg \lambda, \neg(\zeta \wedge (\zeta \rightarrow \xi)), \omega \vee \rho \vee \sigma \vee \alpha, \beta \rightarrow \neg(\eta \wedge \nu), \neg(\zeta \wedge (\zeta \rightarrow \xi) \wedge \pi \wedge (\neg \pi \vee \xi)), \ldots \in \mathcal{L}$.

Three kinds of labels are used in this framework for i) assumptions, ii) deductions and iii) confirmations.

**Definition 2.2.2. Labels.** *Let* $\mathcal{A} = \{a, b, c, \ldots\}$ *be assumption labels,* $\mathcal{B} = \wp(\mathcal{A})$ *be deduction labels and* $\mathcal{C} = \wp(\mathcal{B})$ *be confirmation labels.*

I use the following nomenclature: $\mathcal{B} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \ldots, \mathcal{A}\}$ with equivalent set names of $\mathcal{B} = \{\emptyset, I, J, K, L, M, N, O, \ldots, \mathcal{A}\}$. Consequently $\mathcal{C} = \{\emptyset, \{\emptyset\}, \{I\}, \{J\}, \{K\}, \{I,J\}, \{I,K\}, \{J,K\}, \{J,K,L\}, \ldots, \{\mathcal{A}\}, \mathcal{B}\}$, with equivalent set names of $\mathcal{C} = \{\emptyset, \{\emptyset\}, \mathrm{S}, \mathrm{T}, \mathrm{U}, \mathrm{V}, \mathrm{W}, \mathrm{X}, \mathrm{Y}, \ldots, \mathcal{B}\}$. This detailing of the collection $\mathcal{C}$ is partial as many of its members contain $\emptyset$. Where necessary I also use expansions such as $\mathrm{X} = \{I_1, \ldots, I_n\}, \mathrm{Y} = \{J_1, \ldots, J_m\}$ to show the deduction labels that are members of a confirmation label. All of these labels are unique as they are set members. The empty set is a necessary member of $\mathcal{B}$ for labelling tautologies and of $\mathcal{C}$ for showing the absence of evidence. It follows that $\emptyset \notin \mathcal{A}$, i.e. assumption labels can never be the empty set.

**Definition 2.2.3. Labelled formulae.** *If* $\alpha \in \mathcal{L}$ *is a formula and* $a \in \mathcal{A}$ *is an assumption label, then* $a : \alpha$ *is a labelled assumption. If* $\beta \in \mathcal{L}$ *is a formula and* $I \in \mathcal{B}$ *is a deduction label, then* $I : \beta$ *is a labelled deduction. If* $\beta \in \mathcal{L}$ *is a formula and* $\mathrm{X} \in \mathcal{C}$ *is a confirmation label, then* $\mathrm{X} : \beta$ *is a labelled confirmation.*

It follows that if $\beta \in \mathcal{L}$ is a formula and $b \in \mathcal{A}$ is an assumption label, then $b : \beta$ is a labelled assumption. If $I \in \mathcal{B}$ is a deduction label, where $I = \{b\}$, then $I : \beta$ and $\{b\} : \beta$ are labelled deductions. If $\mathrm{X} \in \mathcal{C}$ is a confirmation label, where $\mathrm{X} = \{I\}$ then $\mathrm{X} : \beta, \{I\} : \beta$ and $\{\{b\}\} : \beta$ are labelled confirmations. Hence $I : \beta = \{b\} : \beta$ and $\mathrm{X} : \beta = \{I\} : \beta = \{\{b\}\} : \beta$. However $b : \beta \neq I : \beta \neq \mathrm{X} : \beta$ as the first is a labelled assumption, the second is a labelled deduction and the third is a labelled confirmation.

**Example 2.2.2. Labelled formulae.** $a : \alpha, b : \beta, c : \beta, d : \beta \wedge \pi$ *are labelled assumptions. Let* $I = \{a\}$. *So* $I : \alpha, \{b\} : \beta, \{a,b\} : \alpha \wedge \beta, \{a,c\} : \alpha \wedge \beta$ *are labelled deductions. Let* $\mathrm{X} = \{I\}$. *Thus* $\mathrm{X} : \alpha, \{I\} : \alpha, \{\{b\}\} : \beta, \{\{a,b\}, \{a,c\}\} : \alpha \wedge \beta$ *are labelled confirmations.*

Clearly these labels have an algebra which could be explored in more detail. I will not be manipulating the labels directly, but rather have them follow, or track, the processes of classical logic and argument aggregation as described below. I also require some housekeeping functions to isolate the parts of labelled formulae. The label function returns just the label and not the formula.

**Definition 2.2.4.** *Let* $a : \alpha$ *be a labelled assumption,* $I : \alpha$ *be a labelled deduction and* $\mathrm{X} : \alpha$ *be a labelled confirmation. The* **label function,** *denoted* $\mathsf{label}()$, *returns just the label of an assumption, deduction or* *confirmation, such that:*

$$\mathsf{label}(a\!:\!\alpha) = a. \qquad \mathsf{label}(I\!:\!\alpha) = I. \qquad \mathsf{label}(\mathrm{X}\!:\!\alpha) = \mathrm{X}.$$

Strip does the opposite of label, giving just the formula and not the label. Clearly here $a \in \mathcal{A}$ is an assumption label, $I \in \mathcal{B}$ is a deduction label and $\mathrm{X} \in \mathcal{C}$ is a confirmation label.

**Definition 2.2.5.** *Let $a : \alpha$ be a labelled assumption, $I : \alpha$ a labelled deduction and $\mathrm{X} : \alpha$ a labelled confirmation. The* **strip functions** *remove the labels from labelled assumptions, labelled deductions and labelled confirmations, such that:*

$$\mathsf{stripAssumption}(a\!:\!\alpha) = \alpha.$$

$$\mathsf{stripDeduction}(I\!:\!\alpha) = \alpha.$$

*If* $\mathrm{X} \neq \emptyset$

*then* $\mathsf{stripConfirmation}(\mathrm{X}\!:\!\alpha) = \alpha,$

*otherwise* $\mathsf{stripConfirmation}(\mathrm{X}\!:\!\alpha) = \emptyset.$

I will be exploring confirmations labelled with the empty set in Definition 2.5.2 et seq. (where I call them unfounded confirmations). Given the strip functions for single formulae, I now extend them to sets of formulae.

**Definition 2.2.6.** *Let $\Phi$ be a set of labelled assumption formulae, a set of labelled deduction formulae or a set of labelled confirmation formulae. The following strip functions remove the label or labels from sets of labelled formulae, such that:*

$$\mathsf{stripAssumptions}(\Phi) = \{\mathsf{stripAssumption}(a\!:\!\alpha) \mid a\!:\!\alpha \in \Phi\},$$

*where $\Phi$ is a set of labelled assumptions.*

$$\mathsf{stripDeductions}(\Psi) = \{\mathsf{stripDeduction}(I\!:\!\alpha) \mid I\!:\!\alpha \in \Psi\},$$

*where $\Psi$ is a set of labelled deductions.*

$$\mathsf{stripConfirmations}(\Upsilon) = \{\mathsf{stripConfirmation}(\mathrm{X}\!:\!\alpha) \mid \mathrm{X}\!:\!\alpha \in \Upsilon\},$$

*where $\Upsilon$ is a set of labelled confirmations.*

If strip is applied to a set that already lacks labels, then the result is the same set, so e.g. $\mathsf{stripDeductions}(\mathsf{stripDeductions}(\Phi)) = \mathsf{stripDeductions}(\Phi)$. If any member of $\Upsilon$ is labelled with the empty set then that stripped confirmation is the empty set and the rest of e.g. $\mathsf{stripDeductions}(\Upsilon)$ is unaffected as, of course, for any set $A \cup \emptyset = A$. Now for a simple example of stripping off labels.

**Example 2.2.3. Stripping off labels.** *Let $\{a:\alpha, b:\beta\}$ be a set of labelled assumption formula, $\{\{a\}:\alpha, \{b\}:\beta\}$ be a set of labelled deduction formulae and $\{\{\{a\}\}:\alpha, \{\{b\}\}:\beta\}$ be a set of labelled confirmation formula. It thus follows that* $\mathsf{stripAssumptions}(\{a:\alpha, b:\beta\}) = \mathsf{stripDeductions}(\{\{a\}:\alpha, \{b\}:\beta\}) = \mathsf{stripConfirmations}(\{\{\{a\}\}:\alpha, \{\{b\}\}:\beta\}) = \{\alpha, \beta\}$.

In the above example, set names could be substituted for the sets, that is for the deduction $I = \{a\}$ and the confirmation $\mathrm{X} = \{I\} = \{\{a\}\}$. The resultant stripped formulae would be unchanged as the labels are removed regardless of whether they are sets or set names.

A popular starting point in the argumentation literature is to assume a knowledgebase of formulae that can be used as assumptions in arguments. I have the same starting point, with the addition of unique labels to the assumption formulae.

I will use the upper case Greek letters to denote possibly-infinite sets of labelled formulae $\Omega, \Psi, \Phi, \Gamma, \ldots$. I will assume the initial **knowledgebase**, denoted $\Delta$, to be finite, where all the members of $\Delta$ are labelled assumption formulae. Thus I have fixed the definition of $\Delta$ to be the starting-point knowledgebase of labelled assumptions, a convention that holds throughout this thesis. If the nature of the set of labelled formulae is different then a different capital Greek letter from $\Delta$ will be used. In some examples in this thesis I also explore the effect of changing or adding to the contents of $\Delta$.

**Proposition 2.2.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae.*

*If* $|\mathsf{stripAssumptions}(\Delta)| < |\Delta|$

*then there exists* $a\!:\!\alpha, b\!:\!\beta \in \Delta$ *s.t.* $a\!:\!\alpha \neq b\!:\!\beta$, $\mathsf{stripAssumption}(a\!:\!\alpha) = \mathsf{stripAssumption}(b\!:\!\beta)$.

*Proof.* If stripping the labels off $\Delta$ reduces its cardinality then at least two members have merged. So if $|\mathsf{stripAssumptions}(\Delta)| < |\Delta|$ it is because an $A, B \in \Delta$ with $A \neq B$ have become $\mathsf{stripAssumption}(A), \mathsf{stripAssumption}(B) \in \mathsf{stripAssumptions}(\Delta)$ with $\mathsf{stripAssumption}(A) = \mathsf{stripAssumption}(B)$. A labelled formula $a : \alpha$ has only two parts, label $a$ and formula $\alpha$, so if the label is removed only the formula remains. Therefore it must be the case that if $a : \alpha \neq b : \beta$, but $\mathsf{stripAssumption}(a\!:\!\alpha) = \mathsf{stripAssumption}(b\!:\!\beta)$ then $\alpha = \beta$. Hence the cardinality reduction. □

So as $\emptyset \notin \mathcal{A}$ and $\emptyset$ is not a formula, if $|\mathsf{strip\ Assumptions}(\Delta)| < |\Delta|$ then $|\Delta| \geq 2$.

**Proposition 2.2.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae.*

$$0 \leq |\mathsf{stripAssumptions}(\Delta)| \leq |\mathsf{label}(\Delta)| = |\Delta| < \infty.$$

*Proof.* Labels in the knowledgebase are by Definition 2.2.2 unique so $|\mathsf{label}(\Delta)| = |\Delta|$. $\Delta$ is by definition finite and can be empty, so $0 \leq |\Delta| < \infty$. Formulae in the knowledgebase have no uniqueness constraint so $|\mathsf{stripAssumptions}(\Delta)| \leq |\Delta|$. Hence the number of formulae in $\Delta$, i.e. $|\mathsf{stripAssumptions}(\Delta)|$ is less than or equal to the number of labels in $\Delta$, i.e. $|\mathsf{stripAssumptions}(\Delta)|$, hence $|\mathsf{stripAssumptions}(\Delta)| \leq |\mathsf{label}(\Delta)|$. □

The unique assumption labels thus allow otherwise indistinguishable formulae to be distinguished. The combination of unique label and non-unique formula allows multiple occurrences of the same formula in the same set. This non-uniqueness, or repetition, of formulae is an interesting feature that is explored throughout the thesis. It is used, for example, to allow the observations of two witnesses $a$ and $b$, who both saw the same event $\alpha$, to be denoted as $a : \alpha$ and $b : \alpha$. The assumption labels on a set of labelled deductions may be non-unique, as a natural consequence of classical deduction, for example, the assumption $a : \alpha \wedge \beta$ yields deductions $\{a\} : \alpha$ and $\{a\} : \beta$. An area for further research is to use a part of each assumption label to track which agent or expert originated that assumption.

## 2.3 Propagating Labels During the Process of Deduction

This section on labelled deduction addresses the behaviour of labels during the process of classical deduction. I propagate the assumption labels through the process of deduction to give labelled deduction formulae. While I could have created an augmented form of $\vdash$, such as $\vdash^*$, to propagate labels, the notation is lighter if I simply extend the standard definition of $\vdash$ in the style of (Gabbay, 1996).

**Definition 2.3.1. Label propagation.** *Let the standard definition of $\vdash$ be extended so that $\vdash$ propagates labels, such that:*

$$\{\mathsf{stripAssumption}(a_1 : \alpha_1), \ldots, \mathsf{stripAssumption}(a_n : \alpha_n)\} \vdash \beta$$
$$\textit{iff } \{a_1 : \alpha_1, \ldots, a_n : \alpha_n\} \vdash \{a_1, \ldots, a_n\} : \beta, \textit{ where } n \geq 0.$$

I now show this propagation of labels during the process of deduction using a set-based notation, an approach I will continue to use throughout the thesis. Here is the set of all deductions which contains every $I : \alpha, J : \beta, K : \gamma, \ldots$ that can be deduced from $\Delta$.

**Definition 2.3.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae. The* **set of labelled deductions**, *denoted* $\mathsf{deductions}(\Delta)$ *that can be deduced from $\Delta$ using classical logic is:*

$$\mathsf{deductions}(\Delta) = \{I : \alpha \mid \Delta \vdash I : \alpha\}.$$

The number of assumptions for a deduction is $n$, where $n = 0$ for tautologies. Although later in this thesis unfounded confirmations are discussed, there are no unfounded deductions, nor unfounded arguments. Unfounded means that there is no evidence to know whether $\alpha$ is true or false. Unfounded confirmations always exist for every formula in $\mathcal{L}$. The last two deductions in the following example are tautologies; I call them tautological deductions.

**Example 2.3.1. Deductions and tautologies.** *Let $\Delta = \{a : \alpha, b : \alpha \rightarrow \beta\}$ and $I = \{a\}$, $J = \{a, b\}$ then it follows that $\{a\} : \alpha, \{b\} : \alpha \rightarrow \beta, \{b\} : \neg\alpha \vee \beta, \{b\} : \neg\alpha \vee \beta \vee \gamma, \{a, b\} : \beta, I : \alpha, J : \beta, \emptyset : \alpha \vee \neg\alpha \in$* $\mathsf{deductions}(\Delta)$.

The only deductions which are labelled with the empty set are where the deduced formula is a tautology. No assumptions are needed to deduce a tautology; some authors write a tautology specifically with no set of supporting assumptions, e.g. just $\vdash \alpha \vee \neg\alpha$. All of the deduction labels, that is members of the set $\mathcal{B}$, are useful, with the empty set label having the use of labelling tautologies. Note also that the strip function behaves as normal for the case of stripping a deduction where the label is the empty set, i.e. $\mathsf{stripDeduction}(\emptyset : \alpha) = \alpha$.

In all cases the set of deductions is infinite, whether $\Delta \nvdash \bot$ or $\Delta \vdash \bot$, as I now show.

**Proposition 2.3.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae.*

$$|\mathsf{deductions}(\Delta)| = \infty.$$

*Proof.* Even if $\Delta = \emptyset, \emptyset \vdash \top$ so there are an infinite number of tautologies such as $\emptyset : \alpha \vee \neg\alpha, \emptyset : \alpha \vee \neg\alpha \vee \beta \vee \neg\beta, \emptyset : \alpha \vee \neg\alpha \vee \gamma, \ldots \in \mathsf{deductions}(\emptyset)$. Furthermore as additional formulae are added to $\Delta$, beyond $\emptyset$, they can only increase $|\mathsf{deductions}(\Delta)|$. So in all cases $|\mathsf{deductions}(\Delta)| = \infty$. □

If the knowledgebase is inconsistent then at least one deduction will have inconsistent premises.

**Example 2.3.2.** *Let* $\Delta = \{a : \phi, b : \psi, c : \neg\psi \wedge (\phi \rightarrow \alpha)\}$. *Certainly there are many deductions from* $\Delta$ *with consistent premises, such as* $\{a, c\} : \alpha, \{c\} : \neg\psi$. *However because there exists at least one deduction with inconsistent premises,* $\{b, c\} : \psi \wedge (\neg\phi \vee \alpha)$, *it follows that* $\Delta \vdash \bot$.

I now need a helper function to give the assumption formulae represented by a deduction label. Whilst defining this function I make it general enough to work with either deduction labels or confirmation labels.

**Definition 2.3.3.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae,* $I$ *be a deduction label and* $\mathrm{X}$ *be a confirmation label. The* **formulae function**, *denoted* $\mathsf{formulae}(I, \Delta)$, *obtains the labelled assumption formulae that are labelled by I, from* $\Delta$, *such that:*

$$\mathsf{formulae}(I, \Delta) = \{a : \alpha \mid a \in I \textit{ and } a : \alpha \in \Delta\}.$$

*Similarly the formulae function* $\mathsf{formulae}(\mathrm{X}, \Delta)$ *obtains the labelled assumption formulae that are labelled by the confirmation label* $\mathrm{X}$, *from* $\Delta$, *such that:*

$$\mathsf{formulae}(\mathrm{X}, \Delta) = \{a : \alpha \mid a \in I, I \in \mathrm{X} \textit{ and } a : \alpha \in \Delta\}.$$

**Example 2.3.3. Formulae function.** *Let* $\{a : \alpha, b : \alpha \rightarrow \beta, c : \zeta\} \subseteq \Delta$, *then* $\{a, b\} : \beta \in \mathsf{deductions}(\Delta)$. *Let* $I = \{a, b\}$, *then* $I : \beta \in \mathsf{deductions}(\Delta)$. *Therefore* $\mathsf{formulae}(I, \Delta) = \{a : \alpha, b : \alpha \rightarrow \beta\}$ *and* $\mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta)) = \{\alpha, \alpha \rightarrow \beta\}$. *While* $c : \zeta \in \Delta$ *it is not in the formulae needed for* $I$.

So now given the formulae helper function I can establish the property that if the knowledgebase is inconsistent then at least one deduction will have inconsistent premises.

**Proposition 2.3.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae.*

*If* $\Delta \vdash \bot$ *then there exists an* $I : \alpha \in \mathsf{deductions}(\Delta)$ *such that* $\mathsf{formulae}(I, \Delta) \vdash \bot$.

*Proof.* $\mathsf{deductions}(\Delta)$ is the set of all possible deductions. Any $\Phi \subseteq \Delta$ can be the premises of a deduction. Thus at least one deduction will have inconsistent premises $\Phi \vdash \bot$ given $\Delta \vdash \bot$. So given an inconsistent knowledgebase, there exists a deduction $I : \alpha \in \mathsf{deductions}(\Delta)$ with inconsistent premises, that is $\Phi = \mathsf{formulae}(I, \Delta) \subseteq \Delta$ with $\mathsf{formulae}(I, \Delta) \vdash \bot$. □

If the knowledgebase is inconsistent then every formula possible in the language $\mathcal{L}$ can be inferred. This is 'ex falso quodlibet', from falsehood anything follows, i.e. $\{\psi, \neg\psi\} \vdash \phi$, also commonly known as the principle of explosion.

**Proposition 2.3.3.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae.*

*If* $\Delta \vdash \bot$ *then* $\mathsf{stripDeductions}(\mathsf{deductions}(\Delta)) = \mathcal{L}$.

*Proof.* Consider $\Delta = \{a : \alpha, b : \neg\alpha\}, I = \{a, b\}$. So by disjunction introduction it follows that $\mathsf{deductions}(\Delta)$ will contain at least the following members: $\{a\} : \alpha, \{b\} : \neg\alpha, \{b\} : \neg\alpha \vee \gamma$. Now apply disjunctive syllogism on $\{a\} : \alpha, \{b\} : \neg\alpha \vee \gamma$ to see that $I : \gamma \in \mathsf{deductions}(\Delta)$. Here $\gamma$ could be any formula $\gamma \in \mathcal{L}$ so given a $\Delta \vdash \perp$ it follows that $\mathsf{stripDeductions}(\mathsf{deductions}(\Delta)) = \mathcal{L}$. $\square$

The formalisation of assumptions without labels renders confirmation invisible, as $\{\alpha\} \cup \{\alpha\} = \{\alpha\}$. I make these multiple occurrences of $\alpha$ distinguishable through the use of labels: $\{a : \alpha\} \cup \{b : \alpha\} = \{a : \alpha, b : \alpha\}$, which is my primary reason for using labels. In the related situation, $\{\alpha\} \cup \{\neg\alpha\} = \{\alpha, \neg\alpha\}$, contradictory formula are visible without the use of labels. A set of labelled formulae can have a formula and its negation as members and thus be inconsistent. For more information on labelling of formulae see chapter five of (Gabbay, 1998) or in greater detail (Gabbay, 1996; Gabbay et al., 2000).

A second reason for using labels is to support an explanation feature, so that the user can see how a certain conclusion was reached. The set of formulae used to deduce a formula, $\beta$, is known as the 'provenance' of $\beta$. The user can examine the label to see the provenance. The proof rules used however cannot be directly seen by this approach, a constraint which is acceptable for most applications. Note that while a set of proof rules that imply a deduction can be determined, it is not always possible to determine the proof rules actually used. For example, one route might be via modus ponens and another via modus tollens. A third benefit of labels is that they can track which agent contributed which formulae.

## 2.4 Arguments from the Knowledgebase

Arguments are a special case of deductions. All arguments are deductions, but not necessarily vice versa. Deduction labels are also used as argument labels. There are no unfounded arguments, but there are tautological arguments. The strip function works for arguments as it does for deductions. I now explain the motivation and formalisation for arguments.

Not all deductions are sensible or useful, from the perspective of practical debates. Two types of deductions that are not useful are i) where the premises are inconsistent and ii) where the premises are not the minimal subset of $\Delta$ necessary for the deduction. As the knowledgebase may not be consistent, it is important to avoid the classical logic problem of 'ex falso quodlibet'.

**Example 2.4.1. Ex falso quodlibet.** *Let* $\Delta = \{a : \alpha, b : \neg\alpha\}$. *The following deductions can be inferred:* $\{\{a\} : \alpha, \{a\} : \alpha \vee \beta, \{a, b\} : \neg\alpha \wedge (\alpha \vee \beta), \{a, b\} : \beta\} \in \mathsf{deductions}(\Delta)$. *So from an inconsistency purely in* $\alpha$, $\beta$ *is derivable. Thus out of inconsistency anything can be inferred.*

I therefore follow the approach of reasoning with minimal consistent subsets (which is a standard approach in much of the literature, e.g. (Doyle, 1979; Benferhat et al., 1993; Elvang-Gøransson & Hunter, 1995; Besnard & Hunter, 2001; Amgoud & Cayrol, 2002)) and augment it with label propagation. Arguments based on this minimal consistent subset approach are quite commonly referred to as *mincons*.

**Definition 2.4.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae. The labelled consistent subsets of $\Delta$, denoted* consistent$(\Delta)$ *are:*

$$\text{consistent}(\Delta) = \{\Pi \mid \Pi \subseteq \Delta \textit{ and } \Pi \nvdash \bot\}.$$

I now define the set of all of the arguments, $I:\alpha, J:\beta, K:\gamma, \ldots$, that follow from $\Delta$:

**Definition 2.4.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae. The set of labelled* **arguments** *that can be deduced from minimal consistent subsets of $\Delta$, denoted* arguments$(\Delta)$ *is:*

$$\begin{aligned}\text{arguments}(\Delta) \quad = \quad & \{I:\alpha \mid \Pi \in \text{consistent}(\Delta) \textit{ and } \Pi \vdash I:\alpha \textit{ and} \\ & \qquad (\textit{there does not exist } J \textit{ such that } J \subsetneq I \textit{ and } \Pi \vdash J:\alpha)\}.\end{aligned}$$

I call any $I:\alpha \in$ arguments$(\Delta)$ an **argument** for $\alpha$ derivable from $\Delta$ and I call $\alpha$ the **claim** of that argument, unless $\alpha$ is a tautology. The label on an argument here is a deduction label that tracks a single minimal consistent set of assumptions for an argument. I call the set of assumption formulae identified by the argument label the **support** of the argument. Not all deductions are valid arguments; they may not have consistent or minimal supports.

**Example 2.4.2. Inconsistent premises.** *Let $\Delta = \{a:\alpha \wedge \neg(\alpha \rightarrow \beta), b:\alpha \rightarrow \beta\}$. Thus $\{a:\alpha \wedge \neg(\alpha \rightarrow \beta), b:\alpha \rightarrow \beta\} \vdash \{a,b\}:\beta$ is a valid deduction but not a valid argument as the set of assumptions is not consistent. Clearly $\{a:\alpha \wedge \neg(\alpha \rightarrow \beta), b:\alpha \rightarrow \beta\} \vdash \bot$. Thus $\{a,b\}:\beta \in$* deductions$(\Delta)$, *but $\{a,b\}:\beta \notin$* arguments$(\Delta)$,

I now give an example of valid arguments, showing the situation of multiple experts.

**Example 2.4.3. Argument, minimal premises.** *Let $\Delta = \{a:\alpha, b:\alpha \rightarrow \beta, c:\alpha \rightarrow \beta\}$. Then $\{a,b\}:\beta, \{a,c\}:\beta \in$* arguments$(\Delta)$. *However $\{a,b,c\}:\beta \notin$* arguments$(\Delta)$ *as the set of assumptions is not the minimal required to deduce $\beta$.*

This example may be interpreted as one expert contributing $b:\alpha \rightarrow \beta$ and the second expert $c:\alpha \rightarrow \beta$. Thus there are two valid arguments for $\beta$. This situation of multiple experts is akin to that of knowledgebase fusion. If each expert is equated to an original knowledgebase, then the set arguments$(\Delta)$ could represent the fusion. However for arguments$(\Delta)$ to always be consistent a further level of judging is needed, as is discussed at the end of this chapter and in subsequent chapters. In merging the knowledge of multiple experts it is possible that something is learned or deduced that is not known to any expert individually. In the above example, 'a' could be viewed as a third expert meaning that $\beta$ was not known to any of them prior to pooling, but became known after the fusion. To consistently track which expert contributed which formulae would require a more sophisticated labelling scheme, such as $a1, a2, \ldots$ for one expert and $b1, b2, \ldots$ for another.

### 2.4.1 Catenate Inferences

Some schemes for describing arguments, e.g. (Walton, 1989; Gordon & Karacapilidis, 1997; Verheij, 2001), have the notion of intermediate conclusions, so that more than one argument is involved in some kind of sequence in order to reach a final conclusion. The paper (Fox et al., 1993b) seeks a 'complete record of the argument structure', which from their context appears to be a reference to the same notion of a sequence of arguments.

In 1930, Wigmore informally introduced this notion of connected sequenced arguments calling it 'catenate inference' (Wigmore, 1937; Twining, 1985). The following definition of catenate inference shows it to be basically just the cut property of classical logic. To tie better to his informal logic I present it without reference to labels or arguments. As *cut* is well known, no formal definition or proof is needed:

Let $\Gamma$ be a set of unlabelled assumption formulae and let $\Phi, \Psi \subseteq \Gamma$.

$$\text{If } (\Phi \vdash \beta \text{ and } (\{\beta\} \cup \Psi) \vdash \alpha) \text{ then } (\Phi \cup \Psi) \vdash \alpha.$$

For this discussion, I call the antecedent, i.e. the left hand side ($\Phi \vdash \beta$ and $(\{\beta\} \cup \Psi) \vdash \alpha$), the sequence of inferences and $\beta$ the intermediate conclusion. Furthermore, I call the consequent, i.e. the right hand side $(\Phi \cup \Psi) \vdash \alpha$, the **catenate inference** and $\alpha$ the final conclusion. Although written here with just two inferences in the sequence, cut could be applied repeatedly to create sequences of arbitrary length. The example below illustrates catenate inference:

**Example 2.4.4. Catenate inference.** *Let* $\Delta = \{a : \gamma, b : \gamma \rightarrow \beta, c : \beta \rightarrow \alpha\}$. *Thus* $\{a, b\} : \beta, \{a, b, c\} : \alpha \in \mathsf{arguments}(\Delta)$. *Here* $\{a, b\} : \beta$ *is the intermediate conclusion, and* $\{a, b, c\} : \alpha$ *the final conclusion. In my nomenclature,* $\{a, b, c\} : \alpha$ *also captures the catenate inference. However, to be able to write the sequence of inferences as two separate steps would require a slightly different notation from mine. The challenge is that for members of* $\mathsf{arguments}(\Delta)$ *items to the left of the colon must be assumption labels, not a mix of assumption and deduction labels.*

Catenate inferences are also unnecessary for documenting practical arguments as my label notation provides 'a complete record of the argument structure' as sought by (Fox et al., 1993b). Thus developing a notation that mixes assumption and deduction labels to the left of the colon is thus, I argue, unnecessary.

There is a problem with catenate inference in that they allow inconsistent reasoning as I now show:

**Example 2.4.5. Inconsistent catenate inference.** *Let* $\Delta = \{a : \neg\gamma \rightarrow \beta, b : \neg\gamma, c : (\neg\gamma \wedge \alpha) \wedge \neg\beta\}$. *So clearly* $\{a, b\} : \beta \in \mathsf{arguments}(\Delta)$. *Let* $\Phi = \{a : \neg\gamma \rightarrow \beta, b : \neg\gamma\} \subsetneq \Delta, \Psi = \{c : (\neg\gamma \wedge \alpha) \vee \neg\beta\} \subsetneq \Delta$. *Thus* $\Phi \vdash \beta$, *i.e. the* $\{a, b\} : \beta \in \mathsf{arguments}(\Delta)$, *and it follows that* $(\{\beta\} \cup \Psi) \vdash \alpha$ *is a valid catenate inference. The problem comes in that* $\Phi \cup \Psi \vdash \bot$. *Consequently while* $\{a, b, c\} : \alpha \in \mathsf{deductions}(\Delta)$ *is a valid deduction, the point is that the final conclusion* $\{a, b, c\} : \alpha \notin \mathsf{arguments}(\Delta)$ *is not a valid argument. Thus* $\{a, b, c\} : \alpha$ *is an inconsistent catenate inference.*

It is also possible for the catenate inference not to be an argument because it is not minimal. My framework, in contrast, does not permit such inconsistent or non-minimal reasoning, making it perhaps a better match to the goals of professional argumentation. Consequently I choose not to adopt the approach of (Walton, 1989; Gordon & Karacapilidis, 1997; Verheij, 2001; Wigmore, 1937; Twining, 1985), but instead opt to employ the approach of putting all of the assumptions of a catenate inference into a single argument support.

### 2.4.2 Argument Properties

I now discuss some of the properties of arguments. Essentially this whole thesis is about properties of arguments, thus here I touch on a few of the most basic. All arguments are deductions, but not necessarily vice versa.

**Proposition 2.4.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae.*

$$\mathsf{arguments}(\Delta) \subseteq \mathsf{deductions}(\Delta).$$

*Proof.* $\mathsf{deductions}(\Delta)$ is just classical deduction plus the propagation of labels, which just tracks the assumptions used. $\mathsf{arguments}(\Delta)$ is $\mathsf{deductions}(\Delta)$ plus two more conditions, consistency and minimality. So all arguments are deductions. Let $I : \alpha \in \mathsf{deductions}(\Delta)$. If $\mathsf{formulae}(I, \Delta) \vdash \bot$ then $I : \alpha \notin \mathsf{arguments}(\Delta)$. Thus not all deductions are arguments. The $=$ in $\subseteq$ is necessary as there exist $\Delta$, e.g. $\{a : \alpha\}$ where no deductions violate the two constraints of arguments. □

How many arguments can be derived from any given $\Delta$, where $\Delta$ is finite? The cardinality of the set $\mathsf{arguments}(\Delta)$ is infinity, as shown in the following proposition.

**Proposition 2.4.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae.*

$$|\mathsf{arguments}(\Delta)| = \infty.$$

*Proof.* The set of arguments is infinite because even if $\Delta = \emptyset$, then there are an infinite number of tautologies $\Delta \vdash \top$ that follow, such as $\Delta \vdash \emptyset : \alpha \vee \neg\alpha, \Delta \vdash \emptyset : \alpha \vee \neg\alpha \vee \xi \vee \neg\xi, \ldots$ and so on ad infinitum. Here the support $\{a : \alpha\}$ in all cases is consistent and minimal. Adding formulae to $\Delta$ would simply make $|\mathsf{arguments}(\Delta)|$ even larger. □

As the set of arguments is always infinite, while $\Delta$ is by definition finite, it follows that $\Delta$ can never be closed under argument. I give an example of this infinity.

**Example 2.4.6. Infinity of arguments.** *Let $\Delta = \{a : \alpha, b : \beta\}$. Conjunction introduction adds an infinite number of formulae, such as $\{\{a\} : \alpha, \{a, b\} : \alpha \wedge \beta, \{a\} : \alpha \wedge \alpha, \{a, b\} : \alpha \wedge \beta \wedge \alpha, \{a\} : \alpha \wedge \alpha \wedge \alpha, \ldots \in \mathsf{arguments}(\Delta)$. Disjunction introduction also adds an infinite number of claims, including $\{a\} : \alpha, \{a, b\} : \alpha \vee \beta, \{a\} : \alpha \vee \alpha, \{a, b\} : \alpha \vee \beta \vee \gamma, \{a\} : \alpha \vee \delta \vee \zeta, \{a\} : \alpha \vee \delta \vee \zeta \vee \lambda \vee \mu \vee \psi \vee \varsigma, \ldots \in$ $\mathsf{arguments}(\Delta)$.*

The members introduced by the disjunction introduction rule are constrained by the set of assumptions. However the members introduced by $\vee$I are constrained only by the language $\mathcal{L}$ and not by $\Delta$. Some readers might think that claims introduced by these rules are a kind of noise or nuisance, however, as will be shown in Chapter 3 on undercuts, a number of them are particularly useful.

While the overall set of arguments is infinite, for a specific claim it is finite. To prove this cardinality matter first requires two helper functions and a new definition to isolate just the arguments for one formula. Truncate delivers the nearest integer $i$ below any positive real number $r$. The output is always $i \geq 0$.

**Definition 2.4.3.** *The* **truncate** *function, denoted* $\mathsf{trunc}(r)$*, is such that:* $i = \mathsf{trunc}(r)$*, where* $i \in \mathbb{N}, r \in \mathbb{R}, i \leq r, (i+1) \geq r$ *and* $r \geq 0$.

A standard result in discrete mathematics is the number of combinations of n objects taken k at a time, with order being irrelevant. The factorial function, n!, has the standard definition $n! = n \cdot (n-1) \cdot \cdots \cdot 1$, where $\cdot$ is multiplication.

**Definition 2.4.4.** *Let* $i, n \in \mathbb{N}$. *The* **combinations** *function, denoted* $\mathsf{combinations}(n, i)$*, gives the number of arrangements of a set of n objects, taken i at a time in a given order.*

$$\mathsf{combinations}(n, i) = \frac{n!}{i! \cdot (n-i)!}.$$

All the subsets of a set, e.g. $\Phi$, that is the power set of $\Phi$, denoted $\wp(\Phi)$, when ordered via subset inclusion can be drawn as a lattice with the complete set, $\Phi$, at the top of the lattice and $\emptyset$ at the bottom. In between the top and the bottom is the body of the lattice holding all the other subsets of $\Phi$. See the example of Section 2.8 to see a visually laid out lattice (akin to a Hasse diagram of order theory). The widest part of the lattice for an even $\Phi$ contains subsets with a cardinality of $|\Phi| \div 2$; for an odd $\Phi$ there are two widest rows and the truncate function, trunc(), is needed. While I do not employ such an ordering in my definitions, the notion of lattices is informally helpful for visualising the characteristics of a power set.

**Definition 2.4.5.** *The* **mid lattice** *function, denoted* $\mathsf{midLattice}(\Phi)$*, is the cardinality of the largest subset of subsets of* $\Phi$ *in which none of the members are strict subsets of other members.*

$$\mathsf{midLattice}(\Phi) = \mathsf{combinations}(|\Phi|, \mathsf{trunc}(|\Phi| \div 2)).$$

If the power set of $\Phi$ was ordered by subset inclusion then this function delivers the number of members in the largest horizontal line across the centre of a lattice of the power set of $\Phi$. If $|\Delta| = 5$ then $\mathsf{midLattice}(\Delta) = 10$. Or if $|\Delta| = 6$ then $\mathsf{midLattice}(\Delta) = 20$. Given these helper functions, I can now return to the questions of cardinality. Only for a rare $\Delta$ does $\mathsf{stripDeductions}(\mathsf{arguments}(\Delta)) = \alpha$, however for any $\Delta$ to see just the arguments for $\alpha$, leaving out those for $\beta, \gamma, \pi, \ldots$ I need the following definition.

**Definition 2.4.6.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\alpha \in \mathcal{L}$. *The set of* **arguments for** $\alpha$*, denoted* $\mathsf{arguments}(\alpha, \Delta)$*, is such that:*

$$\mathsf{arguments}(\alpha, \Delta) = \{A \in \mathsf{arguments}(\Delta) \mid \mathsf{stripDeduction}(A) = \alpha\}.$$

The cardinality of the set of arguments for just one claim is:

**Proposition 2.4.3.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\alpha \in \mathcal{L}$.

$$0 \leq |\mathsf{arguments}(\alpha, \Delta)| \leq \mathsf{midLattice}(\Delta).$$

*Proof.* It could be that there are no arguments for $\alpha$ in a given $\Delta$, if $\alpha \notin \mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$, so for that $\Delta$, $\mathsf{arguments}(\alpha, \Delta) = \emptyset$ and thus $0 \leq |\mathsf{arguments}(\alpha, \Delta)|$. Or it could be that every $\Phi \subseteq \Delta$

is such that there exists an argument $I:\alpha$ in $\mathsf{arguments}(\alpha,\Delta)$, where $\mathsf{formulae}(I,\Delta) = \Phi$. Let one of these subsets be denoted $\Psi \subseteq \Delta$ where there exists an argument $J:\alpha$ in $\mathsf{arguments}(\alpha,\Delta)$, such that $\mathsf{formulae}(J,\Delta) = \Psi$. Not all of these subsets, $\Phi \subseteq \Delta, \Psi \subseteq \Delta$, would yield valid arguments as some of them would be subsets of others, i.e. if $\Phi \subseteq \Psi$, and thus not be minimal. The maximum number of subsets of $\Delta$ that are not subsets of each other are thus given by the midLattice() function. Consequently $|\mathsf{arguments}(\alpha,\Delta)| \leq \mathsf{midLattice}(\Delta)$. □

In stripping labelled-arguments labelled by the empty set the behaviour is that the label goes and the claim (a tautology) remains, as shown in the following example.

**Example 2.4.7. Tautological argument.** *Let $\emptyset:\alpha \in \mathsf{arguments}(\Delta)$ be a tautological argument. Then $\mathsf{stripDeduction}(\emptyset:\alpha) = \alpha$ and $\alpha \vdash \top$. Furthermore if $\Delta = \emptyset$ then for every $I:\alpha \in \mathsf{arguments}(\Delta)$ it is the case that $\alpha \vdash \top$.*

**Proposition 2.4.4.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae.*

$$\mathsf{stripAssumptions}(\Delta) \subseteq \mathsf{stripDeductions}(\mathsf{arguments}(\Delta)).$$

*Proof.* For all $a:\alpha \in \Delta$ it follows that $\{a\}:\alpha \in \mathsf{arguments}(\Delta)$ and $\{a\}:\alpha \vee \beta \in \mathsf{arguments}(\Delta)$. For all $\alpha \in \mathsf{stripAssumptions}(\Delta)$, there exists a label $b$ such that $\{b\}:\alpha \in \mathsf{arguments}(\Delta)$ and $\{b\}:\alpha \vee \beta \in \mathsf{arguments}(\Delta)$. Therefore for all $\alpha \in \mathsf{stripAssumptions}(\Delta)$ it follows that $\alpha \in \mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$ and $\alpha \vee \beta \in \mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$. But, for all $\alpha \in \mathsf{stripAssumptions}(\Delta)$, it is not necessarily the case that $\alpha\vee\beta \in \mathsf{stripAssumptions}(\Delta)$. Even if $\Delta = \emptyset$ it follows that $\mathsf{arguments}(\Delta)$ will contain innumerable tautologies so therefore $\mathsf{stripAssumptions}(\Delta) \subseteq \mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$. □

The majority of authors using mincons write an argument as $\langle\Phi,\alpha\rangle$, (or a similar notation using square brackets instead of angle brackets) where $\Phi$ is a minimal consistent set of formulae implying $\alpha$, i.e. $\Phi$ is the support and $\alpha$ the claim. All $\langle\Phi,\alpha\rangle$ can be written as $I:\alpha$, however the converse does not necessarily hold as my notation allows multiple occurrences of a formulae in $\Delta$. Thus:

$$\text{If } \langle\Phi,\alpha\rangle \text{ then } I:\alpha, \text{ where } \Phi = \mathsf{formulae}(I,\Delta).$$

Another simple feature is that some arguments are reflexive arguments, as defined below (which is different from reflected arguments in Chapter 4). My use of the word reflexive here is conceptually, but not formally, the same as the many uses of the term in grammar, mathematics and computer science. In the following definition $\bigwedge$ provides the conjunction of a members of a set of formulae. Intuitively an argument is a reflexive argument iff the conjunction of its support equals its claim, or more formally:

**Definition 2.4.7.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha \in \mathsf{arguments}(\Delta)$. I call $I:\alpha$ a **reflexive argument** iff*

$$\mathsf{stripDeduction}(I:\alpha) = \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(I:\alpha),\Delta)).$$

With classical logic as the set of inference rules, reflexive arguments always exist, so if $a:\alpha \in \Delta$ then $\{a\}:\alpha \in \mathsf{arguments}(\Delta)$. If the support is only one formula then the conjunction is immaterial, however, if $|\mathsf{formulae}(\mathsf{label}(I:\alpha),\Delta)| > 1$ the $\bigwedge$ is necessary.

## 2.5 Introducing Confirmation

Now that I have established the set of arguments, I need to aggregate together the ones that are for the same claim – which I then call a confirmation. This section formally defines the confirmation as the first novel (albeit simple) idea of the thesis. I put that this formalisation of confirmation is novel relative to the formal argumentation literature; the concept has been present in parts of the informal argumentation literature and in debate in general since at least the 1930s (Wigmore, 1937; Twining, 1985). The argumentation diagramming software tool, Araucaria, for example, building on the work of Wigmore, uses a closed triangular symbol to represent a confirming argument (Reed & Rowe, 2006). The formalisation of confirmation becomes my launching off point for discovering a number of thought-provoking properties of argumentation and debates. I start by defining a single confirmation and then go on to define various sets of confirmations.

**Definition 2.5.1.** *A* **single confirmation** *is of the form* $\mathrm{X}:\alpha$*, where* $\alpha \in \mathcal{L}$ *is a formula and* $\mathrm{X} \in \mathcal{C}$ *is a confirmation label and each* $I \in \mathrm{X}$ *is such that* $I:\alpha \in \mathsf{arguments}(\Delta)$.

A confirmation label, $\mathrm{X} = \{I_1, \ldots, I_n\}$, thus tracks a set of arguments for a single given claim. By definition, a confirmation label does not necessarily track all arguments for a claim, just some of them, as I go on to explore. The number of arguments in a confirmation can be as few as zero.

**Proposition 2.5.1.** *For all* $\alpha$, $\{\emptyset\}:\alpha$ *is a confirmation, iff* $\alpha \vdash \top$.

*Proof.* $\{\emptyset\}:\alpha$ means that the support for the argument for $\alpha$ is the empty set, i.e. that it needs no support and therefore $\alpha$ is always true, i.e. $\alpha \vdash \top$ is a tautology. Thus if $\{\emptyset\}:\alpha$ is a confirmation then $\alpha \vdash \top$. Now starting with $\alpha$ is a tautology posit two arguments $I:\alpha$ and $J:\alpha$ where $I \neq J$. Clearly $I = \emptyset$ as $\emptyset \vdash \top$. If $J \neq \emptyset$ than $J$ would not be minimal and hence $J:\alpha$ not an argument. Therefore such a $J:\alpha$ cannot exist and it must be the case that $\{\emptyset\}:\alpha$ is a confirmation. So if $\alpha \vdash \top$ then $\{\emptyset\}:\alpha$ is a confirmation. Consequently for all $\alpha$, $\{\emptyset\}:\alpha$ is a confirmation, iff $\alpha \vdash \top$. □

**Definition 2.5.2.** *A confirmation of the form* $\{I\}:\alpha$ *is called a* **self confirmation***. A confirmation of the form* $\emptyset:\alpha$ *is called an* **unfounded confirmation***. A confirmation of the form* $\{\emptyset\}:\alpha$ *is called a* **tautological confirmation***.*

In addition to these three special cases for labelled confirmations, coming shortly in Section 2.5.1, is a fourth, the inconsistent confirmation. A self confirmation means that there is only one argument for $\alpha$. In normal English it would not be called 'confirmation', however I call it that as it is clearly a logical member of the family of confirmations. Unfounded means that there is no evidence to know whether $\alpha$ is true or false, or in other words there are no arguments for $\alpha$. I will come on to show the role played by unfounded confirmations within various contradictions. Tautological confirmations such as the confirmation $\{\emptyset\} : \alpha \vee \neg\alpha$ always exist. A tautological confirmation is a self confirmation of a tautological argument; it needs no assumptions to prove its claim.

I now define the confirmations function. While Definition 2.5.1 gives the form of a single confirmation, Definition 2.5.3 introduces the notion of a set of confirmations. I employ $\Omega$ here as any set of labelled arguments, distinguishing it from $\Delta$ which is the full knowledgebase of labelled assumptions.

**Definition 2.5.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $I_1 : \alpha, \dots, I_n : \alpha, I_j : \alpha \in \mathsf{arguments}(\Delta)$, let $\Omega \subseteq \mathsf{arguments}(\Delta)$ and let $j, n \in \mathbb{N}$. The* **set of confirmations** *derivable from a set of arguments $\Omega$, denoted $\mathsf{confirmations}(\Omega)$, is such that:*

$$\mathsf{confirmations}(\Omega) = \{\{I_1, \dots, I_n\} : \alpha \mid \textit{for all } I_j \in \{I_1, \dots, I_n\} \textit{ it is the case that } I_j : \alpha \in \Omega\}.$$

The above definition of a set of confirmations is purposefully flexible to allow for a wide variety of confirmations. Each confirmation for $\alpha$ does not have to contain all the arguments for $\alpha$ and thus the above function builds up the full power set of confirmations for $\alpha$.

**Proposition 2.5.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $I_1 : \alpha, \dots, I_n : \alpha, J_1 : \alpha, \dots, J_m : \alpha \in \mathsf{arguments}(\Delta)$, let $\Omega \subseteq \mathsf{arguments}(\Delta)$ and let $n, m \in \mathbb{N}$.*

$$\begin{aligned} &\{I_1, \dots, I_n, J_1, \dots, J_m\} : \alpha \in \mathsf{confirmations}(\Omega) \\ &\textit{iff} \;\; \{I_1, \dots, I_n\} : \alpha \in \mathsf{confirmations}(\Omega) \textit{ and} \\ &\qquad \{J_1, \dots, J_m\} : \alpha \in \mathsf{confirmations}(\Omega). \end{aligned}$$

*Proof.* If $n = m = 0$ then the starting point is two unfounded confirmations, creating a further unfounded confirmation. For $n = m = 1$ two self confirmations make a confirmation with two arguments. By induction, considering $n = n + 1, m = m + 1$, it can be seen that successive elaborations of $\{I_1, \dots, I_n, J_1, \dots, J_m\}$ can be built up. So if $\{I_1, \dots, I_n\} : \alpha \in \mathsf{confirmations}(\Omega)$ and $\{J_1, \dots, J_m\} : \alpha \in \mathsf{confirmations}(\Omega)$ then $\{I_1, \dots, I_n, J_1, \dots, J_m\} : \alpha \in \mathsf{confirmations}(\Omega)$. Now for the other direction. If $\{I_1, \dots, I_n, J_1, \dots, J_m\} : \alpha \in \mathsf{confirmations}(\Omega)$ then from Definition 2.5.3 any subset of $\{I_1, \dots, I_n, J_1, \dots, J_m\}$ is also the support of a confirmation for $\alpha$. So in conclusion $\{I_1, \dots, I_n\} : \alpha \in \mathsf{confirmations}(\Omega)$ and $\{J_1, \dots, J_m\} : \alpha \in \mathsf{confirmations}(\Omega)$ iff $\{I_1, \dots, I_n, J_1, \dots, J_m\} : \alpha \in \mathsf{confirmations}(\Omega)$. □

To describe the confirmations that follow from a set of labelled assumption formulae $\Delta$, the two functions just defined are nested. Two shorthand notations, using $\Diamond$ to mean confirmation, are now introduced to lighten the notation.

**Definition 2.5.4.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\alpha \in \mathcal{L}$. The* **set of confirmations** *derivable from $\Delta$, denoted $\Diamond(\Delta)$, is:*

$$\Diamond(\Delta) = \mathsf{confirmations}(\mathsf{arguments}(\Delta)).$$

And now the second of the shorthands, this time for a specific formulae (not just claim) as opposed to for all formulae in $\mathcal{L}$.

**Definition 2.5.5.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae. The* **set of confirmations for** $\alpha$ *derivable from $\Delta$, denoted $\Diamond(\alpha, \Delta)$, is:*

$$\Diamond(\alpha, \Delta) = \mathsf{confirmations}(\mathsf{arguments}(\alpha, \Delta)).$$

Naturally it follows that $\Diamond(\alpha, \Delta) = \{\mathrm{X} : \alpha \in \Diamond(\Delta)\} \mid \mathsf{stripConfirmation}(\mathrm{X} : \alpha) = \alpha\}$. Note that $\Diamond(\alpha, \Delta)$ will be likely to contain many confirmations (all of which are useful) and is thus not necessarily a singleton set.

**Example 2.5.1. Ordinary confirmation.** *Let $\Delta = \{a : \beta, b : \beta \to \alpha, c : \gamma, d : \neg\gamma \vee \alpha\}$. Then $\{\{a, b\} : \alpha, \{c, d\} : \alpha\} \subset \mathsf{arguments}(\Delta)$. Consequently $\{\{a, b\}\} : \alpha, \{\{c, d\}\} : \alpha, \{\{a, b\}, \{c, d\}\} : \alpha \in \Diamond(\alpha, \Delta)$. Let $\mathrm{X} = \{\{a, b\}, \{c, d\}\}$. So $\mathrm{X} : \alpha \in \Diamond(\alpha, \Delta)$, but $\mathrm{X} : \alpha \neq \Diamond(\alpha, \Delta)$. It follows that $\mathsf{formulae}(\mathrm{X}, \Delta) = \{a : \beta, b : \beta \to \alpha, c : \gamma, d : \neg\gamma \vee \alpha\}$, $\mathsf{stripAssumptions}(\mathsf{formulae}(\mathrm{X}, \Delta)) = \{\beta, \beta \to \alpha, \gamma, \neg\gamma \vee \alpha\}$, $\mathsf{stripConfirmation}(\mathrm{X} : \alpha) = \alpha$ and $\mathsf{label}(\mathrm{X} : \alpha) = \mathrm{X}$.*

To be more precise, I also refer to $\Diamond(\alpha, \Delta)$ a power set of confirmations for $\alpha$. Proposition 2.5.2 shows how a power set of confirmations is built up. I use the informal shorthand of referring to the minimum cardinality member of the power set as the bottom of the power set and the maximum cardinality member as the top of the power set. Thus the power set's minimum cardinality is the unfounded confirmation, $\emptyset : \alpha \in \Diamond(\alpha, \Delta)$. The top of the power set is the maximum cardinality confirmation, which I will formally define shortly in Definition 2.5.8. To see an example of a power set of confirmations refer to the 'Worked Example of the Basic Argumentation Framework' in Section 2.8.4. As the empty set, $\emptyset$, plays such an influential role in these power sets, I now focus on it.

First, some more small helper functions, the set of confirm functions to describe the mapping between a set of arguments all with the same claim and their equivalent confirmation are needed. I refer to this as a set of functions as normally a function has a fixed arity. Thus to provide the arbitrary arity in the definition below it is necessary to have a set of functions.

**Definition 2.5.6.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $n \in \mathbb{N}$, let $I_1 : \alpha, \ldots, I_n : \alpha \in \mathsf{arguments}(\Delta)$ and let $\mathrm{X} : \alpha \in \Diamond(\Delta)$. Each function in the set of* **confirm** *functions, denoted $\mathsf{confirm}(I_1 : \alpha, \ldots, I_n : \alpha)$, turns a set of arguments all with the same claim into the equivalent confirmation, such that:*

$$\mathrm{X} : \alpha = \mathsf{confirm}(I_1 : \alpha, \ldots, I_n : \alpha), \textit{ where } \mathrm{X} = \{I_1, \ldots, I_n\}.$$

The confirm functions turn a set of arguments, all with the same claim, into a confirmation by making its many deduction labels into a confirmation label. This transformation is done by placing the deduction labels in a set. The difference between confirm() and confirmations() is that confirm gives just one confirmation, with a confirmation label comprising all deductions labels the function is input, whereas confirmations gives all members of the power set that can be built from the deduction labels it is input. A special case confirm function is when the input is a single argument. In that case the resultant confirmation is a self confirmation.

In stripping labelled confirmations there are two contrasting situations relating to the empty set. For $\emptyset : \alpha \in \Diamond(\Delta)$, an unfounded confirmation for $\alpha$, it follows that $\alpha \notin \mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$. Thus for the unfounded confirmation $\emptyset : \alpha \in \Diamond(\Delta)$, it is the case that $\mathsf{stripConfirmation}(\emptyset : \alpha) = \emptyset$ so $\mathsf{stripConfirmation}(\emptyset : \alpha) \neq \alpha$. In contrast, for $\{\emptyset\} : \alpha \in \Diamond(\Delta)$, the tautological confirmation (that is a self confirmation of a tautological argument), then $\mathsf{stripConfirmation}(\{\emptyset\} : \alpha) = \alpha \neq \emptyset$.

If the claim of a confirmation is a tautology then the cardinality of the confirmation label, X will always be one. Or if $\emptyset \in \mathrm{X}$ the claim will be a tautology.

**Proposition 2.5.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae.*

$$\textit{If } \mathrm{X} : \alpha \in \Diamond(\Delta) \textit{ and } \emptyset \in \mathrm{X}, \textit{ then } |\mathrm{X}| = 1 \textit{ and } \{\alpha\} \vdash \top.$$

*Proof.* If $\emptyset \in \mathrm{X}$ then the tautological argument $\emptyset : \alpha \in \mathsf{arguments}(\Delta)$ exists. Thus $\alpha \vdash \top$ so no premises are required to deduce $\alpha$. Conjecture another argument $I \in \mathrm{X}$ with $I \neq \emptyset$ then $I$ would not be the minimal set that $\vdash \alpha$, so such an $I : \alpha$ would not be a valid argument. Therefore the conclusion is that $I : \alpha$ must also be a tautological argument with $I = \emptyset$. A set cannot contain the same member twice, so as $\emptyset \in \mathrm{X}$ then $|\mathrm{X}| = 1$. Thus if $\mathrm{X} : \alpha \in \Diamond(\Delta)$ and $\emptyset \in \mathrm{X}$, it can only be the case that $|\mathrm{X}| = 1$ and $\{\alpha\} \vdash \top$. □

For a given knowledgebase the set of stripped arguments is the same as the set of stripped confirmations.

**Proposition 2.5.4.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae.*

$$\mathsf{stripDeductions}(\mathsf{arguments}(\Delta)) = \mathsf{stripConfirmations}(\Diamond(\Delta)).$$

*Proof.* The general case is that $\mathsf{arguments}(\Delta) = \{I_1 : \phi_1, \ldots, I_n : \phi_n\}, n \in \mathbb{N}$, so $\mathsf{stripDeductions}(\mathsf{arguments}(\Delta)) = \{\phi_1, \ldots, \phi_n\}$. If $\mathsf{arguments}(\Delta) = \{I_1 : \phi_1, \ldots, I_n : \phi_n\}$ then $\Diamond(\Delta) = \{\{I_1, \ldots, I_l\} : \phi_1, \ldots, \{I_k, \ldots, I_n\} : \phi_n\}$. Thus $\mathsf{stripConfirmations}(\Diamond(\Delta)) = \{\phi_1, \ldots, \phi_n\}$. For unfounded confirmations $\emptyset : \phi_j \in \Diamond(\Delta)$, Definition 2.2.5 is clear $\mathsf{stripConfirmation}(\emptyset : \phi_j) = \emptyset$. So although there are many unfounded confirmations in $(\Diamond(\Delta))$ they are not in $\mathsf{stripConfirmations}(\Diamond(\Delta))$. Consequently $\mathsf{stripDeductions}(\mathsf{arguments}(\Delta)) = \mathsf{stripConfirmations}(\Diamond(\Delta))$. □

The confirmations() function only groups labels, it does not manipulate the formulae that are labelled. Therefore when the labels are removed by stripConfirmations() the effects of confirmations() are removed and what is left is just $\mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$.

A feature of this definition of confirmation is that confirmation is not relative to any specific 'proposer'. No one piece of evidence, that is no one argument, has to be taken as the first, and then be confirmed by the others. In my definition all pieces of evidence are equal and no explicit ordering is implied.

Whilst confirmation, with a form like the above, is not explicitly considered in the literature, much has been published on the related topic of 'extensions to a theory' whereby if a particular set of arguments meets certain criteria then the whole is deemed acceptable. Acceptability tends to revolve around notions of conflict and freedom from conflict, with much consideration of sceptical versus credulous views, see for example (Caminada & Amgoud, 2005). This semantics-based approach, typically with abstract forms of argument, is outside the scope of this thesis, however there appear to be parallels and the intersection may be a fruitful area for future research.

As an illustration from medicine, in second surgical opinion, while there is a first and second reviewer, the surgical result would be the same if the sequence of reviewers was reversed. Consequently my scheme would support such medical reasoning.

A small area for future research would be to find a more succinct way of defining confirmations and related aggregations. The above sequence of definitions is good for this thesis as it has a didactic nature to it, but it is not as concise as it could be.

### 2.5.1 Inconsistent Confirmations

I now make the observation that confirmations can be inconsistent.

**Definition 2.5.7.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{V}:\alpha, \mathrm{X}:\alpha \in \Diamond(\Delta)$.

$$\textit{If } \mathrm{X}:\alpha \in \Diamond(\Delta) \textit{ and there exists a } \mathrm{V} \subseteq \mathrm{X} \textit{ such that } \mathsf{formulae}(\mathrm{V},\Delta) \vdash \bot$$
$$\textit{then } \mathrm{X}:\alpha \textit{ is an } \textbf{inconsistent confirmation}.$$

The common definition of a mincon argument, see Definition 2.4.2, prevents an individual support from being inconsistent. The use of labelled mincons also prevents inconsistent catenate inferences. There are parallels between this notion and that of 'Hang Yourself Arguments' in (Caminada, 2004). There are also several differences stemming from Caminada's use of default logic contrasting with my use of classical propositional logic.

**Example 2.5.2. Inconsistent confirmation.** *Let* $\Delta = \{a:\beta, b:\beta\rightarrow\alpha, c:\gamma\wedge\neg\beta, d:\gamma\rightarrow\alpha\}$ *and* $I = \{a,b\}, J = \{c,d\}, \mathrm{X} = \{I,J\}$. *So* $I:\alpha, J:\alpha \in \mathsf{arguments}(\Delta)$ *and* $\mathrm{X}:\alpha \in \Diamond(\Delta)$. *As* $\mathsf{formulae}(\mathrm{X},\Delta) \vdash \bot$ *then* $\mathrm{X}:\alpha$ *is an inconsistent confirmation.*

In informal argumentation it is quite common to hear the accusation 'You just contradicted yourself'. A debating team (i.e. the parties forming the prosecution or proponents or the parties forming the defence or opponents) is usually obligated to have not just its individual arguments be consistent, but also any set of arguments for a claim to be consistent. I suggest that the allowance of inconsistent confirmations in many argumentation frameworks is something of a problem, as it may hinder formal argumentation from emulating debating in professional life. However if the inconsistency is small (such as one witness said four shots were fired, while another witness said five) then professional debaters would probably tolerate it.

**Proposition 2.5.5.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{X}:\alpha \in \Diamond(\Delta)$.

$$\textit{If } \mathrm{X}:\alpha \textit{ is an inconsistent confirmation, then for all } \beta \in \mathcal{L},\ \mathsf{formulae}(\mathrm{X},\Delta) \vdash \beta.$$

*Proof.* Even though no individual argument support $I \in \mathrm{X}$ is inconsistent, i.e. $\mathsf{formulae}(I,\Delta) \nvdash \bot$, the collective beliefs represented by $\mathsf{formulae}(\mathrm{X},\Delta) \vdash \bot$. So even though a constraint in the definition of $I$, Definition 2.4.2, prevents the immediate ex falso quodlibet, that constraint fails to prevent $\mathsf{formulae}(\mathrm{X}, \Delta)$ from being inconsistent so for all $\beta \in \mathcal{L}$ it is the case that $\mathsf{formulae}(\mathrm{X}, \Delta) \vdash \beta$. $\square$

The logic of this chapter allows inconsistent confirmations; a restraint is provided in Section 3.7.

### 2.5.2 Cardinality of Confirmations

I now investigate how many confirmations can be derived from a given knowledgebase $\Delta$, given that the knowledgebase is finite, while the set of arguments is infinite.

**Proposition 2.5.6.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\mathrm{X}:\alpha \in \Diamond(\Delta)$.*

$$0 \leq |\mathsf{label}(\mathrm{X}:\alpha)| \leq \mathsf{midLattice}(\Delta).$$

*Proof.* A confirmation $\mathrm{X} : \alpha \in \Diamond(\alpha, \Delta)$ tracks some or all of the arguments for $\alpha$ in $\mathsf{arguments}(\Delta)$. Each member of $\mathsf{label}(\mathrm{X} : \alpha)$ is the deduction label of an argument for $\alpha$. So, for the maximum end, $|\mathsf{label}(\mathrm{X}:\alpha)| \leq |\mathsf{arguments}(\alpha, \Delta)|$. Proposition 2.4.3 states that $|\mathsf{arguments}(\alpha, \Delta)| \leq \mathsf{midLattice}(\Delta)$. Therefore $|\mathsf{label}(\mathrm{X} : \alpha)| \leq \mathsf{midLattice}(\Delta)$. At the minimum end, there may be no arguments for $\alpha$ derivable from $\Delta$. $\square$

I now move to the question of the number of confirmations for a given claim $\alpha$, something which is different from the number of arguments in a confirmation for $\alpha$. Clearly $|\Diamond(\alpha, \Delta)| \neq |\mathsf{arguments}(\alpha, \Delta)|$. Showing just one exception will prove the inequality. Suppose $I : \alpha, J : \alpha \in \mathsf{arguments}(\Delta)$ then $|\{I:\alpha \in \mathsf{arguments}(\Delta)\}| = 2$, however $\emptyset:\alpha, \{I\}:\alpha, \{J\}:\alpha, \{I, J\}:\alpha \in \Diamond(\alpha, \Delta)$ so $|\Diamond(\alpha, \Delta)| = 4$ hence $|\Diamond(\alpha, \Delta)| \neq |\mathsf{arguments}(\alpha, \Delta)|$. In fact $|\Diamond(\alpha, \Delta)| \geq |\mathsf{arguments}(\alpha, \Delta)|$. To be specific, the cardinality of $\Diamond(\alpha, \Delta)$ is as follows:

**Proposition 2.5.7.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\alpha \in \mathcal{L}$.*

$$1 \leq |\Diamond(\alpha, \Delta)| \leq 2^{\mathsf{midLattice}(\Delta)}.$$

*Proof.* Even if $\Delta \nvdash \alpha$ it is the case that the unfounded confirmation $\emptyset:\alpha \in \Diamond(\alpha, \Delta)$, therefore the minimum cardinality is 1, i.e. $1 \leq |\Diamond(\alpha, \Delta)|$. Now for the maximum cardinality, Proposition 2.4.3 states that the maximum number of arguments for a single claim is given by $\mathsf{midLattice}(\Delta)$. Proposition 2.5.6 states that the largest cardinality confirmation for $\alpha$, let it be $\mathrm{X}:\alpha \in \Diamond(\Delta)$, will contain $|\mathrm{X}| \leq \mathsf{midLattice}(\Delta)$ arguments. All other valid confirmations for $\alpha$ will have labels, $\mathrm{Y}, \mathrm{Z}, \ldots$ which are strict subsets of X, i.e. $\mathrm{Y} \subsetneq \mathrm{X}, \mathrm{Z} \subsetneq \mathrm{X}$. With confirmations there is no constraint that one subset, e.g. Y, cannot be the subset of another subset, e.g. Z. So $\mathrm{Y} \subsetneq \mathrm{Z}$ or $\mathrm{Z} \subsetneq \mathrm{Y}$ are both allowed. Thus the number of subsets of this confirmation label X is up to $2^{\mathsf{midLattice}(\Delta)}$. $\square$

Thus for any $\Delta$ it is the case that $\emptyset:\alpha \in \Diamond(\alpha, \Delta)$ even if there exist arguments for $\alpha$. In the above power set it is not necessary to exclude any element either at the top or bottom. While the number of confirmations for one formula is finite, the number for all formulae is infinite.

**Proposition 2.5.8.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae.*

$$|\Diamond(\Delta)| = \infty.$$

*Proof.* $\Diamond(\Delta)$ contains confirmations for every formula derivable from $\Delta$. As $|\text{arguments}(\Delta)| = \infty$ then so also $|\Diamond(\Delta)| = \infty$. □

### 2.5.3 Maximum Cardinality of Confirmations

Recall that all of the confirmations for a particular claim, $\alpha$, derivable from a given knowledgebase $\Delta$, denoted $\Diamond(\alpha, \Delta)$, if ordered by set inclusion could be arranged as a lattice of confirmations. Likewise all of the confirmations derivable from a set of arguments for a single claim can also be ordered and arranged in a lattice. For either lattice, at the top of the lattice is the maximum cardinality confirmation and at the bottom of the lattice is the unfounded confirmation. To extract this maximum cardinality confirmation requires a helper function:

**Definition 2.5.8.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae. Let $\Psi$ be the set of all the confirmations for $\alpha$ derivable from a set of arguments for $\alpha$ such that $\Psi = \text{confirmations}(\Phi)$ where $\Phi \subseteq \text{arguments}(\alpha, \Delta)$. The* **top** *of confirmation lattice function, denoted $\text{top}(\Psi)$, returns the confirmation for $\alpha$ with the largest label cardinality out of all the confirmations in $\Psi$ such that:*

$$\begin{aligned}\text{top}(\Psi) \quad = \quad & \{X : \alpha \in \Diamond(\Delta) \mid X : \alpha \in \Psi \textit{ and} \\ & \textit{there does not exist a confirmation } Y : \alpha \in \Psi, \\ & \textit{where } \text{label}(X : \alpha) \subsetneq \text{label}(Y : \alpha)\}.\end{aligned}$$

As $\Phi \subseteq \text{arguments}(\alpha, \Delta)$ it follows that $\Psi = \text{confirmations}(\Phi) \subseteq \Diamond(\alpha, \Delta)$. The top of the power set is the one with the largest cardinality. The cardinality of the top() function is always one, as it always returns a single unique confirmation.

**Proposition 2.5.9.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\alpha \in \mathcal{L}$.*

$$|\text{top}(\Diamond(\alpha, \Delta))| = 1.$$

*Proof.* Suppose $\text{arguments}(\Delta) \nvdash \alpha$ so there are no arguments for $\alpha$ in $\text{arguments}(\Delta)$; in that case there is still a confirmation for $\alpha$, i.e. the unfounded confirmation, of the form $\emptyset : \alpha \in \Diamond(\Delta)$. The unfounded confirmation is a single confirmation, even though its label cardinality is zero. For a self confirmation, with the form $\{I\} : \alpha$ there is still a single confirmation albeit with a label cardinality of 1. If $|\Diamond(\alpha, \Delta)| > 1$ then its members will always form a non-trivial power set. A power set of the subsets of X always has a single top member which is X. So in all cases $|\text{top}(\Diamond(\alpha, \Delta)| = 1$. □

Clearly the cardinality of a labelled confirmation is equal to the number of deduction labels in the confirmation label. The following simple relationships also hold; that $|\text{label}(\text{top}(\Diamond(\alpha, \Delta))| = |\text{arguments}(\alpha, \Delta)|$ and also that $\text{top}(\Diamond(\alpha, \Delta)) \neq \emptyset$.

Additionally, this top() function allows clarification of the earlier point relating to the distinction between confirmations and confirm, where $n \in \mathbb{N}$:

$$\text{top}(\text{confirmations}(I_1 : \phi, \ldots, I_n : \phi)) = \text{confirm}(I_1 : \phi, \ldots, I_n : \phi) = X : \phi.$$

I now touch on the cardinality of unfounded and tautological confirmations.

**Example 2.5.3. Special cardinalities.** *For any $\Delta$, if $\Delta \nvdash \alpha$, then the unfounded confirmation $\emptyset : \alpha \in \Diamond(\Delta)$ is the only member of $\Diamond(\alpha, \Delta)$ and in that case $|\mathsf{label}(\mathsf{top}(\Diamond(\alpha, \Delta)))| = 0$. Again for any $\Delta$, if $\emptyset \vdash \alpha$, then $\alpha \vdash \top$ is a tautology, the tautological confirmation $\{\emptyset\} : \alpha \in \Diamond(\alpha, \Delta)$ and $|\mathsf{label}(\mathsf{top}(\Diamond(\alpha, \Delta)))| = 1$.*

This ends the analysis of the cardinality of confirmations, allowing the discussion to move on to the topic of contradictions.

## 2.6 Introducing Contradiction

A contradiction is a further argument aggregation whereby a confirmation for $\alpha$ and a confirmation for $\neg\alpha$ are combined into a new structure. Each contradiction thus captures an area of inconsistency. Later in the thesis, in Chapter 6, I show that contradictions are useful for tracking professional debates.

**Definition 2.6.1.** *A* **single contradiction** *is of the form of a tuple of two confirmation labels labelling a formula $\alpha$, denoted $\langle X, Y \rangle : \alpha$, where $\alpha \in \mathcal{L}$, $X : \alpha$ is a confirmation for $\alpha$ and $Y : \neg\alpha$ is a confirmation for $\neg\alpha$.*

As with confirmation being denoted by $\Diamond$, I use a similar lightweight notation of $\blacklozenge$ to denote contradiction.

**Definition 2.6.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae. Let $\blacklozenge(\Delta)$ be the* **set of all contradictions** *derivable from $\Delta$, such that:*

$$\blacklozenge(\Delta) = \{\langle X, Y \rangle : \alpha \mid X : \alpha \in \Diamond(\Delta), Y : \neg\alpha \in \Diamond(\Delta)\}.$$

The above $\blacklozenge(\Delta)$ contains contradictions for all formulae in $\mathcal{L}$. The next definition is for one specific $\alpha \in \mathcal{L}$.

**Definition 2.6.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\alpha, \neg\alpha \in \mathcal{L}$. The* **set of contradictions for** *$\alpha$ derivable from $\Delta$, denoted $\blacklozenge(\alpha, \Delta)$ is:*

$$\blacklozenge(\alpha, \Delta) = \Diamond(\alpha, \Delta) \times \Diamond(\neg\alpha, \Delta).$$

Thus the set of contradictions for $\alpha$ is the cross product of the set of confirmations for $\alpha$ with the set of confirmations for $\neg\alpha$. At times to be more specific I also refer to $\blacklozenge(\alpha, \Delta)$ as a power set of contradictions. Clearly $\blacklozenge(\alpha, \Delta) \subseteq \blacklozenge(\Delta)$ and that $\blacklozenge(\alpha, \Delta)$ can contain many confirmations and is thus not just a singleton set.

This definition has a flavour of Dung's definition of an argument framework (see Definition 3.9.1 which is from (Dung, 1993; Dung, 1995)). Both use the cross product of sets of arguments. One difference is that my definition of contradiction only considers an argument for $\alpha$ contradicting an argument for $\neg\alpha$. Depending on what assumptions are used, the literature covers other ways that one argument can attack another, notably undercut, which are outside the scope of my current definition. Chapter 3 goes on to address undercut. A second difference with Dung's framework is that his is abstract, whereas mine is limited to only one kind of argument - those whose support is a minimally consistent subset of premises and whose method of inference is classical logic.

**Example 2.6.1. Ordinary contradiction.** *Let* $\Delta = \{a:\beta, b:\beta \rightarrow \alpha, c:\gamma, d:\neg\gamma \vee \beta, e:\neg\alpha\}$. *Then* $\{a,b\}:\alpha, \{c,d\}:\alpha, \{e\}:\neg\alpha \in \text{arguments}(\Delta)$. *Consequently* $\{\{a,b\}\}:\alpha, \{\{c,d\}\}:\alpha, \{\{a,b\},\{c,d\}\}:\alpha, \{\{e\}\}:\neg\alpha \in \Diamond(\Delta)$. *Thus* $\langle\{\{a,b\},\{c,d\}\},\{\{e\}\}\rangle:\alpha \in \blacklozenge(\Delta)$.

An empty contradiction is a contradiction where there is no evidence for either or both $\alpha$ and $\neg\alpha$.

**Definition 2.6.4.** *If* $\langle X, Y\rangle:\alpha \in \blacklozenge(\Delta)$ *is a contradiction and* $(X = \emptyset \text{ or } Y = \emptyset)$, *then* $\langle X, Y\rangle:\alpha$ *is called an* **empty contradiction**.

As the definition contains an 'or', empty contradictions come in three forms: $\langle X,\emptyset\rangle:\alpha$, $\langle\emptyset, Y\rangle:\alpha$ and $\langle\emptyset,\emptyset\rangle:\alpha$. The second and third use the unfounded confirmation $\emptyset:\alpha \in \Diamond(\Delta)$ explicitly stating that there is no evidence for $\alpha$, while the first and third use the unfounded confirmation $\emptyset:\neg\alpha \in \Diamond(\Delta)$ explicitly stating that there is no evidence for $\neg\alpha$.

Contradiction thus provides a distinction between ignorance and negation. This distinction between ignorance and negation also has a flavour of the Dempster-Shafer theory of evidence, where absence of evidence is explicitly recognised.

Although $\langle X, Y\rangle:\alpha$ is a contradiction in $\alpha$, it is just as equally a contradiction in $\neg\alpha$. One implies the other.

**Proposition 2.6.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae. For all* $\alpha, X, Y$ *it is the case that:*

$$\langle X, Y\rangle:\alpha \in \blacklozenge(\Delta) \textit{ iff } \langle Y, X\rangle:\neg\alpha \in \blacklozenge(\Delta).$$

*Proof.* $\langle X, Y\rangle:\alpha \in \blacklozenge(\Delta)$ iff $X:\alpha \in \Diamond(\Delta)$ and $Y:\neg\alpha \in \Diamond(\Delta)$ iff $X:\neg\neg\alpha \in \Diamond(\Delta)$ and $Y:\neg\alpha \in \Diamond(\Delta)$ iff $X:\neg(\neg\alpha) \in \Diamond(\Delta)$ and $Y:(\neg\alpha) \in \Diamond(\Delta)$ iff $\langle Y, X\rangle:\neg\alpha \in \blacklozenge(\Delta)$. □

I now extend the label-removing strip functions to contradictions.

**Definition 2.6.5.** *Let* $\{\langle X_1, Y_1\rangle:\alpha_1, \ldots, \langle X_n, Y_n\rangle:\alpha_n\}$ *be a set of contradictions. The* **contradiction strip functions** *remove the label or labels from a contradiction or set of contradictions, such that:*

$$\text{stripContradiction}(\langle X, Y\rangle:\alpha) = \text{stripConfirmation}(X:\alpha) \cup \text{stripConfirmation}(Y:\neg\alpha).$$

$$\begin{aligned}&\text{stripContradictions}(\{\langle X_1, Y_1\rangle:\alpha_1, \ldots, \langle X_n, Y_n\rangle:\alpha_n\})\\ &\quad = \{\text{stripContradiction}(\langle X_i, Y_i\rangle:\alpha_i) \mid \langle X_i, Y_i\rangle:\alpha_i \in \{\langle X_1, Y_1\rangle:\alpha_1, \ldots, \langle X_n, Y_n\rangle:\alpha_n\}\}.\end{aligned}$$

The following example shows how stripping contradictions works when the empty set is involved.

**Example 2.6.2. Stripped contradictions.** *The stripping of contradictions involving the empty set illustrates that the confirmation* $\emptyset:\alpha$ *represents there being no arguments for* $\alpha$, *while the confirmation* $\{\emptyset\}:\alpha$ *states that* $\alpha$ *is a tautology.*

1. $\mathsf{stripContradiction}(\langle\emptyset,\emptyset\rangle:\alpha)=\emptyset$ *so* $\mathsf{stripContradiction}(\langle\emptyset,\emptyset\rangle:\alpha)\nvdash\bot$,
2. $\mathsf{stripContradiction}(\langle \mathrm{X},\emptyset\rangle:\alpha)=\alpha$ *so* $\mathsf{stripContradiction}(\langle \mathrm{X},\emptyset\rangle:\alpha)\nvdash\bot$,
3. $\mathsf{stripContradiction}(\langle\emptyset,\mathrm{Y}\rangle:\alpha)=\neg\alpha$ *so* $\mathsf{stripContradiction}(\langle\emptyset,\mathrm{Y}\rangle:\alpha)\nvdash\bot$,
4. $\mathsf{stripContradiction}(\langle \mathrm{X},\mathrm{Y}\rangle:\alpha)=\{\alpha,\neg\alpha\}$ *so* $\mathsf{stripContradiction}(\langle \mathrm{X},\mathrm{Y}\rangle:\alpha)\vdash\bot$,
5. $\mathsf{stripContradiction}(\langle\emptyset,\{\emptyset\}\rangle:\alpha)=\{\emptyset,\neg\alpha\}$ *so* $\mathsf{stripContradiction}(\langle \mathrm{X},\{\emptyset\}\rangle:\alpha)\nvdash\bot$,
6. $\mathsf{stripContradiction}(\langle\{\emptyset\},\emptyset\rangle:\alpha)=\{\alpha,\emptyset\}$ *so* $\mathsf{stripContradiction}(\langle\{\emptyset\},\mathrm{Y}\rangle:\alpha)\nvdash\bot$.

In 1. above both confirmations in the contradiction are unfounded confirmations. In 2. and 3. one of the confirmations in the contradiction is an unfounded confirmation. In 5. and 6. one of the confirmations in the contradiction is a tautological confirmation. Note that $\langle\{\emptyset\},\{\emptyset\}\rangle:\alpha\notin\blacklozenge(\Delta)$ as the negation of a tautology is an inconsistency. Thus it is impossible for both confirmations in a contradiction to be tautological confirmations.

The cardinality of $\blacklozenge(\alpha,\Delta)$ is as follows:

**Proposition 2.6.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\alpha\in\mathcal{L}$.*

$$1\le|\blacklozenge(\alpha,\Delta)|\le 2^{\mathsf{trunc}(\mathsf{midLattice}(\Delta)\div 2)}\cdot 2^{\mathsf{midLattice}(\Delta)-\mathsf{trunc}(\mathsf{midLattice}(\Delta)\div 2)}.$$

*Proof.* For the minimum, even if $\Delta=\emptyset$, the empty contradiction $\langle\emptyset,\emptyset\rangle:\alpha\in\blacklozenge(\alpha,\Delta)$ exists, so in all cases $\blacklozenge(\alpha,\Delta)\neq\emptyset$. Now for the maximum. The set of contradictions for $\alpha$, i.e. $\blacklozenge(\alpha,\Delta)$, can be represented as a rectangular array with one axis for members of $\lozenge(\alpha,\Delta)$ and the other axis for members of $\lozenge(\neg\alpha,\Delta)$. Because $\blacklozenge(\alpha,\Delta)=\lozenge(\alpha,\Delta)\times\lozenge(\neg\alpha,\Delta)$, each cell is thus a unique contradiction. Start the case where $\mathsf{midLattice}(\Delta)$ is even. This array of contradictions will have the greatest area when $|\lozenge(\alpha,\Delta)|=|\lozenge(\neg\alpha,\Delta)|$, i.e. that it is square. Now, if a set of premises, $I$, of an argument infers $\alpha$, i.e. $I:\alpha\in\mathsf{arguments}(\Delta)$ then $I:\neg\alpha\notin\mathsf{arguments}(\Delta)$ as otherwise $\mathsf{formulae}(I,\Delta)\vdash\bot$, falsely rendering $I:\alpha\notin\mathsf{arguments}(\Delta)$. If $I:\alpha, J:\neg\alpha\in\mathsf{arguments}(\Delta)$ and $\Psi=\mathsf{formulae}(I,\Delta),\Phi=\mathsf{formulae}(J,\Delta)$, then due to the definition of argument $\Psi\nsubseteq\Phi$ and $\Phi\nsubseteq\Psi$. The largest number of contradictions is obtained if half of the possible confirmations and thus half the possible arguments, i.e. $\mathsf{midLattice}(\Delta)\div 2$, are for $\alpha$ and half for $\neg\alpha$. So taking an optimal $\Delta$ that yields exactly two sets of arguments such that $|\mathsf{arguments}(\alpha,\Delta)|=|\mathsf{arguments}(\neg\alpha,\Delta)|=\mathsf{midLattice}(\Delta)\div 2$. The number of confirmations that can be produced thus has the property $|\lozenge(\alpha,\Delta)|=|\lozenge(\neg\alpha)|=2^{\mathsf{midLattice}(\Delta)\div 2}$. The area of this square is consequently $(2^{\mathsf{midLattice}(\Delta)\div 2})^2$. However, it can also be the case that $\mathsf{midLattice}(\Delta)$ is odd (e.g. if $|\Delta|=7$ then $\mathsf{midLattice}(\Delta)=35$). The rectangular array can no longer be exactly square. Thus the more precise answer is that the maximum number of contradictions is $2^{\mathsf{trunc}(\mathsf{midLattice}(\Delta)\div 2)}\cdot 2^{\mathsf{midLattice}(\Delta)-\mathsf{trunc}(\mathsf{midLattice}(\Delta)\div 2)}$. Clearly if $\mathsf{midLattice}(\Delta)$ is even (e.g. if $|\Delta|=9$ then $\mathsf{midLattice}(\Delta)=126$) the precise equation still works, providing the necessary square array. $\square$

The number of possible contradictions can be large, even for a simple knowledgebase. The compute time to find all of the contradictions derivable from a given $\Delta$ rises exponentially.

**Example 2.6.3. Contradiction size.** *Imagine a naïve debate using just ten labelled assumption formulae. How many contradictions for $\alpha$ can be deduced from these 10? The mid lattice, of the array of* *subsets of* $\Delta$*, contains 252 members or 252 potential arguments, i.e. for* $|\Delta| = 10, \mathsf{midLattice}(\Delta) = 252$. *Let half of these,i.e. 126, be for* $\alpha$ *and the other half, i.e. 126 , be for* $\neg\alpha$. *The largest number of confirmations that can be produced from an* $\mathsf{arguments}(\alpha, \Delta)$ *with* $|\mathsf{arguments}(\alpha, \Delta)| = 126$ *is* $2^{126}$. $2^{126}$ *is of the order* $8.5 \cdot 10^{37}$. *The maximum number of contradictions,* $|\blacklozenge(\alpha, \Delta)|$, *is the square of the number of confirmations. So* $|\blacklozenge(\alpha, \Delta)|$ *is of the order of* $7.2 \cdot 10^{75}$. *To put that in context, if a supercomputer had computed 1 billion contradictions a second since the universe began, it still would not definitely compute the answer to a debate involving just 10 assumptions. The universe is 13.7 billion years old (*$1.37 \cdot 10^{10}$*), there are* $3.16 \cdot 10^7$ *seconds in a year and* $1.0 \cdot 10^9$ *contradictions per second on this state of the art computer, so the computer processes* $4.3 \cdot 10^{26}$ *contradictions, which is a tiny fraction of* $7.2 \cdot 10^{75}$.

These definitions are also somewhat related to Belnap's four valued logic (Belnap, 1977). Belnap's scheme uses a set with four values: neither, true, false and both. In the absence of evidence (for both $\alpha$ and $\neg\alpha$) that formulae ($\alpha$) must be given the value neither. Once the first piece of evidence is available the value must change from neither to either true or false. If a contradictory piece of information arrives the Belnap value moves up to both. In my scheme it could be argued that I introduce a further area in the set above both, that could be described as 'many occurrences of both'. To have a 'many occurrences of both' value would mean that the definition of Belnap's 'both' would be made more specific to be 'one occurrence of both'. My definitions give a precise breakdown of what underlies the 'many occurrences of both' value. I could also add values for 'confirmed true' and 'confirmed false', which would come above 'true' and 'false' but below 'both'. Furthermore summarising functions could be provided to reconstruct the four Belnap values from my seven values.

### 2.6.1 Maximum Cardinality of Contradictions

I now introduce a further helper function that I use later is the thesis. Just as confirmations exist in a power set, providing the need to refer to just the maximum cardinality member or top of the power set, so also contradictions exist in power sets. The following function, which builds on Definition 2.5.8, returns the maximum cardinality contradiction for a power set of contradictions.

**Definition 2.6.6.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae. Let* $\Psi$ *be the set of all the confirmations for* $\alpha$ *derivable from a set of arguments for* $\alpha$ *such that* $\Psi = \mathsf{confirmations}(\Phi)$ *where* $\Phi \subseteq \mathsf{arguments}(\alpha, \Delta)$. *Likewise let* $\Omega$ *be the set of all the confirmations for* $\neg\alpha$ *derivable from a set of arguments for* $\neg\alpha$ *such that* $\Omega = \mathsf{confirmations}(\Pi)$ *where* $\Pi \subseteq \mathsf{arguments}(\neg\alpha, \Delta)$. *Let* $\Upsilon = \Psi \times \Omega$. *The* **top of contradiction lattice** *function, denoted* $\mathsf{top}(\Upsilon)$, *returns the contradiction for* $\alpha$ *with the two largest label cardinalities out of all the contradictions in* $\Upsilon$ *such that:*

$$\begin{aligned} \mathsf{top}(\Upsilon) \quad = \quad & \{\langle X, Y\rangle : \alpha \in \blacklozenge(\Delta) \mid \\ & \langle X, Y\rangle : \alpha \in \Upsilon,\ X : \alpha = \mathsf{top}(\Psi) \text{ and} \\ & Y : \neg\alpha = \mathsf{top}(\Omega)\}. \end{aligned}$$

Clearly the set of contradictions $\Upsilon$, above, is such that $\Upsilon \subseteq \blacklozenge(\alpha, \Delta)$. While this function could have been defined to operate just on $\blacklozenge(\alpha, \Delta)$, its current form is more general and subsequently more useful.

### 2.6.2 Rebuttal

For this discussion of contradiction to be more comprehensive it is helpful to introduce the concept of rebuttal. Rebuttal is the simplest of contradictions, is in widespread use in the literature and is employed later in this thesis as one kind of attacks relationship.

**Definition 2.6.7.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $I : \alpha \in \mathsf{arguments}(\Delta)$. *The set of all* **rebuttals** *of* $I : \alpha$ *derivable from* $\Delta$, *denoted* $\mathsf{rebuttals}(I : \alpha, \Delta)$, *is such that:*

$$\mathsf{rebuttals}(I : \alpha, \Delta) = \{J : \neg\alpha \mid J : \neg\alpha \in \mathsf{arguments}(\neg\alpha, \Delta)\}.$$

The definition ensures that the rebuttals take on a well ordered form as the following example shows.

**Example 2.6.4. Rebuttal.** *Let* $\Delta = \{a : \alpha \wedge \beta \wedge \gamma, b : \neg\alpha, c : \neg\beta, d : \neg\gamma\}$. *Let* $I = \{a\}$ *and* $J = \{b, c, d\}$ *So* $I : \alpha \wedge \beta \wedge \gamma, J : \neg(\alpha \wedge \beta \wedge \gamma), J : \neg(\beta \wedge \alpha \wedge \gamma), J : \neg\alpha \vee \neg\beta \vee \neg\gamma) \in \mathsf{arguments}(\Delta)$. *The definition ensures that* $J : \neg(\beta \wedge \alpha \wedge \gamma), J : \neg\alpha \vee \neg\beta \vee \neg\gamma \notin \mathsf{rebuttals}(I : \alpha \wedge \beta \wedge \gamma, \Delta)$. *Only* $J : \neg(\alpha \wedge \beta \wedge \gamma) \in \mathsf{rebuttals}(I : \alpha \wedge \beta \wedge \gamma, \Delta)$.

Now that the maximum cardinality, contradictions and rebuttals functions have been defined I can show that rebuttals are essentially a kind of contradiction:

**Proposition 2.6.3.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae, let* $I : \alpha, J : \neg\alpha \in \mathsf{arguments}(\Delta)$ *and let* $\langle X, Y\rangle : \alpha \in \blacklozenge(\Delta)$.

$$\begin{aligned} &\langle X, Y\rangle : \alpha = \mathsf{top}(\blacklozenge(\alpha, \Delta)) \\ &\quad \textit{iff} \quad X : \alpha = \mathsf{confirm}(\mathsf{rebuttals}(J : \neg\alpha, \Delta)) \textit{ and} \\ &\qquad\qquad Y : \neg\alpha = \mathsf{confirm}(\mathsf{rebuttals}(I : \alpha, \Delta)). \end{aligned}$$

*Proof.* Given $\langle X, Y\rangle : \alpha = \mathsf{top}(\blacklozenge(\alpha, \Delta))$ then Definition 2.6.6 of the maximum cardinality contradiction informs that $X : \alpha = \mathsf{top}(\lozenge(\alpha, \Delta))$ and $Y : \neg\alpha = \mathsf{top}(\lozenge(\neg\alpha, \Delta))$. Then using Definition 2.5.8 of the maximum cardinality confirmation establishes that every argument $I : \alpha$ for $\alpha$ derivable from $\Delta$ will be in $X : \alpha$ where $I \in X$ and that every argument $J : \neg\alpha$ for $\neg\alpha$ derivable from $\Delta$ will be in $Y : \neg\alpha$ where $J \in Y$. Adding the rebuttals function, Definition 2.6.7 provides every rebuttal for $I : \alpha$ derivable from $\Delta$, so $Y : \neg\alpha = \mathsf{confirm}(\mathsf{rebuttals}(I : \alpha, \Delta))$ and likewise $X : \alpha = \mathsf{confirm}(\mathsf{rebuttals}(J : \neg\alpha, \Delta))$. So if $\langle X, Y\rangle : \alpha = \mathsf{top}(\blacklozenge(\alpha, \Delta)$ then $X : \alpha = \mathsf{confirm}(\mathsf{rebuttals}(J : \neg\alpha, \Delta))$ and $Y : \neg\alpha = \mathsf{confirm}(\mathsf{rebuttals}(I : \alpha, \Delta))$. Now for the other direction. Given $X : \alpha = \mathsf{confirm}(\mathsf{rebuttals}(J : \neg\alpha, \Delta))$ and $Y : \neg\alpha = \mathsf{confirm}(\mathsf{rebuttals}(I : \alpha, \Delta))$ it follows from Definition 2.6.7 that $X : \alpha = \mathsf{top}(\lozenge(\alpha, \Delta))$ and $Y : \neg\alpha = \mathsf{top}(\lozenge(\neg\alpha, \Delta))$. So if $X : \alpha = \mathsf{confirm}(\mathsf{rebuttals}(J : \neg\alpha, \Delta))$ and $Y : \neg\alpha = \mathsf{confirm}(\mathsf{rebuttals}(I : \alpha, \Delta))$ then $\langle X, Y\rangle : \alpha = \mathsf{top}(\blacklozenge(\alpha, \Delta)$. So in conclusion $\langle X, Y\rangle : \alpha = \mathsf{top}(\blacklozenge(\alpha, \Delta))$ iff $X : \alpha = \mathsf{confirm}(\mathsf{rebuttals}(J : \neg\alpha, \Delta))$ and $Y : \neg\alpha = \mathsf{confirm}(\mathsf{rebuttals}(I : \alpha, \Delta))$. □

Clearly rebuttals do not always exist and also an argument can never rebut itself. If the support of an argument is the empty set, that is it is a tautology, then it has no rebuttals.

**Proposition 2.6.4.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I : \alpha \in$* $\mathsf{arguments}(\Delta)$.

$$\textit{If } I = \emptyset \textit{ then } \mathsf{rebuttals}(I : \alpha, \Delta) = \emptyset.$$

*Proof.* If $I = \emptyset$ then, $\{\alpha\} \vdash \top$, the argument is a tautology. The negation of a tautology is an inconsistency, $\neg\top \vdash \bot$. Conjecture the existence of a rebuttal $J : \neg\alpha \in \mathsf{rebuttals}(I : \alpha, \Delta)$. Then $\mathsf{stripAssumptions}(\mathsf{formulae}(J, \Delta)) \vdash \neg\alpha$, hence $\mathsf{formulae}(J, \Delta) \vdash \neg\top$, meaning $\mathsf{formulae}(J, \Delta) \vdash \bot$ so $J$ is inconsistent, which is impossible as it is the support of a valid argument. Thus if $I = \emptyset$ then $\mathsf{rebuttals}(I : \alpha, \Delta) = \emptyset$. □

The vice versa is not always true as for example when $\Delta \nvdash \bot$ and there simply are no rebuttals in the set $\mathsf{arguments}(\Delta)$.

The rebuttals function possesses a form of symmetry, but is not a self-inverse function. A function is self-inverse iff f(f(x)) = x.

**Proposition 2.6.5.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I : \alpha, J : \beta \in$* $\mathsf{arguments}(\Delta)$.

$$I : \alpha \in \mathsf{rebuttals}(J : \beta, \Delta) \textit{ iff } J : \beta \in \mathsf{rebuttals}(I : \alpha, \Delta).$$

*Proof.* Definition 2.6.7 of rebuttal informs that $\mathsf{rebuttals}(I : \alpha, \Delta) = \{J : \neg\alpha \in \mathsf{arguments}(\neg\alpha, \Delta)\}$. If $I : \alpha \in \mathsf{rebuttals}(J : \beta, \Delta)$ then it follows that $\beta = \neg\alpha$. Thus $I : \alpha \in \mathsf{rebuttals}(J : \neg\alpha, \Delta)$. Again, by Definition 2.6.7 any argument with a claim of $\neg\alpha$ will rebut $I : \alpha$, therefore $J : \neg\alpha \in \mathsf{rebuttals}(I : \alpha, \Delta)$. Consequently if $I : \alpha \in \mathsf{rebuttals}(J : \beta, \Delta)$ then $J : \beta \in \mathsf{rebuttals}(I : \alpha, \Delta)$. Now for the other direction. Assuming $J : \beta \in \mathsf{rebuttals}(I : \alpha, \Delta)$, Definition 2.6.7 states that any argument for $\neg\beta$ is a rebuttal for $J : \beta$. Thus if $J : \beta \in \mathsf{rebuttals}(I : \alpha, \Delta)$ then $I : \neg\beta$ is a rebuttal for $J : \beta$, meaning $I : \alpha \in \mathsf{rebuttals}(J : \beta, \Delta)$. So in conclusion $I : \alpha \in \mathsf{rebuttals}(J : \beta, \Delta)$ iff $J : \beta \in \mathsf{rebuttals}(I : \alpha, \Delta)$. □

Rebuttal is not self-inverse as it is not always the case that $\mathsf{rebuttals}(\mathsf{rebuttals}(I : \alpha, \Delta), \Delta) = I : \alpha$. This follows because the rebuttals function only takes one argument as its input, while its output can be many arguments. However, if $|\mathsf{rebuttals}(I : \alpha, \Delta)| = 1$, which occurs for some $\Delta$, then $\mathsf{rebuttals}(\mathsf{rebuttals}(I : \alpha, \Delta), \Delta) = I : \alpha$. The set of rebuttals of a single argument can be represented as a confirmation.

**Proposition 2.6.6.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $I : \alpha, K : \neg\alpha \in$* $\mathsf{arguments}(\Delta)$ *and let* $\mathrm{Z} : \neg\alpha \in \Diamond(\Delta)$.

$$\mathrm{Z} : \neg\alpha = \mathsf{confirm}(\mathsf{rebuttals}(I : \alpha, \Delta))$$
$$\textit{iff for all } K : \neg\alpha \in \mathsf{rebuttals}(I : \alpha, \Delta) \textit{ it is the case that } K \in \mathrm{Z}.$$

*Proof.* Let $\mathrm{Z} = \{K_1, \ldots, K_n\}$. Thus $\mathrm{Z} : \neg\alpha = \{K_1, \ldots, K_n\} : \neg\alpha$. Given that $\mathrm{Z} : \neg\alpha = \mathsf{confirm}(\mathsf{rebuttals}(I : \alpha, \Delta))$ it follows from one of the confirm() functions of Definition 2.5.6 that $\{K_1 : \neg\alpha, \ldots, K_n : \neg\alpha\} = \mathsf{rebuttals}(I : \alpha, \Delta)$. Thus from Definition 2.6.7 of rebuttals it is the case that for every $K \in \mathrm{Z}$ that $K : \neg\alpha \in \mathsf{rebuttals}(I : \alpha, \Delta)$. Now for the other direction. Given that $K : \neg\alpha \in \text{rebuttals}(I : \alpha, \Delta)$, there exists a confirmation $\mathrm{Z} : \neg\alpha$ with $K \in \mathrm{Z}$ that contains all of the rebuttals of $I : \alpha$ such that $\mathrm{Z} : \neg\alpha = \text{confirm}(\text{rebuttals}(I : \alpha, \Delta))$. Therefore $\mathrm{Z}:\neg\alpha = \text{confirm}(\text{rebuttals}(I:\alpha, \Delta))$ iff for all $K:\neg\alpha \in \text{rebuttals}(I:\alpha, \Delta)$ it is the case that $K \in \mathrm{Z}$. □

Thus the definition of rebuttal could just as easily be made using confirmation notation. It is also the case the set of rebuttals of a confirmation (rather than just an individual argument) can be represented as a single confirmation, as I now discuss.

### 2.6.3 Confirmation Rebuttal

The confirmation rebuttals function is a simple elaboration of the above rebuttals function, to show the interaction at the level of confirmations, not just at the level of individual arguments. Working at the level of confirmations shows a greater degree of symmetry and also facilitates later definitions in Chapter 5.

**Definition 2.6.8.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\mathrm{X} : \alpha, \mathrm{Y} : \neg\alpha \in \Diamond(\Delta)$. The set of* **confirmation rebuttals** *of $\mathrm{X} : \alpha$, denoted* confirmationRebuttals$(\mathrm{X} : \alpha, \Delta)$, *is a set of confirmations such that:*

$$\text{confirmationRebuttals}(\mathrm{X}:\alpha, \Delta) = \{\mathrm{Y}:\neg\alpha \in \Diamond(\neg\text{stripConfirmation}(\mathrm{X}:\alpha), \Delta) \mid \mathrm{Y} \neq \emptyset, \mathrm{X} \neq \emptyset\ \}.$$

A confirmation labelled with the empty set means that there are no arguments for the confirmation's claim. Thus valid confirmation rebuttals are never between unfounded confirmations. This set is the full power set of confirmations, less the unfounded confirmation - which is labelled with the emptyset and thus has the minimum cardinality label. Given the maximum cardinality confirmation for a claim the confirmation rebuttals function is self inverse.

**Proposition 2.6.7.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\mathrm{X}:\alpha \in \Diamond(\Delta)$.*

$$\text{top}(\text{confirmationRebuttals}(\text{top}(\text{confirmationRebuttals}(\mathrm{X}:\alpha, \Delta)), \Delta)) = \mathrm{X}:\alpha.$$

*Proof.* The confirmationRebuttals$(\mathrm{X} : \alpha, \Delta)$ function returns many confirmations for $\neg\alpha$, namely $\{\mathrm{Y}_1 : \neg\alpha, \ldots, \mathrm{Y}_n : \neg\alpha\}$, which equals the power set $\Diamond(\neg\alpha, \Delta)$ less the minimum cardinality member, i.e. $\emptyset:\neg\alpha \in \Diamond(\Delta)$. The maximum cardinality member of this power set of rebuttals is provided by top(confirmationRebuttals$(\mathrm{X}:\neg\alpha, \Delta)$). So let the confirmation $\mathrm{Y}_n:\neg\alpha = $ top(confirmationRebuttals$(\mathrm{X}: \alpha, \Delta)$). Now apply the same steps again to see that the maximum cardinality rebuttal of $\mathrm{Y}_n:\neg\alpha$ is $\mathrm{X}:\alpha$. Thus $\mathrm{X}:\alpha = $ top(confirmationRebuttals(top(confirmationRebuttals$(\mathrm{X}:\alpha, \Delta)), \Delta)$). □

This paragraph touches on polymorphism. Polymorphic functions can make a nomenclature more condensed and understandable. For example the + function can add integers or floating point numbers: 1 + 1 = 2 has the data types integer + integer = integer, while 1.2 + 1.2 = 2.4 shows float + float = float. In this analogy, individual arguments are parallel to integers and confirmations are parallel to floats – simply as different data types. I could have written confirmationRebuttals() as a polymorphic version of rebuttals(). The rebuttals function, as defined, has a single argument as input parameter and a set of arguments as its output. I showed, in Proposition 2.6.6, that the argument rebuttals output is equal in concept to, and could just as easily be represented by, a single confirmation. The confirmation rebuttals function, however, has a single confirmation as input and a power set of confirmations as its output. The maximum cardinality confirmation rebuttal has a confirmation as input and a single confirmation as output. I have chosen to not adopt polymorphism here in order to make the exact nature of these functional behaviours clearer.

A confirmation rebuttal can never rebut itself.

**Proposition 2.6.8.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let* $\mathrm{X}:\alpha \in \Diamond(\Delta)$.

$$\mathrm{X}:\alpha \notin \mathsf{confirmationRebuttals}(\mathrm{X}:\alpha, \Delta).$$

*Proof.* If an argument $I:\alpha, I \in \mathrm{X}$, could attack itself its support would be inconsistent, meaning that $\mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta)) \vdash \bot$, and thus it would not be an argument, $I:\alpha \notin (\mathsf{arguments}(\Delta)$. Hence $\mathrm{X}:\alpha \notin \mathsf{confirmationRebuttals}(\mathrm{X}:\alpha, \Delta)$. □

The cardinality of the maximum cardinality confirmation rebuttal is either 0 or 1.

**Proposition 2.6.9.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let* $\mathrm{X}:\alpha \in \Diamond(\Delta)$.

$$0 \leq |\mathsf{top}(\mathsf{confirmationRebuttals}(\mathrm{X}:\alpha, \Delta))| \leq 1.$$

*Proof.* No matter how many rebuttal arguments there are for $\mathrm{X}:\alpha$, that is $|J:\neg\alpha \in \mathsf{arguments}(\Delta)|$ they are all in just one confirmation, $\mathsf{top}(\Diamond(\neg\alpha, \Delta))$. If $\Delta \nvdash \bot$ there are no rebuttals, however that it represented as the unfounded confirmation $\emptyset:\alpha \in \Diamond(\Delta)$. □

## 2.7 Judging the Outcome of Debates

I now come to the sixth and final step of the basic framework, after i) $\Delta$, ii) $\mathsf{deductions}(\Delta)$ iii) $\mathsf{arguments}(\Delta)$, iv) $\Diamond(\Delta)$ and v) $\blacklozenge(\Delta)$. My sixth step is the definition of judge functions, which provide the outcome to a debate. While judges are akin to consequence relationships or inference operations found in the logic literature, e.g. (Gabbay, 1985; Makinson, 1988) they must be of a particular form. I address two categories of judge functions; those that judge a single claim and those that judge multiple claims and hence a whole knowledgebase. I start with the signature, or common form, of a single claim judge for $\alpha$, where $j$ is an index in the form of a decimal number to distinguish between the many possible judge functions that fit this general signature.

**Definition 2.7.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae. A* **judge of** $\alpha$, *denoted* $\mathsf{judge}_j(\alpha, \Delta)$, *is a filtering function, such that:*

1. *The contents of the set* $\mathsf{judge}_j(\alpha, \Delta)$ *conforms to one of the following three cases*
   a) *if* $\alpha \in \mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$ *then it may be the case that* $\mathsf{judge}_j(\alpha, \Delta) = \alpha$, *or*
   b) *if* $\neg\alpha \in \mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$*then it may be the case that* $\mathsf{judge}_j(\alpha, \Delta) = \neg\alpha$, *or*
   c) *regardless of whether* $\neg\alpha, \alpha \in \mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$ *or*
   $\neg\alpha, \alpha \notin \mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$ *it may be the case that* $\mathsf{judge}_j(\alpha, \Delta) = \emptyset$,
2. *and in all cases* $\mathsf{judge}_j(\alpha, \Delta) \nvdash \bot$.

The 'or's here are exclusive ors so clearly $\mathsf{judge}_j(\alpha, \Delta) \subseteq \mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$. The judge for $\alpha$ function delivers a consistent subset of $\mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$. Labels are removed delivering just unlabelled deduction formulae. Thus if $\alpha \notin \mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$ then $\alpha \notin \mathsf{judge}_j(\alpha, \Delta)$. However if $\alpha \notin \mathsf{judge}_j(\alpha, \Delta)$ then it does not follow that $\alpha \notin \mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$, as it could be the case that $\alpha$ is filtered out by the judge. Regardless of whether $\Delta \vdash \perp$ or $\Delta \nvdash \perp$ it is always the case that $\mathsf{judge}_j(\alpha, \Delta) \nvdash \perp$. The signature, or common form, of a multiple claim or what is called a cumulative judge, which covers all claims from a knowledgebase, is as follows:

**Definition 2.7.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae. A* **judge of** $\Delta$*, denoted* $\mathsf{judge}_j(\Delta)$*, is a filtering function, such that:*

$$\mathsf{judge}_j(\Delta) = \mathsf{judge}_j(\phi_1, \Delta) \cup \cdots \cup \mathsf{judge}_j(\phi_n, \Delta),$$
$$\textit{where } \mathsf{stripDeductions}(\mathsf{arguments}(\Delta)) = \{\phi_1, \ldots, \phi_n\}.$$

Thus all of the possible judges will have the above definitions in common. While judges are defined as functions of $\Delta$ here, $\lozenge(\Delta)$ and $\blacklozenge(\Delta)$ are of utility in defining specific judge functions.

The above pair of definitions provides the minimum set of properties of judge functions. If it is required that also $\mathsf{judge}_j(\Delta) \nvdash \perp$ then a further property called 'cumulativity' must be added and that will affect the definition of the judge of $\alpha$, but not necessarily that of the judge of $\Delta$. If a judge forces consistency then there may be side effects and implications of forcing such consistency. Properties of judge functions, particularly cumulativity, are reviewed in (Prakken & Vreeswijk, 2002). Two broader reviews of judge function properties are to be found in (Cayrol & Lagasquie-Schiex, 1995) and (Elvang-Gøransson & Hunter, 1995).

My judge function for $\Delta$ has some resonance with Dung's concept of a semantic extension, (Dung, 1993; Dung, 1995). Both filter a set of arguments to give a desirable subset – although my result is just the claim stripped of support or labels. Just as I have $j$ different judges, so also Dung and those who build on his work have many semantics. Dung and those building on his work do not appear to offer a general definition of a semantic and thus there is no rule to say that they must all be consistent. Exploring differences, similarities and a possible integration would be a useful area for further research.

A judge for $\Delta$ delivers a set of unlabelled claims. Thus repetition of claims is not allowed. The opinions of different experts can be tracked in $\Delta$, say as $a : \alpha, b : \alpha$. The judge, however, requires the removal of the labels and thus the removal of the repetition, which merges the opinions of the different experts.

To make the concept of a judge more concrete I provide two illustrative examples. The example judges employ two helper functions known as pro and con.

### 2.7.1 Definition of Pro and Con Functions

To more simply present various argument aggregation schemes, two further cardinality functions are defined, called $\mathsf{pro}()$ and $\mathsf{con}()$. These are notational conveniences giving the number of arguments for and against a claim.

**Definition 2.7.3.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and* $\langle X, Y\rangle : \alpha \in \blacklozenge(\alpha, \Delta)$. *The* **pro** *and* **con** *functions give the number of arguments for or against* $\alpha$ *respectively in* $\Delta$, *denoted* $\mathsf{pro}(\blacklozenge(\alpha, \Delta))$ *and* $\mathsf{con}(\blacklozenge(\alpha, \Delta))$, *such that:*

$$\begin{aligned}\mathsf{pro}(\blacklozenge(\alpha, \Delta)) &= |\mathsf{label}(\mathsf{top}(\lozenge(\alpha, \Delta)))|.\\ \mathsf{con}(\blacklozenge(\alpha, \Delta)) &= |\mathsf{label}(\mathsf{top}(\lozenge(\neg\alpha, \Delta)))|.\end{aligned}$$

These two functions are sensitive to the use of labels.

**Example 2.7.1. Label sensitivity**. *Let* $\Delta = \{a : \xi, b : \xi \rightarrow \omega, c : \xi\}$. *Thus* $\{a, b\} : \omega, \{c, b\} : \omega \in \mathsf{arguments}(\Delta)$. *Consequently* $\mathsf{pro}(\blacklozenge(\omega, \Delta)) = 2$.

### 2.7.2 Illustrative Judges

So that I can include judgement in the example of the basic framework at the end of this chapter and establish that there can be different types of judges with different behaviours, it is helpful to define two illustrative judges. The first of these judges, drawn from (Benferhat et al., 1993), is defined here and discussed in Chapter 7. It is an existential judge that does not rely on counting arguments, but rather on just whether they exist or not.

**Definition 2.7.4.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae. The* **Benferhat** $\mathsf{judge}_{1.0}(\alpha, \Delta, \emptyset)$ *is a judge such that:*

$$\alpha \in \mathsf{judge}_{1.0}(\alpha, \Delta, \emptyset) \textit{ iff } \mathsf{pro}(\blacklozenge(\alpha, \Delta)) \geq 1 \textit{ and } \mathsf{con}(\blacklozenge(\alpha, \Delta)) = 0.$$

A second example judge is called the Benjamin Franklin method (Willcox & Bridgewater, 1977). This judge relies on the number of arguments, not just their existence, and again is discussed in Chapter 7 under the headings on quantitative judges.

**Definition 2.7.5.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae. The* **Franklin** $\mathsf{judge}_{2.0}(\alpha, \Delta, \emptyset)$ *is a judge such that:*

$$\alpha \in \mathsf{judge}_{2.0}(\alpha, \Delta, \emptyset) \textit{ iff } \mathsf{pro}(\blacklozenge(\alpha, \Delta)) > \mathsf{con}(\blacklozenge(\alpha, \Delta)).$$

The origin of the informal definition of this argument aggregation scheme is commonly attributed to eighteenth century American statesman Benjamin Franklin. He called it Prudential Algebra and documented it in a letter of 19 September 1772 to Joseph Priestly, the chemist.

Judges have many properties, each of which, depending upon a combination of general principles and application specific requirements, can be deemed desirable or undesirable. A recurrent theme in AI, commonly known as 'horses for courses', is that there is no one perfect solution that solves all problems, but rather that different problems require different solutions.

To look specifically at cumulativity in the light of these two judges, consider the following example

**Example 2.7.2. Cumulativity**. *Let* $\Delta = \{a : \beta \wedge \neg\alpha, b : \beta \rightarrow \alpha\}$. *Then* $\{a\} : \beta, \{a\} : \neg\alpha, \{b\} : \beta \rightarrow \alpha \in \mathsf{arguments}(\Delta)$. *Suppose a debate was conducted over* $\beta$. *Judges 1.0 and 2.0 would yield*

*$\beta \in \mathsf{judge}_{1.0}(\beta, \Delta)$ and $\beta \in \mathsf{judge}_{2.0}(\beta, \Delta)$. A second debate for $\beta \rightarrow \alpha$ would result in $\beta \rightarrow \alpha \in \mathsf{judge}_{1.0}(\beta \rightarrow \alpha, \Delta)$ and $\beta \rightarrow \alpha \in \mathsf{judge}_{2.0}(\beta \rightarrow \alpha, \Delta)$. Having established $\beta$ and $\beta \rightarrow \alpha$ it would appear to be reasonable to deduce $\alpha$. However, if a debate is conducted over the question of $\alpha$ the outcome is $\neg\alpha$, i.e. $\neg\alpha \in \mathsf{judge}_{1.0}(\alpha, \Delta)$ and $\neg\alpha \in \mathsf{judge}_{2.0}(\alpha, \Delta)$. Both judges have thus delivered sets of inconsistent results showing that neither of them has the property 'cumulativity'.*

A cumulative judge would thus have to examine all claims in $\mathsf{arguments}(\Delta)$ before it could judge one claim. Possibly, heuristics could be employed to limit the number of additional claims to examine, but all other judge outcomes that are possibly inconsistent with $\alpha$ are relevant. An additional mechanism, beyond those presented in judges 1.0 and 2.0, will be required to resolve inconsistencies so the above example would lead to a decision as either the judge results in ($\beta$ and $\beta \rightarrow \alpha$) or in $\neg\alpha$. To provide cumulativity it would appear that either levels of evidence would have to be established by computation or priorities used.

## 2.8 Worked Example of the Basic Argumentation Framework

This section is a worked example illustrating the flow of the basic argumentation framework leading up and including judging the outcome of a debate. My aim is to go into enough detail to show the workings of the three kinds of labels, a full power set of confirmations and a full power set of contradictions that arise from even a modest $\Delta$.

### 2.8.1 Assumptions

The pattern I now show starts with assumption labels which are set members rather than whole sets. The example begins with the set of assumptions. In this example there are six assumptions and they are labelled with assumption labels *a* through to *f*.

$$\Delta = \{a{:}\alpha,\ b{:}\alpha \rightarrow \beta,\ c{:}\alpha,\ d{:}\gamma \wedge \neg\beta,\ e{:}\neg\beta \rightarrow \lambda,\ f{:}\neg\lambda\}, \text{ so } |\Delta| = 6$$

### 2.8.2 Deductions

From the assumptions, the set of deductions is inferred, using classical deduction. $\{b, c, d\} : \beta \wedge \gamma \in \mathsf{deductions}(\Delta)$ is a valid deduction, but its premises are not consistent and thus it is not an argument. So $\{b, c, d\} : \beta \wedge \gamma \notin \mathsf{arguments}(\Delta)$. Furthermore $\{a, b, c\} : \beta, \{a, c\} : \alpha \in \mathsf{deductions}(\Delta)$ are valid deductions, but are not minimal and thus not arguments. Consequently $\{a, b, c\} : \beta, \{a, c\} : \alpha \notin \mathsf{arguments}(\Delta)$.

### 2.8.3 Arguments

The arguments of relevance in this example are a small subset of all possible arguments. The set $\mathsf{arguments}(\Delta)$ includes:

| | | | | | |
|---|---|---|---|---|---|
| $\{a,b\}{:}\beta$ | $\{b,c\}{:}\beta$ | $\{e,f\}{:}\beta$ | $\{d\}{:}\neg\beta$ | $\{d\}{:}\gamma$ | $\{a\}{:}\alpha$ |
| $\{b\}{:}\alpha \rightarrow \beta$ | $\{b\}{:}\neg\beta \rightarrow \neg\alpha$ | $\{d\}{:}\gamma \wedge \neg\beta$ | $\{e\}{:}\neg\beta \rightarrow \lambda$ | $\{f\}{:}\neg\lambda$ | $\{c\}{:}\alpha$ |
| $\{b\}{:}\neg\alpha \vee \beta$ | $\{e\}{:}\neg\gamma \rightarrow \beta$ | $\{e\}{:}\beta \vee \gamma$ | $\{a\}{:}\alpha \vee \gamma$ | $\{d,e\}{:}\lambda$ | $\{b,d\}{:}\neg\alpha$ |

So far I have shown the arguments as labelled with assumption labels. This is one valid approach, it shows the provenance of each deduction. Now to progress the example and make the notation less verbose, I introduce the necessary deduction labels together with their assumption label equivalencies.

$$I = \{a,b\} \quad J = \{b,c\} \quad K = \{e,f\} \quad L = \{d\} \quad M = \{a\} \quad N = \{b\}$$
$$O = \{c\} \quad P = \{e\} \quad Q = \{f\} \quad R = \{d,e\} \quad S = \{b,d\}$$

Thus I can relabel the same set of arguments. They are now labelled with deduction labels, rather than assumption labels. The notation is more condensed, but the provenance is no longer visible. The provenance is still accessible however as the label equivalencies just defined provide traceability.

$$\begin{array}{llllll} I:\beta & J:\beta & K:\beta & L:\neg\beta & L:\gamma & M:\alpha \\ N:\alpha\rightarrow\beta & N:\neg\beta\rightarrow\neg\alpha & L:\gamma\wedge\neg\beta & P:\neg\beta\rightarrow\lambda & Q:\neg\lambda & O:\alpha \\ N:\neg\alpha\vee\beta & P:\neg\gamma\rightarrow\beta & P:\beta\vee\gamma & M:\alpha\vee\gamma & R:\lambda & S:\neg\alpha \end{array}$$

Of course there are innumerable additional arguments I could show, such as others added by disjunction introduction, e.g. $L:\gamma\vee\neg\beta\vee\psi$, $J:\beta\vee\zeta\vee\neg\xi \in \mathsf{arguments}(\Delta)$, but I won't clutter the example with them.

### 2.8.4 Confirmations

Given the set of arguments, the confirmations become clear. Rather than describe all of the confirmations, this example focuses on the confirmations in $\beta$ and also the confirmations in $\neg\beta$. Other confirmations exist (e.g. in $\alpha,\gamma$ and $\lambda$), but they are mostly self confirmations and would clutter the example.

$$\begin{array}{rcl} \Diamond(\beta,\Delta) & = & \{\emptyset:\beta,\{I\}:\beta,\{J\}:\beta,\{K\}:\beta,\{I,J\}:\beta,\{J,K\}:\beta,\{I,K\}:\beta,\{I,J,K\}:\beta\} \\ \Diamond(\neg\beta,\Delta) & = & \{\emptyset:\neg\beta,\{L\}:\neg\beta\} \end{array}$$

The confirmations in each of $\beta$ and $\neg\beta$ each form separate power sets. Here is the power set of confirmations for $\beta$, employing sets of deduction labels to show the pattern. The ordering criterion is confirmation label set inclusion.

$$\begin{array}{ccc} & \{I,J,K\}:\beta & \\ \{I,J\}:\beta & \{J,K\}:\beta & \{I,K\}:\beta \\ \{I\}:\beta & \{J\}:\beta & \{K\}:\beta \\ & \emptyset:\beta & \end{array}$$

The lattice of confirmations for $\neg\beta$, shown with deduction labels, is much simpler:

$$\begin{array}{c} \{L\}:\neg\beta \\ \emptyset:\neg\beta \end{array}$$

Here the $|\mathsf{label}(\mathsf{top}(\Diamond(\beta,\Delta)))| = 3$ and $|\mathsf{label}(\mathsf{top}(\Diamond(\neg\beta,\Delta)))| = 1$. Thus $\mathsf{pro}(\blacklozenge(\beta,\Delta)) = 3$ and $\mathsf{con}(\blacklozenge(\beta,\Delta)) = 1$.

### 2.8.5 Contradictions

Now that all of the confirmations in $\beta$ and $\neg\beta$ have been listed, I can enumerate the resultant contradictions and draw them as if ordered by set inclusion in a lattice. The ordering criteria I use to draw this contradiction lattice, relevant only to this example, is $\langle \mathrm{X},\mathrm{Y}\rangle:\beta$ *'subset or equal to'* $\langle \mathrm{X}',\mathrm{Y}'\rangle:\beta$ *iff* $\mathrm{X}\subseteq \mathrm{X}'$ *and* $\mathrm{Y}\subseteq\mathrm{Y}'$. The deduction labels show which arguments the confirmations and contradictions arise from. This power set derives from the above, as the pairing of all the confirmations for $\beta$ first with the single $\{L\} : \neg\beta$ and again with the empty set based $\emptyset : \neg\beta$. Two power set of confirmations, for $\beta$ and $\neg\beta$, can be captured in just one integrated power set of contradictions:

$$\langle\{I,J,K\},\{L\}\rangle : \beta$$

$$\langle\{I,J,K\},\emptyset\rangle : \beta \quad \langle\{I,J\},\{L\}\rangle : \beta \quad \langle\{I,K\},\{L\}\rangle : \beta \quad \langle\{J,K\},\{L\}\rangle : \beta$$

$$\langle\{I,J\},\emptyset\rangle : \beta \quad \langle\{I,K\},\emptyset\rangle : \beta \quad \langle\{J,K\},\emptyset\rangle : \beta \quad \langle\{I\},\{L\}\rangle : \beta \quad \langle\{J\},\{L\}\rangle : \beta \quad \langle\{K\},\{L\}\rangle : \beta$$

$$\langle\{I\},\emptyset\rangle : \beta \quad \langle\{J\},\emptyset\rangle : \beta \quad \langle\{K\},\emptyset\rangle : \beta \quad \langle\emptyset,\{L\}\rangle : \beta$$

$$\langle\emptyset,\emptyset\rangle : \beta$$

It can be seen that a portion of this power set of contradictions is empty-contradictions and the compliment portion the non-empty contradictions. Such power sets will not necessarily be symmetrical, for example this one has nine empty contradictions and seven non-empty ones.

### 2.8.6 Judgement

For the claim $\beta$, applying $\mathsf{judge}_{1.0}(\beta, \Delta, \emptyset)$ there exist arguments for both $\beta$ and $\neg\beta$ so:

$$\beta, \neg\beta \notin \mathsf{judge}_{1.0}(\beta, \Delta, \emptyset).$$

However instead of yielding a stalemate, $\mathsf{judge}_{2.0}(\beta, \Delta, \emptyset)$ resolves the inconsistency to declare a clear winner, such that:

$$\beta \in \mathsf{judge}_{2.0}(\beta, \Delta, \emptyset) \qquad \neg\beta \notin \mathsf{judge}_{2.0}(\beta, \Delta, \emptyset).$$

Thus $\mathsf{judge}_{1.0}(\beta, \Delta, \emptyset) = \emptyset$. Both $\mathsf{judge}_{1.0}(\beta, \Delta, \emptyset) \nvdash \bot$, $\mathsf{judge}_{2.0}(\beta, \Delta, \emptyset) \nvdash \bot$. This example of $\mathsf{judge}_{2.0}$ is one simple way to resolve inconsistency, but as I go on to explain it is a naïve one. By naïve I mean that that the judge suffers from a number of undesirable properties – using as yet undefined terms, such as being too eager (i.e. letting through weakly argued claims or noise, see Chapter 7), insensitive to undercut (Chapters 3 and 6), insensitive to reflection (Chapters 4 and 5) and insensitive to assisting arguments (Section 8.4).

Broadening my observation it can be seen that there are other inconsistencies such that:

$$\lambda, \neg\lambda \notin \mathsf{judge}_{1.0}(\lambda, \Delta, \emptyset) \qquad \alpha, \neg\alpha \notin \mathsf{judge}_{1.0}(\alpha, \Delta, \emptyset).$$

However there are further claims in $\mathsf{stripDeductions}(\mathsf{arguments}(\Delta))$ which are not in conflict, for example:

$$\gamma \in \mathsf{judge}_{1.0}(\gamma, \Delta, \emptyset) \qquad \neg\gamma \notin \mathsf{judge}_{1.0}(\gamma, \Delta, \emptyset).$$

So for a debate judged by judge 1.0 with four motions $\alpha, \beta, \gamma$ and $\lambda$ the outcome would be that motion $\gamma$ would pass, but that $\alpha, \beta$ and $\lambda$ would fail to pass. That completes my example showing the six steps of the basic framework, namely knowledgebase, deductions, arguments, confirmations, contradictions and judgement.

## 2.9 Conclusions for Basic Argumentation Framework

This chapter on my basic argumentation framework has formally defined the first two foundations I need to track and judge debates, namely confirmation and rebuttal.

In addition to the definition of the knowledgebase $\Delta$, the eight key argument aggregations introduced in this chapter and reused throughout the thesis are:

| Aggregation | For One Formula | Definition | For All Formulae | Definition |
|---|---|---|---|---|
| Arguments: | $\mathsf{arguments}(\alpha, \Delta)$ | 2.4.6 | $\mathsf{arguments}(\Delta)$ | 2.4.2 |
| Confirmations: | $\lozenge(\alpha, \Delta)$ | 2.5.5 | $\lozenge(\Delta)$ | 2.5.4 |
| Contradictions: | $\blacklozenge(\alpha, \Delta)$ | 2.6.3 | $\blacklozenge(\Delta)$ | 2.6.2 |
| Judgement: | $\mathsf{judge}_j(\alpha, \Delta)$ | 2.7.1 | $\mathsf{judge}_j(\Delta)$ | 2.7.2 |

Table 2.1: Summary of Key Argument Aggregation Functions

Contradictions can be thought of as a general representation of the common concept of rebuttal. Contradictions are simply groups of arguments for the same claim and its negation, a construct which turns out to be useful for representing and thinking about professional debates.

A contribution of the chapter is the definition of the inconsistent confirmation – one where the supports of the arguments for a single claim contradict each other. Additionally I examine the literature's concept of catenate inference and find that it allows inconsistent catenate inferences – a problem addressed by my framework. Another contribution of the chapter is in clarifying different types of argument and confirmation, namely the definitions of terms: reflexive argument, tautological argument, self confirmation, unfounded confirmation, tautological confirmation and empty contradiction. Confirmation is not just a mechanical grouping of arguments for a claim, but rather a mindful and useful device with a number of interesting properties, a good correlation to professional debate and is helpful for building more refined debating constructs.

In addition, my helper functions will continue to be useful in manipulating the argument aggregations in subsequent chapters, they are: stripAssumption(), stripAssumptions(), stripDeduction(), stripDeductions(), stripConfirmation(), stripConfirmations(), stripContradiction(), stripContradictions(), label(), formulae(), midLattice(), confirm(), top(), pro() and con(). A number of these are polymorphic functions. In a category of their own, the functions rebuttals() and confirmationRebuttals() help in linking my work to the literature and then extend it. The many propositions and examples of this chapter clarify the behaviours of each of these argument aggregations. Included in this analysis is the cardinality track with its Propositions 2.4.3, 2.5.7 and 2.6.2. Now the basic argumentation framework has been established I can move in the direction of determining how to properly track a debate, so that the whole debate and nothing but the debate is tracked, and judge its outcome.

# Chapter 3

# Undercut Framework

## 3.1 Overview of the Chapter

This undercut framework chapter analyses four types of undercut (Wigmore, Pollock, canonical and preclusive) over eight sections. The first section covers both Wigmore and Pollock, the second is devoted to canonicals and the last six introduce my new preclusive undercut.

I should emphasise at the outset that in rendering the undercuts of others into my formal framework these undercuts become no longer absolutely identical to their originals. I deem it vital to do this rendering as it allows me to compare and contrast the various undercuts in the literature, and I hold that the distinctions exposed are due to the original works and do not stem from my rendering. Nonetheless what I call a Pollock undercut is not literally identical to that published by Pollock. To a certain degree this difference is at the philosophical level of rationalising the connection between premises and claims. Likewise for Wigmore and for Besnard and Hunter. This point of parallelism, but not identicality, also applies to those tree and judge functions which I take from the literature. So although my rendering approach has a weakness of not being a carbon-copy of the original, it is a contribution in that it allows the various approaches to be compared and contrasted.

**Introductory undercuts** The first three types of undercut are from the literature, however many of the properties I explore are new – and found wanting for professional debate.

**Wigmore undercut** Section 3.2.1, is an attack that states that one of the assumptions supporting an argument is wrong, affects debate outcomes, see Example 3.2.2, and resolves inconsistency, see Example 3.2.3.

**Pollock undercut** Section 3.2.4, argues that one or more of the assumptions supporting an argument is wrong. This is the most widely used undercut in the literature.

**Canonical undercut** Section 3.3, means that one or other of the assumptions supporting an argument is wrong. I formalise properties, showing it to be better behaved than the first two undercuts.

**Preclusive undercut** In this second half of this chapter, I introduce a new form of undercut. Section 3.4 defines preclusion as one or other of the attacked arguments is wrong.

**Motivation** Section 3.5, preclusive undercuts are much used in professional debate, so I give examples from politics, detective work and court-room cross examination.

**Canonical subsumption** Section 3.6, here I show how preclusive undercuts cover and extend the ground of canonicals to better mirror professional debate.

**Tautological preclusions** Section 3.7, these novel undercuts prevent a debating team from being inconsistent when arguing a claim.

**Non-unique formulae** Section 3.8, goes more deeply into the properties of preclusive undercuts, examining the effects of labels and non-unique formulae on cardinality.

Finally, Section 3.9, relates the definitions of undercut and rebuttal discussed so far with the notion of abstract attack.

This chapter builds my set of argument interactions to the principal three – confirmation, rebuttal and undercut, providing a strong foundation for subsequent analysis for tracking and judging debates.

## 3.2 The Concept of Undercut

I start the body of this chapter by introducing the concept of undercut as described in the literature. There are two ways that one argument can **attack** another argument; rebuttal and undercut. The first way, **rebuttal**, is where the conclusion of one argument is the negation of the conclusion of the other argument. The second way, **undercut**, involves the conclusion of one argument attacking the support of the other argument. I refer to the argument that is rebutted or undercut as the **defending argument**. Consequently I can refer to the undercutting or rebutting arguments as the **attacking** arguments.

Work on inconsistency has generally focussed on the rebuttal form of inconsistency, however undercut also merits attention as it is arguably a form of inconsistency and can effect the outcome of debates (as shown later in the example of Section 3.2.2). Each of the judge functions presented so far (which have excluded undercuts) could, broadly speaking, be repeated with the single difference of incorporating undercuts. There is, however, little consensus in the literature on a) the exact definition of undercut and b) how undercuts should be incorporated into argument aggregation, making it possible to have not just one, but several variations on each of the judges presented so far. In Section 7.6, I propose a way to incorporate undercuts into judgement.

The location of the first informal description of undercut in the literature is unclear, as a number of authors, including (Wigmore, 1937; Toulmin, 1958; Walton, 1989), explore the field that surrounds and includes undercut with a variety of vocabularies. The formalisation of undercut is generally attributed to (Pollock, 1970). Argument aggregation schemes not encompassing undercut include the various judges from (Manor & Rescher, 1970) and (Benferhat et al., 1993), the Franklin approach (Willcox & Bridgewater, 1977) and net support (Fox & Das, 2000). It seems fair to suggest, given that undercuts can effect the outcomes of debates, to exclude them from argument aggregation schemes is somewhat naïve.

The formal use of undercut within argument aggregation has been further developed notably by (Elvang-Gøransson et al., 1993b; Elvang-Gøransson & Hunter, 1995; Besnard & Hunter, 2001). Today undercut is well established in the logic literature with quite a number of authors, in addition to the above mentioned, also writing on the topic, including many papers by Amgoud, Nute, Parsons, Prakken, Shroeder, Verheij and Vreeswijk; in particular I cite (Prakken, 1997; Vreeswijk, 1997; Amgoud & Cayrol, 2002). For reviews covering this developing aspect of argumentation see (Hunter, 2001; Chesñevar et al., 2000; Prakken & Vreeswijk, 2002).

At least three definitions of undercut are used in the literature:

1. What I call a Wigmore undercut, e.g. (Elvang-Gøransson & Hunter, 1995; Dung et al., 2006).
2. Pollock undercut, (Pollock, 1970).
3. Canonical undercut (Besnard & Hunter, 2001).

.

Several rather more complicated defeasible-logic based definitions are given by Hage and by Prakken with applicability to law, (Hage, 1996; Prakken & Sartor, 1997; Verheij, 2001). Limiting my analysis to classical propositional logic with minimal consistent arguments I now define three kinds of undercut; Wigmore, Pollock and canonical and then add one of my own.

### 3.2.1 Wigmore Undercut

Wigmore was the first to introduce the concept of undercut into informal logic in his writings on jurisprudence in 1937 (Wigmore, 1937; Twining, 1985). Some attribute undercut to Toulmin, but a) Wigmore predates Toulmin by two decades and b) the literature is clear that Toulmin did not precisely describe undercut, but rather what is more commonly known as the more general or related concept of defeat. See for example, the commentary of (Verheij, 1999) stating that Toulmin does not make the distinction between rebuttal and undercut.

Wigmore calls the construct 'explanatory evidence' and shows how it reduces the strength of effect of the argument it attacks. Thus if a witness said they saw a red wall and their barrister argues that therefore the wall is red, an explanatory evidence would be to say actually a red light was or could have been shining on that wall thereby making it only appear to be red.

I take Wigmore's description as attacking an assumption and formalise it as Definition 3.2.1 below. This same definition, but without labels appears in a number of published sources, e.g. (Elvang-Gøransson & Hunter, 1995; Amgoud & Cayrol, 2002; Dung et al., 2006), but without the attribution to Wigmore. This definition may be considered to be simpler than Pollock's definition (to be covered in Section 3.2.4 below) which is slightly more predominant in later literature. I now provide this definition in my nomenclature, so that I can show the relationships between the different kinds of undercut.

**Definition 3.2.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I : \alpha, J : \neg\phi \in$* $\mathsf{arguments}(\Delta)$.

$$J : \neg\phi \textit{ is a } \textbf{Wigmore undercut} \textit{ of } I : \alpha \textit{ iff } \phi \in \mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta)).$$

Now I provide an example of a Wigmore undercut.

**Example 3.2.1. Wigmore undercut.** *Let $\Delta = \{a : \alpha, b : \alpha \rightarrow \beta, c : \neg\alpha)\}$. Then $\{a, b\} : \beta, \{c\} : \neg\alpha \in$* $\mathsf{arguments}(\Delta)$. *It follows that the argument $\{c\} : \neg\alpha$ is a Wigmore undercut of $\{a, b\} : \beta$.*

The Wigmore undercut of (Elvang-Gøransson & Hunter, 1995) is expressed as follows: 'Let $\langle \Pi, \phi \rangle$ and $\langle \Theta, \psi \rangle$ be any arguments constructed from $\Delta$. If $\gamma \in \Theta$ and $\vdash \phi \leftrightarrow \neg\gamma$, then $\langle \Pi, \phi \rangle$ is an undercutting defeater of $\langle \Theta, \psi \rangle$'. I now discuss how this definition is equivalent to my definition. $\vdash \phi \leftrightarrow \neg\gamma$ means that $\phi \leftrightarrow \neg\gamma$ is a tautology. Consequently there is another argument, $\langle \Pi, \neg\gamma \rangle$ that can also be derived from $\Delta$, that is not a tautology and is a Wigmore undercut of $(\Theta, \psi)$. Because $\phi \leftrightarrow \neg\gamma$ is a tautology, no further formulae need to be added to $\Pi$, over and above those used to deduce $\phi$, in order to deduce $\neg\gamma$ (as would have been the case if $\phi \rightarrow \neg\gamma$ had been used instead of $\phi \leftrightarrow \neg\gamma$ in their definition). Similarly the ABA attack can be seen to be a generalisation of the Wigmore undercut; see Definition 2.5 in (Dung et al., 2006) where it states that argument A attacks argument B if 'A *attacks an assumption in the set of assumptions on which* B *is based*'.

My use of labels makes Wigmore undercuts able to attack more arguments than would be the case without labels. For the Wigmore undercut without labels within the example below I employ the definition of (Elvang-Gøransson & Hunter, 1995) and the notation of (Besnard & Hunter, 2001).

**Example 3.2.2. Wigmore and labels.** *Let* $\Delta = \{a : \alpha, b : \alpha \rightarrow \beta, c : \neg\alpha, d : \alpha, e : \alpha \rightarrow \beta\}$. *Then* $\{a, b\} : \beta, \{d, e\} : \beta, \{c\} : \neg\alpha \in \text{arguments}(\Delta)$. *With labels, it follows that the argument* $\{c\} : \neg\alpha$ *is a Wigmore undercut of* $\{a, b\} : \beta$ *and also of* $\{d, e\} : \beta$. *Now without labels, let* $\Gamma = \{\alpha, \alpha \rightarrow \beta, \neg\alpha\}$. *Consequently there are two arguments* $\langle \{\alpha, \alpha \rightarrow \beta\}, \beta \rangle$ *and* $\langle \{\neg\alpha\}, \neg\alpha \rangle$. *Therefore,* $\langle \{\neg\alpha\}, \neg\alpha \rangle$ *is a Wigmore undercut of only one argument* $\langle \{\alpha, \alpha \rightarrow \beta\}, \beta \rangle$.

Now here is an example showing how undercuts (even simple Wigmore undercuts) can make a difference to the outcome of a debate.

### 3.2.2 Undercuts Can Change Debate Outcomes

This subsection is built around a somewhat informal example to provide motivation for subsequent formalisms in this thesis. It also places this motivation in the context of the literature. The discussion builds on the earlier definition of judge 2.0, see Definition 2.7.5.

**Example 3.2.3. Changed debate.** *Let the initial* $\Delta_i = \{a : \beta, b : \gamma, c : \neg\lambda, d : \neg\alpha \rightarrow \neg\gamma, e : \neg\alpha \rightarrow \neg\beta, f : \neg\alpha \vee \lambda\}$. *Then* $\{a, e\} : \alpha$, $\{b, d\} : \alpha$, $\{c, f\} : \neg\alpha \in \text{arguments}(\Delta_i)$ *are the arguments for* $\alpha$ *and* $\neg\alpha$ . *Let* $\{a, e\} = I, \{b, d\} = J$ *and* $\{c, f\} = K$, *so* $\{I, J\} : \alpha \in \lozenge(\alpha, \Delta_i)$ *and* $\{K\} : \neg\alpha \in \lozenge(\neg\alpha, \Delta_i)$. *Then the debate can be summarised as the maximum cardinality contradiction* $\langle \{I, J\}, \{K\} \rangle : \alpha \in \blacklozenge(\alpha, \Delta_i)$. *Using* $\text{judge}_{2.0}(\Delta_i)$ *it is clear that* $\alpha$ *wins. The outcome of the debate is* $\alpha$ *because* $\text{pro}(\blacklozenge(\alpha, \Delta_i)) = 2$, $\text{con}(\blacklozenge(\alpha, \Delta_i)) = 1$ *and as* $2 > 1$ *so* $\alpha \in \text{judge}_{2.0}(\Delta_i)$ *and* $\neg\alpha \notin \text{judge}_{2.0}(\Delta_i)$. *A property of* $\text{judge}_{2.0}$ *is that it is always consistent, so* $\alpha \in \text{judge}_{2.0}(\Delta_i)$ *implies* $\neg\alpha \notin \text{judge}_{2.0}(\Delta_i)$. *Now add two undercuts to the debate. Let the new* $\Delta = \Delta_i \cup \{g : \neg\beta, h : \neg\gamma\}$. *Let* $L = \{g\}, M = \{h\}$. *So now* $L : \neg\beta, M : \neg\gamma \in \text{arguments}(\Delta)$. *Note that any attempt to combine* $g : \neg\beta$ *with* $e : \neg\alpha \rightarrow \neg\beta$ *implies nothing about* $\alpha$*; as* $g : \neg\beta$ *only affirms the claim of* $e : \neg\alpha \rightarrow \neg\beta$. *Similarly* $h : \neg\gamma$ *with* $d : \neg\alpha \rightarrow \neg\gamma$ *also implies nothing about* $\alpha$. *So no new confirmations or rebuttals of* $\alpha$ *are introduced. What exists is a pair of undercuts, namely the argument* $L : \neg\beta$ *is an undercut of* $I : \alpha$ *and* $M : \neg\gamma$ *is an undercut of* $J : \alpha$. *While there is more consensus in the literature to the effect that* $L : \neg\beta$

*attacks $I : \alpha$, there is less agreement that $L : \neg\beta$ defeats $I : \alpha$.*

So given this example, I can now examine the three possibilities of the impact of $L : \neg\beta$ on $I : \alpha$ as full defeat, graded (or partial) defeat and no defeat (i.e. no effect).

1. **No effect**. Attack by an undercut can be taken to not imply defeat and have no effect. Papers taking this position include (Manor & Rescher, 1970), (Benferhat et al., 1993) and (Fox & Das, 2000). 'No effect' is the assumption implicit in $\mathsf{judge}_{2.0}$. In this case the outcome of the above debate is $\alpha$, namely $\alpha \in \mathsf{judge}_{2.0}(\Delta)$.

2. **Full defeat**. Alternatively, attack by an undercut can be taken to imply total defeat, i.e., to fully nullify the effect of the attacked argument. Papers broadly following this approach include (Dung, 1995), (Prakken & Vreeswijk, 2002) and (Amgoud & Cayrol, 2002). So far I have not formally defined this possibility, but intuitively, for the above example, the value of an undercut sensitive pro function should now reduce to zero. I say zero because the two arguments for $\alpha$, i.e. $I : \alpha$, $J : \alpha$ are now defeated by the two undercuts $L : \neg\beta, M : \neg\gamma$. Thus by modifying the pro and con functions to be undercut aware and using them in an otherwise unmodified $\mathsf{judge}_{2.0}$ (call it $\mathsf{judge}_{2.1}$, which is not formally defined, although practical judges akin to it are defined in Chapter 7) would yield $\neg\alpha \in \mathsf{judge}_{2.1}(\Delta)$. These modified pro and con functions should still deliver natural numbers in the same range as the original pro and con functions. Using this assumption reverses the outcome of the debate – from $\alpha \in \mathsf{judge}_{2.0}(\Delta)$ to $\neg\alpha \in \mathsf{judge}_{2.1}(\Delta)$ – showing that, given certain assumptions, undercuts can affect debate outcomes.

3. **Graded defeat**. The third option is to say that attack by an undercut implies a gradation of defeat, that is anything in between the above two extremes of full and no defeat. This perspective is described in (Wigmore, 1937; Besnard & Hunter, 2001) and touched on in (Verheij, 1996).

   - **Half**. I examine first the situation where attack by an undercut halves the strength of the attacked argument. While I have not yet defined strength it is possible to explore it as an intuitive notion, and say that the strength of $I : \alpha$, initially 1 is reduced by the attack of $L : \neg\beta$ to $\frac{1}{2}$. Similarly the attack of $M : \neg\gamma$ reduces the strength of $J : \alpha$ to $\frac{1}{2}$. Thus the collective strength of the confirmation $\{I, J\} : \alpha$ is reduced to 1. Consequently $\alpha \notin \mathsf{judge}_{2.1}(\Delta)$ and also $\neg\alpha \notin \mathsf{judge}_{2.1}(\Delta)$. To formalise this situation will require modified pro and con functions that deliver rational numbers rather than natural numbers. Neither $\alpha$ nor $\neg\alpha$ wins. I call this situation a **stalemate** because the inconsistency is not resolved and there is equal evidence for $\alpha$ and $\neg\alpha$.

   - **Over half**. Continuing the above example, if the attacking strength of $L : \neg\beta$ on $I : \alpha$ is over half, then the residual strength of $I : \alpha$ is reduced to under a half. The same applies for the $M : \neg\gamma$ attack on $J : \alpha$. Consequently the overall residual strength for $\alpha$ is under one and $\alpha$ is defeated. The effect of the debate is reversed and $\neg\alpha \in \mathsf{judge}_{2.1}(\Delta)$.

   - **Under half**. Now for the contrary situation. If the attacking strength of $L : \neg\beta$ on $I : \alpha$ is under half, then the residual strength of $I : \alpha$ is reduced but is still over a half. Consider the same applies for the $M:\neg\gamma$ attack on $J:\alpha$. Consequently the overall residual strength for $\alpha$ is over one and $\alpha$ is still victorious. The effect of the debate is unchanged so $\alpha \in \mathsf{judge}_{2.1}(\Delta)$ just as $\alpha \in \mathsf{judge}_{2.0}(\Delta)$.

This example covering i) no effect, ii) full and iii) graded defeat shows that undercuts can make a major difference to the outcome of debates. The word 'can' is used due to two reasons: one is undercuts only make a difference if certain assumptions are made and the other is that even then the difference is seen only with certain $\Delta$. One of these assumptions is my use of labels. As shown in Example 3.2.2, working with labels can alter the number of undercuts and thus can also affect debate outcomes.

Clearly there are some debates to which incorporating undercut does not effect the outcome. For instance, consider a $\Delta$ that implies the contradiction $\langle\{I, J, K\}, \{L\}\rangle : \alpha$ and also just one undercut, the argument $M:\beta$, such that $\neg\beta \in$ strip Assumptions(formulae($I, \Delta$)) with the same $\mathsf{judge}_j$ as above. Then even if I assume $M:\beta$ fully defeats $I:\alpha$ it follows that there are still two arguments for and only one against so $\alpha \in \mathsf{judge}_{2.1}(\Delta)$. The outcome of this debate would not be changed.

To generalise from the above example there is nothing unique about the $\frac{1}{2}$ in its ability to infer stalemate or inconsistency. For example, the rational numbers $\frac{1}{3}$ or $\frac{2}{3}$ for defeating strength of an undercut could create stalemate for other $\Delta$. Similarly it is not always true that full defeat reverses the outcome of debates; this too depends upon $\Delta$. What is generally true, however, is that there are three possible effects of including undercut on the outcome of a debate, that is a) no effect, b) reverse the outcome, and c) stalemate where inconsistency is not resolved.

Before progressing to counting undercuts and rebuttals of arguments (in Section 7.2) it will be necessary to further evolve the understanding of patterns of repetition when arguments attack each other. To introduce an indication of the puzzles ahead consider the following example that shows that one undercut can undercut two arguments.

**Example 3.2.4. Number of Wigmore undercuts.** *Let* $\Delta = \{a:\alpha, b:\alpha \to \beta, c:\gamma, d:\gamma \to \beta, e:\neg(\alpha \vee \gamma)\}$. *Then* $\{a,b\}:\beta, \{c,d\}:\beta, \{e\}:\neg(\alpha \vee \gamma), \{e\}:\neg\alpha, \{e\}:\neg\gamma \in$ arguments($\Delta$). *It follows that* $\{e\}:\neg(\alpha \vee \gamma)$ *is a Wigmore undercut of* $\{a,b\}:\beta$ *and also of* $\{c,d\}:\beta$. *Furthermore* $\{e\}:\neg\alpha$ *is another Wigmore undercut of* $\{a,b\}:\beta$ *and* $\{e\}:\neg\gamma$ *is of* $\{c,d\}:\beta$.

When tracking and judging a debate, should one, two or three of these undercuts be counted? Rather than counting all possible undercuts that can be found in arguments($\Delta$) instead I need some subset of well ordered undercuts that represents all others.

### 3.2.3 Undercuts, Confirmations and Rebuttals Resolve Inconsistency

So far this chapter has shown that undercuts, confirmations and rebuttals are interrelated and have a bearing on inconsistency. I now provide an example to demonstrate that each of these three interactions can be used to resolve an inconsistency. I argue that because all three can be used to achieve the same effect then all three are interrelated.

Let the initial $\Delta_i = \{a:\beta,\ b:\neg\alpha \to \neg\beta,\ c:\neg\lambda,\ d:\neg\alpha \vee \lambda\}$. Then $\{\{a,b\}:\alpha,\ \{c,d\}:\neg\alpha\} \subseteq$ arguments($\Delta_i$) are the arguments for $\alpha$ and $\neg\alpha$ . Let $\{a,b\} = I$ and $\{c,d\} = K$. Therefore the debate can be summarised as $\langle\{I\},\{K\}\rangle : \alpha \in \blacklozenge(\alpha, \Delta_i)$. Thus $\Delta_i \vdash \bot, \alpha \notin \mathsf{judge}_j(\Delta_i)$ and $\neg\alpha \notin \mathsf{judge}_j(\Delta_i)$, where $\mathsf{judge}_j$ is akin to $\mathsf{judge}_{2.0}$ but also incorporates undercuts in some as yet undefined way.

There are three ways to resolve this inconsistency. All three approaches involve adding formulae $\Gamma$ to $\Delta_i$ to create a new $\Delta$, that is $\Delta = \Delta_i \cup \Gamma$. My goal, of resolving the inconsistency, is to have $\alpha \in \mathsf{judge}_j(\Delta)$ or $\neg\alpha \notin \mathsf{judge}_j(\Delta)$. Even with the judge resolving the inconsistency it can still be the case that $\Delta \vdash \bot$.

The three ways to resolve the inconsistency, which I describe relative to $\alpha$ are:

1. Add a confirmation to resolve the inconsistency. $\Delta = \Delta_i \cup \{e : \pi, f : \pi \rightarrow \alpha\}$, I add another argument for $\alpha$, namely $L : \alpha$, where $\{e, f\} = L$ so now $\langle\{I, L\},\{K\}\rangle : \alpha \in \blacklozenge(\alpha, \Delta)$, and $\alpha \in \mathsf{judge}_j(\Delta)$.

2. Add a rebuttal. Alternatively the inconsistency can be resolved using rebuttal of the argument for $\alpha$, that is an argument for $\neg\alpha$. $\Delta = \Delta_i \cup \{g : \phi, h : \phi \rightarrow \neg\alpha\}$ let $\{g, h\} = M$ and now $\langle\{I\},\{K, M\}\rangle : \alpha \in \blacklozenge(\alpha, \Delta)$, so $\neg\alpha \in \mathsf{judge}_j(\Delta)$.

3. Add an undercut as a third way to resolve the inconsistency. $\Delta = \Delta_i \cup \{j : \lambda\}$. If I assume full defeat then the argument $\{j\} : \lambda$ defeats $\{c, d\} : \neg\alpha$, leaving one argument for $\alpha$ and no viable argument against $\alpha$, so $\alpha \in \mathsf{judge}_j(\Delta)$.

So given an inconsistency that is based on a stalemate – basically the worst kind of inconsistency – then three separate ways to resolve it are to add a confirmation, a rebuttal or an undercut to the debate.

### 3.2.4 Pollock Undercut

The concept of undercut was added to the formal literature by Pollock with key papers on the subject stemming from 1970 (Pollock, 1970; Pollock, 1974; Pollock, 1987). To relate Pollock's defeasible logic description to the classical logic approach of this thesis, consider the deduction $\{P, P \rightarrow Q\} \vdash Q$. Here Pollock refers to $P$ as a 'premise', $Q$ as a 'conclusion' and $P \rightarrow Q$ as the 'connection between premise and conclusion'. He states that his undercut defeats this 'connection between premise and conclusion' so that '$P$ does not guarantee $Q$'. He writes the defeated $P \rightarrow Q$ as $P \otimes Q$. In my framework both $P$ and $P \rightarrow Q$ are assumptions and I allow for either to be subject to undercut. The form of a Pollock undercut with many assumptions is overtly documented in (Pollock, 1992) showing that the claim of the undercutter is the negation of the conjunction of premises. Thus a Pollock undercut rebuts some, i.e. one or more, assumptions.

While this style of undercut originated with Pollock it has been more recently and perhaps more precisely documented by others, for example (Elvang-Gøransson et al., 1993a) and (Schroeder, 1999). This is arguably the main definition in the literature for undercut, but to suggest a consensus would be inaccurate. The Pollock undercut has also been called defeat, e.g. (Krause & Clark, 1993) p231, however the more common definition of a defeater is that it is a collective term for undercuts and rebuttal, see for example the formalisation of defeaters in (Besnard & Hunter, 2001). Recall that Wigmore undercuts, in contrast, only allow undercuts that rebut one and only one assumption. Pollock's own definition does not involve labels or consistent or minimal premises, however in my framework the definition is as follows.

**Definition 3.2.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, and let $I : \alpha, J : \neg(\phi_1 \wedge \ldots \wedge \phi_n) \in \mathsf{arguments}(\Delta)$.*

*$J : \neg(\phi_1 \wedge \ldots \wedge \phi_n)$ is a* **Pollock undercut** *of $I : \alpha$ iff $\{\phi_1, \ldots, \phi_n\} \subseteq \mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta))$.*

I now give an example of an undercut that is a Pollock undercut, but not a Wigmore undercut.

**Example 3.2.5. Pollock undercut.** *Let $\Delta = \{a : \alpha, b : \alpha \to \gamma, c : \gamma \to \beta, d : \neg(\alpha \wedge (\gamma \to \beta))\}$. Then $\{a, b, c\} : \beta, \{d\} : \neg\alpha \vee \neg(\gamma \to \beta), \{d\} : \neg(\alpha \wedge (\gamma \to \beta)) \in \mathsf{arguments}(\Delta)$. The argument $\{d\} : \neg(\alpha \wedge (\gamma \to \beta))$ is a Pollock undercut, but not a Wigmore undercut, of the assumptions $a : \alpha$ and $c : \gamma \to \beta$ in the argument $\{a, b, c\} : \beta$.*

This example shows that the set of Pollock undercuts for an argument can be a superset of the set of Wigmore undercuts. All Wigmore undercuts are also Pollock undercuts, but not vice versa.

As shown with Wigmore undercuts, my use of labels can affect the number of Pollock undercuts, as illustrated in the following example. For the Pollock undercut without labels below I employ the definition of (Elvang-Gøransson et al., 1993a) combined with the notation of (Besnard & Hunter, 2001).

**Example 3.2.6. Pollock and labels.** *Let $\Delta = \{a : \alpha, b : \alpha \to \gamma, c : \gamma \to \beta, d : \neg(\alpha \wedge (\gamma \to \beta)), e : \alpha\}$. Then $\{a, b, c\} : \beta, \{e, b, c\} : \beta, \{d\} : \neg(\alpha \wedge (\gamma \to \beta)) \in \mathsf{arguments}(\Delta)$. With labels, it follows that the argument $\{d\} : \neg(\alpha \wedge (\alpha \to \beta))$ is a Pollock undercut of $\{a, b, c\} : \beta$ and also of $\{e, b, c\} : \beta$. Now without labels, let the knowledgebase of unlabelled formulae be $\Gamma = \{\alpha, \alpha \to \gamma, \gamma \to \beta, \neg(\alpha \wedge (\gamma \to \beta))\}$. Hence the two arguments of relevance are $\langle\{\alpha, \alpha \to \gamma, \gamma \to \beta, \}, \beta\rangle$ and $\langle\{\neg(\alpha \wedge (\gamma \to \beta))\}, \neg(\alpha \wedge (\gamma \to \beta))\rangle$. Therefore, $\langle\{\neg(\alpha \wedge (\gamma \to \beta))\}, \neg(\alpha \wedge (\gamma \to \beta))\rangle$ is a Pollock undercut of $\langle\{\alpha, \alpha \to \gamma, \gamma \to \beta, \}, \beta\rangle$. So with labels one attacking argument provides a Pollock undercut of two defending arguments. Without labels, however, the one attacking argument only undercuts only one defending argument.*

Thus the use of labels makes Pollock undercuts able to attack more arguments than would be the case without labels. I now turn to two problems with Pollock undercuts, namely that of multiple subsets and that of multiple claim orderings. I illustrate each of these problems with an example.

**Example 3.2.7. Multiple subsets.** *Let $\Delta = \{a : \alpha, b : \beta, c : (\alpha \vee \beta) \to \gamma, d : \neg\alpha\}$. The defending argument is $\{a, b, c\} : \gamma$. How many Pollock undercuts of $\{a, b, c\} : \gamma$ can be inferred from $\Delta$? There are multiple subsets to $\{a, b, c\}$, so all of $\{d\} : \neg\alpha$, $\{d\} : \neg(\alpha \wedge \beta)$, $\{d\} : \neg(\alpha \wedge \beta \wedge (\alpha \vee \beta) \to \gamma)$ are valid Pollock undercuts of the defending argument.*

How many of the above Pollock undercuts should be counted within a judge that counts arguments? Judge 2.0 is a judge that counts arguments, so how should it be extended to consider Pollock undercuts in its deliberations? Intuitively the above three Pollock undercuts are three manifestations of the same one underlying argument, so if all are counted would the error that accountants call 'double counting', whereby an item is counted more than once, occur?

**Example 3.2.8. Multiple orderings.** *Continuing with the same $\Delta$ and defending argument $\{a, b, c\} : \gamma$ as above. Even with the one attack $\{d\} : \neg(\alpha \wedge \beta \wedge (\alpha \vee \beta) \to \gamma)$, classical logic allows for many orderings* *of the formulae within the claim. Thus* $\{d\}:\neg(\beta \wedge \alpha \wedge (\alpha \vee \beta) \rightarrow \gamma), \{d\}:\neg(\alpha \wedge (\alpha \vee \beta) \rightarrow \gamma) \wedge \beta$ *are also valid Pollock undercuts of* $\{a, b, c\}:\gamma$.

The above example deepens the concerns raised by Example 3.2.7. In conclusion, this review of Wigmore and Pollock undercuts shows that when aggregating arguments in order to avoid double counting requires some well ordered or standard subset of undercuts that properly represents all other undercuts.

## 3.3 Canonical Undercut

I now come onto a third definition of undercut. It is more specific than the Pollock undercut, with the value of this specificity being to give more appropriate results in argument aggregation schemes. It cannot represent new undercuts beyond those expressible in the Pollock form. However there are situations where many well-formed Pollock undercuts are subsumed into a single canonical undercut. The aim of the canonical undercut is to provide an undercut that represents a variety of other, logically-equivalent, undercuts.

The definition of canonical undercut requires a definition of canonical enumeration, which is a minor syntactical device that is necessary (but not sufficient) to prevent double counting or redundancy in subsequent definitions.

**Definition 3.3.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae. Let every subset of* $\mathsf{stripAssumptions}(\Delta)$ *have an enumeration* $\langle \alpha_1, \ldots, \alpha_n \rangle$ *of its elements. This specific ordering, or enumeration, is called the* **canonical enumeration** *of the subset.*

This constraint of enumeration is satisfied whenever I impose an arbitrary total ordering over $\Delta$ and is thus not an onerous constraint. It is only a convenient way to indicate the order in which I assume the formulae in any subset of $\Delta$ are conjoined to make a formula logically equivalent to that subset.

This minor syntactical ordering is not intended to relay that some formulae are preferred over others. Preference-based ordering is a different topic, as addressed in such papers as (Cayrol et al., 1992; Amgoud & Cayrol, 1998), and is outside the scope of this thesis.

**Example 3.3.1. Canonical enumeration.** *The knowledgebase* $\Delta = \{a : \alpha, b : \beta, c : \gamma\}$ *has only one canonical form* $\langle a : \alpha, b : \beta, c : \gamma \rangle$. *This ordering over* $\Delta$ *dictates the ordering over all subsets of* $\Delta$. *The one canonical form is used to stand for all of the many other possible sequences* $\langle a : \alpha, c : \gamma, b : \beta \rangle, \ldots, \langle c:\gamma, b:\beta, a:\alpha \rangle$ *etc.*

Having defined canonical ordering I can now define canonical undercut. The definition, below, while based on (Besnard & Hunter, 2001), is in my notation and incorporates the use of labels. The claim of the undercutting argument is the negation of the conjunction of the assumptions of the undercut argument. In this approach a broader set of undercuts are allowed than with Wigmore as it includes cases where exactly which assumption or assumptions are negated is unknown.

**Definition 3.3.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $I : \alpha, J : \neg(\phi_1 \wedge \ldots \wedge \phi_n) \in \mathsf{arguments}(\Delta)$. *The set of* **canonical undercuts** *of* $I:\alpha$ *derivable from* $\Delta$, *denoted*

$\mathsf{canonicalUndercuts}(I\!:\!\alpha, \Delta)$, *is such that:*

$$\mathsf{canonicalUndercuts}(I\!:\!\alpha, \Delta) = \{J\!:\!\neg(\phi_1 \wedge \ldots \wedge \phi_n) \mid$$
$$\{\phi_1, \ldots, \phi_n\} = \mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta)) \textit{ and}$$
$$\langle\phi_1, \ldots, \phi_n\rangle \textit{ is the canonical enumeration of } \mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta))\}.$$

I call an argument $J : \neg(\phi_1 \wedge \ldots \wedge \phi_n)$ a **canonical undercut** of an argument $I : \alpha$ iff $J : \neg(\phi_1 \wedge \ldots \wedge \phi_n) \in \mathsf{canonicalUndercuts}(I\!:\!\alpha, \Delta)$. I call $I\!:\!\alpha$ the **defending argument** and $J\!:\!\neg(\phi_1 \wedge \ldots \wedge \phi_n)$ the **attacking argument**.

I now give an example of the increased generality relative to Wigmore undercut and increased precision relative to Pollock undercut. In the following examples I assume that the canonical enumeration is the same as the sequence that the set members are written in each definition of $\Delta$. While this detail could be added to each example it would make the text quite a bit longer and more cluttered without adding significant clarity.

**Example 3.3.2. Canonical undercut.** *Let* $\Delta = \{a : \alpha, b : \alpha \to \beta, c : \neg\alpha \vee \neg(\alpha \to \beta)\}$. *Let* $I = \{a, b\}, J = \{c\}$ *Then* $I\!:\!\beta, J\!:\!\neg(\alpha \wedge (\alpha \to \beta)) \in \mathsf{arguments}(\Delta)$. *The argument* $J\!:\!\neg(\alpha \wedge (\alpha \to \beta)) \in \mathsf{canonicalUndercuts}(I : \beta, \Delta)$ *is a canonical undercut, but not a Wigmore undercut, of the defending argument* $I : \beta$. *The argument* $J : \neg(\alpha \wedge (\alpha \to \beta))$ *is also one of many Pollock undercuts of the same defending argument* $I\!:\!\beta$.

The disjunction present in $\neg\alpha \vee \neg(\alpha \to \beta)$ means it could be one or the other, or both assumptions that are undercut.

**Proposition 3.3.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $J : \gamma, I : \alpha, J : \beta, K\!:\!\beta \in \mathsf{arguments}(\Delta)$.

*If* $J\!:\!\gamma$ *is a Wigmore undercut of* $I\!:\!\alpha$

*then there exists a* $K\!:\!\beta \in \mathsf{canonicalUndercuts}(I\!:\!\alpha, \Delta)$, *where* $K \subseteq J$,

*but not necessarily vice versa.*

*Proof.* Definition 3.2.1 states that $J\!:\!\neg\phi$ is a Wigmore undercut of $I\!:\!\alpha$ iff $\phi \in \mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta))$. Definition 3.3.2 gives the end point: $\mathsf{canonicalUndercuts}(I : \alpha, \Delta) = \{K : \neg(\phi_1 \wedge \ldots \wedge \phi_n) \mid \{\phi_1, \ldots, \phi_n\} = \mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta))$ and $\langle\phi_1, \ldots, \phi_n\rangle$ is the canonical enumeration of $\mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta))\}$. Let $\phi_i \in \{\phi_1, \ldots, \phi_n\} = \mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta))$. So $\gamma = \neg\phi_i$ and $J : \neg\phi_i$ is a Wigmore undercut of $I : \alpha$. Using $\vee$I and De Morgan $J : \neg\phi_i$ implies $J : \neg\phi_i \vee \phi_j$ implies $J : \neg\phi_i \vee \neg\phi_j$ implies $J : \neg\phi_1 \vee \cdots \vee \neg\phi_n$ implies $J : \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. However, $\Delta$ may be such that $K \subseteq J$ also implies $\neg(\phi_1 \wedge \cdots \wedge \phi_n)$ yielding $K\!:\!\neg(\phi_1 \wedge \cdots \wedge \phi_n)$. Such a $\Delta$ is $\Delta = \{a\!:\!\pi, b\!:\!\beta, c\!:\!\neg\pi \vee \neg\beta\}$ with $I\!:\!\alpha = \{a, b\}\!:\!\pi \wedge \beta$, a Wigmore undercut of $\{b, c\} : \neg\pi$ and a canonical undercut of $\{c\} : \neg(\pi \wedge \beta)$. Lastly, of the many orderings of $J : \neg(\phi_1 \wedge \cdots \wedge \phi_n)$ such as $J : \neg(\phi_2 \wedge \phi_1 \wedge \cdots \wedge \phi_n)$ or $J : \neg(\phi_n \wedge \cdots \wedge \phi_1)$ one of them will have the required canonical ordering, so a canonical undercut is guaranteed. Now for the vice versa. So although $\{\gamma\} \vdash \beta$, the rule $\{\psi \vee \pi\} \nvdash \psi$ means that, unless $|\mathsf{formulae}(I, \Delta)| = 1$, it is not the case that $\{\beta\} \vdash \gamma$. Thus, given $J:\beta \in \mathsf{canonicalUndercuts}(I:\theta, \Delta)$ is not always possible to infer a Wigmore undercut $J:\gamma$. So in conclusion if $J:\gamma$ is a Wigmore undercut of $I:\alpha$ then there exists a $K:\beta \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta)$, where $K \subseteq J$, but not necessarily vice versa. □

Likewise, given a Pollock undercut it is always possible to infer the existence of a canonical undercut and vice versa.

**Proposition 3.3.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $J:\alpha, I:\theta, K:\beta \in \mathsf{arguments}(\Delta)$.*

*There exists a Pollock undercut $J:\alpha$ of $I:\theta$*

*iff there exists $K:\beta \in \mathsf{canonicalUndercuts}(I:\theta, \Delta)$ where $K \subseteq J$.*

*Proof.* Contrasting Definition 3.2.2 of Pollock undercut with Definition 3.3.2 of canonical undercut shows two differences: i) canonicals must attack the full support and ii) the canonical ordering. So given the Pollock undercut $J:\alpha$ it is known that $\alpha = \neg \bigwedge \Phi$, where $\Phi \subseteq \mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta))$. Which ever subset $\Phi$ is started with it is always possible to use $\vee$I to extend $\alpha$ into $\beta$ where $\beta = \neg \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta))$. It should be noted (see the proof of Proposition 3.3.1) that as the full support is weaker than the subset it may not require all of $J$ to infer it, hence $K \subseteq J$, meaning $J:\beta \notin \mathsf{arguments}(\Delta)$ and $K:\beta \in \mathsf{arguments}(\Delta)$. Furthermore there are many orderings for the members of $\mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta))$, however one of them will always exist that will satisfy the canonical ordering of Definition 3.3.1. So if $J:\alpha$ a Pollock undercut of $I:\theta$ then $K:\beta \in \mathsf{canonicalUndercuts}(I:\theta, \Delta)$, where $K \subseteq J$. Now for the other direction: given $K:\beta \in \mathsf{canonicalUndercuts}(I:\theta, \Delta)$ it immediately follows that $K:\beta$ is also a Pollock undercut of $I:\theta$. Hence there exists $J:\alpha$ a Pollock undercut of $I:\theta$ iff there exists $K:\beta \in \mathsf{canonicalUndercuts}(I:\theta, \Delta)$ where $K \subseteq J$. □

The number of canonical undercuts is as follows:

**Proposition 3.3.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha \in \mathsf{arguments}(\Delta)$.*

$$0 \leq |\mathsf{canonicalUndercuts}(I:\alpha, \Delta)| < \mathsf{midLattice}(\Delta).$$

*Proof.* Some $\Delta$'s will not produce any attacks, for example when $\Delta \nvdash \bot$, so the lower bound is zero. All canonical undercuts of $I:\alpha$ will have the same claim, namely $\phi$, equal to $\neg(\bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta)))$. Thus for two canonical undercuts of $I:\alpha$, i.e. $K:\phi, L:\phi \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta)$, to be distinct they must different supports, so $K \neq L$. The number of different deduction supports derivable from $\Delta$ is $2^{|\Delta|}$; not all of these supports belong to valid arguments. The support of one undercut $K:\phi$ cannot be a subset of the support of another $L:\phi$, that is $K \not\subseteq L$ and $L \not\subseteq K$. The largest number of subsets of $\Delta$ with this property is given by the $\mathsf{midLattice}()$ function. $I:\phi$ can never be a canonical undercut of any argument $I:\alpha$, as then $\mathsf{formulae}(I, \Delta) \vdash \bot$ which is not allowed. The attacked support $I$ could occur anywhere in the power set of $2^{\Delta}$ attacking supports, that is a) in the bottom portion between $\emptyset$ and below midLattice(), b) in the upper portion above midLattice() up to $\Delta$, or c) on the midLattice() line. If $I$ is in a) it would remove many members of midLattice(). In b) members of midLattice() would invalidate $I$, or consequently many members of midLattice(). Thus the smallest reduction from midLattice() is case c) and the reduction is 1. Therefore the maximum number of canonical undercuts is $<$ midLattice($\Delta$). □

As an argument cannot attack itself, so $I:\alpha \notin$ canonicalUndercuts($I:\alpha, \Delta$). Now another property: if an argument A is undercut by B then B cannot undercut A's rebuttal C.

**Proposition 3.3.4.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\neg\alpha, K:\phi \in$ arguments($\Delta$).*

$$\textit{If } K:\phi \in \text{canonicalUndercuts}(I:\alpha,\Delta) \textit{ then } K:\phi \notin \text{canonicalUndercuts}(J:\neg\alpha,\Delta).$$

*Proof.* Clearly stripAssumptions(formulae($I,\Delta$)) $\vdash \alpha$, stripAssumptions(formulae($K,\Delta$)) $\vdash \phi$ and stripAssumptions(formulae($J,\Delta$)) $\vdash \neg\alpha$. Because $K:\phi \in$ canonicalUndercuts($I:\alpha,\Delta$) it is the case that $\{\neg\phi\} \vdash \alpha$. If the above proposition was false then it would also be the case that $\{\neg\phi\} \vdash \neg\alpha$. As $\neg\phi$ is equal to the negation of the conjunction of the members of stripAssumptions(formulae($I,\Delta$) it would follow that $J = I$ and $I:\alpha, I:\neg\alpha \in$ arguments($\Delta$), which is impossible, so the proposition is correct. □

Clearly the set of canonical undercuts of a single argument can be represented as a confirmation, where $Z:\phi =$ confirm(canonicalUndercuts($I:\alpha,\Delta$)) iff ($K:\phi \in$ canonicalUndercuts($I:\alpha,\Delta$) and $K \in Z$). Therefore the original definition of canonical undercut, Definition 3.3.2, could just as easily be made using confirmation notation.

In the definition of canonical undercut note the equality sign, meaning that every assumption within the undercut argument must appear in the claim of the undercutting argument. Pollock could also have every assumption in the claim, but the Pollock definition *only requires a subset.* What is interesting, however, is that even if one or more assumption is missing from the claim of an argument (that is the reference to the assumption is missing relative to a valid canonical undercut), these formulae can be added by $\vee$I. I call this process of adding formulae by $\vee$I, **growing** the claim. This 'growing' behaviour extends the disjunction of formulae in the claim of the attacking argument in order to round it out to match and attack the full support of the defending argument.

The support of a canonical undercut cannot be the empty set.

**Proposition 3.3.5.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\gamma \in$ arguments($\Delta$).*

$$\textit{If } J:\gamma \in \text{canonicalUndercuts}(I:\alpha,\Delta) \textit{ then } J \neq \emptyset.$$

*Proof.* If the support of the canonical undercut was the empty set, $J = \emptyset$, then the claim would be a tautology $\{\gamma\} \vdash \top$. From Definition 3.3.2 it is clear that $\neg\gamma$ is the conjunction of the members of stripAssumptions(formulae($I,\Delta$)). So, because $\neg\top \vdash \bot$, it follows that if $J = \emptyset$ then formulae($I,\Delta$) $\vdash$

$\perp$ which is impossible given $I:\alpha \in$ arguments($\Delta$). So the support of a canonical undercut can never be the empty set. □

### 3.3.1 The Effect on Labels in Canonical Undercut

I now turn to the use of labels in canonical undercut, looking at their role and effects. The use of labels can increase the number of canonical undercuts (and similarly the number of Wigmore and Pollock undercuts) compared to the situation without labels, as shown in the next two examples. The following three examples illustrate the effects of labels.

**Example 3.3.3. Labels add supports.** *Let* $\Delta = \{a:\alpha, b:\alpha \rightarrow \beta, c:\neg\alpha, d:\neg(\alpha \rightarrow \beta), e:\neg\alpha\}$. *Let* $I = \{a,b\}, J = \{c\}, K = \{d\}, L = \{e\}, M = \{e,d\}$. *Thus* $I:\beta, J:\neg(\alpha \wedge (\alpha \rightarrow \beta)), K:\neg(\alpha \wedge (\alpha \rightarrow \beta)), L:\neg\alpha, L:\neg(\alpha \wedge (\alpha \rightarrow \beta)), M:\neg\alpha \wedge \neg(\alpha \rightarrow \beta) \in$ arguments($\Delta$). *Then additional undercuts of* $I:\beta$ *include the Wigmore and Pollock undercut* $L:\neg\alpha$ *and the canonical undercut* $L:\neg(\alpha \wedge (\alpha \rightarrow \beta)) \in$ canonicalUndercuts($I:\beta, \Delta$). *Thus* $J:\neg(\alpha \wedge (\alpha \rightarrow \beta)), K:\neg(\alpha \wedge (\alpha \rightarrow \beta)), L:\neg(\alpha \wedge (\alpha \rightarrow \beta)) \in$ canonicalUndercuts($I:\beta, \Delta$). *These undercuts would not all occur without the use of labels, that is because* stripAssumption($c:\neg\alpha$) = stripAssumption($e:\neg\alpha$) = $\neg\alpha$.

The above example shows labels increasing the number of canonical undercuts by introducing more than one support. It is also possible for labels to increase the number of canonical undercuts by introducing more than one claim, as shown in the next example.

**Example 3.3.4. Labels add arguments.** *Let* $\Delta = \{a:\alpha, b:\alpha, c:\alpha \rightarrow \beta, d:\neg\alpha\}$. *Let* $I = \{a,c\}, J = \{b,c\}, K = \{d\}$. *Then* $I:\beta, J:\beta, K:\neg(\alpha \wedge (\alpha \rightarrow \beta)) \in$ arguments($\Delta$). *Consequently the one assumption* $d:\neg\alpha$*, which gives rise to one argument* $K:\neg(\alpha \wedge (\alpha \rightarrow \beta))$*, produces two canonical undercuts, namely* $K:\neg(\alpha \wedge (\alpha \rightarrow \beta)) \in$ canonicalUndercuts($I:\beta, \Delta$) *and* $K:\neg(\alpha \wedge (\alpha \rightarrow \beta)) \in$ canonicalUndercuts($J:\beta, \Delta$).

It is also possible for the labels to add no additional arguments relative to the non-label situation, however the labels could still be used to track which expert contributed which items of knowledge.

### 3.3.2 Reflexivity and Canonical Undercut

Can an argument be both a rebuttal and an undercut? The answer, somewhat surprisingly, is yes as it occurs for reflexive arguments.

**Example 3.3.5. Rebuttal = undercut.** *Let* $\Delta = \{a:\alpha, b:\neg\gamma, c:\alpha \rightarrow \gamma\}$. *Let* $I = \{a\}, J = \{b,c\}$. *Consequently* $I:\alpha, J:\neg\alpha \in$ arguments($\Delta$). *It is possible for an argument to be both a rebuttal and an undercut if the defending argument is a reflexive argument. Here* $I:\alpha$ *is reflexive. Thus* $J:\neg\alpha \in$ rebuttals($I:\alpha, \Delta$) *and* $J:\neg\alpha \in$ canonicalUndercuts($I:\alpha, \Delta$).

Intuitively one might think that undercut is an asymmetric relationship, i.e. if attacked, the attacked argument cannot attack back. Clearly rebuttal is always symmetric, i.e. if attacked by a rebuttal there is always a rebuttal counter attack. Can undercut ever be symmetric? The answer is yes as it occurs for reflexive arguments. Now, unlike Example 3.3.5, both arguments have to be reflexive.

**Example 3.3.6. Reflexive counterattack.** *Let* $\Delta = \{a : \alpha, b : \neg\alpha\}$. *Thus* $\{a\} : \alpha, \{b\} : \neg\alpha \in$ arguments$(\Delta)$. *It is possible for undercut to be reflexive;* $\{a\} : \alpha \in$ canonicalUndercuts$(\{b\} : \neg\alpha, \Delta)$ *and* $\{b\} : \neg\alpha \in$ canonicalUndercuts$(\{a\} : \alpha, \Delta)$.

So reflexive undercuts can act like rebuttals. These observations do raise the interesting question that if one was somehow tracking rebuttals and undercuts in a debate would one show an attack on a reflexive argument as both an undercuts and a rebuttal?

That concludes this discussion of canonical undercuts. The next section introduces an even more general definition of undercut, namely preclusive undercut. This definition continues the progression of generalisation described so far.

## 3.4 Introducing and Defining Preclusive Undercut

I now provide a new and arguably a somewhat more fundamental and general, definition of undercut. Just as canonical undercut provides a generalisation on the Wigmore undercut so also preclusive undercut provides a generalisation on canonical undercut. An informal way to describe preclusive undercuts is this: just as a canonical undercut is one argument undercutting another argument, so also a preclusive undercut is one confirmation undercutting another confirmation. Given that a confirmation is simply a set of arguments for the same claim and just as a canonical undercut is one argument attacking one argument, so also a preclusive undercut is a set of arguments attacking a set of arguments. Furthermore, in simple terms, just as a canonical undercut says 'one or more of your assumptions is wrong' so also a preclusive undercut says 'one or more of your arguments is wrong'.

The word preclusive is used because just as a canonical undercut precludes one or more premises, so also a preclusive undercut precludes one or more arguments. My definition, below, of preclusive undercut has two principle changes from canonical undercut. Firstly, I define the attacker to be a confirmation rather than limiting it to just an individual argument. Secondly I broaden the definition of the claim of the undercut, so that I accept undercuts that attack more than one argument, as shown in Example 3.4.1. Such undercuts identify that the targeted arguments are incompatible with each other, given the existence of the undercutter. The definition of a set of preclusive undercuts is closely related to i) the definition of canonical undercut, which is Definition 3.3.2 and ii) the definition of the set $\Diamond(\alpha, \Delta)$ of confirmations which is Definition 2.5.5. I now give the formal definition of preclusive undercut in two steps.

**Definition 3.4.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{X} : \alpha$, $\mathrm{Y} : \alpha$, $\mathrm{Z} : \neg(\phi_1 \wedge \ldots \wedge \phi_n) \in \Diamond(\Delta)$. *The set of* **candidate preclusive undercuts** *of* $\mathrm{Y} : \alpha$ *derivable from* $\Delta$, *denoted* candidatePreclusiveUndercuts$(\mathrm{Y} : \alpha, \Delta)$, *is such that:*

$$\begin{aligned}
&\text{candidatePreclusiveUndercuts}(\mathrm{Y}:\alpha,\Delta)\\
&= \{\langle \mathrm{Z}:\neg(\phi_1 \wedge \ldots \wedge \phi_n), \mathrm{X}, \mathrm{Y}:\alpha\rangle \mid \mathrm{Z} \neq \emptyset,\\
&\quad \emptyset \neq \mathrm{X} \subseteq \mathrm{Y}, \text{stripAssumptions}(\text{formulae}(\mathrm{X},\Delta)) = \{\phi_1,\ldots,\phi_n\} \textit{ and}\\
&\quad \textit{the canonical enumeration of } \{\phi_1,\ldots,\phi_n\} \textit{ is } \langle\phi_1,\ldots,\phi_n\rangle\}.
\end{aligned}$$

So now given the form of the candidate preclusive undercut I take the one with the minimal claim to be the preclusive undercut.

**Definition 3.4.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{W} : \alpha,\ \mathrm{X} : \alpha,\ \mathrm{Y} : \alpha,\ \mathrm{Z} : \phi, \mathrm{Z} : \psi \in \Diamond(\Delta)$. *The set of* **preclusive undercuts** *of* $\mathrm{Y} : \alpha$ *derivable from* $\Delta$, *denoted* $\mathsf{preclusiveUndercuts}(\mathrm{Y}:\alpha, \Delta)$, *is such that:*

$$\begin{aligned} &\mathsf{preclusiveUndercuts}(\mathrm{Y}:\alpha, \Delta) \\ &\quad = \{\langle \mathrm{Z}:\phi, \mathrm{X}, \mathrm{Y}:\alpha\rangle \in \mathsf{candidatePreclusiveUndercuts}(\mathrm{Y}:\alpha, \Delta) \mid \\ &\qquad\quad \textit{if}\ \langle \mathrm{Z}:\psi, \mathrm{W}, \mathrm{Y}:\alpha\rangle \in \mathsf{candidatePreclusiveUndercuts}(\mathrm{Y}:\alpha, \Delta) \\ &\qquad\qquad \textit{then}\ \mathrm{X} \subseteq \mathrm{W}\}. \end{aligned}$$

For a preclusive undercut between two confirmations, which is of the form of a 3-tuple $\langle \mathrm{Z} : \neg(\phi_1 \wedge \ldots \wedge \phi_n), \mathrm{X}, \mathrm{Y} : \alpha\rangle$, I call $\mathrm{Z} : \neg(\phi_1 \wedge \ldots \wedge \phi_n)$ the **attacking confirmation**, I call X the **target confirmation label** and I call $\mathrm{Y} : \alpha$ the **defending confirmation**. Clearly for any preclusive undercut, $\langle \mathrm{Z}:\neg(\phi_1 \wedge \ldots \wedge \phi_n), \mathrm{X}, \mathrm{Y}:\alpha\rangle$, it is always the case that $\mathrm{X} \subseteq \mathrm{Y}$. Given the target confirmation label is X and the defending confirmation is $\mathrm{Y} : \alpha$ then $\mathrm{X} : \alpha \in \Diamond(\Delta)$ is the **target confirmation**. At times I use the word 'preclusion' as a shortened form of 'preclusive undercut'.

Preclusive undercuts are more general than canonical undercuts as they allow undercuts which could not be previously represented, as shown in the following example. As before, I will continue to assume that the canonical enumeration condition is met, unless otherwise stated.

**Example 3.4.1. A preclusive undercut.** *Let* $\Delta = \{a : \alpha, b : \alpha \rightarrow \beta, c : \gamma, d : \gamma \rightarrow \beta, e : \neg(\alpha \wedge \gamma)\}$. *Then* $\{a, b\} : \beta, \{c, d\} : \beta, \{e\} : \neg\alpha \vee \neg\gamma, \{e\} : \neg(\alpha \wedge (\alpha \rightarrow \beta) \wedge \gamma \wedge (\gamma \rightarrow \beta)) \in \mathsf{arguments}(\Delta)$. *Let* $\langle \alpha, \alpha \rightarrow \beta, \gamma, \gamma \rightarrow \beta\rangle$ *be the canonical enumeration of* $\{\alpha, \alpha \rightarrow \beta, \gamma, \gamma \rightarrow \beta\}$. *Let* $\{a, b\} = I, \{c, d\} = J, \{e\} = K$, *then* $\{I, J\} : \beta, \{K\} : \neg(\alpha \wedge (\alpha \rightarrow \beta) \wedge \gamma \wedge (\gamma \rightarrow \beta)) \in \Diamond(\Delta)$. *A preclusive undercut of* $\{I, J\} : \beta$ *is* $\{K\} : \neg(\alpha \wedge (\alpha \rightarrow \beta) \wedge \gamma \wedge (\gamma \rightarrow \beta))$, *or, put more formally, that is* $\langle\{K\} : \neg(\alpha \wedge (\alpha \rightarrow \beta) \wedge \gamma \wedge (\gamma \rightarrow \beta)), \{I, J\}, \{I, J\} : \beta\rangle \in \mathsf{preclusiveUndercuts}(\{I, J\} : \beta, \Delta)$. *However* $\{K\} : \neg(\alpha \wedge (\alpha \rightarrow \beta) \wedge \gamma \wedge (\gamma \rightarrow \beta)) \notin \mathsf{canonicalUndercuts}(I : \beta, \Delta)$ *and likewise* $\{K\} : \neg(\alpha \wedge (\alpha \rightarrow \beta) \wedge \gamma \wedge (\gamma \rightarrow \beta)) \notin \mathsf{canonicalUndercuts}(J : \beta, \Delta)$ *as it is not possible to say that just one of the arguments* $I : \beta, J : \beta$ *is undercut. Thus* $\{K\} : \neg(\alpha \wedge (\alpha \rightarrow \beta) \wedge \gamma \wedge (\gamma \rightarrow \beta))$ *is not a Wigmore undercut of either* $I : \alpha$ *or* $J : \alpha$.

The second definition is necessary because given a candidate preclusive undercut that attacks X, $\vee$I will ensure many additional candidate preclusive undercuts that attack supersets of X. Because there is no requirement that a preclusive undercut is a maximal cardinality confirmation then it is possible to have many preclusive undercuts attacking the same target X, hence the need for the $\mathrm{X} = \mathrm{W}$ part of $\mathrm{X} \subseteq \mathrm{W}$.

The second definition covers the process of **invalidation**. In the above definition, if there exists a $\langle \mathrm{Z} : \psi, \mathrm{W}, \mathrm{Y} : \alpha\rangle$ such that $\mathrm{X} \subseteq \mathrm{W}$ then I call $\langle \mathrm{Z} : \phi, \mathrm{X}, \mathrm{Y} : \alpha\rangle$ an invalidated candidate preclusive undercut, because it is not a preclusive undercut. I call $\langle \mathrm{Z} : \psi, \mathrm{W}, \mathrm{Y} : \alpha \rangle$ the **invalidating preclusive undercut**.

Nowhere in the above two definitions does it insist that $\mathrm{Z}:\phi$ is a maximum cardinality confirmation. Hence if $\mathrm{Z} : \phi$ is an attacking confirmation then each $\mathrm{V} : \phi$, where $\emptyset \neq \mathrm{V} \subsetneq \mathrm{Z}$, is also an attacking confirmation. Also for a single defending confirmation there can be many target confirmations. For a single target confirmation X there will be one attacking claim, i.e. $|\mathsf{stripConfirmation}(\mathrm{Z} : \phi)| = 1$. Clearly, however, it is possible for $|\mathsf{label}(\mathrm{Z}:\phi)| \geq 1$.

From a perspective of the cardinality of the attack and the target, I identify four types of preclusive undercut:

- a **mono-target** preclusive undercut iff target $|\mathrm{X}| = 1$,
- a **multi-target** preclusive undercut iff target $|\mathrm{X}| > 1$,
- a **mono-attack** preclusive undercut iff attacking $|\mathrm{Z}| = 1$ and
- a **multi-attack** preclusive undercut iff attacking $|\mathrm{Z}| > 1$.

Referring to Definition 2.5.2 of self confirmation, all self-confirmed canonical undercuts are mono-target mono-attack preclusions, see Proposition 3.6.1. The self confirmation of a set containing many canonical undercuts, all with the same target, yields a mono-target multi-attack preclusion, see Proposition 3.6.2.

Preclusive undercuts, by definition (see Definitions 3.4.1 and 3.4.2), allow that the attacking, defending and target confirmations can be tautological, but not unfounded. Unfounded confirmations with $\mathrm{X} = \emptyset, \mathrm{Z} = \emptyset, \mathrm{W} = \emptyset, |\mathrm{X}| \neq 0, |\mathrm{Z}| \neq 0, |\mathrm{W}| \neq 0$ always exist and are used in professional debate only to make clear where there are no arguments for a claim, but by definition they are not parts of preclusions.

Only multi-target preclusive undercut candidates are subject to invalidation by invalidating confirmations. A multi-target preclusive undercut candidate may be invalidated either by another multi-target preclusion of lower target cardinality or by a mono-target preclusion (all subject to constraints being met).

Because $\mathrm{X} \subseteq \mathrm{Y}$, the defending confirmation $\mathrm{Y}:\alpha$ can be viewed as the whole front line of soldiers, perhaps many miles long and the target confirmation $\mathrm{X}:\alpha$ as the small section of the front that is actually targeted for attack and attacked. Thus, the arguments in the defending confirmation can all be potentially targeted, while the ones in the target confirmation are the ones that are effectively (or actually) targeted and attacked. It is the relationship between the attacking confirmation and the defending confirmation that is the preclusive undercut.

Having defined and given an example of preclusive undercut I now move immediately to their practical motivation.

## 3.5 Practical Motivation for Preclusive Undercut

I have refrained from giving practical motivation for preclusive undercut until now as I first needed the formal definition. The use of preclusive undercut is common in various professional activities such as in courts of law, political debates and detective investigations. My examples mostly come from current events, or news, at the time of writing. I am not intending to pass comment on the content of the arguments, but rather focus just on the form of the arguments. My focus is logical structure and in particular the presence of particular kinds of argument that arise in the real world and can be naturally modelled by preclusive undercut, but not by earlier forms of undercut.

Preclusive undercuts are often identifiable from the phrase 'they can't all be true', which is a succinct English description of the claim of a preclusive undercut. Here 'they' refers to the supports of the arguments in question, which equates to the set of assumptions $\{\phi_1, \ldots, \phi_n\}$. 'Can't' is the negation at the front of the claim. 'All' refers to the conjunctions joining the assumptions. Thus 'they can't all be true' means $\neg(\phi_1 \wedge \ldots \wedge \phi_n)$. Sometimes the larger phrase 'at a minimum they can't all be true' is used, as if more information were known it would be possible to say specifically which arguments were not true. Other phrases that identify preclusive undercuts are 'you can't have it both ways' and 'this constraint $x$ means that both $y$ and $z$ can't be the case'. The constraint $x$ is often time or money. Thus, for example, the witness's story cannot be true as there was not enough time $x$ for them to do both $y$ and $z$ as they claimed. Or, for example, that some political manifesto is not viable as there is not enough money $x$ to pay for both $y$ and $z$.

I now give motivating examples.

### 3.5.1 Incompatibilities due to Unsatisfied Constraints

A common place to find the use of preclusive undercuts is when someone's proposals are disputed on the grounds that they do not fit with constraints of time or money.

For example at the 2003 Conservative Party conference, their leader at that time, Mr Iain Duncan Smith made two key proposals that he argued together would lead to winning the next general election, as reported by the BBC and others on 9 October 2003. He stated that upon election he would a) definitely lower taxes rather than just maybe lower taxes and b) raise pensions by £11 a week. He argued that these promises would lead to his winning the election. I can refer to lowering taxes as $\beta$, winning the election as $\alpha$ and raising pensions as $\gamma$. Thus he has two consistent arguments $\{\beta, \beta \rightarrow \alpha\} \vdash \alpha$ and $\{\gamma, \gamma \rightarrow \alpha\} \vdash \alpha$, resulting in a confirmed argument for winning the election.

Subsequently his critics pointed out that the overall constraint of the government's financial budget meant that he could not pursue both promises. If I let the budget be $\theta$ then the critics, notably Work and Pensions Secretary Andrew Smith, are adding the formula of world knowledge $\theta \rightarrow \neg(\beta \wedge \gamma)$. Stating this more formally: I now have $\Delta = \{a : \beta, b : \beta \rightarrow \alpha, c : \gamma, d : \gamma \rightarrow \alpha, e : \theta, f : \theta \rightarrow \neg(\beta \wedge \gamma)\}$. Thus $\{a,b\} : \alpha, \{c,d\} : \alpha, \{e,f\} : \neg(\beta \wedge \gamma), \{e,f\} : \neg(\beta \wedge (\beta \rightarrow \alpha) \wedge \gamma \wedge (\gamma \rightarrow \alpha)) \in \text{arguments}(\Delta)$. The defending confirmation is $\{\{a,b\},\{c,d\}\} : \alpha$. The attacking confirmation is $\{\{e,f\}\} : \neg(\beta \wedge (\beta \rightarrow \alpha) \wedge \gamma \wedge (\gamma \rightarrow \alpha))$. The target and defending confirmations are $\{\{a,b\},\{c,d\}\} : \alpha$. So the preclusive undercut is:

$$\langle\{\{e,f\}\}:\neg(\beta \wedge (\beta \rightarrow \alpha) \wedge \gamma \wedge (\gamma \rightarrow \alpha)), \{\{a,b\},\{c,d\}\}, \{\{a,b\},\{c,d\}\}:\alpha\rangle$$

$$\in \textsf{preclusiveUndercuts}(\{\{a,b\},\{c,d\}\}:\alpha,\Delta).$$

A similar example of a preclusive undercut was raised at Governor of California Arnold Schwarzenegger after he proposed he would reduce the state's deficit and also reduce taxes. He promised to repeal a proposed increase in car tax with an implied cost to the state of $4Bn. As reported by the BBC and others on 8 October 2003, California State Treasurer Phil Angelides made the preclusive undercut: given that the state deficit is $8Bn today, repealing the car tax would inevitably raise rather than lower the deficit, indicating that Schwarzenegger could not successfully lower taxes and lower the deficit.

### 3.5.2 Sherlock Holmes Relies on Preclusive Undercuts

The underlying plot of many of the Sherlock Holmes detective stories, written by Sir Arthur Conan Doyle during the Victorian era, revolves around the existence of a preclusive undercut. They progress like this:

Sherlock Holmes is called in to investigate a murder. Holmes invariably starts by casting his net wide, gathering police interviews from all possible suspects. Naturally each of the, let us say three, suspects professes their innocence. The first suspect provides their evidence $b:\pi \in \Delta$, which when combined with world knowledge $c:\pi \rightarrow \alpha \in \Delta$, where $\alpha$ means innocence (effectively 'we are all innocent'), leads to the argument that the first suspect is innocent $\{b,c\}:\alpha \in \textsf{arguments}(\Delta)$. Similarly the second suspect gives their evidence to the police, $d:\psi \in \Delta$, which again combined with world knowledge $e:\psi \rightarrow \alpha \in \Delta$ shows their innocence $\{d,e\}:\alpha \in \textsf{arguments}(\Delta)$. Similarly for the third suspect, $f:\phi, g:\phi \rightarrow \alpha \in \Delta$ so $\{f,g\}:\alpha \in \textsf{arguments}(\Delta)$. Thus early in the investigation Holmes will frame the confirmation $\{\{b,c\},\{d,e\},\{f,g\}\}:\alpha$, i.e. the witness statements and world knowledge forming the defending confirmation.

What Holmes also knows from his evaluations is that this list of suspects is exhaustive and thus he knows $a:\neg(\pi \wedge \psi \wedge \phi) \in \Delta$. Therefore Holmes knows $\Delta = \{a:\neg(\pi \wedge \psi \wedge \phi), b:\pi, c:\pi \rightarrow \alpha, d:\psi, e:\psi \rightarrow \alpha, f:\phi, g:\phi \rightarrow \alpha\}$. From this constraint, combined with $\vee$I, the preclusive undercut arises, namely:

$$\langle\{\{a\}\}:\neg(\pi\wedge(\pi \rightarrow \alpha)\wedge\psi\wedge(\psi \rightarrow \alpha)\wedge\phi\wedge(\phi \rightarrow \alpha)), \{\{b,c\},\{d,e\},\{f,g\}\}, \{\{b,c\},\{d,e\},\{f,g\}\}:\alpha\rangle$$

$$\in \textsf{preclusiveUndercuts}(\{\{b,c\},\{d,e\},\{f,g\}\}:\alpha,\Delta).$$

At this stage he does not know which suspect is lying. Using his keen powers of observation, coupled with his various pieces of world knowledge (concerning the behaviour of cigar ash, strides, mud on shoes, writing on walls, etc.) he gathers some additional pieces of evidence (which he adds to $\Delta$). The way he describes his next step of logic is 'eliminate all the other factors and the one that remains must the truth'. In other words, the standard logic of $\vee$E: given $\neg\pi \vee \neg\psi \vee \neg\phi$, then establishing independent arguments confirming $\psi$ and $\phi$ implies that $\neg\pi$ must be the case.

In the murder mystery Holmes quietly gathers small details that eliminate two of the three suspects; the police then arrest the remaining suspect, who provided $b : \pi$. Subsequently Holmes explains to Watson that the matter was elementary.

### 3.5.3 Cross Examination Uses Tautological Preclusive Undercuts

In Section 2.5.1 I introduced the inconsistent confirmation and pointed out that it is problematic if used in professional debate. I argued the need for some construct to disallow the use of inconsistent confirmations. Preclusive undercut is the mechanism exactly fitting that requirement. It is, in fact, just a particular form of preclusive undercut that repudiates inconsistent confirmations, namely the tautological preclusion. I start this motivating example by showing the inconsistent confirmation.

During the Hutton enquiry in September 2003, (as reported by the BBC and others), a witness for the UK Government who worked in Human Relations was arguing that her conduct was in order, call it the claim $\alpha$. At one point she said that there was no 'naming procedure' for releasing the names of civil servants to the press, so she could be excused for not following such a procedure. Let this naming procedure be $\beta$ and thus her piece of evidence is $\neg\beta$. Her logic, building on an implied piece of world knowledge $\neg\beta \rightarrow \alpha$ was thus $\{\neg\beta, \neg\beta \rightarrow \alpha\} \vdash \alpha$.

Under cross examination at another point the same witness said that Dr Kelly would have known of the Government's naming procedure, call it claim $\gamma$ (i.e. $\gamma =$ 'Kelly knew $\beta$') and thus there was no need for her to brief him about his naming. Common sense or world knowledge provides $\gamma \rightarrow \beta$ (i.e. 'if Kelly knew the naming procedure then that implies a naming procedure exists') and thus the argument $\{\gamma, \gamma \rightarrow \beta\} \vdash \beta$. Her implied reasoning is still that her conduct was in order, this time implying the world knowledge $\beta \rightarrow \alpha$ and thus she is arguing $\{\gamma, \gamma \rightarrow \beta, \beta \rightarrow \alpha\} \vdash \alpha$.

Each of this witness's arguments has a valid minimal consistent subset of premises that deductively proves its conclusion. In lieu of undercuts the witness may even feel (naïvely) pleased that she has two arguments for $\alpha$ and thus $\alpha$ is confirmed. Also the barrister is not able to prove his goal of $\neg\alpha$. To formalise this reasoning, adding labels, $a{:}\neg\beta, b{:}\neg\beta \rightarrow \alpha, c{:}\gamma, d{:}\gamma \rightarrow \beta, e{:}\beta \rightarrow \alpha \in \Delta$ gives the defending confirmation is $\{\{a, b\}, \{c, d, e\}\} : \alpha$. What is clear, however, is that $\mathsf{formulae}(\{\{a, b\}, \{c, d, e\}\}, \Delta) \vdash \bot$. Cross examination has exposed that, taken together, her arguments are inconsistent, even though no individual argument is inconsistent.

Classical logic shows that not an inconsistency is a tautology. Therefore a preclusive undercut that attacks an inconsistent confirmation [i.e. that argues for $\neg(\bigwedge \mathsf{formulae}(\{\{a, b\}, \{c, d, e\}\}, \Delta))$] needs no assumptions. It is a self evident truth that the witness has contradicted herself. Thus:

$$\langle \emptyset{:}\neg(\neg\beta \wedge (\neg\beta \rightarrow \alpha) \wedge \gamma \wedge (\gamma \rightarrow \beta) \wedge (\beta \rightarrow \alpha)), \{\{a, b\}, \{c, d, e\}\}, \{\{a, b\}, \{c, d, e\}\}{:}\alpha\rangle$$

$$\in \mathsf{preclusiveUndercuts}(\{\{a, b\}, \{c, d, e\}\}{:}\alpha, \Delta).$$

What the barrister has done is undermine both her arguments. A common phrase is that both of the witnesses arguments have been brought into question, and usually both arguments are treated as defeated.

When a barrister cross examines a helpful witness (one on his own side) he aims to do the opposite, that is to introduce confirmations of the key points thereby bolstering the main arguments supporting his case. If a third argument for $\alpha$ was introduced, then the defeat of the above two would render $\alpha$ undefeated.

This discussion of cross examination has focussed on a contradiction in evidence from one witness. If more than one party is accused, or if the accused has witnesses to support their case, then labels are necessary. In this multiple-source labelled situation, contradictions in evidence, in practice, are even more likely to occur and thus preclusive undercuts are all the more common. Again in the Hutton enquiry, under cross examination from the barrister Gompertz, Queens Council for the Kelly family, incompatibilities were exposed between the evidence of the Defence Secretary Mr Geoffrey Hoon and that of his aides on who was present at the meeting that decided on the Dr Kelly naming strategy. The logic is fundamentally the same as for the single witness case.

### 3.5.4 Tautological Preclusions are 'Just Asking Questions'

To conclude this section on motivation for preclusive undercuts I observe that there is an interesting connection between preclusive undercuts and the asking of questions.

A tautological preclusive undercut can be phrased as a question. For example one could ask: 'please can you clarify how, as you say, on the one hand Dr Kelly would have known the Government's naming procedure, while on the other hand you say that such a procedure did not exist?' Such a question tends to have an innocent tone as it requires no evidence to argue its claim, but rather is pointing out the self-evident truth that an inconsistency is present.

Politicians and debaters often prefer asking questions to making direct statements of attack, because a question cannot be rebutted or undercut, whereas as a direct statement can. This observation is present in the classical logic as not a tautology is a contradiction. Thus, any attempt to rebut or undercut a tautological preclusion would require an inconsistency and hence would not be a valid argument. If the topic of debate is highly controversial and opposition is not well tolerated or well received then asking questions is all the more common.

It can be the case that the barrister has unearthed an apparent contradiction, but one that, in truth, has a rational explanation. Even this situation can be turned to advantage as such an answer provides further formulae in $\Delta$, with the possible result that inconsistency is introduced. It is not uncommon that while a witness tries to hide an inconsistency by introducing falsehoods they 'tie themselves in knots' and supply further contradictions.

Two examples of political commentators 'just innocently asking questions' on highly controversial topics are 1) the article in the Guardian on 6 September 2003 by Michael Meacher, Member of Parliament and former Minister for the Environment, entitled 'This war on terrorism is bogus' and 2) the first chapter of the book by Michael Moore 'Dude, Where's My Country' regarding events of 11 September 2001. I cite these two controversial examples because of their precise use of logic in 'just asking questions' to imply preclusive undercuts attacking the claims of their opponents.

Each of these questions stems from an inconsistency found between the opponent's statements and reports in reputable news sources. These question takes the form of 'please explain this apparent inconsistency', thereby politely allowing for the case that there is a reasonable explanation that the questioner was unaware of. Also as the subject matter of the questioning is so politically contentious the questioner avoids being subject to criticism as they are making no direct accusations.

## 3.6 Preclusive Undercut Generalises Canonical Undercut

Canonical undercuts are a special case of preclusive undercuts, or to reverse that, preclusive undercuts are a generalisation of canonical undercuts – a relationship which I now clarify. Every canonical undercut has an equivalent preclusive undercut, but not always vice versa. I examine this relationship in four stages:

1. a mono-attack mono-target preclusive undercut has an equivalent canonical undercut,
2. a multi-attack mono-target preclusive undercut has an equivalent set of canonical undercuts,
3. a mono-attack multi-target preclusive undercut has no equivalent canonical undercuts and
4. a multi-attack multi-target preclusive undercut has no equivalent canonical undercuts.

So first, the mono-target mono-attack preclusive undercut situation:

**Proposition 3.6.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $I : \alpha, J : \phi \in$ $\mathsf{arguments}(\Delta)$ and let $\{I\}:\alpha, \{J\}:\phi \in \Diamond(\Delta)$.*

$$J:\phi \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta)$$
$$\textit{iff } \langle\{J\}:\phi, \{I\}, \{I\}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\{I\}:\alpha, \Delta).$$

*Proof.* Given that $J:\phi \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta)$, each condition of Definitions 3.4.1 and 3.4.2 of preclusive undercut will have to be met for $\langle\{J\}:\phi, \{I\}, \{I\}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\{I\}:\alpha, \Delta)$. A canonical undercut, by Definition 3.3.2, is one argument $J : \phi$ attacking one argument $I : \alpha$. Let $\mathrm{Z} = \{J\}$ and $\mathrm{Y} = \{I\}$. So $\mathsf{confirm}(J : \phi) = \{J\} : \phi = \mathrm{Z} : \phi$ is a self confirmation, which means $|\mathsf{label}(\mathrm{Z} : \phi)| = 1$ and $\mathrm{Z} \neq \emptyset$, thereby meeting the mono-attack condition. Likewise for the defending confirmation $\mathsf{confirm}(I : \alpha) = \{I\} : \alpha = \mathrm{Y} : \alpha$ is another self confirmation, which means also $|\mathsf{label}(\mathrm{Y} : \alpha)| = 1$. Thus the target X must be $\mathrm{X} = \mathrm{Y}$ as that is the only value that satisfies $\emptyset \neq \mathrm{X} \subseteq \mathrm{Y}$, meaning mono-target. Consequently there cannot exist any invalidating confirmation as this candidate meets the $\mathrm{X} \subseteq \mathrm{W}$ condition, so the conditions of Definition 3.4.2 do not exclude any candidate preclusion. All that remains now is the latter part of of Definitions 3.4.1. Given that $\mathsf{stripAssumptions}(\mathsf{formulae}(\mathrm{X}, \Delta)) = \mathsf{stripAssumptions}(\mathsf{formulae}(\{J\}, \Delta)) =$ $\mathsf{stripAssumptions}(\mathsf{formulae}(J, \Delta))$ is the same for confirmations and individual arguments, the strip and canonical ordering conditions are identical for both canonical and preclusive undercuts; so there is no need to analyse the expansion of $\phi$. Thus given $J : \phi \in \mathsf{canonicalUndercuts}(I : \alpha, \Delta)$ it follows that $\langle\mathsf{confirm}(J : \phi), \mathsf{label}(\mathsf{confirm}(I : \alpha)), \mathsf{confirm}(I : \alpha)\rangle \in \mathsf{preclusiveUndercuts}(\mathsf{confirm}(I : \alpha), \Delta)$ so $\langle\{J\}:\phi, \{I\}, \{I\}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\{I\}:\alpha, \Delta)$. Given the proof so far, the other *iff* direction is simple. The attacking argument inside $\mathrm{Z}:\phi$ is $J:\phi$ and the defending argument inside $\mathrm{Y}:\alpha$ is $I:\alpha$. Because the strip and canonical ordering conditions of Definition 3.4.1 are met it follows that the claim and enumeration conditions for canonical undercut are also met. Thus $J:\phi \in \mathsf{canonicalUndercuts}(I:\alpha,\Delta)$ iff $\langle\{J\}:\phi,\{I\},\{I\}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\{I\}:\alpha,\Delta)$. □

Thus every canonical undercut has one and only one equivalent preclusive undercut, and that preclusive undercut will be a mono-attack mono-target preclusive undercut. This equivalent preclusive undercut may not be a maximum cardinality preclusive undercut, but all the same it will be a perfectly valid preclusive undercut.

Now I look at the second of the four situations, that of multi-attack mono-target preclusive undercuts. Every mono-target preclusive undercut has an equivalent set of canonical undercuts all with the same claim. In other words, there exists a mono-target preclusive undercut iff there exists a set of canonical undercuts all with the same claim.

**Proposition 3.6.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $I:\alpha, J_1:\phi,\ldots,J_n:\phi \in \mathsf{arguments}(\Delta)$ and let $n \in \mathbb{N}$.*

$$\langle\{J_1,\ldots,J_n\}:\phi,\{I\},\{I\}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\{I\}:\alpha,\Delta)$$
$$\textit{iff}\ \ J_1:\phi,\ldots,J_n:\phi \in \mathsf{canonicalUndercuts}(I:\alpha,\Delta).$$

*Proof.* Proposition 3.6.1 has established that each $J_i : \phi \in \{J_1 : \phi,\ldots,J_n : \phi\}$, where $J_i : \phi \in \mathsf{canonicalUndercuts}(I : \alpha,\Delta)$, will mandate the existence of $\langle\{J_i\} : \phi,\{I\},\{I\} : \alpha\rangle \in \mathsf{preclusiveUndercuts}(\{I\} : \alpha,\Delta)$ and vice versa. The next step is to use Proposition 2.5.2 that many self-confirming confirmations for the same claim, i.e. $\{J_1\}:\phi,\ldots,\{J_n\}:\phi$ are equivalent to a single confirmation, $\{J_1,\ldots,J_n\}:\phi$, for that same claim and vice versa. Thus $\langle\{J_1,\ldots,J_n\}:\phi,\{I\},\{I\}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\{I\} : \alpha,\Delta)$ iff $\langle\{J_1\} : \phi,\{I\},\{I\} : \alpha\rangle,\ldots,\langle\{J_n\} : \phi,\{I\},\{I\} : \alpha\rangle \in \mathsf{preclusiveUndercuts}(\{I\}:\alpha,\Delta)$ iff $J_1:\phi,\ldots,J_n:\phi \in \mathsf{canonicalUndercuts}(I:\alpha,\Delta)$. □

Multi-target preclusive undercuts are the preclusive undercuts that do not have equivalent canonical undercuts. The contrary, that mono-target preclusive undercuts have equivalent canonical undercuts, is Proposition 3.6.1.

**Proposition 3.6.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $I:\alpha, J_1:\phi,\ldots,J_n:\phi \in \mathsf{arguments}(\Delta)$, let $\mathrm{X}:\alpha, \mathrm{Y}:\alpha \in \lozenge(\Delta)$ and let $n \in \mathbb{N}$.*

$$\textit{If}\ \langle\{J_1,\ldots,J_n\}:\phi,\mathrm{X},\mathrm{Y}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Y}:\alpha,\Delta)\ \textit{such that}\ |\mathrm{X}| > 1$$
$$\textit{then}\ \ J_1:\phi,\ldots,J_n:\phi \notin \mathsf{canonicalUndercuts}(I:\alpha,\Delta)\ \textit{where}\ I \in \mathrm{X}.$$

*Proof.* If the target confirmation $\mathrm{X}:\alpha$ is such that $|\mathsf{label}(\mathrm{X}:\alpha)| > 1$ then there does not exist an $I:\alpha$ such that $\mathsf{confirm}(I:\alpha) = \mathrm{X}:\alpha$. Instead it will be the case that $\mathrm{X} = \{I_1,\ldots,I_m\}$ where $m \geq 2$. It also follows, from Definition 3.4.2, that $\phi = \phi_1\wedge\ldots\wedge\phi_p = \neg(\bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\{I_1,\ldots,I_n\},\Delta)))$ with $m \geq 2$ and $p \geq 2$. So any argument or arguments with this claim of $\phi = \phi_1 \wedge \ldots \wedge \phi_m$ will be attacking more than one argument and consequently cannot comply with Definition 3.3.2 of canonical undercut. So if $\langle\{J_1,\ldots,J_n\} : \phi,\mathrm{X},\mathrm{Y} : \alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Y} : \alpha,\Delta)$, where $|\mathrm{X}| > 1$ then $J_1:\phi,\ldots,J_n:\phi \notin \mathsf{canonicalUndercuts}(I:\alpha,\Delta)$ where $I \in \mathrm{X}$. □

In the above proposition if $n = 1$ I have mono-attack, multi-target preclusions. If $n > 1$ I have multi-attack, multi-target preclusions. Thus I have just addressed both the third and fourth situation outlined at the start of this section.

A preclusive undercut is either mono-target or multi-target (it cannot be both or neither) so this proposition makes is clear exactly which preclusive undercuts have equivalent canonicals and which do not. The first input parameter to a canonical undercut is a single argument, not a confirmation. So, while a canonical undercut can only attack one argument, a preclusive undercut has the added ability to attack multiple argument. Nonetheless, preclusive undercut still retains all of the properties of canonical undercut – none are lost in this generalisation. It is this ability to attack multiple arguments that allows preclusive undercut to attack inconsistent confirmations.

## 3.7 Tautological Preclusions Attack Inconsistent Confirmations

Preclusive undercuts prevent a debater or debating team from being inconsistent. Exposing of inconsistencies in a debater's arguments is part of cross-examination. Specifically, it is the tautological preclusive undercut which attack the inconsistent confirmations of Section 2.5.1. The following example shows that it is possible to start with a tautology, extend it with $\vee$I and end up with the full claim of a multi-target preclusive undercut that attacks an inconsistent confirmation.

**Example 3.7.1. Tautological undercut** *Let* $\Delta = \{a:\beta, b:\beta \rightarrow \alpha, c:\pi \rightarrow \alpha, d:\neg\beta \wedge \pi\}$. *Let* $I = \{a,b\}, J = \{c,d\}, K = \emptyset$. *Thus* $I:\alpha, J:\alpha, K:\beta \vee \neg\beta, K:\neg\beta \vee \beta \vee \neg\pi, K:\neg\beta \vee \neg(\neg\beta \wedge \pi), K:\neg(\beta \wedge (\beta \rightarrow \alpha) \wedge (\pi \rightarrow \alpha) \wedge (\neg\beta \wedge \pi)) \in \text{arguments}(\Delta)$. *Therefore* $\{I,J\}:\alpha, \{K\}:\neg(\beta \wedge (\beta \rightarrow \alpha) \wedge (\pi \rightarrow \alpha) \wedge (\neg\beta \wedge \pi)) \in \Diamond(\Delta)$. *Consequently* $\langle\{K\}:\neg(\beta \wedge (\beta \rightarrow \alpha) \wedge (\pi \rightarrow \alpha) \wedge (\neg\beta \wedge \pi)), \{I,J\}, \{I,J\}:\alpha\rangle \in \text{preclusiveUndercuts}(\{I,J\}:\alpha, \Delta)$. *Here* $\{I,J\}:\alpha$ *is an inconsistent confirmation and* $\langle\{K\}:\neg(\beta \wedge (\beta \rightarrow \alpha) \wedge (\pi \rightarrow \alpha) \wedge (\neg\beta \wedge \pi)), \{I,J\}, \{I,J\}:\alpha\rangle$ *is a tautological preclusive undercut.*

A tautological preclusive undercut must have an attacking confirmation containing only one argument. That one argument must have a support containing no assumptions. Hence for an attacking confirmation $\mathrm{Z}:\phi$ it is not the case that $\mathrm{Z} \neq \emptyset$, but rather $\mathrm{Z} = \{\emptyset\} = \{\emptyset\}$ so $|\mathrm{Z}| = 1$ and not the zero of $|\emptyset| = 0$. Using this approach a debating team can argue that it is self evident that the other side has contradicted itself. It was established in Proposition 2.5.1 that for all $\alpha, \{\emptyset\}:\alpha$ is a confirmation, iff $\alpha \vdash \top$. Therefore given that $\Delta$ is a knowledgebase of labelled assumption formulae, $J \in \text{arguments}(\Delta)$ and $\mathrm{W}:\alpha, \mathrm{Y}:\alpha, \mathrm{Z}:\gamma \in \Diamond(\Delta)$, then it follows immediately that if $\langle\mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\alpha\rangle \in \text{preclusiveUndercuts}(\mathrm{Y}:\alpha, \Delta)$ with $J \in \mathrm{Z}$ and $J = \emptyset$ then $|\text{label}(\mathrm{Z}:\gamma)| = 1$.

A tautological undercut must thus be attacking a target confirmation of at least two arguments.

**Proposition 3.7.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{W}:\alpha, \mathrm{Y}:\alpha, \mathrm{Z}:\gamma \in \Diamond(\Delta)$.

*If* $\langle\mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\alpha\rangle \in \text{preclusiveUndercuts}(\mathrm{Y}:\alpha, \Delta)$, *where* $\mathrm{Z} = \{\emptyset\}$ *then* $|\text{label}(\mathrm{W}:\alpha)| \geq 2$.

*Proof.* Suppose instead that $|\mathsf{label}(\mathrm{W} : \alpha)| = 1$. Clearly $\gamma$ is a tautology. Also known is that $\gamma$ is the negation of the conjunction of $\mathsf{stripAssumptions}(\mathsf{formulae}(\mathrm{W}, \Delta))$. Clearly $\{\neg\top\} \vdash \bot$ so if indeed $|\mathsf{label}(\mathrm{W}:\alpha)| = 1$ then $\mathsf{stripAssumptions}(\mathsf{formulae}(\mathrm{W}, \Delta)) \vdash \bot$. If that was the case then the support of an argument $J : \alpha$ where $J \in \mathrm{W}$, would be inconsistent, $\mathsf{stripAssumptions}(\mathsf{formulae}(J, \Delta)) \vdash \bot$ and thus $J : \alpha$ would not be a valid argument. So $|\mathsf{label}(\mathrm{W} : \alpha)| = 1$ cannot be the case. Any other value of $|\mathsf{label}(\mathrm{W}:\alpha)|$ would not require any individual argument $J \in \mathrm{W}$ to be inconsistent so it can be concluded that $|\mathsf{label}(\mathrm{W}:\alpha)| \geq 2$. □

Thus tautological undercuts cannot be canonical undercuts and must be preclusive undercuts. This point, that the support of any canonical undercut $J : \gamma \in \mathsf{canonicalUndercuts}(I : \alpha, \Delta)$ is such that $J \neq \emptyset$, was established earlier in Proposition 3.3.5. Thus I can unambiguously use the term tautological undercut to refer to a tautological preclusive undercut.

From the definition of preclusive undercut, Definition 3.4.2, the terminology has been established that the target confirmation label $\mathsf{label}(\mathrm{X} : \alpha)$ is always a subset or equal to the defending confirmation label $\mathsf{label}(\mathrm{Y} : \alpha)$, i.e. $\mathrm{X} \subseteq \mathrm{Y}$. Therefore a tautological undercut must be attacking a defending confirmation of at least two arguments.

To further the analysis of preclusive undercut it is useful to have two helper functions as follows:

**Definition 3.7.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\langle \mathrm{Z} : \phi, \mathrm{X}, \mathrm{Y} : \alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Y} : \alpha, \Delta)$. *The attacker functions, denoted* $\mathsf{attacker}(\langle \mathrm{Z} : \phi, \mathrm{X}, \mathrm{Y} : \alpha\rangle)$ *and* $\mathsf{attacker}(\{\langle \mathrm{Z}:\phi, \mathrm{X}, \mathrm{Y}:\alpha\rangle\})$, *given a preclusive undercut or set of preclusive undercuts return the attacking confirmation or set of attacking confirmations, such that:*

$$\begin{aligned}\mathsf{attacker}(\langle \mathrm{Z}:\phi, \mathrm{X}, \mathrm{Y}:\alpha\rangle) &= \mathrm{Z}:\phi.\\ \mathsf{attacker}(\{\langle \mathrm{Z}:\phi, \mathrm{X}, \mathrm{Y}:\alpha\rangle\}) &= \{\mathrm{Z}:\phi\}.\end{aligned}$$

If there exists an inconsistency between two arguments of one confirmation then the existence of the tautological undercut of those two inconsistent undercuts is independent of $\Delta$. Thus given an inconsistent confirmation, the tautological preclusive undercut is automatically and always introduced into the debate without needing to reference $\Delta$. It happens for all knowledgebases.

One might be tempted to record the following relationship using iff: there exists an inconsistent confirmation $\mathrm{Y} : \alpha$ iff there exists a tautological preclusion $\mathrm{Z} : \phi \in \{\mathsf{attacker}(\langle \mathrm{Z} : \phi, \mathrm{X}, \mathrm{Y} : \alpha\rangle) \mid \langle \mathrm{Z} : \phi, \mathrm{X}, \mathrm{Y}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Y}:\alpha, \Delta), J \in \mathrm{Y} \text{ and } J = \emptyset\}$. However, such a biconditional is not helpful as it has $\mathrm{Y} : \alpha$ on both sides. More realistically, the confirmation $\mathrm{Z} : \phi$ always exists as it is a tautology, but the confirmation $\mathrm{Y}:\alpha$ only sometimes exists – it is a matter of $\Delta$. So such a biconditional would only be saying that if $\mathrm{Y} : \alpha$ exists it will always be subject to undercut – which is a conditional, not a biconditional.

**Proposition 3.7.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{V}:\alpha, \mathrm{W}:\alpha, \mathrm{Y}:\alpha, \mathrm{Z}:\gamma \in \Diamond(\Delta)$.

*For any confirmation* $\mathrm{Y} : \alpha$, *if there exists a* $\mathrm{W} \subseteq \mathrm{Y}$ *such that* $\mathsf{formulae}(\mathrm{W}, \Delta) \vdash \bot$ *and there* *does not exist a* $\mathrm{V} \subseteq \mathrm{W}$ *such that* $\mathsf{formulae}(\mathrm{V}, \Delta) \vdash \perp$*, then there exists a* $\langle \mathrm{Z} : \gamma, \mathrm{W}, \mathrm{Y} : \alpha \rangle \in$ $\mathsf{preclusiveUndercuts}(\mathrm{Y} : \alpha, \Delta)$.

*Proof.* The proposition starts with by ensuring $\mathrm{Y} : \alpha$ is an inconsistent confirmation. However, the set $\mathsf{formulae}(Y, \Delta)$ may not be minimal with respect to $\mathsf{formulae}(\mathrm{Y}, \Delta) \vdash \perp$. Thus it is necessary to ensure the non-existence of $\mathrm{V} \subseteq \mathrm{W}$ with $\mathsf{formulae}(V, \Delta) \vdash \perp$, so it follows that $\mathsf{formulae}(\mathrm{V}, \Delta)$ is minimal for proving $\perp$. Establishing this minimality is essential as otherwise $\mathrm{V} : \alpha$ would be an invalidating target confirmation for the preclusive undercut $\langle \mathrm{Z} : \gamma, \mathrm{W}, \mathrm{Y} : \alpha \rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Y} : \alpha, \Delta)$. So given that the target confirmation $\mathrm{W} : \alpha$ is a minimal inconsistent confirmation it follows that the tautological preclusive undercut $\langle \mathrm{Z} : \gamma, \mathrm{W}, \mathrm{Y} : \alpha \rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Y} : \alpha, \Delta)$ must exist. □

One might think that the supports of only two arguments would combine to make an inconsistency, but that is not the case. The supports of any number of arguments can combine and be necessary to make an inconsistency. However that number must be greater than one because the definition of mincon arguments prevents having an inconsistent support.

Once a debating team has established the other side has contradicted itself then it is an interesting question as to the interpretation of that result. All of the arguments in the target confirmation have been brought into question, however no single one of them has been shown to be defeated. A common theme in legal argumentation is that the entire set of inconsistent arguments is defeated. An even more extreme legal position, albeit a popular one, is that the tautological undercut shows that everything the other team says is untrustworthy. This extreme position is not logically justifiable as it is an argument that generalises from the specific to the universal.

A tautological confirmation (which can act as the attacking confirmation in a tautological preclusive undercut) cannot be subject to a preclusive undercut.

**Proposition 3.7.3.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{Y} : \alpha \in \Diamond(\Delta)$.

$$\textit{If } \{\alpha\} \vdash \top \textit{ then } \mathsf{preclusiveUndercuts}(\mathrm{Y} : \alpha, \Delta) = \emptyset.$$

*Proof.* If $\mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(\mathrm{Y} : \alpha), \Delta))) \vdash \top$ then it follows that $\mathrm{Y} : \alpha$ is a tautological confirmation, such that $I \in \mathrm{Y}, I = \emptyset$ and $|\mathrm{Y}| = 1$, i.e. $\vdash \alpha$. Consequently any preclusive undercut with a target or defending confirmation of this $\mathrm{Y} : \alpha$ will require a claim of $\neg(\bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta)))$. However such a claim is not allowed for any argument as that argument would be inconsistent. Thus if $\alpha$ is a tautology then $\mathsf{preclusiveUndercuts}(\mathrm{Y} : \alpha, \Delta) = \emptyset$. □

Thus for a debater, the tautological preclusion is a useful tool. It has the unique benefits that a) it can never be counter attacked and b) it is always available regardless of the contents of $\Delta$. It is thus effective in preventing an opposing team from using inconsistent confirmations. So while one could directly constrain confirmations to be consistent, the preclusive undercut does exactly that job and does it more elegantly. The preclusion is more elegant as a) it directly mirrors professional debating and b) it has additional purposes in also undercutting consistent confirmations.

## 3.8 Non-unique Formulae, Labels and Preclusive Undercut

I now discuss cardinality of the preclusive undercut function as a starting point for looking at the effect of labels and non-unique assumption formulae.

Whereas with canonical undercuts the statement $J : \beta \in \mathsf{canonicalUndercuts}(I : \alpha, \Delta)$ specifies one and only one canonical undercut, the situation with preclusive undercuts is different. The statement $\langle \mathrm{Z}:\beta, \mathrm{X}, \mathrm{Y}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Y}:\alpha, \Delta)$ specifies one or more preclusive undercuts. The distinguishing factor between these many preclusive undercuts is that they have different target confirmations, X. Thus while a canonical undercut is a relationship between two arguments, a preclusive undercut is a relationship between three confirmations.

**Proposition 3.8.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let* $\mathrm{X}:\alpha, \mathrm{Y}:\alpha \in \Diamond(\Delta)$.

$$0 \leq |\mathsf{preclusiveUndercuts}(\mathrm{Y}:\alpha, \Delta))\}| \leq (2^{|\mathrm{Y}|} - 1) \cdot (2^{\mathsf{midLattice}(\Delta)} - 1).$$

*Proof.* The label of the defending confirmation $\mathrm{Y} : \alpha$ can contain many target confirmation labels X as $\mathrm{X} \subseteq \mathrm{Y}$. The Definition 3.4.1 of preclusive undercut requires $\emptyset \neq \mathrm{X}$. Thus the maximum number of target confirmations for a given defending confirmation $\mathrm{Y}:\alpha$ is $2^{|\mathrm{Y}|} - 1$, with the $-1$ due to the $\emptyset$ constraint. Given one target confirmations $\mathrm{X}:\alpha$ the next challenge is to find the number of attacking confirmations $\mathrm{Z} : \phi$. So how many confirmations are derivable from $\Delta$ that imply this $\phi$? Proposition 2.5.7 states that $1 \leq |\Diamond(\phi, \Delta)| \leq 2^{\mathsf{midLattice}(\Delta)}$. A refinement is needed because the unfounded confirmation is not allowed, again by Definition 3.4.1 , to ensure $\emptyset : \phi \neq \mathrm{Z} : \phi$ and thus it is necessary to deduct 1. Combining these two results means that the maximum number of preclusive undercuts for $\mathrm{Y} : \alpha$ is $(2^{|\mathrm{Y}|} - 1) \cdot (2^{\mathsf{midLattice}(\Delta)} - 1)$. The smallest number of attacking confirmations is zero if $\Delta \nvdash \bot$, so the lower limit is zero. Furthermore, if non-unique formulae exist in $\Delta$ it will not effect X, Y or Z as X, Y and Z are purely labels. Thus $0 \leq |\mathsf{preclusiveUndercuts}(\mathrm{Y}:\alpha, \Delta))\}| \leq (2^{|\mathrm{Y}|} - 1) \cdot (2^{\mathsf{midLattice}(\Delta)} - 1)$. □

I now look at the impact of non-unique formulae, labels and the stripping of within the definition of preclusive undercut, Definitions 3.4.1 and 3.4.2. This stripping is necessary as, in my representation for confirmation, labels only exist on the left hand side of the colon. Claims are unlabelled formulae. When forming the claim of the preclusive undercut I make one conjunction out of the premises of the attacked arguments using the unlabelled premise formulae. Thus, strip is needed in the definition to remove the labels to allow the claim of the undercut to match the support of the defending argument.

Unlike canonical, Pollock and Wigmore undercuts, preclusive undercuts do not have an unlabelled form in the literature to be contrasted with. The following example shows the role of labels in preclusive undercuts, demonstrating that two confirmations (i.e. a defending confirmation and an attacking confirmation) related by preclusive undercut can have many target confirmations, due to labels.

**Example 3.8.1. Two target confirmations.** *Let* $\Delta = \{a:\alpha, b:\alpha \rightarrow \beta, c:\alpha, d:\gamma, e:\neg\gamma \vee \beta, f:\neg\alpha\}$. *Let* $I = \{a, b\}, J = \{b, c\}, K = \{d, e\}, L = \{f\}$. *Then* $I : \beta, J : \beta, K : \beta, L : \neg\alpha, L : \neg(\alpha \wedge (\alpha \rightarrow \beta)) \in \mathsf{arguments}(\Delta)$ *and* $\{I, J, K\} : \beta, \{L\} : \neg\alpha, \{L\} : \neg(\alpha \wedge (\alpha \rightarrow \beta)) \in \Diamond(\Delta)$. *Consequently* $\langle\{L\} : \neg(\alpha \wedge (\alpha \rightarrow \beta)), \{I\}, \{I, J, K\} : \beta\rangle, \langle\{L\} : \neg(\alpha \wedge (\alpha \rightarrow \beta)), \{J\}, \{I, J, K\} : \beta\rangle \in$

preclusiveUndercuts$(\{I, J, K\} : \beta, \Delta)$. *Furthermore* $\langle\{L\} : \neg(\alpha \wedge (\alpha \rightarrow \beta)), \{I, J\}, \{I, J, K\} : \beta\rangle \in$ preclusiveUndercuts$(\{I, J, K\} : \beta, \Delta)$ *as* $\{I, J\} : \beta$ *cannot be a target confirmation because there exist two invalidating target confirmations, both that happen to be self confirmations.*

Imagine the defending confirmation as a vertex in a tree, with its children being the attacking confirmations of its preclusive undercuts. The interesting question for the above example is then should there be one or two children? [I later go on to adopt having two children in the tree in this situation.] If two children then the attacking confirmations would be identical – without some syntactic device, such as edge labels, they would be indistinguishable.

This Example 3.8.1 also raises the question of how much defeating power is provided. Is it one unit of defeat as there is only one confirmation $\{L\} : \neg(\alpha \wedge (\alpha \rightarrow \beta))$? Or, as there are two target confirmations, is it two units of defeat? Does counting target confirmations introduce double counting? I argue no, that it is more intuitive to count this situation as two attacks and that the target confirmations have merit in tracking what is happening. But first I must relay more about the behaviour of the various confirmations involved in preclusion.

Now I turn to the role of $\vee$I in extending claims of candidate preclusive undercuts.

### 3.8.1 Growing Claims and Invalidation

Earlier with canonical undercuts, in Section 3.3, it was clear that $\vee$I can be safely used to 'grow' the claim so that the full support of the undercut argument is matched. This is simply a property of classical logic. Similarly with preclusive undercuts $\vee$I can play an influential role, however here it is a slightly more novel behaviour. The claim of the preclusive undercut can be 'grown' to fill in a variety of missing parts. If an undercut attacks just one premise each for several arguments, then $\vee$I can be used to grow the claim to attack all of the premises of several arguments. However, the invalidating aspect of the definition of preclusive undercut specifically prohibits growing a claim to attack arguments for which one has no assumptions.

While a conjectured undercut can fail to be a preclusive undercut if it attacks the support of something less than a whole number of arguments, (as shown in the following example), there will always be a logically equivalent argument (following from $\vee$I) that is valid preclusive undercut. Continuing this series of examples:

**Example 3.8.2. The role of $\vee$I.** *Let* $\Delta = \{a : \beta, b : \beta \rightarrow \sigma, c : \gamma, d : \neg\gamma \vee \sigma, e : \lambda, f : \neg\sigma \rightarrow \neg\lambda, g : \neg\beta \vee \neg\gamma\}$. *Let* $I = \{a, b\}, J = \{c, d\}, K = \{e, f\}, L = \{g\}$ *and* $\mathrm{R} = \{L\}, \mathrm{V} = \{I, J, K\}$. *Therefore* $I : \sigma, J : \sigma, K : \sigma, L : \neg(\beta \wedge \gamma), L : \neg(\beta \wedge (\beta \rightarrow \sigma) \wedge \gamma \wedge (\neg\gamma \vee \sigma)), L : \neg(\beta \wedge (\beta \rightarrow \sigma) \wedge \gamma \wedge (\neg\gamma \vee \sigma) \wedge \lambda \wedge (\neg\sigma \rightarrow \neg\lambda)) \in$ arguments$(\Delta)$ *and* $\mathrm{R}{:}\neg(\beta \wedge (\beta \rightarrow \sigma) \wedge \gamma), \mathrm{R}{:}\neg(\beta \wedge (\beta \rightarrow \sigma) \wedge \gamma \wedge (\neg\gamma \vee \sigma)), \mathrm{R}{:}\neg(\beta \wedge (\beta \rightarrow \sigma) \wedge \gamma \wedge (\neg\gamma \vee \sigma) \wedge \lambda \wedge (\neg\sigma \rightarrow \neg\lambda)) \in \Diamond(\Delta)$. *It can be concluded that given* $\Delta_5$, *a valid preclusive undercut for* $\mathrm{V}{:}\sigma$ *is* $\langle \mathrm{R}{:}\neg(\beta \wedge (\beta \rightarrow \sigma) \wedge \gamma \wedge (\neg\gamma \vee \sigma))), \{I, J\}, \mathrm{V}{:}\sigma\rangle$. *However, the valid confirmation* $R{:}\neg(\beta \wedge (\beta \rightarrow \sigma) \wedge \gamma \wedge (\neg\gamma \vee \sigma) \wedge \lambda \wedge (\neg\sigma \rightarrow \neg\lambda))$ *with a target of* $\{I, J, K\}$ *is not a valid preclusive undercut as invalidation occurs.*

In this example I started with pieces of the attack on the assumption sets of two arguments; namely

i) $\neg\beta$ is a partial attack on $\{a:\beta, b:\beta \rightarrow \sigma\}$ and ii) $\neg\neg\gamma$ is a partial attack on $\{c:\gamma, d:\neg\gamma \vee \sigma\}$. The logic allowed 'growing' the formula $\beta \rightarrow \sigma$ in the first case and $\neg\gamma \vee \sigma$ in the second. This part of the example shows the valid growing of assumptions to round out or complete the attack on multiple arguments.

However the attempt to further grow the attack to take on a completely new argument is prohibited. That is, even though $\lambda \wedge (\neg\sigma \rightarrow \neg\lambda)$ can be grown as an argument and also as a confirmation, it is not allowed as a preclusive undercut.

Given the definition of $\Diamond(\alpha, \Delta)$, that if $|\mathsf{label}(\mathrm{X} : \alpha)| > 1$, (i.e. for everything other than self confirmations), then a $\mathrm{W}:\alpha$ with $\mathrm{W} \subseteq \mathrm{X}$ will always exist in $\Diamond(\alpha, \Delta)$ as $\Diamond(\alpha, \Delta)$ is a power set with a minimum cardinality member of $\emptyset:\alpha$.

This concludes the properties of preclusive undercuts. While I have given a comprehensive analysis, more research could be done further comparing and contrasting canonical and preclusive undercuts.

## 3.9 Attack, Abstract Attack and Attack Graphs

Now that definitions for undercut and rebuttal are established, both at the level of individual arguments and confirmations, the simplifying construct of 'attack' can be introduced. The concept of attack is central to the abstract framework of (Dung, 1993; Dung, 1995) and the semi-abstract ABA framework (Bondarenko et al., 1997; Dung et al., 2006). The attack construct has assisted Dung in deriving several important results about logic programming and game playing and facilitated Bondarenko et al in showing the unity of a wide variety of non-monotonic logics.

To build on Dung's concept of abstract attack I will describe bridging assumptions that map between the concrete world of mincon argumentation in the bulk of this thesis and the abstract Dung framework. This building on concepts from the abstract and the ABA work aims to provide a more concrete interpretation of these abstract frameworks – continuing the broad theme of others' research, notably (Amgoud & Cayrol, 1998; Amgoud & Cayrol, 2002), (Baroni et al., 2002; Baroni & Giacomin, 2003) and (Prakken & Vreeswijk, 2002). Thus I start with a reminder of Dung's definition of his Dung argumentation framework, which is comprised solely of abstract arguments and abstract attacks, with a slight refinement of symbol use to avoid any subsequent ambiguity:

**Definition 3.9.1.** *A* **Dung argumentation framework,** *denoted* $\mathsf{dungFramework} = \langle\mathsf{dungArguments}, \mathsf{attacks}_D(\mathsf{dungArguments})\rangle$, *showing the relationships within a set of arguments, is a pair such that:*

$$\mathsf{dungFramework} = \langle\mathsf{dungArguments}, \mathsf{attacks}_D(\mathsf{dungArguments})\rangle, \textit{ where}$$

1. $\mathsf{dungArguments}$ *is a set of abstract arguments and*
2. $\mathsf{attacks}_D(\mathsf{dungArguments}) \subseteq \mathsf{dungArguments} \times \mathsf{dungArguments}$

   *is an abstract attacks relationship between arguments.*

The tuple $\langle\mathcal{X}, \mathcal{Y}\rangle \in \mathsf{attacks}_D(\mathsf{dungArguments})$ where $\mathcal{X}, \mathcal{Y} \in \mathsf{dungArguments}$, denotes that the argument $\mathcal{X}$ attacks argument $\mathcal{Y}$, or in other words that $\mathcal{Y}$ is attacked by $\mathcal{X}$.

To map Dung's simple definition back to the mincon world requires *two* bridging assumptions. While a number of bridging assumptions exist in the literature, the analysis of which I leave as an area for further research, the original bridging assumptions come from (Amgoud & Cayrol, 1998). While I build on (Amgoud & Cayrol, 1998) in this section, I will be using a different definition of undercut and be adding labels.

I start with the argument assumption. I hold that to examine a concrete instance of the Dung or ABA frameworks, that could be implemented or used in a professional debate, it is necessary to say or define what would constitute an argument. Thus given two arbitrary items A and B, what are the criteria to say whether A and B are arguments or not, separate from any declaration that A and B simply are arguments? Clearly Dung gives no indication of what constitutes an argument; that is exactly his foundation assumption. If, however, a more direct comparison of abstract and concrete argumentation is desired then some bridging is needed.

I would suggest that the ideal bridging would allow a) a concrete implementation in a professional debate setting, while at the same time b) providing overall behaviours in keeping with motivations or common sense intuitions for Dung's framework. The paper (Besnard & Hunter, 2001), in looking at this area of bridging, shows how a collapse might occur. Most would probably agree that their collapse is or should be seen as more a feature of the choice of assumptions and not some flaw in the Dung framework.

I have not placed my argument assumption in a formal definition as I do not use it any further in the formalisms of this thesis. My first assumption is thus the argument assumption in which I suggest an instance of the Dung framework where I let $\{\mathcal{X} \mid \mathcal{X} \in \mathsf{dungArguments}\} = \{I\!:\!\alpha \mid I\!:\!\alpha \in \mathsf{arguments}(\Delta)\}$ iff $\mathsf{formulae}(\mathsf{label}(I : \alpha), \Delta)$ is the support of $\mathcal{X}$, $\mathsf{stripDeduction}(I : \alpha)$ is the claim of $\mathcal{X}$ and classical logic is used to infer the claim from the support. Clearly Dung never made any such statements, but this understanding does facilitate comparison.

My second assumption is to clarify what is meant by the attacks relationship. Because (Amgoud & Cayrol, 1998) do not use canonical undercuts from the more recent paper (Besnard & Hunter, 2001), the below is not literally their definition. However, the below is a close adaptation of the (Amgoud & Cayrol, 1998) attack assumption. This simple definition plays a central role in my thesis.

**Definition 3.9.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $I : \alpha \in \mathsf{arguments}(\Delta)$. *An* **argument attack**, *denoted* $\mathsf{argumentAttacks}(I : \alpha, \Delta)$, *is the set of arguments that attack* $I\!:\!\alpha$ *such that:*

$$\mathsf{argumentAttacks}(I\!:\!\alpha, \Delta) = \mathsf{rebuttals}(I\!:\!\alpha, \Delta) \cup \mathsf{canonicalUndercuts}(I\!:\!\alpha, \Delta).$$

A sequence of two or more arguments $\langle A, B, C, \ldots, N\rangle$, where $A, B, C, \ldots, N \in \mathsf{arguments}(\Delta)$, that each attack the adjacent neighbour next in the row, such that $B \in \mathsf{argumentAttacks}(A, \Delta), C \in \mathsf{argumentAttacks}(B, \Delta), \ldots$, I call an **attack chain** of arguments or just a **chain** for short. I call the first member, $A$, which is attacked, the **head** of the chain and the last member, $N$, which attacks, the **tail** of the chain. I call the direction from tail to head **up** the chain, and from head to tail **down** the chain. In this way small pieces of a graph of arguments can be analysed while avoiding the complexity of the whole graph. As trees are a kind of graphs, chains also help with the analysis of trees of arguments.

In Section 4.8 and Definition 4.8.2, for reasons to be made clear in the next chapter, I refine my attacks assumption to that I call "direct attack". An area for further research would be to examine more deeply the topic of collapse from (Besnard & Hunter, 2001) and see how it is affected by using that direct attack rather than this more general attack.

### 3.9.1 Argument Graphs, Confirmation Graphs and Contradiction Graphs

To allow this discussion of attack chains to be more robust and extensive I now define the argument graph, which is akin to Dung's graphs, see for example the many diagrams in (Prakken & Vreeswijk, 2002), as follows. Later in the thesis where I provide argument graphs, these commonly show just the subset of relevant vertices and edges. Often the graphs are annotated to provide further explanation.

**Definition 3.9.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and $I : \phi, J : \psi \in$* $\mathsf{arguments}(\Delta)$*. An* **argument graph**, *denoted* $\mathsf{argumentGraph}(\Delta) = \langle \mathcal{V}, \mathcal{E} \rangle$*, where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges, is a directed graph such that:*

$$\begin{aligned} \mathcal{V} &= \{I : \phi \mid I : \phi = \mathsf{arguments}(\Delta)\}, \\ \mathcal{E} &= \{\langle I : \phi, J : \psi \rangle \mid I : \phi, J : \psi \in \mathcal{V}, \textit{ where } I : \phi \in \mathsf{argumentAttacks}(J : \psi, \Delta)\}. \end{aligned}$$

One implication of these argument and argument attacks assumptions is that self-defeat is ruled out, or in other words that an argument cannot attack itself. Thus there are no loops in my argument graphs. A claim such as 'I am a liar' is an example of a self defeating argument and hence of an argument that cannot be represented as a mincon. Thus it is known that $I : \alpha \notin \mathsf{argumentAttacks}(I : \alpha, \Delta)$, or in other words $\mathcal{X} \notin \mathsf{argumentAttacks}(\mathcal{X}, \Delta)$. The Dung argumentation framework does not rule out self-defeat, that is an argument which attacks itself, such as $\langle \mathcal{X}, \mathcal{X} \rangle$. Self defeat is not dwelt on in (Dung, 1993; Dung, 1995), however it is extensively discussed in (Prakken & Vreeswijk, 2002) and mentioned in (Cayrol & Lagasquie-Schiex, 2002).

Given $\mathsf{argumentAttacks}()$ and $\mathsf{argumentGraph}(\Delta)$, it is also reasonable to extend the analysis from the level of individual arguments to the level of confirmations. In the next definition I extend the attacks assumption. Preclusive undercuts are defined in Definition 3.4.2 and confirmation rebuttals in Definition 2.6.8.

**Definition 3.9.4.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let* $\mathrm{Y}:\alpha, \mathrm{Z}:\phi \in \Diamond(\Delta)$. *The* **confirmation attacks** *function, denoted* $\mathsf{confirmationAttacks}(\mathrm{Y}:\alpha, \Delta)$, *is the set of attacks on* $\mathrm{Y}:\alpha$ *derivable from* $\Delta$ *such that:*

$$\begin{aligned}&\mathsf{confirmationAttacks}(\mathrm{Y}:\alpha, \Delta) = \mathsf{preclusiveUndercuts}(\mathrm{Y}:\alpha, \Delta)\\ &\cup \quad \{\langle \mathrm{Z}:\phi, \mathsf{label}(\mathrm{Y}:\alpha), \mathrm{Y}:\alpha\rangle \mid \mathrm{Z}:\phi \in \mathsf{confirmationRebuttals}(\mathrm{Y}:\alpha, \Delta)\}.\end{aligned}$$

I call an individual $\langle \mathrm{Z}:\phi, \mathrm{X}, \mathrm{Y}:\alpha\rangle \in \mathsf{confirmationAttacks}(\mathrm{Y}:\alpha, \Delta)$ a **targeted attack** tuple, where $\mathrm{X}, \mathrm{Y} \in \mathcal{C}, \mathrm{X} \subseteq \mathrm{Y}$. As with preclusive undercuts, $\mathrm{Z} : \phi$ is the attacking confirmation or attacker, X the target confirmation label and $\mathrm{Y} : \alpha$ the defending confirmation. Clearly neither the attacker nor the attacked confirmation can be labelled with the empty set.

One minor issue in forming this definition of confirmation attack is what should the target confirmation be for a confirmation attack that is a rebuttal? If $\mathrm{W}:\neg\alpha \in \mathsf{confirmationRebuttals}(\mathrm{Y}:\alpha, \Delta)$ then I define the target confirmation as the whole confirmation $\mathrm{Y}:\alpha$ as clearly every argument in it is attacked (which could not be said for the whole defending confirmation in the case of preclusive undercuts, but is the case for the target confirmation for preclusion).

A sequence of two or more confirmations $\langle A, B, C, \ldots, N\rangle$, where $A, B, C, \ldots, N \in \Diamond(\Delta)$, that each attack the adjacent neighbour next in the line, such that $B \in \mathsf{confirmationAttacks}(A, \Delta), C \in \mathsf{confirmationAttacks}(B, \Delta), \ldots$, I call an **attack chain** of confirmations. I use the terms head, tail, up and down as established for individual arguments. Consequently pieces of a graph or tree of confirmations can be studied in isolation. Again to make the discussion more robust I define the confirmation graph as follows. In an edge labelled graph, an edge is represented by a tuple $\langle \mathrm{Z} : \phi, \mathrm{X}, \mathrm{Y} : \alpha\rangle$ where $\mathrm{Z} : \phi$ is the vertex the edge is incident from, i.e. the attacking confirmation, $\mathrm{Y}:\alpha$ the vertex the edge is incident on, i.e. the defending confirmation, and X the edge label, i.e. the target confirmation label. Later in the thesis where I provide confirmation graphs these usually show just the subset of relevant vertices and edges. Often I annotate the graphs with text to provide further explanation.

**Definition 3.9.5.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and* $\mathrm{W}:\psi, \mathrm{X}:\alpha, \mathrm{Y}:\alpha, \mathrm{Z}:\phi \in \Diamond(\Delta)$. *A* **confirmation graph**, *denoted* $\mathsf{confirmationGraph}(\Diamond(\Delta)) = \langle \mathcal{V}, \mathcal{E}\rangle$, *where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges, is an edged-labelled directed graph such that:*

$$\begin{aligned}\mathcal{V} &= \{\mathrm{W}:\psi \mid \mathrm{W}:\psi = \mathsf{top}(\Diamond(\psi, \Delta))\},\\ \mathcal{E} &= \{\langle \mathrm{Z}:\phi, \mathrm{X}, \mathrm{Y}:\alpha\rangle \mid \mathrm{Z}:\phi, \mathrm{Y}:\alpha \in \mathcal{V} \textit{ and } \langle \mathrm{Z}:\phi, \mathrm{X}, \mathrm{Y}:\alpha\rangle \in \mathsf{confirmationAttacks}(\mathrm{Y}:\alpha, \Delta)\}.\end{aligned}$$

To complete this section I introduce a third kind of graph, the contradiction graph, building on the earlier argument graph and confirmation graph.

**Definition 3.9.6.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae,* $\langle \mathrm{S}, \mathrm{T}\rangle : \psi, \langle \mathrm{U}, \mathrm{V}\rangle : \phi, \langle \mathrm{X}, \mathrm{Y}\rangle : \alpha \in \blacklozenge(\Delta)$ *and* $\mathrm{W} \in \mathcal{C}$. *A* **contradiction graph**, *denoted* $\mathsf{contradictionGraph}(\blacklozenge(\Delta)) = \langle \mathcal{V}, \mathcal{E}\rangle$, *where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges, is an edged-labelled directed graph such* *that:*

$$\begin{aligned}
\mathcal{V} &= \{\langle \mathrm{W}:\psi \mid \mathrm{W}:\psi = \mathsf{top}(\blacklozenge(\psi,\Delta))\}, \\
\mathcal{E} &= \{\langle\langle \mathrm{U},\mathrm{V}\rangle:\phi, \mathrm{X}, \langle \mathrm{W},\mathrm{Y}\rangle:\alpha\rangle \mid \langle \mathrm{U},\mathrm{V}\rangle:\phi, \langle \mathrm{X},\mathrm{Y}\rangle:\alpha \in \mathcal{V} \textit{ and either} \\
&\qquad \langle \mathrm{U}:\phi, \mathrm{W}, \mathrm{X}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{X}:\alpha,\Delta) \textit{ or} \\
&\qquad \langle \mathrm{U}:\phi, \mathrm{W}, \mathrm{Y}:\neg\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Y}:\neg\alpha,\Delta)\}.
\end{aligned}$$

Clearly, in the case that $\mathrm{U}:\phi$ is attacking $\mathrm{X}:\alpha$ then $\mathrm{W} \subseteq \mathrm{X}$. Otherwise $\mathrm{U}:\phi$ is attacking $\mathrm{Y}:\neg\alpha$ then $\mathrm{W} \subseteq \mathrm{Y}$. The contradiction graph is first used in Section 5.7.1 where it is helpful in illustrating some of the behaviours of reflection. In this third graph, attacks are occurring both between and within vertices. As all vertices are maximum cardinality contradictions each rebuttal $\mathrm{V}:\neg\phi$ is fully defined.

That concludes the laying of the foundations on attack, abstract attack and attack graphs as well as this whole chapter on undercut.

## 3.10 Conclusions for Undercut Framework

This chapter on my undercut framework has formally defined the third foundation, or type of argument interaction, beyond confirmation and rebuttal, needed to track and judge professional debates. While undercut is widely discussed in the literature there is a lack of consensus on what exactly constitutes an undercut, how to represent them and how they behave. Undercuts, according to a portion of the literature and my own view, must be considered in debate outcomes as they definitely help resolve inconsistency. Consequently there is a clear need for a formally defined undercut that better matches professional debate.

There are four undercut definitions in the chapter, starting with Wigmore undercut and Pollock undercut, leading to the two key undercut functions, which are shown in Table 3.1.

| **Key Undercut Function** | **Definition** |
|---|---|
| $J:\phi \in \mathsf{canonicalUndercuts}(I:\alpha,\Delta)$ | 3.3.2 |
| $\langle \mathrm{Z}:\phi, \mathrm{X}, \mathrm{Y}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Y}:\alpha,\Delta)$ | 3.4.2 |

Table 3.1: Summary of Key Undercut Functions

There are four points to emphasise about preclusive undercuts.

1. Preclusive undercuts comprise a confirmation attacking a confirmation, rather than just an argument attacking an argument – as was the case for the other three undercuts.

2. They cover, or subsume, all the exact same behaviour of canonical undercuts – because confirmations can be self confirmations.

3. Preclusive undercuts also extend canonical undercut to cover the recurrent situation in professional debate where it is put that 'one or other of your arguments is false'. Motivating examples quote real life situations where preclusions have been central to the refuting of Iain Duncan Smith and Arnold

Schwarzenegger, central to Sherlock Holmes proving a case and central to cross examinations in the UK's Hutton Enquiry.

4. A particular form of preclusive undercut is called a tautological preclusion. They are used quite often in professional debate, especially to ask difficult questions. Tautological preclusions have three key properties. Firstly, tautological preclusions enable inconsistent confirmations to be attacked and effectively removed from a debate. Secondly, tautological preclusions always exist, because they require no premises. Finally, they cannot be attacked back, as any conjectured rebuttal or undercut attacking them has to be inconsistent and thus cannot be a valid argument.

This chapter employs propositions and examples to explore and draw out the specific properties of preclusive undercut – showing where they faithfully reproduce established good undercut behaviours, where they remedy the short comings of other approaches and where they extend the behaviour of undercuts in the literature to better match professional debate. The point about this extension of behaviour is that it doesn't just cover arguments put, but also removes inconsistencies from debate. In this way I suggest that preclusion is, for avoiding inconsistency in argumentation, on a par with the use of mincon arguments and the avoidance of catenate inferences.

A final section in this chapter leverages the work of Dung to provide two attacks functions. The first of these, at the level of individual arguments, is based on the work of Amgoud and Cayrol. In the second I extend the literature to show how preclusive undercut is part of a confirmation attack. These two attacks functions are summarised in Table 3.2.

| **Attacks Function** | **Definition** |
|---|---|
| $J:\phi \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$ | 3.9.2 |
| $\langle Z:\phi, X, Y:\alpha \rangle \in \mathsf{confirmationAttacks}(Y:\alpha, \Delta)$ | 3.9.4 |

Table 3.2: Summary of Key Attacks Functions

These two attacks functions are used in my subsequent analysis of the novel concept of reflection. The chapter concludes by providing three attack graphs which aid the illustration of various behaviours later in the thesis. These attack graphs are summarised in Table 3.3.

| **Attacks Graph** | **Vertices** | **Edges** | **Definition** |
|---|---|---|---|
| $\mathsf{argumentGraph}(\Delta)$ | arguments | argument attacks | 3.9.3 |
| $\mathsf{confirmationGraph}(\lozenge(\Delta))$ | confirmations | confirmation attacks | 3.9.5 |
| $\mathsf{contradictionGraph}(\blacklozenge(\Delta))$ | contradictions | preclusive undercuts | 3.9.6 |

Table 3.3: Summary of Three Attacks Graphs

These three graphs provide different and complimentary ways of illustrating the relationships between the various levels of argument aggregation and attacks between those aggregations.

# Chapter 4

# Reflected and Direct Arguments

## 4.1 Overview of the Chapter

This short preamble provides an overview of the chapter, together with its context within the thesis. This chapter introduces the core novel notion of the thesis – reflection. The thesis covers four types of reflection: Type I is defined in Section 4.2.4 and analysed throughout this chapter ; the next chapter defines Type II in Section 5.5, Type III in Section 5.8 and Type IV in Section 5.9. Within the current chapter four kinds of Type I reflection are analysed, which I call Base Propositions One, Two, Three and Four.

### 4.1.1 Sections on Reflected and Direct Arguments

I develop this chapter's analysis of reflected and direct arguments and attacks in seven stages. The stages progress from a definition and analysis of the problem I call 'reflection' to my proposed solution of only employing 'direct' arguments. The sections of this chapter are:

**Abstract reflection** Section 4.2 identifies the predictable presence of reflected arguments and attacks. This analysis, relating to Dung's work, demonstrates that attack engenders attack.

**Reflected canonical undercuts** Section 4.4 studies these reflected attacks that are also canonical undercuts. They can be reflected off rebuttals or off other undercuts.

**Direct canonical undercuts** Section 4.5 covers those attacks that are not reflected and must be isolated to properly track a debate. These are the arguments and attacks of value.

**Mono-pair enlarged reflection** Section 4.6 establishes that there can be many more reflections than might be intuitively thought. Reflections for $\alpha$ and $\neg\alpha$ have inexact symmetry.

**Reflected argument rebuttals** Section 4.7 shows that rebuttals can be reflected off canonical undercuts or other rebuttals. Their properties differ markedly from reflected canonical undercuts.

**Direct argument rebuttals** Section 4.8 shows that tracking such rebuttals that are not reflected is vital to capture a complete debate – a point that has been missed in the literature.

While the chapters prior to this one have broadened the focus from individual arguments to sets of arguments, this chapter steps back to single arguments again to establish the basic principles of reflection.

The next chapter, Chapter 5, while closely allied to this one, extends the analysis back up to sets of arguments. That said, the properties unfolded in the current chapter, when subjected to the confirm() functions, nevertheless also apply to self-confirmed confirmation rebuttals and to self-confirmed mono-attack mono-target preclusions.

## 4.2 Introducing Attack and Reflection

Conceptually the starting point for understanding reflection is attack, which is described in an abstract form by (Dung, 1993), see Definition 3.9.1. This section builds on the concept of attack, abstract and concrete, to introduce a new and related concept I call reflection. In order to define reflection in terms of attack it is necessary to first establish two base propositions which simplify subsequent proofs.

### 4.2.1 Base Proposition One: Rebuttal Engenders Undercut

My first base proposition is that rebuttal engenders undercut.

**Proposition 4.2.1. Base Proposition One.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $I:\alpha, J:\beta, K:\phi \in \mathsf{arguments}(\Delta)$.

$$\textit{If there exists } J:\beta \in \mathsf{rebuttals}(I:\alpha,\Delta)$$
$$\textit{then there exists } K:\phi \in \mathsf{canonicalUndercuts}(J:\beta,\Delta)$$
$$\textit{where } K \subseteq I.$$

*Proof.* For $J:\beta$ to be a rebuttal of $I:\alpha$ it must be the case that $\beta = \neg\alpha$. Let $\mathsf{stripAssumptions}(\mathsf{formulae}(J,\Delta)) = \{\phi_1,\ldots,\phi_n\}$. So $J:\beta$ equates to $(\phi_1 \wedge \ldots \wedge \phi_n) \rightarrow \neg\alpha$ and thus by contraposition $\alpha \rightarrow \neg(\phi_1 \wedge \ldots \wedge \phi_n)$. Let $\mathsf{stripAssumptions}(\mathsf{formulae}(I,\Delta)) = \{\psi_1,\ldots,\psi_m\}$, so similarly $I:\alpha$ equates to $(\psi_1 \wedge \ldots \wedge \psi_m) \rightarrow \alpha$. Combining $(\psi_1 \wedge \ldots \wedge \psi_m) \rightarrow \alpha$ with $\alpha \rightarrow \neg(\phi_1 \wedge \ldots \wedge \phi_n)$, by hypothetical syllogism, yields the valid deduction $(\psi_1 \wedge \ldots \wedge \psi_m) \rightarrow \neg(\phi_1 \wedge \ldots \wedge \phi_n)$. The support $(\psi_1 \wedge \ldots \wedge \psi_m)$ may not be minimal, so given a $\{\psi_p,\ldots,\psi_q\} \subseteq \{\psi_1,\ldots,\psi_m\}$, it follows that $(\psi_p \wedge \ldots \wedge \psi_q) \rightarrow \neg(\phi_1 \wedge \ldots \wedge \phi_n)$ is a minimal deduction. The support $(\psi_p \wedge \ldots \wedge \psi_q) \nvdash \bot$ as its superset is consistent, being as it is the support of the valid argument $I:\alpha$. Moving to label nomenclature, let the support $\{\psi_p,\ldots,\psi_q\} = \mathsf{stripAssumptions}(\mathsf{formulae}(K,\Delta))$, hence $K \subseteq I$. Consequently there must exist $K:\neg(\phi_1 \wedge \ldots \wedge \phi_n) \in \mathsf{arguments}(\Delta)$. Given the equality $\phi = \neg(\phi_1 \wedge \ldots \wedge \phi_n)$ it follows that $K:\phi \in \mathsf{arguments}(\Delta)$. Selecting $\langle\phi_1,\ldots,\phi_n\rangle$ as the canonical ordering of $\{\phi_1,\ldots,\phi_n\}$, it follows that $K:\phi$ is a canonical undercut for $J:\beta$. In conclusion it is clear that this $K:\phi \in \mathsf{canonicalUndercuts}(J:\beta,\Delta)$, given the stated prerequisites, always exists. □

The proof shows that the support of the argument $K:\phi$ is known from the support of the defending argument $I:\alpha$, as $K \subseteq I$. Furthermore the claim of $K:\phi$ is known from the support of the attacking argument $J:\beta$, as $\phi = \neg(\bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(J,\Delta)))$. The prerequisites referred to are simply the first part of the proposition, namely that $I:\alpha, J:\beta \in \mathsf{arguments}(\Delta)$ and $J:\beta \in \mathsf{rebuttals}(I:\alpha,\Delta)$. Thus if argument B rebuts argument A, then argument B is always subject to canonical undercut from an argument C.

### 4.2.2 Base Proposition Two: Undercut Engenders Rebuttal

Turning now to my second base proposition it is the case that undercut always engenders rebuttal.

**Proposition 4.2.2. Base Proposition Two.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\beta, K:\phi \in \mathsf{arguments}(\Delta)$.*

$$\textit{If there exists } J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha,\Delta)$$

$$\textit{then there exists } K:\phi \in \mathsf{rebuttals}(J:\beta,\Delta)$$

$$\textit{where } K = I.$$

*Proof.* From the existence of $J:\beta$ and $I:\alpha$ it is always possible to infer the construct $K:\phi$. Using the definition of canonical undercut, for $J:\beta$ to be a canonical undercut of $I:\alpha$ it must be the case that $\beta = \neg(\beta_1 \wedge \ldots \wedge \beta_r)$ where $\{\beta_1,\ldots,\beta_r\} = \mathsf{stripAssumptions}(\mathsf{formulae}(I,\Delta))$. By conjunction introduction, $\{\beta_1,\ldots,\beta_r\} \vdash \beta_1 \wedge \ldots \wedge \beta_r$ is a valid deduction. It cannot be the case that a strict subset of $\{\beta_1,\ldots,\beta_r\}$ implies $\beta_1 \wedge \ldots \wedge \beta_r$ because if it did the argument $I:\alpha$ would not be minimal. Inconsistency is also impossible as $K = I$ and $I:\alpha \in \mathsf{arguments}(\Delta)$. Thus $I:\beta_1 \wedge \ldots \wedge \beta_r \in \mathsf{arguments}(\Delta)$. Clearly for the two arguments $J:\beta$ and $K:\phi$ to rebut it must be the case that $\neg\beta = \phi$. Thus the reflexive argument (reflexive by Definition 2.4.7) $I:\beta_1 \wedge \ldots \wedge \beta_r = I:\neg\beta = I:\phi = K:\phi$, which is a rebuttal of $J:\beta$,therefore $K:\phi \in \mathsf{rebuttals}(J:\beta,\Delta)$, always exists. □

The support of this argument $K:\phi$ is known from the support of the defending argument $I:\alpha$. The claim of the argument $K:\phi$ is also known, this time from the claim of the attacking argument $J:\beta$. Clearly these two base propositions are closely related to each other, however they also contain a number of differences which affect the properties of this field.

### 4.2.3 Attack Engenders Attack

Rather than immediately analyse these two base propositions, I first show where they fit in the larger field of argumentation. I thus introduce a general (i.e. abstract or generic) definition of reflection. My starting point is the attacks function, introduced earlier in Definition 3.9.2, which provides all the arguments derivable from $\Delta$ that either rebut or undercut an argument $I:\alpha$. To recap, in that definition, $\Delta$ is a set of labelled assumption formulae and $I:\alpha \in \mathsf{arguments}(\Delta)$. The Amgoud-Cayrol attack function, from Definition 3.9.2, is such that:

$$\mathsf{argumentAttacks}(I:\alpha,\Delta) = \mathsf{rebuttals}(I:\alpha,\Delta) \cup \mathsf{canonicalUndercuts}(I:\alpha,\Delta).$$

For a rebuttal, the attacks relationship (following Amgoud-Cayrol) is bidirectional, while for an undercut it is unidirectional. Rebuttal was defined back in Definition 2.6.7 and canonical undercut in Definition 3.3.2. An alternative way of stating the definition of attacks is that:

$$\begin{aligned}&J:\beta \in \mathsf{argumentAttacks}(I:\alpha,\Delta)\\ &\qquad \text{iff } (J:\beta \in \mathsf{rebuttals}(I:\alpha,\Delta) \text{ or } J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha,\Delta)).\end{aligned}$$

Thus an argument $J:\beta$ attacks an argument $I:\alpha$ iff $J:\beta$ is a rebuttal of $I:\alpha$ or $J:\beta$ is a canonical undercut of $I:\alpha$. I now establish the somewhat startling proposition that attack always engenders attack.

**Proposition 4.2.3.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $I:\alpha, J:\beta \in$ $\mathsf{arguments}(\Delta)$.

$$\textit{If } J:\beta \in \mathsf{argumentAttacks}(I:\alpha,\Delta)$$
$$\textit{then } \mathsf{argumentAttacks}(J:\beta,\Delta) \neq \emptyset.$$

*Proof.* It is a given that $J:\beta \in \mathsf{argumentAttacks}(I:\alpha,\Delta)$, so using the definition of attack, Definition 3.9.2, either $J:\beta \in \mathsf{rebuttals}(I:\alpha,\Delta)$ or $J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha,\Delta)$. If $J:\beta \in \mathsf{rebuttals}(I:\alpha,\Delta)$ then from Base Proposition One, (Proposition 4.2.1), this rebuttal always engenders the undercut $K:\phi \in \mathsf{canonicalUndercuts}(J:\beta,\Delta)$. If, however, $J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha,\Delta)$ then from Base Proposition Two, (Proposition 4.2.2), this undercut always engenders the rebuttal $K:\phi \in \mathsf{rebuttals}(J:\beta,\Delta)$. Thus $K:\phi$ is either a rebuttal or an undercut of $J:\beta$. So in either case, the definition of attack, Definition 3.9.2, can be reapplied to show that it is inevitable that $K:\phi \in \mathsf{argumentAttacks}(J:\beta,\Delta)$. Thus it is always that case that $\mathsf{argumentAttacks}(J:\beta,\Delta) \neq \emptyset$. $\square$

In addition to knowing that there always is an engendered argument, quite a lot can be predicted about the support of this engendered argument, as I now show.

**Proposition 4.2.4.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $I:\alpha, J:\beta, K:\phi \in \mathsf{arguments}(\Delta)$.

$$\textit{If } J:\beta \in \mathsf{argumentAttacks}(I:\alpha,\Delta)$$
$$\textit{then there exists a } K:\phi \in \mathsf{argumentAttacks}(J:\beta,\Delta)$$
$$\textit{such that } K \subseteq I.$$

*Proof.* The above Proposition 4.2.3 established that the engendered argument $K:\phi$ always exists. Furthermore, Proposition 4.2.1 established that if the first attack is a rebuttal, then there exists an undercut in the role of the second and that $K \subseteq I$ follows. Or from Proposition 4.2.2, if the first attack is an undercut then there exists a rebuttal in the role of the second argument and thus $K = I$ holds. As $K = I$ is a special case of $K \subseteq I$ it follows that in all cases $K \subseteq I$ holds. $\square$

### 4.2.4 Type I Reflection – The Reflected Attacks Function

The first reflected attacks function, Type I Reflection, which I now define, follows on directly from the attacks function. Three other types of reflection functions, called Type II, III and IV, are covered in this thesis, so this is the first of four. I start with a concise description, supported by the argument graph shown below as Figure 4.1 and then give the formal definition.

Given a pair of arguments $I:\alpha$ and $J:\beta$ where $J:\beta$ attacks $I:\alpha$ then there always exists another argument $K:\phi$ that attacks $J:\beta$. Thus if there are two arguments in a chain, where chain was defined in Section 3.9, and argument attack in Definition 3.9.2, then there always exists a third. Stating that formally:

$$\begin{tikzcd} \text{Reflecting Argument} & I:\alpha & \\ & J:\beta \arrow[u, "J:\beta \in \mathsf{argumentAttacks}(I:\alpha,\Delta)"'] & \\ \text{Reflected Argument} & K:\phi \arrow[u, "K:\phi \in \mathsf{argumentAttacks}(J:\beta,\Delta)"'] & K \subseteq I \end{tikzcd}$$

Figure 4.1: Type I Reflection is Provided by $\mathsf{reflectedAttacks}(J:\beta, I:\alpha, \Delta)$

**Definition 4.2.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I : \alpha,\ J : \beta \in \mathsf{arguments}(\Delta)$. The set of* **reflected attacks** *arising from the attack of $J : \beta$ on $I : \alpha$, denoted $\mathsf{reflectedAttacks}(J:\beta, I:\alpha, \Delta)$, is such that:*

$$\mathsf{reflectedAttacks}(J:\beta, I:\alpha, \Delta) = \{K:\phi \in \mathsf{arguments}(\Delta) \mid$$
$$J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta), K:\phi \in \mathsf{argumentAttacks}(J:\beta, \Delta) \textit{ and } K \subseteq I\}.$$

I call the head of the chain, $I:\alpha$, the **reflecting argument** or **reflector** for short, $J:\beta$ the attacking argument, $K:\phi$ a **reflected argument** and $\{K:\phi\}$ the **set of reflected arguments**. Furthermore, I call the attack $K:\phi \in \mathsf{argumentAttacks}(J:\beta, \Delta)$ a **reflected attack**. If the support, $K$, of the reflected argument, $K:\phi$, is a strict subset of the support $I$ of the reflector $I:\alpha$, i.e. that $K \subsetneq I$, then I say **support shortening** has occurred.

This reflection definition could be applied recursively, suggesting an infinite chain of arguments arising from just one pair of arguments. I clarify and refine that notion over this and the next two chapters, taking the analysis one step at a time. I now prove that reflected arguments, given their prerequisites, always exist.

**Proposition 4.2.5.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I : \alpha, J : \beta \in \mathsf{arguments}(\Delta)$.*

*If* $J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$

*then* $\mathsf{reflectedAttacks}(J:\beta, I:\alpha, \Delta) \neq \emptyset$,

*otherwise* $\mathsf{reflectedAttacks}(J:\beta, I:\alpha, \Delta) = \emptyset$.

*Proof.* It is a given that $J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$ so either $J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta)$ or $J:\beta \in \mathsf{rebuttals}(I:\alpha, \Delta)$. Thus the proof where prerequisites exist can be taken in two halves. The first half is the situation $J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta)$. Base Proposition One, Proposition 4.2.1, demonstrates that if $J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta)$ then there always exists a rebuttal of $J:\beta$. Now the second half, which is the other situation for $J:\beta \in \mathsf{rebuttals}(I:\alpha, \Delta)$. Base Proposition

Two, Proposition 4.2.2, states that if $J:\beta \in \mathsf{rebuttals}(I:\alpha,\Delta)$ then there always exists a canonical undercut of $J:\beta$. So regardless of which of the two situations ($J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha,\Delta)$ or $J:\beta \in \mathsf{rebuttals}(I:\alpha,\Delta)$) is the case, when the prerequisites are present it is always true that $\mathsf{reflectedAttacks}(J:\beta, I:\alpha,\Delta) \neq \emptyset$. If on the other hand, $I:\alpha \notin \mathsf{arguments}(\Delta)$ or $J:\beta \notin \mathsf{arguments}(\Delta)$ or $J:\beta \notin \mathsf{argumentAttacks}(\Delta)$ then the proof so far would not follow and thus $\mathsf{reflectedAttacks}(J:\beta, I:\alpha,\Delta) = \emptyset$. □

Stating the proposition in English: 'if argument A attacks argument B then there always exists an argument C that attacks argument B' shows that in practice the prerequisites are quite naturally fulfilled and are therefore not onerous.

### 4.2.5 Seeing What is Reflected

The reflected attacks function as defined is abstract or general in that it is not specific about what kind of attack (i.e. rebuttal or undercut) is involved. Making attack specific allows certain observations to be made, including the following Propositions 4.2.6 and 4.2.7 and subsequent 4.2.9 and 4.2.11. The first of these four is that one of the reflected arguments from a rebuttal is a canonical undercut.

**Proposition 4.2.6.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\beta, K:\phi \in \mathsf{arguments}(\Delta)$.*

$$\textit{If there exists } J:\beta \in \mathsf{rebuttals}(I:\alpha,\Delta) \textit{ then}$$
$$\textit{there exists a } K:\phi \in \mathsf{reflectedAttacks}(J:\beta, I:\alpha,\Delta)$$
$$\textit{such that } K:\phi \in \mathsf{canonicalUndercuts}(J:\beta,\Delta).$$

*Proof.* Given that $J:\beta \in \mathsf{rebuttals}(I:\alpha,\Delta)$ then by Base Proposition One there exists $K:\phi \in \mathsf{canonicalUndercuts}(J:\beta,\Delta)$ where $K \subseteq I$. Definition 3.9.2 of argument attack establishes that given $J:\beta \in \mathsf{rebuttals}(I:\alpha,\Delta)$ then also $J:\beta \in \mathsf{argumentAttacks}(I:\alpha,\Delta)$. Likewise given $K:\phi \in \mathsf{canonicalUndercuts}(J:\beta,\Delta)$ then $K:\phi \in \mathsf{argumentAttacks}(J:\beta,\Delta)$. Furthermore from Definition 4.2.1 of reflected attack, given this chain of two argument attacks plus the fact that $K \subseteq I$ proves that $K:\phi \in \mathsf{reflectedAttacks}(J:\beta, I:\alpha,\Delta)$. So if there exists $J:\beta \in \mathsf{rebuttals}(I:\alpha,\Delta)$ then there exists a $K:\phi \in \mathsf{reflectedAttacks}(J:\beta, I:\alpha,\Delta)$ such that $K:\phi \in \mathsf{canonicalUndercuts}(J:\beta,\Delta)$. □

This proposition is close to Base Proposition One, Proposition 4.2.1, with the added clarity that this engendered argument may be categorised as a reflected argument and also as a canonical undercut. It is also interesting to see, building on the generic form of reflection, that if the attack is a canonical undercut then there exists a reflection which is a rebuttal.

**Proposition 4.2.7.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\beta, K:\phi \in \mathsf{arguments}(\Delta)$.*

$$\textit{If there exists } J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha,\Delta)$$
$$\textit{then there exists a } K:\phi \in \mathsf{reflectedAttacks}(J:\beta, I:\alpha,\Delta)$$
$$\textit{such that } K:\phi \in \mathsf{rebuttals}(J:\beta,\Delta).$$

*Proof.* Given that $J : \beta \in$ canonicalUndercuts$(I : \alpha, \Delta)$ then by Base Proposition Two there exists $K : \phi \in$ rebuttals$(J : \beta, \Delta)$ where $K = I$. Definition 3.9.2 of argument attack establishes that given $J : \beta \in$ canonicalUndercuts$(I : \alpha, \Delta)$ then also $J : \beta \in$ argumentAttacks$(I : \alpha, \Delta)$. Likewise given $K : \phi \in$ rebuttals$(J : \beta, \Delta)$ then $K : \phi \in$ argumentAttacks$(J : \beta, \Delta)$. Furthermore from Definition 4.2.1 of reflected attack, given this chain of two argument attacks plus the fact that $K = I$, and thus that $K \subseteq I$, proves that $K : \phi \in$ reflectedAttacks$(J : \beta, I : \alpha, \Delta)$. So if there exists $J : \beta \in$ canonicalUndercuts$(I : \alpha, \Delta)$ then there exists a $K : \phi \in$ reflectedAttacks$(J : \beta, I : \alpha, \Delta)$ such that $K : \phi \in$ rebuttals$(J : \beta, \Delta)$. □

### 4.2.6 Base Proposition Three: Rebuttal Engenders Rebuttal

I now present two further base propositions, Base Propositions Three and Four, as two further instances or consequences of reflectedAttacks(). Three is perhaps simple, but the fourth is more profound.

Base Proposition Three is the seemingly trivial observation that rebuttal always engenders rebuttal. I use the same format as used for Base Propositions One, Definition 4.2.1 and Two, Definition 4.2.2, to show this observation:

**Proposition 4.2.8. Base Proposition Three.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I : \alpha, J : \beta, K : \phi \in$ arguments$(\Delta)$.*

*If there exists* $J : \beta \in$ rebuttals$(I : \alpha, \Delta)$

*then there exists* $K : \phi \in$ rebuttals$(J : \beta, \Delta)$

*such that* $K = I$.

*Proof.* For these rebuttals it is a clear that $\alpha = \neg\beta$, $\phi = \neg\beta$ and thus $\alpha = \phi$. Proposition 2.6.5 establishing that rebuttal possesses a form of symmetry, states that $I : \alpha \in$ rebuttals$(J : \neg\alpha, \Delta)$ iff $J : \neg\alpha \in$ rebuttals$(I : \alpha, \Delta)$. The solution is that $K : \phi = I : \alpha$ so thus $K = I$. Therefore a rebuttal attack always engenders another rebuttal attack. □

This simple proposition is interesting as it fits with the general reflection function, of Definition 4.2.1. The engendered rebuttal is in fact a reflected argument.

**Proposition 4.2.9.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I : \alpha, J : \beta, K : \phi \in$ arguments$(\Delta)$.*

*If there exists* $J : \beta \in$ rebuttals$(I : \alpha, \Delta)$

*then there exists a* $K : \phi \in$ reflectedAttacks$(J : \beta, I : \alpha, \Delta)$

*such that* $K : \phi \in$ rebuttals$(J : \beta, \Delta)$.

*Proof.* Given that $J : \beta \in$ rebuttals$(I : \alpha, \Delta)$ then by Base Proposition Three there exists $K : \phi \in$ rebuttals$(J : \beta, \Delta)$ where $K = I$. Definition 3.9.2 of argument attack establishes that given $J : \beta \in$ rebuttals$(I : \alpha, \Delta)$ then also $J : \beta \in$ argumentAttacks$(I : \alpha, \Delta)$. Likewise given $K : \phi \in$ rebuttals$(J : \beta, \Delta)$ so $K : \phi \in$ argumentAttacks$(J : \beta, \Delta)$. Furthermore from Definition 4.2.1 of reflected attack, given this chain of two argument attacks plus the fact that $K = I$ so $K \subseteq I$ proves that $K : \phi \in$ reflectedAttacks$(J : \beta, I : \alpha, \Delta)$. So if there exists $J : \beta \in$ rebuttals$(I : \alpha, \Delta)$ then there exists a $K : \phi \in$ reflectedAttacks$(J : \beta, I : \alpha, \Delta)$ such that $K : \phi \in$ rebuttals$(J : \beta, \Delta)$. □

In this situation the reflected argument turns out to be the same as the original defending argument. What is different or generated and also predictable by this theory of reflection, however, is the attack. In contrast with earlier examples of reflection (Base Propositions One and Two) those base propositions had a new argument plus a new attack. For Base Proposition Three there certainly is a new attack, but the difference is in not having a new argument. If Base Propositions One, Two and Four did not exist, then Base Proposition Three would hardly be worth mentioning, but what is interesting is how it still fits with general reflection. It also is a necessary small step in showing further properties, as I now cover in Base Proposition Four.

### 4.2.7 Base Proposition Four: Undercut Engenders Undercut

Here I show that undercut engenders undercut. This observation employs the earlier Base Propositions One, Two and Three to draw this fresh conclusion. I use the same format as used for Base Propositions One to Three to show the observation.

**Proposition 4.2.10. Base Proposition Four.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I : \alpha$, $J : \beta$, $K : \phi \in$ arguments$(\Delta)$.*

*If there exists $J : \beta \in$ canonicalUndercuts$(I : \alpha, \Delta)$*

*then there exists $K : \phi \in$ canonicalUndercuts$(J : \beta, \Delta)$*

*such that $K \subseteq I$.*

*Proof.* First apply Base Proposition Two, Proposition 4.2.2: given $J : \beta$ is a canonical undercut of $I : \alpha$ then it follows that there exists a $I : \neg\beta \in$ arguments$(\Delta)$ which is a rebuttal of $J : \beta$. Now apply Base Proposition Three, Proposition 4.2.8, to give a rebuttal attack of that rebuttal argument, i.e. $J : \beta \in$ rebuttals$(I : \neg\beta, \Delta)$. This has just created a new attack, but not a new argument. Now apply Base Proposition One, Proposition 4.2.1, given this rebuttal attack, $J : \beta \in$ argumentAttacks$(I : \neg\beta, \Delta)$, there exists $K : \phi$ a canonical undercut of $J : \beta$. So given there exists $J : \beta \in$ canonicalUndercuts$(I : \alpha, \Delta)$ then there always exists $K : \phi \in$ canonicalUndercuts$(J : \beta, \Delta)$ where $K \subseteq I$. □

This base proposition, given the context of a chain of arguments could be referred to as the grandparent reflection. It is also the case that this reflected canonical undercut is a reflected argument, again fitting exactly with the earlier definition of reflection, Definition 4.2.1.

**Proposition 4.2.11.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I : \alpha$, $J : \beta$, $K : \phi \in$ arguments$(\Delta)$.*

*If there exists $J : \beta \in$ canonicalUndercuts$(I : \alpha, \Delta)$*

*then there exists a $K : \phi \in$ reflectedAttacks$(J : \beta, I : \alpha, \Delta)$*

*such that $K : \phi \in$ canonicalUndercuts$(J : \beta, \Delta)$.*

*Proof.* Given that $J : \beta \in$ canonicalUndercuts$(I : \alpha, \Delta)$ then by Base Proposition Four there exists $K : \phi \in$ canonicalUndercuts$(J : \beta, \Delta)$ where $K \subseteq I$. Definition 3.9.2 of argument attack establishes that given $J : \beta \in$ canonicalUndercuts$(I : \alpha, \Delta)$ then also $J : \beta \in$ argumentAttacks$(I : \alpha, \Delta)$. Likewise given $K : \phi \in$ canonicalUndercuts$(J : \beta, \Delta)$ so $K : \phi \in$ argumentAttacks$(J : \beta, \Delta)$. Furthermore from Definition 4.2.1 of reflected attack, given this chain of two argument attacks plus the fact that $K \subseteq I$ proves that $K : \phi \in$ reflectedAttacks$(J : \beta, I : \alpha, \Delta)$. So if there exists $J : \beta \in$ canonicalUndercuts$(I : \alpha, \Delta)$ then there exists a $K : \phi \in$ reflectedAttacks$(J : \beta, I : \alpha, \Delta)$ such that $K : \phi \in$ canonicalUndercuts$(J : \beta, \Delta)$. □

These four base propositions show that whenever there are there are two arguments in an argument chain, then there will always be a combination of reflected rebuttals and reflected undercuts extending this attack chain. Thus both types of attack, rebuttal and undercut, are seen to be reflected off any attack, be it rebuttal or undercut.

### 4.2.8 Reflexive Arguments and Reflection

To show how reflexive arguments fit with reflection, it helps first to clarify some behaviours of reflexive arguments. This clarification requires not just the definitions of reflexivity (Definition 2.4.7), rebuttal (Definition 2.6.7), canonical undercut (Definition 3.3.2), but also attack (Definition 3.9.2). If a reflexive argument A is attacked by an argument B, then B will be a member of the canonical undercuts of A *and* B will be a member of the rebuttals of A.

**Proposition 4.2.12.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I : \alpha, J : \beta \in$* arguments$(\Delta)$.

$$(J : \beta \textit{ is a reflexive argument and } I : \alpha \in \text{argumentAttacks}(J : \beta, \Delta))$$
$$\textit{iff } (I : \alpha \in \text{canonicalUndercuts}(J : \beta, \Delta) \textit{ and } I : \alpha \in \text{rebuttals}(J : \beta, \Delta)).$$

*Proof.* Because $J : \beta$ is a reflexive argument it follows that stripDeduction$(J : \beta) = \bigwedge$ stripAssumptions(formulae(label$(J : \beta), \Delta))$, i.e. that its claim formula is the same as the conjunction of the members of its support formulae. Clearly $J : \beta$ is attacked, so the attack could be $I : \alpha \in$ canonicalUndercuts$(J : \beta, \Delta)$ or $I : \alpha \in$ rebuttals$(J : \beta, \Delta)$, with the emphasis on the word 'or'. However, as $J : \beta$ is reflexive, with its claim a conjunction of its support, anything which is capable of attacking its support will also attack its claim and vice versa. So if one is attacked, both are attacked. It cannot be the case that there exists any $K : \alpha \in$ argumentAttacks$(J : \beta, \Delta)$ where $K \subsetneq I$ as then $I : \alpha$ would not be minimal and hence not an argument, so it is $I : \alpha$ that both rebuts and undercuts $J : \beta$ and not any such $K : \alpha$. Now for the reverse direction of the biconditional. As $I : \alpha$ can undercut the support and rebut the claim of $J : \beta$, it means that stripDeduction$(J : \beta) = \bigwedge$ stripAssumptions(formulae(label$(J : \beta), \Delta))$ so $J : \beta$ is reflexive and is attacked by $I : \alpha$. So in conclusion ($J : \beta$ is a reflexive argument and $I : \alpha \in$ argumentAttacks$(J : \beta, \Delta)$) iff ($I : \alpha \in$ canonicalUndercuts$(J : \beta, \Delta)$ and $I : \alpha \in$ rebuttals$(J : \beta, \Delta)$). □

Thus if an argument A is subject to both rebuttal from an argument B and also canonical undercut from an argument B, then A must be reflexive. If, however, A is subject to an undercut from B, and A is reflexive then A must also be subject to a rebuttal from B. Additionally, if A is subject to rebuttal from B and A is reflexive then A must also be subject to a canonical undercut from B. Thus an attack can be both a canonical undercut and a rebuttal.

Now, here is an example of reflection and reflexivity:

**Example 4.2.1. Reflection and reflexivity.** *Let* $\Delta = \{a:\alpha,\ b:\gamma,\ c:\gamma \to \neg\alpha\}$. *Thus* $\{a\}:\alpha,\ \{b,c\}: \neg\alpha,\ \{a\}:\neg(\gamma \wedge (\gamma \to \neg\alpha)) \in$ arguments$(\Delta)$. *The attack chain for this example has* $\{b,c\}:\neg\alpha$ *at the head of the chain, attacked by the reflexive argument* $\{a\}:\alpha$. *The tail of the chain is the reflected* $\{b,c\}:\neg\alpha \in$ reflectedAttacks$(\{a\}:\alpha,\ \{b,c\}:\neg\alpha,\ \Delta)$. *Observe that* $\{b,c\}:\neg\alpha \in$ rebuttals$(\{a\}:\alpha, \Delta)$ *and also* $\{b,c\}:\neg\alpha \in$ canonicalUndercuts$(\{a\}:\alpha, \Delta)$. *Later in this chapter the reflected argument rebuttal and the reflected canonical undercut are formally defined.*

Thus reflexive arguments are no exception in giving rise to reflections – which need to be understood in assessing debates. A point to draw from the above example is that by removing reflected arguments one also removes some of the 'double counting' arising from reflexive arguments.

Recalling the terms defined immediately after Definition 4.2.1, a three member attack chain has the reflector argument at its head, a mid-chain argument and reflected argument at its tail; reflexive arguments may reside in any of these three places. Looking first at the reflected argument: all reflected argument rebuttals are reflexive arguments. In practical debate most reflected canonical undercuts are not reflexive arguments, but there can exist reflected canonical undercuts that are reflexive. Consider $\Delta = \{a:\alpha, b:\neg\alpha\}$. Later, in Section 4.4 these reflexivity roles are revisited.

Any reflexive argument may or may not take the role of reflecting argument - depending on whether it is attacked or not. Thus any reflecting argument may or may not be reflexive. If the defending argument is a reflexive argument then there exists a reflected argument that is the same as the defending argument.

Any attacking argument (occupying the mid-point in the three member attack chain describing a Type I Reflection) may or may not be reflexive. If this attacking argument is reflexive then the reflected argument rebuttal will be the same argument as the reflected canonical undercut.

When counting arguments in judgements, care should be taken not to count the attacks on a reflexive argument twice, regardless of the presence or absence of reflection. This consideration is addressed in Section 6.7.

## 4.3 Reflection in Other Argumentation Frameworks

In the introduction I stated that there are four main formal approaches to defining an individual argument, namely i) minimal consistent subset arguments following classical logic (mincons), ii) defeasible logic argumentation schemes, iii) semi-abstract and iv) fully abstract. I have established that reflection occurs with the first one, so the question arises does it occur with the other three?

Given that the proofs presented in the semi-abstract approach papers, notably (Bondarenko et al., 1997), show it to encompass many of the defeasible logic argumentation schemes in the literature, it follows that this list of four formalisms is not a simple subdivision of the field of argumentation into equal quarters. The defeasible approach is noteworthy in its own right because of the central position in the argumentation aggregation literature of works of Pollock (Pollock, 1970; Pollock, 1974; Pollock, 1987; Pollock, 1992; Pollock, 2000) and of Chesñevar, García, Loui and Simari (Simari & Loui, 1992; Simari et al., 1994; Chesñevar et al., 2000; García & Simari, 2004; Chesñevar et al., 2005).

I now establish that the semi-abstract approach is so defined as to encompass and allow reflection. Some of the concrete manifestations of the semi-abstract approach, however, may move far enough away from classical logic, by imposing proof rule restrictions, to no longer contain reflection - hence an area deserving further research.

An argument in the ABA framework is defined as follows. I refer to Definitions 2.1, 2.2, 2.3 and 2.4 in (Dung et al., 2006) and have packaged four original definitions into one here for brevity.

**Definition 4.3.1.** *An assumption-based argument (ABA) argument is defined as follows:*

1. *A deductive system is a pair $\langle \mathcal{L}, \mathcal{R} \rangle$ where*
   (a) *$\mathcal{L}$ is a formal language consisting of countably many sentences,*
   (b) *$\mathcal{R}$ is a countable set of inference rules of the form $\frac{\alpha_1,\ldots,\alpha_n}{\alpha}$,*
   (c) *$\alpha \in \mathcal{L}$ is called the conclusion of the inference rule and*
   (d) *$\alpha_1, \ldots, \alpha_n$ are called the premises of the inference rule and $n \geq 0$.*
2. *A deduction of a conclusion $\alpha$ based on a set of premises P is a sequence $\beta_1, \ldots, \beta_m$ of sentences in $\mathcal{L}$, where $m > 0$ and $\alpha = \beta_m$, such that, for all $i = 1, \ldots, m$ :*
   (a) *$\beta_i \in P$, or*
   (b) *there exists $\frac{\alpha_1,\ldots,\alpha_n}{\beta_i} \in \mathcal{R}$ such that $\alpha_1, \ldots, \alpha_n \in \{\beta_1, \ldots, \beta_{i-1}\}$.*
3. *An assumption-based framework is a tuple $\langle \mathcal{L}, \mathcal{R}, \mathcal{K}, ^- \rangle$ where:*
   (a) *$\langle \mathcal{L}, \mathcal{R} \rangle$ is a deductive system.*
   (b) *$\mathcal{K} \subseteq \mathcal{L}, \mathcal{K} \neq \emptyset$. $\mathcal{K}$ is the set of candidate assumptions.*
   (c) *If $\alpha \in \mathcal{K}$, then there is no inference rule of the form $\frac{\alpha_1,\ldots,\alpha_n}{\alpha} \in \mathcal{R}$.*
   (d) *$^-$is a (total) mapping from $\mathcal{K}$ into $\mathcal{L}$. $\bar{\alpha}$ is the contrary of $\alpha$.*
4. *An ABA argument is a deduction whose premises are all assumptions.*

To refer to the parts of an individual ABA argument I reuse the unlabelled notation described following Proposition 2.4.4, where $\langle \Phi, \alpha \rangle$ is an argument. It can be observed that my language $\mathcal{L}$ is an instance of the more general $\mathcal{L}$ above and that $\mathcal{K}$ is an ABA knowledgebase of assumption formulae. The original uses $\mathcal{A}$ not $\mathcal{K}$ for its knowledgebase, but I have changed it to avoid a clash with the set of assumption labels defined in Chapter 2. Observe that the set of proof rules which make up classical logic is an instance of $\mathcal{R}$. Thus mincons are a kind of ABA argument, but not all ABA arguments are mincons. Example 1.1 in (Dung et al., 2006) presents arguments with minimal consistent sets of premises that infer their claims using classical logic.

An attack in the ABA framework (their Definition 2.5) is defined as follows: An ABA argument A ABA attacks an ABA argument B iff A contradicts an assumption in the set of assumptions on which B is based. I present that as a function now and limit it to classical logic, so that I can more easily reuse it in a further definition and proposition.

**Definition 4.3.2.** *Let $\mathcal{R}$ be classical logic, the ABA $\mathcal{L}$ be equal to the $\mathcal{L}$ of Definition 2.2.1, $\mathcal{K}$ be an ABA knowledgebase of assumptions and $^-$be $\neg$. Let $\langle\Phi,\alpha\rangle, \langle\Psi,\beta\rangle$ be ABA arguments derivable from $\mathcal{K}$.* **ABA attack**, *denoted* $\mathsf{attacks}_{ABA}(\langle\Phi,\alpha\rangle,\mathcal{K})$, *is the set of arguments that attack $\langle\Phi,\alpha\rangle$ such that:*

$$\langle\Psi,\beta\rangle \in \mathsf{attacks}_{ABA}(\langle\Phi,\alpha\rangle,\mathcal{K}) \text{ iff } \neg\beta \in \Phi.$$

The ABA attack is thus a generalisation of my Wigmore undercut, see Definition 3.2.1. I now show that reflection exists in this assumption-based framework. If exactly the same definition of reflection as used earlier in this chapter, Definition 4.2.1, is used then the second argument in the attack chain of ABA arguments must be reflexive. Thus if the unlabelled knowledgebase $\mathcal{K} = \{\beta, \beta \rightarrow \alpha, \neg\alpha\}$, then the ABA arguments include $\langle\{\beta, \beta \rightarrow \alpha\}, \alpha\rangle, \langle\{\neg\alpha\}, \neg\alpha\rangle$. Consequently the head of the attack chain is $\langle\{\beta, \beta \rightarrow \alpha\}, \alpha\rangle$, which is subject to Wigmore undercut by the reflexive $\langle\{\neg\alpha\}, \neg\alpha\rangle$. Consequently $\langle\{\beta, \beta \rightarrow \alpha\}, \alpha\rangle$ is a reflected Wigmore undercut of $\langle\{\neg\alpha\}, \neg\alpha\rangle$, so using exactly the same reflection definition is unsatisfactory due to its triviality of only working for reflexive arguments. What is needed is a slightly modified or generalised definition of reflection, which I call a reflected ABA attack, to see that, given certain assumptions, reflection always occurs in the ABA framework regardless of whether the ABA attack is reflexive or not. In the following definition I limit the analysis to classical logic.

**Definition 4.3.3.** *Let $\mathcal{R}$ be classical logic, the ABA $\mathcal{L}$ be equal to the $\mathcal{L}$ of Definition 2.2.1, $\mathcal{K}$ be an ABA knowledgebase of assumptions and $^-$be $\neg$. Let $\langle\Phi,\alpha\rangle, \langle\Psi,\neg\phi_i\rangle$ be ABA arguments derivable from $\mathcal{K}$. The set of* **reflected ABA attacks** *arising from the ABA attack of $\langle\Psi,\neg\phi_i\rangle$ on $\langle\Phi,\alpha\rangle$, denoted* $\mathsf{reflectedAttacks}_{ABA}(\langle\Psi,\neg\phi_i\rangle,\langle\Phi,\alpha\rangle,\mathcal{K})$ *is such that:*

$$\begin{aligned}
&\langle\Gamma,\neg\psi_j\rangle \in \mathsf{reflectedAttacks}_{ABA}(\langle\Psi,\neg\phi_i\rangle,\langle\Phi,\alpha\rangle,\mathcal{K})\\
&\quad \text{iff } \langle\Psi,\neg\phi_i\rangle \in \mathsf{attacks}_{ABA}(\langle\Phi,\alpha\rangle,\mathcal{K}),\\
&\qquad\quad \langle\Gamma,\neg\psi_j\rangle \text{ is an ABA argument derivable from } \mathcal{K} \text{ and}\\
&\qquad\quad \Gamma = \Lambda \cup \Psi \backslash \psi_j, \text{ where } \Lambda \subseteq \Phi.
\end{aligned}$$

The above definition is a generalisation of the basic reflected attacks function, Definition 4.2.1, that I examine throughout much of this thesis. While the reflection function I examine in detail is for canonical undercuts, the above is also viable for Wigmore and Pollock undercuts. An area for further research would be to generalise all of the reflection functions in this thesis to directly cover Wigmore and Pollock undercuts as well as canonical undercuts.

Given their prerequisites, reflected ABA attacks always exist, as I now prove.

**Proposition 4.3.1.** *Let $\mathcal{R}$ be classical logic, the ABA $\mathcal{L}$ be equal to the $\mathcal{L}$ of Definition 2.2.1, $\mathcal{K}$ be an*

*ABA knowledgebase of assumptions and $^-$ be $\neg$. Let $\langle\Phi,\alpha\rangle$, $\langle\Psi,\neg\phi_i\rangle$ be ABA arguments derived from $\mathcal{K}$.*

$$\textit{If } \langle\Psi,\neg\phi_i\rangle \in \mathsf{attacks}_{ABA}(\langle\Phi,\alpha\rangle,\mathcal{K})$$

$$\textit{then } \mathsf{reflectedAttacks}_{ABA}(\langle\Psi,\neg\phi_i\rangle,\langle\Phi,\alpha\rangle,\mathcal{K}) \neq \emptyset,$$

$$\textit{otherwise } \mathsf{reflectedAttacks}_{ABA}(\langle\Psi,\neg\phi_i\rangle,\langle\Phi,\alpha\rangle,\mathcal{K}) = \emptyset.$$

*Proof.* Clearly for $\langle\Psi,\neg\phi_i\rangle$ to be an ABA attack on $\langle\Phi,\alpha\rangle$, by the definition of ABA attack and given $\Phi = \{\phi_1,\ldots,\phi_n\}$, it must be the case that $\phi_i \in \Phi$. Also if $\langle\Gamma,\neg\psi_j\rangle$ is an ABA attack on $\langle\Psi,\neg\phi_i\rangle$, letting $\Psi = \{\psi_1,\ldots,\psi_m\}$, it follows that $\psi_j \in \Psi$. Given that $\langle\Psi,\neg\phi_i\rangle$ is an ABA argument then by $\vee$I so is $\langle\Psi,\neg\phi_i \vee \neg\phi_j\rangle$, which is also the case when $\phi_j \in \Phi$. Thus $\langle\Psi,\neg(\phi_1 \wedge \ldots \wedge \phi_n)\rangle$ is also an ABA argument that exists if $\langle\Psi,\neg\phi_i\rangle$ exists. Now using $\wedge$I it follows that $\psi_1 \wedge \ldots \wedge \psi_m \rightarrow \neg(\phi_1 \wedge \ldots \wedge \phi_n)$. By contraposition $\phi_1 \wedge \ldots \wedge \phi_n \rightarrow \neg(\psi_1 \wedge \ldots \wedge \psi_m)$. Furthermore by $\vee$E, if assumptions are added to the antecedent then their negations can be ruled out from the consequent, i.e. if $\phi_1 \wedge \ldots \wedge \phi_n \rightarrow \neg(\psi_j \wedge \psi_k)$ then $\{\phi_1,\ldots,\phi_n\} \cup \{\psi_k\} \vdash \neg\psi_j$. So by $\vee$E it follows that $\Phi \cup \Psi\backslash\psi_j \vdash \neg\psi_j$, where the $\backslash\psi_j$ prevents ex falso quodlibet. This statement can be simplified as not every member of the set $\Phi$ may be necessary to infer $\neg(\phi_1 \wedge \ldots \wedge \phi_m)$. Thus if $\langle\Phi \cup \Psi\backslash\psi_j,\neg\psi_j\rangle$ and $\Lambda \subseteq \Phi$ then $\langle\Lambda \cup \Psi\backslash\psi_j,\neg\psi_j\rangle$ is also an ABA argument. Thus with $\Gamma = \Lambda \cup \Psi\backslash\psi_j$, where $\Lambda \subseteq \Phi$, it follows, given the prerequisites, that the reflected ABA argument $\langle\Gamma,\neg\psi_j\rangle$ always exists. Absent any one of the prerequisites then $\langle\Gamma,\neg\psi_j\rangle$ will not exist. Thus if $\langle\Psi,\neg\phi_i\rangle \in \mathsf{attacks}_{ABA}(\langle\Phi,\alpha\rangle,\mathcal{K})$ then $\mathsf{reflectedAttacks}_{ABA}(\langle\Psi,\neg\phi_i\rangle,\langle\Phi,\alpha\rangle,\mathcal{K}) \neq \emptyset$, otherwise $\mathsf{reflectedAttacks}_{ABA}(\langle\Psi,\neg\phi_i\rangle,\langle\Phi,\alpha\rangle,\mathcal{K}) = \emptyset$. □

What is seen here is undercut to undercut reflection with a similar, but not identical form, to that covered in Base Proposition Four, see Proposition 4.2.10. Base Proposition Four addresses the situation with canonical undercuts and the above proposition describes essentially the same situation, but this time with a form of Wigmore undercuts.

This proposition shows that if A ABA attacks B then there always exists a counter ABA attack of C ABA attacking B. Thus if A is defeated by B, then B is defeated by C and consequently A is reinstated by C. This inevitable reinstatement raises some interesting issues around some of the semantics used in the ABA framework. Questions suggestive by this initial analysis, each of which is a topic for further research, include the following.

- What other forms of reflection exist within the ABA framework?
- What are the minimum sets of proof rules needed for each form of reflection?
- What alternative minimum sets of proof rules can be used to show each reflection?
- Which of the non-monotonic logics that are special cases of the ABA framework permit which forms of reflection?
- Which semantics are affected by the observation that given the right prerequisites each attacked argument is automatically subject to counterattack and thus reinstatement?

- Which of the semantics referred to in ABA papers are affected by the existence of which forms of reflection and what is the effect?

Referring back to the question of reflection with defeasible logic, most defeasible logics do not have contraposition so that would appear to rule out some forms of reflection. Even without contraposition, however, some other forms of reflection, such as the reflected rebuttal (see Definition 4.7.1 and Proposition 4.7.2) are possible. Therefore it would appear that defeasible logics may also contain reflection and warrant further research.

In the fully abstract approach of (Dung, 1993) there is full clarity that the arguments and deduction rules are undefined and hence there can be no assumption of classical logic or even of assumptions and a claim. So here it cannot be said whether reflection occurs or not.

In conclusion, reflection is not unique to minimal consistent arguments with rebuttal and canonical undercut, but rather is also a feature of other definitions of argument and attack in the literature.

## 4.4 Reflected Canonical Undercuts

I now define the reflected canonical undercut and go more deeply into its properties. This function outputs a set of arguments, all of which are canonical undercuts, that arise through reflection, from a single pair of attacking arguments. The attack could either be a rebuttal or an undercut. The reflected attacks function is Definition 4.2.1 and the canonical undercuts function is Definition 3.3.2.

**Definition 4.4.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha,\ J:\beta \in$* arguments($\Delta$). *The set of* **reflected canonical undercuts**, *denoted* reflectedCanonicalUndercuts($J:\beta, I:\alpha, \Delta$), *attacking $J:\beta$, reflected off $I:\alpha$, is such that:*

$$\begin{aligned}
&\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)\\
&= \{K:\phi \in \mathsf{arguments}(\Delta) \mid\\
&\qquad K:\phi \in \mathsf{reflectedAttacks}(J:\beta, I:\alpha, \Delta) \textit{ and}\\
&\qquad K:\phi \in \mathsf{canonicalUndercuts}(J:\beta, \Delta)\}.
\end{aligned}$$

I now give an example of a reflected canonical undercut. As in earlier chapters the sake of brevity I skip the repeated restatement of the canonical ordering constraint.

**Example 4.4.1. Reflected canonical undercut.** *Let $\Delta = \{a:\alpha, b:\alpha \rightarrow \beta, c:\neg\beta\}$. Let $J = \{a,b\}, I = \{c\}$. It follows that $J:\beta, I:\neg\beta,\ \emptyset:(\alpha \wedge (\alpha \rightarrow \beta)) \rightarrow \beta,\ \emptyset:\neg\beta \rightarrow \neg(\alpha \wedge (\alpha \rightarrow \beta)), I:\neg(\alpha \wedge (\alpha \rightarrow \beta)), \in$* arguments($\Delta$). *Thus $I:\neg(\alpha \wedge (\alpha \rightarrow \beta)) \in$* reflectedCanonicalUndercuts($J:\beta, I:\neg\beta, \Delta$). *This follows as clearly $I \subseteq I$ and thus $I:\neg(\alpha \wedge (\alpha \rightarrow \beta))$ is a reflected canonical undercut.*

The above is a simple example where the support of the undercut is the same as the support of the rebuttal, i.e. no strict subsets are involved. I have shown some of the tautologies which exist in the set of arguments as an aid to seeing the steps of logic that lead to the reflected undercut. To infer the existence of the argument $J:\neg(\alpha \wedge (\alpha \rightarrow \beta))$ only requires the assumption $c:\neg\beta$. However, to deduce that this same argument $J : \neg(\alpha \wedge (\alpha \rightarrow \beta))$ is a reflected canonical undercut requires three assumptions $a:\alpha, b:\alpha \rightarrow \beta$ and $c:\neg\beta$.

So a reflected canonical undercut is simply a canonical undercut that is also a reflected argument. Every member of the set of arguments $\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)$ has the same claim and thus can always be represented as a confirmation. Thus $Z:\phi = \mathsf{confirm}(\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta))$ where for each $K \in Z$ the constraints of the above definition apply. To be more specific, the set of reflected canonical undercut arguments of a given rebutted argument can be precisely represented as a mono-target preclusive undercut. Depending upon $\Delta$ this preclusion may be multi-attack or mono-attack.

Given a single contradiction, comprising of an argument for $\alpha$ and an argument for $\neg\alpha$, it is possible to deduce the existence of at least two canonical undercuts (one attacking the argument for $\alpha$ and the other attacking the argument for $\neg\alpha$). The reflected canonical undercuts function is given a plural name because combining just one attacking argument with just one reflector argument can give rise to many reflected canonical undercuts (that is many arguments even just attacking $\alpha$).

### 4.4.1 Properties of Reflected Canonical Undercuts

A simple corollary to the Proposition 4.2.5 that reflected arguments always exist is the following. Given a defending argument $I:\alpha$ and its attacker $J:\beta$ then reflected canonical undercuts always exist, as shown in the following proposition.

**Proposition 4.4.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\beta \in$* $\mathsf{arguments}(\Delta)$.

*If* $J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$

*then* $\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta) \neq \emptyset$,

*otherwise* $\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta) = \emptyset$.

*Proof.* The attack of $J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$ could be either a rebuttal or a canonical undercut. If $J:\beta \in \mathsf{rebuttals}(I:\alpha, \Delta)$ then Base Proposition One shows that there exists $K:\phi \in \mathsf{canonicalUndercuts}(J:\beta, \Delta)$ with $K \subseteq I$. If $J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta)$ then Base Proposition Four establishes that there exists $K:\phi \in \mathsf{canonicalUndercuts}(J:\beta, \Delta)$ with $K \subseteq I$. Thus in either case there exists $K:\phi \in \mathsf{canonicalUndercuts}(J:\beta, \Delta)$ with $K \subseteq I$. It also follows, by Definition 4.2.1 of reflected attack that $K:\phi \in \mathsf{reflectedAttacks}(J:\beta, \Delta)$. Thus by Definition 4.4.1 of reflected canonical undercut, as $K:\phi$ is both a reflected attack and a canonical undercut it will be the case that $K:\phi \in \mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)$ and so $\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta) \neq \emptyset$. However if $I:\alpha \notin \mathsf{arguments}(\Delta)$ or $J:\beta \notin \mathsf{arguments}(\Delta)$ or $J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$ then $\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta) = \emptyset$. □

The support of a reflected canonical undercut is always equal to the support of the reflector argument, or a strict subset of the support of the reflector argument, regardless of whether the attack on the reflector is a rebuttal or an undercut.

**Proposition 4.4.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\beta \in$* $\mathsf{arguments}(\Delta)$. *Let $J:\beta \in$* $\mathsf{canonicalUndercuts}(I:\alpha,\Delta)$ *or $J:\beta \in$* $\mathsf{rebuttals}(I:\alpha,\Delta)$.

$$\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta) = \{K:\phi \in \mathsf{arguments}(\Delta) \mid K \subseteq I\}.$$

*Proof.* By Definition 4.4.1, members of $\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha,\Delta)$, must be reflected attacks $\mathsf{reflectedAttacks}(J:\beta, I:\alpha,\Delta)$ that are also $\mathsf{canonicalUndercuts}(J:\beta,\Delta)$. Definition 4.2.1 of reflected attack and Proposition 4.2.5 (reflections always exist) establish that a $\mathsf{reflectedAttacks}(J:\beta, I:\alpha,\Delta)$ always arises from any $\mathsf{argumentAttacks}(I:\alpha,\Delta)$. From Definition 3.9.2, that argument attack on the reflector $I:\alpha$, could be either a member of $\mathsf{rebuttals}(I:\alpha,\Delta)$ or of $\mathsf{canonicalUndercuts}(I:\alpha,\Delta)$. Base Proposition One, Proposition 4.2.1, states that a reflected attack off the rebuttal is a canonical undercut $K:\phi \in \mathsf{canonicalUndercuts}(J:\beta,\Delta)$ with $K \subseteq I$. Base Proposition Four, Proposition 4.2.10, states that a reflected attack off the canonical undercut is another canonical undercut $K:\phi \in \mathsf{canonicalUndercuts}(J:\beta,\Delta)$ again with $K \subseteq I$. Thus in either case of reflected canonical undercut, it follows that $K \subseteq I$. So in conclusion $\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha,\Delta) = \{K:\phi \in \mathsf{arguments}(\Delta) \mid (J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha,\Delta)$ or $J:\beta \in \mathsf{rebuttals}(I:\alpha,\Delta))$ and $K \subseteq I\}$. □

The cardinality of the reflected canonical undercuts function is as follows.

**Proposition 4.4.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\beta \in$* $\mathsf{arguments}(\Delta)$.

$$0 \leq |\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)| \leq \mathsf{midLattice}(I).$$

*Proof.* If the prerequisites don't exist, e.g. if either $I:\alpha, J:\beta \notin \mathsf{arguments}(\Delta)$, then there are no reflected canonical undercuts and $\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha,\Delta) = \emptyset$. For the upper limit consider two arguments $K:\phi, L:\phi \in \mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha,\Delta)$. Every $K:\phi \in \mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha,\Delta)$ must be such that $K \subseteq I$ and must have the same claim. The number of subsets of $I$ is $2^{|I|}$. However it cannot be the case that $K \subseteq L$, nor $L \subseteq K$. Consequently the mid lattice function must be used to give the maximum number of subsets that are not subsets of each other. Subsets of assumption labels in $I$ yield the supports of all the possible reflected arguments. □

Furthermore, reflected canonical undercuts are always canonical undercuts, but not necessarily vice versa.

**Proposition 4.4.4.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha,\ J:\beta \in$* $\mathsf{arguments}(\Delta)$.

$$\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta) \subseteq \mathsf{canonicalUndercuts}(J:\beta,\Delta).$$

*Proof.* The function $\mathsf{canonicalUndercuts}(J:\beta,\Delta)$ is defined in Definition 3.3.2. Definition 4.4.1 defines $\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha,\Delta)$ as a canonical undercut that is also a reflected attack, for which see Definition 4.2.1. The additional conditions to be a reflected attack are that $J : \beta$ argument attacks $I : \alpha$ and that $K \subseteq I$. Thus for a canonical undercut canonicalUndercuts($J : \beta, \Delta$) to also be a reflected canonical undercut reflectedCanonicalUndercuts($J : \beta, I : \alpha, \Delta$) there are additional conditions to meet. Thus for any set of canonical undercuts some of the set members will not meet the additional constraints. Therefore reflectedCanonicalUndercuts($J : \beta, I : \alpha, \Delta$) $\subseteq$ canonicalUndercuts($J : \beta, \Delta$).

□

A simple corollary follows from the logic of the proof:

$$|\text{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)| \leq |\text{canonicalUndercuts}(J:\beta, \Delta)|.$$

### 4.4.2 A Motivational Example of Reflected Undercut

I now give a motivational example of reflected canonical undercut where I provide linguistic gloss for the individual formulae and discuss the common sense meaning and implication of this reflection. This subsection continues with the approach of Example 4.4.1 and expands on it by including gloss, as follows. The formulae in the knowledgebase are shown in Table 4.1.

| **Formula** | **Structure** | **Assumption** | **Informal Gloss** |
|---|---|---|---|
| $\alpha$ | atomic | $a : \alpha$ | It's raining. |
| $\beta$ | atomic | not assumed | Bring an umbrella. |
| $\gamma$ | atomic | $b : \gamma$ | Wear a good raincoat. |
| $\alpha \to \beta$ | compound | $c : \alpha \to \beta$ | If it's raining then bring an umbrella. |
| $\gamma \to \neg\beta$ | compound | $d : \gamma \to \neg\beta$ | If one wears a good raincoat then one need not bring an umbrella. |

Table 4.1: Motivational Example of Reflected Undercut

Thus the assumptions in this small debate are just four formulae $\Delta = \{a : \alpha, b : \gamma, c : \alpha \to \beta, d : \gamma \to \neg\beta\}$. Thus the arguments include $\{a, c\} : \beta, \{b, d\} : \neg\beta, \{a, c\} : \neg(\gamma \wedge (\gamma \to \neg\beta)), \{b, d\} : \neg(\alpha \wedge (\alpha \to \beta)) \in$ arguments($\Delta$). The relevant resultant arguments and their interactions are shown in the argument graph of Figure 4.2 below.

This pair of rebutting arguments yields two reflected canonical undercuts: the first is $\{a, c\} : \neg(\gamma \wedge (\gamma \to \neg\beta))$ with a gloss of '*It's not the case that (one is wearing a good raincoat and that if one wears a good raincoat then one need not bring an umbrella). This follows because it's raining and if it's raining then one brings an umbrella.*' The second reflected canonical undercut is $\{b, d\} : \neg(\alpha \wedge (\alpha \to \beta))$ with a gloss of '*Either it's not raining or it's not the case that when its raining you should bring an umbrella. This follows because one has a raincoat and if one wears a raincoat then one need not bring an umbrella.*' While these are logically valid arguments, the reflections would not normally appear in professional debate.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\{a,c\}:\beta \arrow[r, shift right, "\{a,c\}:\beta \in \mathsf{rebuttals}(\{b,c\}:\neg\beta,\Delta)"'] & \{b,d\}:\neg\beta \arrow[l, shift right, "\{b,d\}:\neg\beta \in \mathsf{rebuttals}(\{a,c\}:\beta,\Delta)"'] \\
\{b,d\}:\neg(\alpha \wedge (\alpha \rightarrow \beta)) \arrow[u, "\{b,d\}:\neg(\alpha\wedge(\alpha\rightarrow\beta)) \in \mathsf{canonicalUndercuts}(\{a,c\}:\beta,\Delta)"] & \{a,c\}:\neg(\gamma \wedge (\gamma \rightarrow \neg\beta)) \arrow[u, "\{a,c\}:\neg(\gamma\wedge(\gamma\rightarrow\neg\beta)) \in \mathsf{canonicalUndercuts}(\{b,d\}:\neg\beta,\Delta)"']
\end{tikzcd}$$

$$\{b,d\}:\neg(\alpha \wedge (\alpha \rightarrow \beta)) \in \mathsf{reflectedCanonicalUndercuts}(\{a,c\}:\beta, \{b,d\}:\neg\beta, \Delta)$$
$$\{a,c\}:\neg(\gamma \wedge (\gamma \rightarrow \neg\beta)) \in \mathsf{reflectedCanonicalUndercuts}(\{b,d\}:\neg\beta, \{a,c\}:\alpha, \Delta)$$

Figure 4.2: Reflected Canonical Undercuts Shown in an Argument Graph

## 4.5 Direct Canonical Undercuts

Not all canonical undercuts are reflected canonical undercuts. I now define the direct canonical undercut simply as those canonical undercuts that are not reflected. I will show that they do exist, but do not always exist. I now build on Definition 3.3.2 of canonical undercut and 4.4.1 of reflected canonical undercut.

**Definition 4.5.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $I : \alpha, J : \beta \in$ $\mathsf{arguments}(\Delta)$. *The set of* **direct canonical undercuts** *of* $J : \beta$, *not reflected off* $I : \alpha$, *denoted* $\mathsf{directCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)$, *is such that:*

$$\begin{aligned}
&\mathsf{directCanonicalUndercuts}(J:\beta, I:\alpha, \Delta) \\
&\quad = \mathsf{canonicalUndercuts}(J:\beta, \Delta) \setminus \mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta).
\end{aligned}$$

I call $K : \phi$ a **direct** (or unreflected) argument and as before the head of the chain, $I : \alpha$ is the reflecting argument or reflector. There are two situations where reflections do not occur; either there is no reflector or the reflector exists but is not attacked. For direct canonical undercuts, predictions can be made about the support of the attacking argument:

**Proposition 4.5.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae, let* $I:\alpha$, $J:\beta$, $K:\phi \in$ $\mathsf{arguments}(\Delta)$ *and let* $J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$.

$$\textit{If } K:\phi \in \mathsf{directCanonicalUndercuts}(J:\beta, I:\alpha, \Delta) \textit{ then } K \not\subseteq I.$$

*Proof.* Proposition 4.4.2 establishes that if $K : \phi \in \mathsf{reflectedCanonicalUndercuts}(J : \beta, I : \alpha, \Delta)$ then $K \subseteq I$. Definition 3.3.2 of canonical undercut imposes no constraint on the support of the attacking argument, $K : \phi$, relative to any reflector, $I : \alpha$, so in that case for any $I : \alpha$ some rebuttals may have $K \subseteq I$ and others $K \not\subseteq I$. The definition of direct canonical undercut, Definition 4.5.1 immediately above, uses set difference to remove the reflected canonical undercuts from the full set of canonical undercuts, thus the only undercuts remaining in directCanonicalUndercuts$(J : \beta, I : \alpha, \Delta)$ must have $K \not\subseteq I$. So in conclusion if $K : \phi \in$ directCanonicalUndercuts$(J : \beta, I : \alpha, \Delta)$ then $K \not\subseteq I$. □

It will also be the case that $K \not\subseteq J$ as if it were then, as for any attack, it would infer $J \vdash \bot$; likewise for $J \not\subseteq I$. So far, this concept of direct undercut is weak as the existence of a specific direct canonical undercut depends on the choice of reflector, i.e. $I : \alpha$. For example, there could exist an $L : \alpha \in$ rebuttals$(J : \neg\alpha, \Delta)$ such that $K \subseteq L$ and thus if $L : \alpha$ were passed in as the reflecting argument it would render $K : \phi$ reflected. This potential weakness is removed once I progress to the consideration of sets of maximal direct arguments in Sections 5.3 and 5.6.

A particular canonical undercut (which is a relationship between two arguments) must be either direct or reflected. A canonical undercut cannot be both. An individual argument, however, can be both direct and reflected at the same time as shown by the following example.

**Example 4.5.1. Both direct and reflected argument.** *Let* $\Delta = \{a : \pi,\ b : \pi \to \alpha,\ c : \gamma \wedge \neg\alpha,\ d : \lambda,\ e : \lambda \to \neg\gamma\}$. *Let* $I = \{a, b\}$, $J = \{c\}$, $L = \{d, e\}$. *Thus* $I : \alpha$, $J : \neg\alpha$, $J : \gamma$, $L : \neg(\gamma \wedge \neg\alpha)$, $L : \neg\gamma \in$ arguments$(\Delta)$. *This example involves two chains, each of three arguments. The first chain, from head to tail is* $I : \alpha, J : \neg\alpha, L : \neg(\gamma \wedge \neg\alpha)$. *The second chain, again from head to tail is* $L : \neg\gamma, J : \gamma, L : \neg(\gamma \wedge \neg\alpha)$. *For the first chain the defending argument is* $I : \alpha$. *The attacking argument is* $J : \neg\alpha$ *where* $J : \neg\alpha \in$ rebuttals$(I : \alpha, \Delta)$. *This attacking argument is in turn attacked by a canonical undercut, namely* $L : \neg(\gamma \wedge \neg\alpha) \in$ canonicalUndercuts$(J : \neg\alpha, \Delta)$. *As* $\{d, e\} \not\subseteq \{a, b\}$ *so* $L \not\subseteq I$ *then* $L : \neg(\gamma \wedge \neg\alpha) \notin$ reflectedCanonicalUndercuts$(J : \neg\alpha, I : \alpha, \Delta)$. *So it follows that* $L : \neg(\gamma \wedge \neg\alpha) \in$ directCanonicalUndercuts$(J : \neg\alpha, I : \alpha, \Delta)$. *Turning now to the second chain, the defending argument is* $L : \neg\gamma$. *The attacking argument is* $J : \gamma$ *and as clearly* $L \subseteq L$, $L : \neg(\gamma \wedge \neg\alpha)$ *is a reflected canonical undercut. So one argument can be both direct and reflected, that is* $L : \neg(\gamma \wedge \neg\alpha) \in$ reflectedCanonicalUndercuts$(J : \gamma, L : \neg\gamma, \Delta)$ *and also* $L : \neg(\gamma \wedge \neg\alpha) \in$ directCanonicalUndercuts$(J : \neg\alpha, I : \alpha, \Delta)$.

Thus it is the *context* which determines whether a particular argument is direct or reflected. Therefore reflection is a relative property of arguments and not an absolute property. This point emphasises the need for placing arguments in suitable graphs or trees to provide the right context for determining which arguments are then reflected and which are direct. It also shows the importance of passing in the reflector, $J : \beta$, in Definition 4.5.1.

Thus there are two situations where direct canonical undercuts are found: one is where there is no reflector; the other is where there is a reflector, but the undercut in question is not reflected. I now give an example of such a direct canonical undercut where the argument $I : \alpha$ has no reflector.

**Example 4.5.2. Direct canonical undercut without reflector.** *Let* $\Delta = \{a : \alpha \wedge \beta, b : \neg\beta\}$. *Then* $\{a\} : \alpha, \{a\} : \beta, \{b\} : \neg\beta, \{b\} : \neg\beta \vee \neg\alpha, \{b\} : \neg(\alpha \wedge \beta) \in$ arguments$(\Delta)$. *Thus the defending argument* $\{a\} : \alpha$ *has a canonical undercut of* $\{b\} : \neg(\alpha \wedge \beta)$, *but no rebuttal. It has no rebuttal because* $\Delta \nvdash \neg\alpha$. *So* $\{b\} : \neg(\alpha \wedge \beta) \in$ canonicalUndercuts$(\{a\} : \alpha, \Delta)$, *but* $\{c\} : \neg\alpha \notin$ arguments$(\Delta)$ *Thus* $\{b\} : \neg(\alpha \wedge \beta) \in$ directCanonicalUndercuts$(\{a\} : \alpha, \{c\} : \neg\alpha, \Delta)$ *is a direct canonical undercut.*

The second situation is when the reflector argument $I:\alpha$ exists, but the undercut in question, $K:\phi$, is not a reflected undercut. This second situation is more interesting as given the result of Proposition 4.4.1, that reflected canonical undercuts always exist, it is perhaps somewhat unexpected. I now give an example to show that not all canonical undercuts are reflected canonical undercuts.

**Example 4.5.3. Direct canonical undercut with reflector.** *Let* $\Delta = \{a:\alpha \wedge \beta, b:\neg\beta, c:\neg\alpha\}$. *Then* $\{a\}:\alpha, \{c\}:\neg\alpha, \{b\}:\neg\beta, \{b\}:\neg\beta \vee \neg\alpha, \{b\}:\neg(\alpha \wedge \beta), \{c\}:\neg\beta \vee \neg\alpha, \{c\}:\neg(\alpha \wedge \beta) \in \mathsf{arguments}(\Delta)$. *Thus the defending argument* $\{a\}:\alpha$ *has two canonical undercuts namely* $\{b\}:\neg(\alpha\wedge\beta), \{c\}:\neg(\alpha\wedge\beta) \in \mathsf{canonicalUndercuts}(\{a\}:\alpha, \Delta)$. *If follows that* $\{c\}:\neg(\alpha \wedge \beta) \in \mathsf{reflectedCanonicalUndercuts}(\{a\}:\alpha, \{c\}:\neg\alpha, \Delta)$, *but* $\{b\}:\neg(\alpha \wedge \beta) \notin \mathsf{reflectedCanonicalUndercuts}(\{a\}:\alpha, \{c\}:\neg\alpha, \Delta)$ *as* $\{b\} \not\subseteq \{c\}$. *Thus* $\{b\}:\neg(\alpha \wedge \beta) \in \mathsf{directCanonicalUndercuts}(\{a\}:\alpha, \{c\}:\neg\alpha, \Delta)$ *is a direct canonical undercut.*

Clearly in the above discussion a canonical undercut $K:\neg(\phi_1 \wedge \ldots \wedge \phi_n)$ is either a member of reflectedCanonicalUndercuts or it is not. To emphasise the key point: for a given reflector (which could be the empty set) a canonical undercut is either direct or reflected; it cannot be both. So some canonical undercuts are a function of a reflector, whilst others are not. The number of direct canonical undercuts that an argument has is clearly unrelated to any reflectors and is instead purely a function of the contents of $\Delta$.

As the number of direct canonical undercuts is not constrained by any reflector argument, it is the same as for canonical undercuts, Proposition 3.3.3:

**Proposition 4.5.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $I:\alpha, J:\beta \in \mathsf{arguments}(\Delta)$.

$$0 \;\leq\; |\mathsf{directCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)| \;<\; \mathsf{midLattice}(\Delta).$$

*Proof.* The proof is the same as for the cardinality of canonical undercuts, Proposition 3.3.3. An undercut of $J:\beta$ given reflector $I:\alpha$ is either direct or reflected. If none of them are reflected then all canonical undercuts are direct. □

This analysis thus shows that a single argument $I:\alpha$, that attacks a reflector $J:\beta$, can be attacked by a mixture of direct and reflected canonical undercuts.

## 4.6 Mono-Pair Enlarged Reflections

I have shown that given an attacking argument and a reflector then a reflected canonical undercut always exists. While this is true, there is more to the matter: the number of reflected canonical undercuts is one or more than one. There are two reasons why there may be more reflected arguments than expected. The first stems from support shortening resulting in what I call a mono-pair enlarged reflection (Definition 4.6.1 below); the second reason I address shortly (in Section 5.7.4 on multi-pair enlarged reflections).

To introduce mono-pair enlarged reflections I start by giving an example that demonstrates the case where the support of the reflected canonical undercut is a strict subset of the support of the reflector.

**Example 4.6.1. Support shortening.** *Let* $\Delta = \{a : \beta, b : \beta \rightarrow \alpha, c : \neg\beta, d : \neg\beta \rightarrow \neg\alpha\}$. *Let* $P = \{a, b\}, Q = \{c, d\}, R = \{c\}$. *Therefore* $P:\alpha, Q:\neg\alpha, R:\neg\beta, R:\neg\beta \vee (\beta \rightarrow \alpha), R:\neg(\beta \wedge (\beta \rightarrow \alpha)) \in \mathsf{arguments}(\Delta)$. *The defending argument is* $P : \alpha$. *The reflector is the rebuttal* $Q : \neg\alpha$. *One might think that* $Q : \neg(\beta \wedge (\beta \rightarrow \alpha)) \in \mathsf{reflectedCanonicalUndercuts}(P : \alpha, Q : \neg\alpha, \Delta)$ *would be a reflected canonical undercut, however it is not even a valid argument as the support is not minimal. Rather* $R:\neg(\beta \wedge (\beta \rightarrow \alpha)) \in \mathsf{reflectedCanonicalUndercuts}(P:\alpha, Q:\neg\alpha, \Delta)$ *is a reflected canonical undercut. In this valid case the support of the reflected canonical undercut* $R$ *is a strict subset of the support of the rebuttal* $Q$.

There can be more than one possible minimal consistent subset that implies $\neg(\phi_1 \wedge \ldots \wedge \phi_n)$. In other words, because support shortening can happen in more than one way, reflected canonical undercuts are not necessarily unique, as shown below:

**Proposition 4.6.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $I : \alpha,\ J : \beta \in \mathsf{arguments}(\Delta)$.

$$\text{If } J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$$
$$\text{then } |\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)| \geq 1.$$

*Proof.* Given Proposition 4.4.1, plus that $I:\alpha, J:\beta \in \mathsf{arguments}(\Delta)$ and $J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$, it follows that $\mathsf{reflectedCanonicalUndercuts}(J : \beta, I : \alpha, \Delta) \neq \emptyset$. For an argument $K : \gamma$ to be a member of $\mathsf{reflectedCanonicalUndercuts}(J : \beta, I : \alpha, \Delta)$, by Definition 4.4.1, requires that $K : \gamma \in \mathsf{reflectedAttacks}(J : \beta, I : \alpha, \Delta)$, which in turn requires $K \subseteq I$. For any argument $I : \alpha$, because $|\mathsf{label}(I:\alpha)| \geq 0$, the argument can have many assumptions, and thus the set of assumptions can have many subsets. Therefore if $J : \beta \in \mathsf{argumentAttacks}(I : \alpha, \Delta)$ then $|\mathsf{reflectedCanonicalUndercuts}(J : \beta, I:\alpha, \Delta)| \geq 1$. □

I now give an example that illustrates the above proposition.

**Example 4.6.2. Mono-pair enlarged reflection.** *Let* $\Delta = \{a:\beta, b:\gamma, c:\neg\beta, d:\neg\gamma, e:(\beta \wedge \gamma) \rightarrow \alpha, f:(\neg\beta \wedge \neg\gamma) \rightarrow \neg\alpha\}$. *Let* $P = \{a, b, e\}, Q = \{c, d, f\}, R = \{c\}, S = \{d\}$. *Therefore* $P:\alpha, Q:\neg\alpha, R:\neg\beta, R:\neg\beta \vee \neg\gamma \vee \neg((\beta \wedge \gamma) \rightarrow \alpha), R:\neg(\beta \wedge \gamma \wedge ((\beta \wedge \gamma) \rightarrow \alpha)), S:\neg\gamma, S:\neg\gamma \vee \neg\beta \vee \neg((\beta \wedge \gamma) \rightarrow \alpha), S:\neg(\beta \wedge \gamma \wedge ((\beta \wedge \gamma) \rightarrow \alpha)) \in \mathsf{arguments}(\Delta)$. *One might think that* $Q:\neg(\beta \wedge \gamma \wedge ((\beta \wedge \gamma) \rightarrow \alpha))$ *is a valid canonical undercut of* $P:\alpha$, *however it is not even a valid argument as it is not minimal. The two arguments* $R:\neg(\beta \wedge \gamma \wedge ((\beta \wedge \gamma) \rightarrow \alpha))$ *and* $S:\neg(\beta \wedge \gamma \wedge ((\beta \wedge \gamma) \rightarrow \alpha))$ *are valid canonical undercuts of* $P:\alpha$ *and also members of* $\mathsf{reflectedCanonicalUndercuts}(P:\alpha, Q:\neg\alpha, \Delta)$. *These two valid arguments show the situation of multiple reflected canonical undercuts arising from a single defending argument and its rebuttal.*

In contrast to the above example, Example 4.6.1, while showing support shortening (where the support of the reflected canonical undercut is a strict subset of the support of the reflector) still only yields one reflected canonical undercut, so no mono-pair enlarged reflection occurred there. Given this conceptual introduction, I now define the mono-pair enlarged reflection.

The following definition, Definition 4.6.1, provides the introduction to support shortening, strict subsets, multiple undercuts and mono-pair enlarged reflection. The pair of arguments referred to is the attacking argument and the argument which it attacks, also known as the reflector.

**Definition 4.6.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae, let* $I : \alpha, J : \beta \in$ $\mathsf{arguments}(\Delta)$ *and let* $Z : \phi \in \Diamond(\Delta)$. *A* **mono-pair enlarged reflection**, *where a single pair of arguments give rise to more than one reflected argument, denoted* $\mathsf{monoPairEnlargedReflection}(J : \beta, I : \alpha, \Delta)$ *is such that:*

$$\begin{aligned}&\mathsf{monoPairEnlargedReflection}(J : \beta, I : \alpha, \Delta)\\ =\ &\{Z : \phi = \mathsf{confirm}(\mathsf{reflectedCanonicalUndercuts}(J : \beta, I : \alpha, \Delta)) \mid |Z| > 1\}.\end{aligned}$$

Because the set $\mathsf{reflectedCanonicalUndercuts}(J : \beta, I : \alpha, \Delta)$ contains all such reflections, the $\mathsf{confirm}()$ of that is the maximum cardinality confirmation $Z : \phi$. If $|Z| = 1$ then $\mathsf{monoPairEnlargedReflection}(J : \beta, I : \alpha, \Delta) = \emptyset$. Mono-pair enlarged reflections only occur when there is support shortening, which in turn can permit more than one subset of the support of the defending argument to be the support of an attacking argument. Not every attack chain of two arguments gives rise to a mono-pair enlarged reflection, even though it is the case that every attack chain of a pair of arguments gives rise to a reflected canonical undercut. Support shortening can also result in two previous distinct supports becoming indistinguishable - this is **support merging** - where two or more distinct reflectors $L : \alpha, M : \alpha$ with $N \subsetneq L, N \subsetneq M$ yield fewer reflections $N : \phi$.

The maximum number of mono-pair reflected arguments is as follows. The mid lattice function, Definition 2.4.5, returns an integer.

**Proposition 4.6.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $I : \alpha,\ J : \beta \in$ $\mathsf{arguments}(\Delta)$.

$$0 \leq |\mathsf{monoPairEnlargedReflection}(J : \beta, I : \alpha, \Delta)| \leq \mathsf{midLattice}(I).$$

*Proof.* The cardinality can be zero, if for example $J : \beta \notin \mathsf{argumentAttacks}(I : \alpha, \Delta)$, which sets the lower limit as $0 \leq$. Now for the upper limit. As it's possible to have many arguments making the same point [i.e. the point $\neg(\bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(J, \Delta)))$ ], all the different canonical undercuts $K : \neg(\phi_1, \ldots, \phi_n) \in \mathsf{canonicalUndercuts}(J : \alpha, \Delta)$ will have the same claim but different supports. Take one of these undercuts with support $K$. Any strict subset, $K$, of the support of the reflector $I$, that is $K \subsetneq I$, can be the support of a reflected canonical undercut attacking $J : \beta$ as long as there does not exist an $L \subsetneq K$ such that $L : \neg(\phi_1 \wedge \ldots \wedge \phi_n)$ is also a canonical undercut of $J : \beta$. If there were an $L \subsetneq K$ then $L : \neg(\phi_1, \ldots, \phi_n)$ would be the valid argument and $K : \neg(\phi_1, \ldots, \phi_n)$ would not be minimal and therefore not a valid argument. The canonical ordering condition will always be met by just one conjunction of the members of $J$ and thus does not affect the cardinality. The subsets of $I$ (which are deduction labels) form a power set with $I$ as its maximum cardinality member and $\emptyset$ as its minimum cardinality member. The largest set of subsets of $I$, (call it X as it will be a set of deduction labels and thus a confirmation label), that does not include subsets of other members of X [i.e. $L, M \in$ X iff

($L \not\subseteq M$ and $M \not\subseteq L$)] has a cardinality of $\mathsf{midLattice}(I)$. Hence the maximum number of reflected canonical undercuts attacking a single argument $J:\beta$ arising from its interaction with the single reflector $I:\alpha$, i.e. the maximum cardinality of mono-pair enlarged reflection, is $\mathsf{midLattice}(I)$. $\square$

Even for a relatively small number of premises in $J:\beta$ the number of reflected canonical undercuts of $I:\alpha$ can be large. For example if $|J| = 10$ then the number of mono-pair enlarged reflected canonical undercuts can be up to 252.

As a final comment, given their prerequisites, the smallest number of reflections is 1, which is not surprising due to the inevitability of reflection. Shortly I will discuss a phenomena that arises with confirmations, preclusions and reflection where there are fewer reflected arguments than one might expect; this I will call reduced reflection.

### 4.6.1 Reflection has an Inexact Symmetry

The reflected attacks function has an inexact symmetry about a pair of rebutting arguments for the claims $\alpha, \neg\alpha$. While I use the word symmetry in the title this is not a perfect symmetry as mono-pair enlarged reflections may or may not occur for each side of $\alpha$ and $\neg\alpha$. Also if mono-pair enlarged reflections do occur for both $\alpha$ and $\neg\alpha$, the number of reflected arguments on each side may differ. However there is still a certain undeniable symmetry as I now show:

**Symmetry of existence.** In this analysis of symmetry of reflected canonical undercuts I focus on the case where the reflector is a rebuttal.

**Proposition 4.6.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\neg\alpha \in$* $\mathsf{arguments}(\Delta)$.

$$\mathsf{reflectedCanonicalUndercuts}(I:\alpha, J:\neg\alpha, \Delta) \neq \emptyset$$
$$\textit{iff } \mathsf{reflectedCanonicalUndercuts}(J:\neg\alpha, I:\alpha, \Delta) \neq \emptyset.$$

*Proof.* If $\mathsf{reflectedCanonicalUndercuts}(I:\alpha, J:\neg\alpha, \Delta) \neq \emptyset$ then it must be the case that $I:\alpha, J:\neg\alpha \in \mathsf{arguments}(\Delta)$ and $I:\alpha \in \mathsf{argumentAttacks}(J:\neg\alpha, \Delta)$. The claims of $I:\alpha, J:\neg\alpha$ confirm the attack and also make it clear that $I:\alpha \in \mathsf{rebuttals}(J:\neg\alpha, \Delta)$. Using Definition 2.6.7 of rebuttal, it is the case that $J:\neg\alpha \in \mathsf{rebuttals}(I:\alpha, \Delta)$ and thus that $J:\neg\alpha \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$. Therefore, using Proposition 4.4.1, it follows that $\mathsf{reflectedCanonicalUndercuts}(J:\neg\alpha, I:\alpha, \Delta) \neq \emptyset$. Now for the other direction. If $\mathsf{reflectedCanonicalUndercuts}(J:\neg\alpha, I:\alpha, \Delta) \neq \emptyset$ then it must be the case that $I:\alpha, J:\neg\alpha \in \mathsf{arguments}(\Delta)$ and $J:\neg\alpha \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$. The claims of $I:\alpha, J:\neg\alpha$ confirm the attack and also make it clear that $J:\neg\alpha \in \mathsf{rebuttals}(I:\alpha, \Delta)$. Using Definition 2.6.7 of rebuttal, it follows that $I:\alpha \in \mathsf{rebuttals}(J:\neg\alpha, \Delta)$ and thus that $I:\alpha \in \mathsf{argumentAttacks}(J:\neg\alpha, \Delta)$. Therefore, using Proposition 4.4.1, it follows that $\mathsf{reflectedCanonicalUndercuts}(I:\alpha, J:\neg\alpha, \Delta) \neq \emptyset$. So in conclusion $\mathsf{reflectedCanonicalUndercuts}(I:\alpha, J:\neg\alpha, \Delta) \neq \emptyset$ iff $\mathsf{reflectedCanonicalUndercuts}(J:\neg\alpha, I:\alpha, \Delta) \neq \emptyset$. $\square$

While I have shown that there is a symmetry of existence, in that both sides will have reflected canonical undercuts, this does not mean that the number of reflected canonical undercuts is symmetrical.

**Asymmetry of cardinality.** I now show by example, there is no guaranteed symmetry in the number of reflected canonical undercuts. The following example shows:

$$|\text{reflectedCanonicalUndercuts}(I:\alpha, J:\neg\alpha, \Delta)| \neq |\text{reflectedCanonicalUndercuts}(J:\neg\alpha, I:\alpha, \Delta)|$$

**Example 4.6.3. Asymmetry of cardinality.** *Let* $\Delta = \{a:\alpha,\ b:\alpha \rightarrow \beta,\ c:\gamma\wedge\neg\alpha,\ d:(\gamma \rightarrow \neg\beta)\wedge\neg\alpha\}$. *Let* $I = \{a, b\}, J = \{c, d\}, K = \{a\}, L = \{b\}, M = \{c\}, N = \{d\}$. *Therefore* $I:\beta,\ J:\neg\beta,\ M:\neg\alpha,\ N:\neg\alpha,\ M:\neg(\alpha \wedge (\alpha \rightarrow \beta)),\ N:\neg(\alpha \wedge (\alpha \rightarrow \beta)),\ K:\alpha, K:\alpha \vee \neg\gamma,\ K:\alpha \vee \neg\gamma \vee \neg(\gamma \rightarrow \neg\beta) \vee \alpha,\ K:\neg(\gamma \wedge \neg\alpha \wedge (\gamma \rightarrow \neg\beta) \wedge \neg\alpha),\ L:\beta \vee \neg\alpha,\ L:\neg\beta \rightarrow \neg\alpha,\ M:\gamma,\ M:\neg\alpha, N:\gamma \rightarrow \neg\beta \in \text{arguments}(\Delta)$. *The defending argument is* $I:\beta$ *and its rebuttal is* $J:\neg\beta$. *One might think that* $J:\neg(\alpha \wedge (\alpha \rightarrow \beta))$ *was a reflected canonical undercut of* $I:\beta$*, however it is not even a valid argument as it is not minimal. Standing in the place of the invalid argument* $J:\neg(\alpha \wedge (\alpha \rightarrow \beta))$ *are two reflected canonical undercuts of* $I:\beta$*, namely* $M:\neg(\alpha \wedge (\alpha \rightarrow \beta))$ *and* $N:\neg(\alpha \wedge (\alpha \rightarrow \beta))$*, as clearly* $M \subsetneq J$ *and* $N \subsetneq J$*. That has addressed the* $\beta$ *side of the debate (showing two reflected undercuts), so now consider the* $\neg\beta$ *side of the debate. One might think that* $I:\neg((\gamma \wedge \neg\alpha) \wedge (\gamma \rightarrow \neg\beta) \wedge \neg\alpha)$ *was the only reflected canonical undercut on this side of the debate, however it is not even a valid argument as it is not minimal. Rather, on this side, it is* $K:\neg(\gamma \wedge \neg\alpha \wedge (\gamma \rightarrow \neg\beta) \wedge \neg\alpha)$ *that is the only reflected canonical undercut that exists here. Thus there are two reflections on one side and one on the other, making the result asymmetrical.*

The above example is symmetrical at the level of 'one argument and its one rebuttal', which is at the level of pro and con functions covered so far, and would thus yield a stalemate. It would seem preferable to have a judge function that considered the effects of undercut and reflection before reaching a conclusion. With an undercut aware, reflection unaware judge for this same debate intuitively no stalemate would be seen.

That completes the analysis of reflected and direct canonical undercuts, with the interesting property of mono-pair enlarged reflections. What makes reflected canonical undercuts particularly interesting is that they can be reflected off either an undercut or a rebuttal.

## 4.7 Reflected Argument Rebuttals

The final part of this chapter is on rebuttals: reflected and direct argument rebuttals. Although there are close similarities between these rebuttals and the behaviour of undercut analysed above, a key point is that there are differences. These subtle differences are relevant when tracking debates and thus it is necessary to analyse them now. I have already established that a reflected argument rebuttal can be reflected off a canonical undercut or off another rebuttal. While it is true that the rebuttal to rebuttal reflection is a genuine reflection it creates no new argument. Consequently the focus of the rest of this chapter is on rebuttals reflected off canonical undercuts. Even so I start this analysis in a general way and then home in on the specific.

I now define the reflected argument rebuttal as a special case or instance of abstract reflection (Definition 4.2.1). Here I am defining a set of arguments, which are all rebuttals, that arise through reflection, from a single defending argument and its attacking argument.

**Definition 4.7.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I : \alpha,\ J : \beta \in$ arguments$(\Delta)$. The set of* **reflected argument rebuttals**, *denoted* reflectedArgumentRebuttal$(J : \beta, I : \alpha, \Delta)$, *attacking $J : \beta$, reflected off $I : \alpha$, is such that:*

$$\begin{aligned}
&\mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta)\\
= \quad &\{K:\neg\beta \in \mathsf{arguments}(\Delta) \mid \\
&\quad K:\neg\beta \in \mathsf{reflectedAttacks}(J:\beta, I:\alpha, \Delta) \textit{ and}\\
&\quad K:\neg\beta \in \mathsf{rebuttals}(J:\beta, \Delta)\}.
\end{aligned}$$

The function is defined with a singular name, unlike the related functions, because as I prove in Proposition 4.7.5 it is a singleton set. Here is an example of a reflected argument rebuttal.

**Example 4.7.1. Reflected argument rebuttal.** *Let* $\Delta = \{a : \gamma, b : \gamma \rightarrow \alpha, c : \theta, d : \theta \rightarrow \neg\gamma\}$. *Thus* $\{a, b\} : \alpha, \{c, d\} : \neg(\gamma \wedge (\gamma \rightarrow \alpha)), \{a, b\} : \gamma \wedge (\gamma \rightarrow \alpha) \in$ arguments$(\Delta)$. *So* $\{c, d\} : \neg(\gamma \wedge (\gamma \rightarrow \alpha)) \in$ canonicalUndercuts$(\{a, b\} : \alpha, \Delta)$. *The reflected argument rebuttal is* $\{a, b\} : \gamma \wedge (\gamma \rightarrow \alpha)$, *in other words* $\{a, b\} : \gamma \wedge (\gamma \rightarrow \alpha) \in$ reflectedArgumentRebuttal$(\{c, d\} : \neg(\gamma \vee (\gamma \rightarrow \alpha)), \{a, b\} : \alpha, \Delta)$. *Clearly* label$(\{a, b\} : \gamma \wedge (\gamma \rightarrow \alpha)) \subseteq$ label$(\{a, b\} : \alpha))$.

### 4.7.1 A Motivational Example of Reflected Rebuttal

A practical question to ask about reflected arguments is 'what do they mean?', i.e. when the informal gloss of the direct and reflected arguments are studied would it commonly be agreed that the reflected arguments are superfluous or redundant. Therefore, I now provide a motivational example of reflected argument rebuttal where I give gloss to the formulae and arguments. The formulae and assumptions involved are as shown in Table 4.2.

| Symbol | Formula | Assumption | Informal Gloss |
|---|---|---|---|
| $\alpha$ | atomic | $a : \alpha$ | It's raining. |
| $\beta$ | atomic | not assumed | Bring an umbrella. |
| $\theta$ | atomic | $e : \theta$ | Water is visible on the window pane. |
| $\pi$ | atomic | $f : \pi$ | The neighbour's hosepipe is wetting the glass. |
| $\alpha \rightarrow \beta$ | compound | $c\!: \alpha \rightarrow \beta$ | If it's raining then bring an umbrella. |
| $(\theta \wedge \pi) \rightarrow \neg\beta$ | compound | $g\!: (\theta \wedge \pi) \rightarrow \neg\alpha$ | If water is visible on the window pane and the neighbour's garden hose is wetting the glass then it's not raining. |

Table 4.2: Motivational Example of Reflected Rebuttal

The assumptions in this small debate are just five formulae $\Delta = \{a\!: \alpha, c\!: \alpha \rightarrow \beta, e\!: \theta, f : \pi, g : (\theta \wedge \pi) \rightarrow \neg\alpha\}$. I have picked labels such that they do not clash with the previous motivational debate of Example 4.4.1 and Section 4.4.2, but I still conduct this current example as an isolated debate. Thus the arguments include $\{a, c\} : \beta, \{a, c\} : \alpha \wedge (\alpha \rightarrow \beta), \{e, f, g\} : \neg\alpha, \{e, f, g\} : \neg\alpha \vee \neg(\alpha \rightarrow \beta), \{e, f, g\} : \neg(\alpha \wedge (\alpha \rightarrow \beta)) \in$ arguments$(\Delta)$. The relevant chain of arguments is shown in the Figure 4.3.

The reflector argument, $\{a, c\} : \beta$, means *'it's raining','if it's raining then bring an umbrella', therefore 'bring an umbrella'*. The canonical undercut argument which attacks the reflector has the gloss *'water is visible on the window pane', 'the neighbour's hosepipe is wetting the glass', 'if water is visible on the window pane and the neighbour's garden hose is wetting the glass then it's not raining', therefore it's not the case that ('it's raining' and 'if it's raining then bring an umbrella')*. Consequently there exists a reflected argument rebuttal, $\{a, c\} : \alpha \wedge (\alpha \rightarrow \beta)$ with the gloss of *'given the facts that 'it is raining' and 'if it's raining then bring an umbrella', then it can be concluded that 'it is raining' and 'if it's raining then bring an umbrella'*. Thus it can be seen from an informal perspective that this reflected argument adds nothing sensible to the debate even though from a formal logic view it is correct.

$$
\begin{tikzcd}
\text{Reflector Argument} \quad \{a,c\}:\beta \\
\{e,f,g\}:\neg(\alpha \wedge (\alpha \rightarrow \beta)) \arrow[u, "{\{e,f,g\}:\neg(\alpha \wedge (\alpha \rightarrow \beta)) \in \mathsf{canonicalUndercuts}(\{a,c\}:\beta, \Delta)}"'] \\
\text{Reflected Rebuttal} \quad \{a,c\}:\alpha \wedge (\alpha \rightarrow \beta) \arrow[u, "{\{a,c\}:\alpha \wedge (\alpha \rightarrow \beta) \in \mathsf{rebuttals}(\{e,f,g\}:\neg(\alpha \wedge (\alpha \rightarrow \beta)), \Delta)}"']
\end{tikzcd}
$$

$$\{a,c\}:\alpha \wedge (\alpha \rightarrow \beta) \in \mathsf{reflectedArgumentRebuttals}(\{e,f,g\}:\neg(\alpha \wedge (\alpha \rightarrow \beta)), \{a,c\}:\alpha, \Delta)$$

Figure 4.3: Motivational Example of Reflected Rebuttal Shown in an Argument Graph

### 4.7.2 Properties of Reflected Argument Rebuttals

Having defined reflected argument rebuttals in Definition 4.7.1, it is appropriate to go on to establish their properties.

The support of this type of reflected argument only uses the $=$ part of the $\subseteq$ sign. Thus while Definition 4.7.1 is perfectly valid, it turns out that the $\subsetneq$ part of $\subseteq$ in the reflected attacks function is not used in this context. The following 'or' is a normal logical or, not an exclusive or, as for a reflexive $I:\alpha$ then sides of the 'or' can be true.

**Proposition 4.7.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I : \alpha, J : \beta \in$* $\mathsf{arguments}(\Delta)$.

$$
\begin{aligned}
& \mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta) \\
= \quad & \{K:\neg\beta \in \mathsf{arguments}(\Delta) \mid \\
& \quad (J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta) \text{ or} \\
& \quad J:\beta \in \mathsf{rebuttals}(I:\alpha, \Delta)) \text{ and } K = I\}.
\end{aligned}
$$

*Proof.* By Definition 4.7.1, members of $\mathsf{reflectedArgumentRebuttal}(J : \beta, I : \alpha, \Delta)$, must be reflected attacks $\mathsf{reflectedAttacks}(J : \beta, I : \alpha, \Delta)$ that are also $\mathsf{rebuttals}(J : \beta, \Delta)$. Definition 4.2.1 and Proposition 4.2.5 establish that a $\mathsf{reflectedAttacks}(J:\beta, I:\alpha, \Delta)$ always arises from any $\mathsf{argumentAttacks}(I:\alpha, \Delta)$. From Definition 3.9.2, that argument attack on the reflector $I:\alpha$, could be either a member of $\mathsf{rebuttals}(I:\alpha, \Delta)$ or of $\mathsf{canonicalUndercuts}(I:\alpha, \Delta)$. Base Proposition Two, Proposition 4.2.2, states that a reflected attack off the canonical undercut is a rebuttal $K:\phi \in \mathsf{rebuttals}(J:\beta, \Delta)$ with $K = I$. Base Proposition Three, Proposition 4.2.8, states that a reflected attack off the rebuttal is another rebuttal $K:\phi \in \mathsf{rebuttals}(J:\beta, \Delta)$ again with $K = I$. Thus in either case of reflected rebuttal, it follows that $K = I$. So in conclusion $\mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta) = \{K:\neg\beta \in \mathsf{arguments}(\Delta) \mid (J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta)$ or $J:\beta \in \mathsf{rebuttals}(I:\alpha, \Delta))$ and $K = I\}$. □

Given the preconditions of existence of i) a defending argument and ii) an attacking argument of that defending argument, then reflected argument rebuttals always exist. A simple corollary to the earlier proposition that reflected arguments always exist, is the following more specific one. Reflected argument rebuttals, given their prerequisites always exist.

**Proposition 4.7.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\beta \in$* $\mathsf{arguments}(\Delta)$.

*If* $J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$

*then* $\mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta) \neq \emptyset$,

*otherwise* $\mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta) = \emptyset$.

*Proof.* The attack of $J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$ could be either a rebuttal or a canonical undercut. If $J:\beta \in \mathsf{rebuttals}(I:\alpha, \Delta)$ then Base Proposition Two shows that there exists $K:\phi \in \mathsf{rebuttals}(J:\beta, \Delta)$ with $K = I$. If $J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta)$ then Base Proposition Three establishes that there exists $K:\phi \in \mathsf{rebuttals}(J:\beta, \Delta)$ again with $K = I$. Thus in either case there exists $K:\phi \in \mathsf{rebuttals}(J:\beta, \Delta)$ with $K = I$. It also follows, by Definition 4.2.1 of reflected attack that $K:\phi \in \mathsf{reflectedAttacks}(J:\beta, \Delta)$. Thus by Definition 4.7.1 of reflected argument rebuttal, as $K:\phi$ is both a reflected attack and a rebuttal it will be the case that $K:\phi \in \mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta)$ and so $\mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta) \neq \emptyset$. However, if $I:\alpha \notin \mathsf{arguments}(\Delta)$ or $J:\beta \notin \mathsf{arguments}(\Delta)$ or $J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$ then $\mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta) = \emptyset$. □

Building on the Proposition 4.2.12 which shows that reflexive arguments if attacked are attacked twice, it follows that:

**Proposition 4.7.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\beta, K:\phi \in$* $\mathsf{arguments}(\Delta)$.

*If* ($J:\beta$ *is a reflexive argument and* $J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$)

*then* $K:\phi \in \mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)$ *and*

$K:\phi \in \mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta)$.

*Proof.* Proposition 4.2.12 shows that if a reflexive argument is either undercut or rebutted then it is in fact both undercut and rebutted. The argument chain, of $J:\beta$ attacking $I:\alpha$ as defined here, always gives rise to reflection, hence if the attacking argument (the one in the middle of the chain) is a reflexive argument there will always exist one reflected argument rebuttal $J:\phi \in \mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta)$ and one reflected canonical undercut ($J:\phi \in \mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)$. □

The fact that a reflected canonical undercut of a reflexive argument cannot be subject to mono-pair enlarged reflection is also noteworthy. That must be the case as the same argument is also a reflected argument rebuttal – which can only give rise to a reflection of one argument and not more than one argument, a point that is formally proved in Proposition 4.7.5.

To allow the next proposition, Proposition 4.7.4, to be clear I need to first provide a new definition, the strict subset reflected argument rebuttal.

**Definition 4.7.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $I:\alpha$, $J:\beta \in \mathsf{arguments}(\Delta)$. *The* **strict subset reflected argument rebuttals**, *denoted* $\mathsf{strictSubsetReflectedRebuttals}(J:\beta, I:\alpha, \Delta)$, *are the rebuttals of* $J:\beta$ *that can be inferred from the existence of* $I:\alpha$ *and* $J:\beta$, *such that:*

$$\begin{aligned}&\mathsf{strictSubsetReflectedRebuttals}(J:\beta, I:\alpha, \Delta)\\ &\quad = \{K:\neg\beta \in \mathsf{arguments}(\Delta) \mid J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta) \textit{ and } K \subsetneq I\}.\end{aligned}$$

I now show that strict subset reflected argument rebuttals never exist.

**Proposition 4.7.4.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae, let* $I:\alpha, J:\beta \in \mathsf{arguments}(\Delta)$ *and let* $J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta)$.

$$\mathsf{strictSubsetReflectedRebuttals}(J:\beta, I:\alpha, \Delta) = \emptyset.$$

*Proof.* Suppose $K:\neg\beta \in \mathsf{strictSubsetReflectedRebuttals}(J:\beta, I:\alpha, \Delta)$. There always exists an $M:\neg\beta \in \mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta)$. As both $M:\neg\beta$ and $K:\neg\beta$ are rebuttals of $J:\beta$ they must have the same claim. If $K \subsetneq I$ and $M = I$ then $K \subsetneq M$. If $K \subsetneq M$ and the two arguments have the same claim, then $M$ is not a minimal support and therefore $M:\neg\beta$ is not a valid argument. This is a contradiction, so consequently no such $K:\neg\beta$ can exist and thus $\mathsf{strictSubsetReflectedRebuttals}(I:\alpha, J:\beta, \Delta) = \emptyset$. □

What is interesting here is not only that such arguments are not reflected, but also that they simply never exist. This point may seem counterintuitive or somewhat surprising because with reflected canonical undercuts, the reflected arguments arise from both parts of the subset or equal to sign (the $\subsetneq$ and $=$ parts of $\subseteq$). This simple result has a key effect on the number of reflected rebuttals and thus on the structure of debates, as I will shortly show.

Because of the $=$ sign there can be no multiple subsets, which would require support shortening but not support merging, and therefore no mono-pair enlarged reflections. There is no need to define such a thing as mono-pair enlarged reflected rebuttals because they do not exist. Thus the function $\mathsf{reflectedArgumentRebuttal}()$ is a singleton set and thus has been named in the singular.

**Proposition 4.7.5.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\beta \in$* $\mathsf{arguments}(\Delta)$.

$$\textit{If } J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$$
$$\textit{then } |\mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta)| = 1,$$
$$\textit{otherwise } |\mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta)| = 0.$$

*Proof.* If it were the case that subset reflected argument rebuttals, i.e. a $K:\phi$ where $K \subsetneq J$, existed (see earlier Proposition 4.7.4) then there could be many subsets $K$ of $I$. However, as such subsets are not allowed then there is only one way that the support of the reflected argument $K:\phi$ can equal the support of the defending argument $I:\alpha$ such that $K = I$. Consequently for reflected rebuttals there will be one and only one reflected argument. If the prerequisites of $I:\alpha, J:\beta \in \mathsf{arguments}(\Delta)$ and $J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$ are not met then there are no reflected argument rebuttals. □

Any reflected canonical undercut $K:\phi$ has a support that is a subset of the support of the reflecting argument $J:\beta, K \subseteq J$. However, if the attacking argument (the one in the middle of the attack chain) is reflexive then the reflected canonical undercut reflected canonical undercut $K:\phi$ will have the identical support to its reflector.

**Proposition 4.7.6.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and $I:\alpha, J:\beta, K:\phi \in$* $\mathsf{arguments}(\Delta)$.

$$\textit{If } J:\beta \textit{ is a reflexive argument, } J:\beta \in \mathsf{argumentAttacks}(I:\alpha)$$
$$\textit{and } K:\phi \in \mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)$$
$$\textit{then } K = I.$$

*Proof.* Because $K:\phi \in \mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)$ it must be the case that $K:\phi \in \mathsf{canonicalUndercuts}(J:\beta, \Delta)$. Now $J:\beta$ is a reflexive, so $K:\phi \in \mathsf{rebuttals}(J:\beta, \Delta)$. So $K:\phi$ is also a reflected argument rebuttal, $K:\phi \in \mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta)$. Now for reflected argument rebuttals $K = I$. So even though reflected canonical undercuts allow $K \subseteq I$ it must be the case that here $K = I$. □

It will also be the case for the above proposition that there is only one reflected canonical undercut not a set of many of them. The number of reflected arguments arising from a single attack pair is as follows.

**Proposition 4.7.7.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\beta \in$* $\mathsf{arguments}(\Delta)$.

$$0 \leq |\mathsf{reflectedAttacks}(J:\beta, I:\alpha, \Delta)| \leq (\ |\ \mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)\ |\ + 1).$$

*Proof.* If $\Delta \nvdash \bot$ then $\Delta$ implies no undercuts, thus for that $\Delta$, $\mathsf{reflectedAttacks}(J:\beta, I:\alpha, \Delta) = \emptyset$, so the lower cardinality limit is zero. If $J:\beta$ is a reflexive argument then $\mathsf{rebuttals}(J:\beta, \Delta) =$

canonicalUndercuts($J : \beta, \Delta$) and thus the upper limit is not reached. For a non-reflexive argument the reflected argument rebuttal is not a subset of or equal to a reflected canonical undercut, i.e. reflectedArgumentRebuttal($J : \beta, I : \alpha, \Delta$) $\not\subseteq$ reflectedCanonicalUndercuts($J : \beta, I : \alpha, \Delta$). Consequently the cardinalities of these two functions must be added together to get the upper limit. Therefore $0 \leq |$reflectedAttacks($J{:}\beta, I{:}\alpha, \Delta$)$| \leq (\ |$ reflectedCanonicalUndercuts($J{:}\beta, I{:}\alpha, \Delta$) $|\ + 1)$. □

That completes the analysis of reflected argument rebuttals.

## 4.8 Direct Argument Rebuttals

Not all argument rebuttals are reflected argument rebuttals. The pattern here is similar to direct canonical undercuts in many ways (Definition 4.5.1), but also different as I now show and discuss.

**Definition 4.8.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I : \alpha, J : \beta \in$* arguments($\Delta$). *The set of* **direct argument rebuttals** *of $J : \beta$, not reflected off $I : \alpha$, denoted* directArgumentRebuttals($J{:}\beta, I{:}\alpha, \Delta$), *is such that:*

directArgumentRebuttals($J{:}\beta, I{:}\alpha, \Delta$)

$=$ rebuttals($J{:}\beta, \Delta$) $\setminus$ reflectedArgumentRebuttal($J{:}\beta, I{:}\alpha, \Delta$).

In this analysis, I continue with the attack chain of three arguments, i.e. $L{:}\gamma \in$ argumentAttacks($J : \beta, \Delta$), $J : \beta \in$ argumentAttacks($I : \alpha, \Delta$), and make it specific so that reflected argument $L : \gamma$ is a rebuttal, hence $L{:}\neg\beta$ is used in subsequent analysis. With direct argument rebuttals, predictions can be made about the support of the attacking argument:

**Proposition 4.8.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $I{:}\alpha, J{:}\beta, L{:}\neg\beta \in$* arguments($\Delta$) *and let $J{:}\beta \in$* argumentAttacks($I{:}\alpha, \Delta$).

*If $L{:}\neg\beta \in$* directArgumentRebuttals($J{:}\beta, I{:}\alpha, \Delta$) *then $L \neq I$.*

*Proof.* Proposition 4.7.1 establishes that if $L : \neg\beta \in$ reflectedArgumentRebuttals($J : \beta, I : \alpha, \Delta$) then $L = I$. Definition 2.6.7 of argument rebuttal imposed no constraint on the support of the attacking argument, $L : \neg\beta$, relative to any reflector, $I : \alpha$, so in that case for any $I : \alpha$ some rebuttals may have $L = I$ and others $L \neq I$. The definition of direct argument rebuttal, Definition 4.8.1 immediately above, uses set difference to remove the reflected argument rebuttals from the full set of argument rebuttals, thus the only rebuttals remaining in directArgumentRebuttals($J : \beta, I : \alpha, \Delta$) must have $L \neq I$. So in conclusion if $L{:}\neg\beta \in$ directArgumentRebuttals($J{:}\beta, I{:}\alpha, \Delta$) then $L \neq I$. □

Direct argument rebuttals may exist, dependant upon the reflecting argument passed in, here is an example.

**Example 4.8.1. Direct argument rebuttal.** *Let $\Delta = \{a{:}\pi,\ b{:}\pi \to \alpha,\ c{:}\neg\pi,\ d{:}\gamma,\ e{:}\gamma \to (\pi \wedge (\pi \to \alpha))\}$. Let $I = \{a, b\}$, $J = \{c\}$, $L = \{d, e\}$. Thus $I{:}\alpha$, $J{:}\neg(\pi \wedge (\pi \to \alpha))$, $I{:}\pi \wedge (\pi \to \alpha)$, $L{:}\pi \wedge (\pi \to \alpha)$, $L{:}\alpha \in$* arguments($\Delta$). *So $J : \neg(\pi \wedge (\pi \to \alpha)) \in$* canonicalUndercuts($I : \alpha, \Delta$) *and $I{:}\pi \wedge (\pi \to \alpha)$, $L{:}\pi \wedge (\pi \to \alpha) \in$* rebuttals($J : \neg(\pi \wedge (\pi \to \alpha)), \Delta$). *Consequently as $L \not\subseteq I$ and* *clearly* $I \subseteq I$, *so* $I:\pi \wedge (\pi \rightarrow \alpha) \in \mathsf{reflectedArgumentRebuttal}(J:\neg(\pi \wedge (\pi \rightarrow \alpha)), I:\alpha, \Delta)$ *and* $L:\pi \wedge (\pi \rightarrow \alpha) \in \mathsf{directArgumentRebuttals}(J:\neg(\pi \wedge (\pi \rightarrow \alpha)), I:\alpha, \Delta)$. *Note* $J:\neg(\pi \wedge (\pi \rightarrow \alpha)) \notin \mathsf{canonicalUndercuts}(L:\alpha, \Delta)$ *so* $L:\pi \wedge (\pi \rightarrow \alpha) \notin \mathsf{reflectedArgumentRebuttal}(J:\neg(\pi \wedge (\pi \rightarrow \alpha)), L:\alpha, \Delta)$.

Not every $\Delta$ will give rise to direct argument rebuttals, so they do not always exist; they may exist. For the same input parameters, a particular argument can never be in both direct and reflected argument rebuttals, so clearly:

$$\mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta) \cap \mathsf{directArgumentRebuttals}(J:\beta, I:\alpha, \Delta) = \emptyset.$$

In other words:

If $K:\phi \in \mathsf{directArgumentRebuttals}(J:\beta, I:\alpha, \Delta)$

then $K:\phi \notin \mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta)$.

If $K:\phi \in \mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta)$

then $K:\phi \notin \mathsf{directArgumentRebuttals}(J:\beta, I:\alpha, \Delta)$.

How many direct argument rebuttals can there be? Depending upon the given $\Delta$ it is quite possible to have large numbers of direct argument rebuttals that are unconstrained by any encumbrances of reflection.

**Proposition 4.8.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $I:\alpha, J:\beta \in \mathsf{arguments}(\Delta)$.

$$0 \leq |\mathsf{directArgumentRebuttals}(J:\beta, I:\alpha, \Delta)| < \mathsf{midLattice}(\Delta).$$

*Proof.* It is possible that a given defending argument $J:\beta$ can be subject to rebuttal by every argument that is of the form $K:\neg\beta \in \mathsf{arguments}(\Delta)$. Clearly $\mathsf{formulae}(K, \Delta) \subseteq \Delta$. However not every subset of $\Delta$ can be a valid support for a $K:\neg\beta$ as arguments must be minimal. The largest number of subsets of $\Delta$ that are also not subsets of themselves is given by the midLattice function, $\mathsf{midLattice}(\Delta)$. However as $K, J \subseteq \Delta$ and for the support to be consistent $J:\neg\beta \notin \mathsf{arguments}(J:\beta, \Delta)$ it is necessary to take one argument off the total and use $<$ . The minimum number is zero as $\Delta$ may simply contain no formulae that such rebuttals can be derived from. □

Now that direct argument rebuttals are defined I can add one final property, requiring first the definition of direct attack. My starting point for this analysis of reflection was the attack function, Definition 3.9.2. I have shown that that definition always yields reflection. If reflections are problematic then the definition is arguably the cause or root of the problem. In that case the solution is to use direct attack, as defined below, in lieu of attack. I leave as an area of future research further analysis of bridging assumptions between the work of Dung and mincon argumentation, both with and without a consideration of reflection.

**Definition 4.8.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and $I : \alpha,\ J : \beta,\ \in$* arguments($\Delta$). *The set of* **direct attacks** *on $J:\beta$ which are not subject to reflection off $I:\alpha$, denoted* directAttacks $(J:\beta, I:\alpha, \Delta)$*, is such that:*

$$\text{directAttacks}(J:\beta, I:\alpha, \Delta) = \text{argumentAttacks}(J:\beta, \Delta) \setminus \text{reflectedAttacks}(J:\ \beta, I:\alpha, \Delta).$$

As defined, this direct attack is relative to the reflector passed in. One could conjecture a kind of direct attack between two arguments that is absent from reflections not just from one reflector, but rather from many or all possible reflectors. That generalisation is done in the next chapter which generalises reflection from individual arguments to sets of arguments. So now the final property.

**Proposition 4.8.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and $I:\alpha\ J:\beta \in$* arguments($\Delta$).

$$\begin{aligned} &\text{directAttacks}(J:\beta, I:\alpha, \Delta) \\ &\quad = \text{directArgumentRebuttals}(J:\beta, I:\alpha, \Delta) \\ &\qquad\qquad \cup\ \text{directCanonicalUndercuts}(J:\beta, I:\alpha, \Delta). \end{aligned}$$

*Proof.* Direct attack, in Definition 4.8.2, is defined as all attacks argumentAttacks$(J:\beta, \Delta)$ less the reflected arguments reflectedAttacks$(J:\beta, I:\alpha, \Delta)$. Attacks is defined, in Definition 3.9.2, as canonical undercuts set-unionised with rebuttals. Direct canonical undercuts, by Definition 4.5.1 are canonical undercuts less reflected canonical undercuts. Direct argument rebuttals, from Definition 4.8.1, are rebuttals less reflected argument rebuttals. Therefore the set of direct attacks directAttacks$(J:\beta, I:\alpha, \Delta)$ is equal to the set of direct argument rebuttals $\cup$ the set of direct canonical undercuts. □

This completes analysis of reflected argument rebuttals and also of reflected and direct arguments.

## 4.9 Conclusion for Reflected and Direct Arguments

This chapter on reflected and direct arguments and attacks introduces this thesis' main contribution – namely reflection. Reflected arguments, given their prerequisites, always exist. The process of reflection always involves a reflecting argument and one or more reflected arguments. As reflections are predictable they introduce redundancy or duplication and therefore make any counting of arguments problematic.

The following table shows the six new key definitions presented in the chapter, together with previous ones to provide context. These nine direct and reflected attacking functions are shown in Table 4.3 below.

Note that all of the above table is at the level of individual arguments. The dimension of confirmations is added in the next chapter. Of the above nine, only reflectedArgumentRebuttal$(J:\beta, I:\alpha, \Delta)$ is always a singleton set. Three discoveries in this chapter are i) the existence of reflected arguments with either rebuttal or undercut as the attack on the reflector and ii) again with either rebuttal or undercut as the reflected attacks by the reflected arguments and iii) the mono-pair enlarged reflection. The following table, Table 4.4, summarises the reflections that occur with individual arguments.

| Attack | Reflection | Argument Attacks Function | Definition |
|---|---|---|---|
| Rebuttal | Reflected ∪ Direct | $\mathsf{rebuttals}(I:\alpha, \Delta)$ | 2.6.7 |
| Undercut | Reflected ∪ Direct | $\mathsf{canonicalUndercuts}(I:\alpha, \Delta)$ | 3.3.2 |
| Attacks | Reflected ∪ Direct | $\mathsf{argumentAttacks}(I:\alpha, \Delta)$ | 3.9.2 |
| Rebuttal | Reflected Only | $\mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta)$ | 4.7.1 |
| Undercut | Reflected Only | $\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)$ | 4.4.1 |
| Attacks | Reflected Only | $\mathsf{reflectedAttacks}(J:\beta, I:\alpha, \Delta)$ | 4.2.1 |
| Rebuttal | Direct Only | $\mathsf{directArgumentRebuttals}(J:\beta, I:\alpha, \Delta)$ | 4.8.1 |
| Undercut | Direct Only | $\mathsf{directCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)$ | 4.5.1 |
| Attacks | Direct Only | $\mathsf{directAttacks}(J:\beta, I:\alpha, \Delta)$ | 4.8.2 |

Table 4.3: Summary of Reflected and Direct Argument Attacking Functions

| Reflected | Enlarged | Exact | Reduced |
|---|---|---|---|
| Canonical Undercut: | Exist: Mono-Pair, Section 4.6 | Exist: Prop 4.6.1 | Don't Exist: Prop 4.6.1 |
| Argument Rebuttal: | Don't Exist: Prop 4.7.5 | Exist: Prop 4.7.5 | Don't Exist: Prop 4.7.5 |

Table 4.4: Summary of Scaling Factors Found With Argument Reflections

Of particular interest with mono-pair enlarged reflections are a) the fact that there can be more reflected arguments than might be anticipated intuitively, b) that the number of reflections can be large and c) that while the reflection has a basic symmetry of existence it d) does not have symmetry of cardinality. There are no reduced reflections from mono-pair reflection, however these are encountered through another mechanism in the next chapter. These complications with mono-pair enlarged reflections are a harbinger for the deeper challenges to be unfolded in the next chapter. These scaling properties of reflection show that the approach of using reflected arguments as proxies or substitutes for direct arguments is questionable.

That completes my analysis of single arguments attacking single arguments. In the next chapter I extend this discussion by moving up to the consideration of confirmations attacking confirmations.

# Chapter 5

# Reflected and Direct Confirmations

## 5.1 Overview of the Chapter

I develop this chapter's analysis of reflected and direct confirmations and attacks in seven stages. The chapter retraces the previous chapter's coverage of Type I Reflection attacks (with its four forms) this time considering *sets* of arguments. It uncovers novel properties and three further types of reflection: Types II, III and IV. These properties, particularly distorted reflection, impact the tracking of debates. Reflected confirming arguments, where knowledge of arguments nearer a tree's leaves allow prediction of arguments nearer the root, come near the end of this chapter, however they influence debate tracking as much as the other types of reflection. The sections of this chapter are:

**Reflected Preclusive Undercuts** Section 5.2 identifies the form and properties of reflected preclusive undercuts.

**Direct Preclusive Undercuts** Section 5.3 shows that not all preclusive undercuts are reflected. While reflected preclusive undercuts are always mono-attack, direct preclusive undercuts can be multi-attack.

**Mixed Preclusive Undercuts and Invalidation** Section 5.4 illustrates that a single confirmation may contain a mixture of reflected and direct preclusive undercuts. Features of the process of invalidation are described.

**Reflected Confirmation Rebuttals** Section 5.5 introduces a second kind of reflection, Type II Reflection, where a different mechanism from reflectedAttacks() is involved, due to multi-target preclusions, to create reflections arising off more than one argument.

**Direct Confirmation Rebuttals** Section 5.6 establishes that confirmation rebuttals exist which are not reflected. They are half of what needs to be tracked for a complete debate. The other half is the direct preclusions.

**Scaled and Distorted Reflections** Section 5.7 defines reflection scaling factors and shows that enlarged, reduced, exact and distorted reflections exist. Also a second mechanism for enlargement is shown.

**Reflected Confirming Arguments** Section 5.8 covers a third and Section 5.9 a fourth type of reflection, i.e Type III and IV Reflection, as distinct from the forms addressed so far in this and the previous chapter. These reflections can be understood as flowing up attack chains from tail to head, in contrast to the other types which flow down.

This chapter argues that there is no simple practical way to have reflected arguments stand in for, or represent, direct arguments. Attempts to count the reflected arguments, as a substitute for counting the direct ones, are likely to create erroneous, if not random, results. Even if existential schemes are used instead of argument counting the same problem appears to arise. The ground work is laid here to reason that all reflected arguments should be removed from the tracked debate and all direct ones included. Whether the nomenclature uses confirmations or some other way to aggregate arguments is relatively unimportant – the exclusion of reflection and inclusion of direct arguments is the focus. The confirmation notation, however, facilitates both the description and comprehension of these phenomena.

## 5.2 Reflected Preclusive Undercuts

Just as there exist reflected canonical undercuts arising from a pair of arguments where one attacks the other, so also there exist reflected preclusive undercuts arising from a pair of confirmations where one attacks the other. These two different undercut reflect functions, $\mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)$ and $\mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$, take slightly different input parameters: the former takes an attack chain of two arguments while the latter takes a attack chain of two confirmations (with its target confirmation label). Both result in attack chains of three items. The former is of three arguments, which I have referred to as the reflector $I:\alpha$ attacked by $J:\beta$, attacked by the reflected $K:\gamma$. The latter is of three confirmations. See Figure 5.1 for an annotated confirmation graph illustrating reflected preclusive undercut.

Reflector Confirmation
is Target $\mathrm{V}:\alpha$, where $\mathrm{V} \subseteq \mathrm{X}$

$$\begin{tikzcd} \mathrm{X}:\alpha \\ \mathrm{Y}:\beta \arrow[u, "\mathrm{V}"] \\ \mathrm{Z}:\gamma \arrow[u, "\mathrm{W}"] \end{tikzcd}$$

$\langle \mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha\rangle \in \mathsf{confirmationAttacks}(\mathrm{X}:\alpha, \Delta)$

$\langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\beta\rangle \in$
$\mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$

Reflected Confirmation

For each $K \in \mathrm{Z},\ K \subseteq I, I \in \mathrm{V}$

Figure 5.1: Reflected Preclusive Undercut is a Form of Type I Reflection

I now define the reflected preclusive undercut and its associated terminology:

**Definition 5.2.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $I:\alpha,\ K:\gamma \in$ $\mathsf{arguments}(\Delta)$ and let $\mathrm{V}:\alpha,\ \mathrm{W}:\beta,\ \mathrm{X}:\alpha,\ \mathrm{Y}:\beta,\ \mathrm{Z}:\gamma \in \Diamond(\Delta)$. The reflected preclusive undercuts func-*

*tion, denoted* reflectedPreclusiveUndercuts$(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$*, provides the set of* **reflected preclusive undercuts** *of* $\mathrm{Y}:\beta$ *that can be inferred from the existence of* $\mathrm{V}:\alpha$ *and* $\mathrm{Y}:\beta$ *such that:*

$$\begin{aligned}&\text{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)\\ &= \{\langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\beta\rangle \in \text{preclusiveUndercuts}(\mathrm{Y}:\beta, \Delta) \mid\\ &\qquad \langle \mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha\rangle \in \text{confirmationAttacks}(\mathrm{X}:\alpha, \Delta) \textit{ and}\\ &\qquad \textit{for every } K \in \mathrm{Z} \textit{ there exists an } I \in \mathrm{V} \textit{ where } K \subseteq I\}.\end{aligned}$$

I call the attacked target confirmation at the head of the chain, i.e. $\mathrm{V}:\alpha$, the reflecting confirmation or **reflector**. The reflector is attacked by $\mathrm{Y}:\beta$, which is attacked by the reflected confirmation $\mathrm{Z}:\gamma$. If a targeted attack tuple $\langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\beta\rangle \in$ reflectedPreclusiveUndercuts$(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$ then I call $\langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\beta\rangle$ a **reflected preclusive undercut**. Each argument in the reflected preclusive undercut is reflected off one and only one reflector argument.

Now here is an example of a reflected preclusive undercut, where the reflection is two mono-target mono-attack preclusions.

**Example 5.2.1. Reflected preclusive undercut.** *Let* $\Delta = \{a:\beta,\ b:\beta \to \alpha,\ c:\gamma,\ d:\gamma \to \neg\alpha,\ e: \pi \vee \neg\alpha,\ f:\neg\pi\}$*. Let* $I = \{a,b\}, J = \{c,d\}, K = \{e,f\}, \mathrm{U} = \{J\}, \mathrm{V} = \{K\}, \mathrm{X} = \{I\}, \mathrm{Y} = \{J,K\}, \mathrm{Z} = \{I\}$*. So* $I:\alpha, J:\neg\alpha, K:\neg\alpha, I:\neg(\gamma \wedge (\gamma \to \neg\alpha)), I:\neg((\pi \vee \neg\alpha) \wedge \neg\pi) \in$ arguments$(\Delta)$*. Consequently* $\mathrm{U}:\neg\alpha, \mathrm{V}:\neg\alpha, \mathrm{X}:\alpha, \mathrm{Y}:\neg\alpha, \mathrm{Z}:\neg(\gamma \wedge (\gamma \to \neg\alpha)), \mathrm{Z}:\neg((\pi \vee \neg\alpha) \wedge \neg\pi) \in \Diamond(\Delta)$*. The head of this attack chain is* $\mathrm{X}:\alpha$*, so defending confirmation* $\mathrm{X}:\alpha =$ *target confirmation* $\mathrm{X}:\alpha$*, so it is this confirmation which acts in the role of reflector. Next in the attack chain is* $\mathrm{Y}:\neg\alpha$*, as* $\mathrm{Y}:\neg\alpha \in$ confirmationRebuttals$(\mathrm{X}:\alpha, \Delta)$*. Two reflected preclusive undercuts follow, namely* $\langle \mathrm{Z}: \neg(\gamma \wedge (\gamma \to \neg\alpha)), \mathrm{U}, \mathrm{Y}:\neg\alpha\rangle \in$ reflectedPreclusiveUndercuts$(\mathrm{Y}:\neg\alpha, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$ *and* $\langle \mathrm{Z}: \neg((\pi \vee \neg\alpha) \wedge \neg\pi), \mathrm{V}, \mathrm{Y}:\neg\alpha\rangle \in$ reflectedPreclusiveUndercuts$(\mathrm{Y}:\neg\alpha, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$*. Clearly there are other members of* arguments$(\Delta)$ *such as* $\{c,d,e\}:\pi$*, but these do not partake in the reflections.*

Commenting on Figure 5.1, clearly the confirmation attack by $\mathrm{Y}:\beta$ on the head of the chain $\mathrm{X}:\alpha$ could be either a confirmation rebuttal or a preclusive undercut. In either case the target confirmation label V will be a subset of the defending confirmation label X, i.e. $\mathrm{V} \subseteq \mathrm{X}$. The next attack in the chain here is $\mathrm{Z}:\gamma$ attacking $\mathrm{Y}:\beta$. It is a preclusive undercut and thus $\mathrm{W} \subseteq \mathrm{Y}$.

Therefore in the above definition there is one confirmation, namely $\mathrm{V}:\alpha$, in the role of the reflector. A second confirmation $\mathrm{Y}:\beta$ is the attacking (i.e. rebutting or undercutting) confirmation of $\mathrm{X}:\alpha$. The reflected preclusive undercut function does not just yield a single tuple $\langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\beta\rangle$, rather, it yields a *set* of tuples $\{\langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\beta\rangle\}$, where each tuple in the set is a reflected preclusive undercut. The definition does not mandate that either $\mathrm{V}:\alpha$ or $\mathrm{Y}:\beta$ are maximum cardinality confirmations. Any cardinality is allowed and thus knowledge of both $\mathrm{V}:\alpha$ and $\mathrm{Y}:\beta$ need to be explicitly passed in.

### 5.2.1 Properties of Reflected Preclusive Undercuts

Reflected preclusive undercuts, given their prerequisites, always exist. This proposition uses Definition 3.9.4 of confirmation attacks.

**Proposition 5.2.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{V}:\alpha, \mathrm{X}:\alpha, \mathrm{Y}:\beta \in \Diamond(\Delta)$.

$$\textit{If } \langle \mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha\rangle \in \mathsf{confirmationAttacks}(\mathrm{X}:\alpha, \Delta)$$

$$\textit{then } \mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta) \neq \emptyset,$$

$$\textit{otherwise } \mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta) = \emptyset.$$

*Proof.* Consider an $I \in \mathrm{V}$, a $J \in \mathrm{Y}$ and a $K \in \mathrm{Z}$ where $\langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\beta\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Y}:\beta, \Delta)$, where $\mathrm{W}:\beta, \mathrm{Z}:\gamma \in \Diamond(\Delta)$. For $\langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\beta\rangle$ to be in $\mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$, where $\mathrm{W} \subseteq \mathrm{Y}$, then every argument $K:\gamma$ must be reflected. Because $\mathrm{V}:\alpha, \mathrm{Y}:\beta \in \Diamond(\Delta)$ it follows that $I:\alpha, J:\beta \in \mathsf{arguments}(\Delta)$. Furthermore given $\langle \mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha\rangle \in \mathsf{confirmationAttacks}(\mathrm{X}:\alpha, \Delta)$ then $J:\beta \in \mathsf{argumentAttacks}(I:\alpha, \Delta)$, where V is any $\mathrm{V} \subseteq \mathrm{X}$. Proposition 4.4.1 establishes that given a pair of attacking arguments then a reflected canonical undercut always exists. These conditions are met here, so such a reflection exists. Consequently for every $K \in \mathrm{Z}$ there exists an $I \in \mathrm{V}$ where $K \subseteq I$ and thus the set $\mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$ cannot be the $\emptyset$. However if any prerequisite is not met then the reflection cannot occur. Therefore if $\langle \mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha\rangle \in \mathsf{confirmationAttacks}(\mathrm{X}:\alpha, \Delta)$ then $\mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta) \neq \emptyset$, otherwise $\mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta) = \emptyset$. □

The target cardinality of reflected preclusive undercuts can only ever be one. The reflection is thus always a mono-target preclusive undercut and never multi-target.

**Proposition 5.2.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{V}:\alpha, \mathrm{W}:\beta, \mathrm{X}:\alpha, \mathrm{Y}:\beta, \mathrm{Z}:\gamma \in \Diamond(\Delta)$.

$$\textit{If } \langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\beta\rangle \in \mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta) \textit{ then } |\mathrm{W}| = 1.$$

*Proof.* Definition 5.2.1 establishes that each argument $K:\gamma$ which is in the reflected preclusive undercut attacking confirmation $\mathrm{Z}:\gamma$, i.e. where $K \in \mathrm{Z}$, has a support which is a subset or equal to the support of an argument $I:\alpha$ that is in the reflector confirmation $\mathrm{V}:\alpha$, i.e. $I \in \mathrm{V}$, such that $K \subseteq I$. Definition 5.2.1 does not explicitly constrain the claim of $\mathrm{Z}:\gamma$ apart from stipulating that it must be a preclusive undercut attacking confirmation. Now hypothesise that there exists a $\langle \mathrm{Z}:\neg(\gamma_l \wedge \ldots \wedge \gamma_m), \mathrm{U}, \mathrm{Y}:\beta\rangle \in \mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$ where $\mathrm{U} \in \mathcal{C}, |\mathrm{U}| > 1$. Such a target of U could not exist as their exists an invalidating preclusive undercut $\langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\beta\rangle$ (see Definition 3.4.2 of preclusive undercut and the definitions of terms in the text below it) where each argument $K:\gamma$ with $K \in \mathrm{Z}$ is a reflected canonical undercut (see Definition 4.4.1) and hence any reflected preclusive undercut has a target of $\mathrm{W}:\beta$ where $|\mathrm{W}| = 1$. Thus the attacking confirmation with target $\mathrm{U}:\beta, |\mathrm{U}| > 1$ would be invalidated by the one with $\mathrm{W}:\beta, |\mathrm{W}| = 1$. So all reflected preclusive undercuts have a target cardinality of 1, they are mono-target. So if $\langle \mathrm{Z}:\gamma, \mathrm{V}, \mathrm{Y}:\beta\rangle \in \mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$ then $|\mathrm{W}| = 1$. □

Thus a reflected preclusive undercut never means *'I attack this argument or that argument'* – instead it only attacks one argument: *'I attack this argument'*. In analysing the patterns created by these various reflections the above proposition plays a prominent role. It follows that reflected preclusive undercuts are comprised only of reflected canonical undercuts.

**Proposition 5.2.3.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae, let* $I:\alpha$, $J:\beta$, $K:\gamma \in \mathsf{arguments}(\Delta)$ *and let* $\mathrm{V}:\alpha, \mathrm{W}:\beta, \mathrm{X}:\alpha, \mathrm{Y}:\beta, \mathrm{Z}:\gamma \in \Diamond(\Delta)$.

$$\begin{aligned}
&\mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)\\
&= \{\langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\beta\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Y}:\beta, \Delta) \mid\\
&\qquad \textit{for every } K \in \mathrm{Z} \textit{ there exists an } I \in \mathrm{V} \textit{ and a } J \in \mathrm{Y}\\
&\qquad \textit{such that } K:\gamma \in \mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)\}.
\end{aligned}$$

*Proof.* Proposition 5.2.2 establishes that the target confirmation, which is $\mathrm{W}:\beta$, of each reflected preclusive undercut $\langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\beta\rangle$ in the set $\mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$ can only have a label cardinality of one, that is where $|\mathsf{label}(\mathrm{W}:\beta)| = 1$ and $\mathrm{W} \subseteq \mathrm{Y}$. Hence, using Definition 4.4.1 of reflected canonical undercut, each argument $K:\gamma$ within a reflected preclusive undercut attacker $\mathrm{Z}:\gamma$, where $K \in \mathrm{Z}$, can be represented as a canonical undercut, i.e. $K:\gamma \in \mathsf{canonicalUndercuts}(J:\beta, \Delta)$ so it is always the case that this $K:\gamma \in \mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)$ with no loss of information. Therefore it follows that $\mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta) = \{\langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\beta\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Y}:\beta, \Delta) \mid$ for every $K \in \mathrm{Z}$ there exists an $I \in \mathrm{V}$ and a $J \in \mathrm{Y}$ such that $K:\gamma \in \mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)\}$. □

In defining the reflected preclusive undercut an alternative approach could have started by defining them as comprised of reflected canonical undercuts - in which case Proposition 5.2.3 would become the definition and an adaptation of the above Definition 5.2.1 would follow as a proposition.

I now show that the number of reflected preclusive undercuts attacking any given defending confirmation is between one and the cardinality of that defending confirmation.

**Proposition 5.2.4.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{V}:\alpha, \mathrm{X}:\alpha, \mathrm{Y}:\beta \in \Diamond(\Delta)$.

$$0 \leqslant |\mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)| \leqslant |\mathrm{Y}|.$$

*Proof.* If a prerequisite, e.g. $\langle \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha\rangle \in \mathsf{confirmationAttacks}(\mathrm{X}:\alpha, \Delta)$, is absent then reflected preclusive undercuts do not exist, so the set's minimum cardinality is 0. Proposition 5.2.2 shows that each reflected preclusive undercut has a target cardinality of 1. Thus each argument $I \in \mathrm{Y}$, where $\mathrm{Y}:\beta$ is the defending confirmation of the reflected attack, may be subject to a unique reflected preclusive undercut, i.e. to a maximum of one preclusive undercut. Consequently there can only be as many reflected preclusive undercuts as there are deduction labels in Y. Therefore $0 \leqslant |\mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)| \leqslant |\mathrm{Y}|$. □

Each of these up to $|\mathrm{Y}|$ reflected preclusive undercuts, arising from the interaction of $\mathrm{X}:\alpha$ with $\mathrm{Y}:\beta$, will have a different claim and will contain one or more attacking arguments.

## 5.3 Direct Preclusive Undercuts

A direct preclusive undercut is a preclusive undercut that contains no reflected canonical undercut arguments, relative to a specific reflector. Thus all of its arguments are direct and none of them are reflected. Furthermore, a direct preclusive undercut is allowed to be a multi-target preclusive undercut – as is standard for preclusive undercuts – and therefore not every argument in a direct preclusive undercut can be represented as a direct canonical undercut, nor even as a canonical undercut. It may be the case that the input to a direct preclusive undercut is a confirmation attack $\langle \mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha\rangle \in \mathsf{confirmationAttacks}(\mathrm{X}:\alpha, \Delta)$ or it may not, in which case $\mathrm{V} = \mathrm{X} = \emptyset$ and only $\mathrm{Y}:\beta$ is non-empty. The definition builds on Definition 3.4.2 for preclusive undercut and Definition 4.4.1 for reflected canonical undercut.

**Definition 5.3.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $I:\alpha,\ J:\beta,\ K:\gamma \in$ $\mathsf{arguments}(\Delta)$ and let $\mathrm{V}:\alpha, \mathrm{W}:\beta, \mathrm{X}:\alpha, \mathrm{Y}:\beta, \mathrm{Z}:\gamma \in \Diamond(\Delta)$. The* **direct preclusive undercuts** *function, denoted $\mathsf{directPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$, provides the set of preclusive undercuts of $\mathrm{Y}:\beta$ that are not reflected off $\mathrm{V}:\alpha$ such that:*

$$
\begin{aligned}
&\mathsf{directPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)\\
&= \{\langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\beta\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Y}:\beta, \Delta) \mid\\
&\qquad (\textit{either } \langle \mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha\rangle \in \mathsf{confirmationAttacks}(\mathrm{X}:\alpha, \Delta) \textit{ or } \mathrm{V} = \mathrm{X} = \emptyset)\\
&\qquad \textit{and for every } K \in \mathrm{Z} \textit{ there does not exist a } J \in \mathrm{W} \textit{ and an } I \in \mathrm{V}\\
&\qquad \textit{where } K:\gamma \in \mathsf{reflectedCanonicalUndercuts}(J:\beta, I:\alpha, \Delta)\}.
\end{aligned}
$$

Here $\mathrm{Z}:\gamma$ is the direct attacking confirmation and $\mathrm{V}:\alpha$ is the reflecting confirmation. Clearly it is the case that i) $\mathrm{W}:\beta$ is such that $\emptyset \neq \mathrm{W} \subseteq \mathrm{Y}$, ii) that the negation of the conjunction of the members of $\mathsf{stripAssumptions}(\mathsf{formulae}(\mathrm{W}, \Delta))$ is equal to $\gamma$ and iii) as always, canonical ordering is followed. Also note that the attack of $\mathrm{Y}:\beta$ on $\mathrm{X}:\alpha$ can be either a confirmation rebuttal or a preclusive undercut. It is not possible to use a definition akin to Definition 4.5.1 of direct canonical undercut as $\mathsf{directPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta) \neq \mathsf{preclusiveUndercuts}(\mathrm{Y}:\beta, \Delta) \setminus \mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$ for reasons addressed in Section 5.4 (i.e. the existence of 'mixed preclusive undercuts', see Definition 5.4.1).

The cardinality of the target confirmation $\mathrm{W}:\beta$ for direct preclusive undercuts can be greater than one, $1 \leq |\mathrm{W}| \leq |\mathrm{Y}|$, unlike the situation with reflected preclusive undercuts where the cardinality of the target confirmation can only be one, $|\mathrm{W}| = 1$. In the above, Definition 5.3.1, the last two lines remove reflections and as such all removed arguments have the form of a reflected canonical undercut with a target cardinality of one.

I now give an example of a direct preclusive undercut, taking the opportunity to show an attacking confirmation, here $\mathrm{Z}:\neg(\beta \wedge (\beta \rightarrow \alpha))$, that has a support cardinality of greater than one, i.e. $|\mathrm{Z}| > 1$, making it a multi-attack direct preclusive undercut.

**Example 5.3.1. Direct preclusive undercut.** *Let $\Delta = \{a:\beta,\ b:\beta \rightarrow \alpha,\ c:\neg\beta,\ d:\neg\beta\}$. So $\{a, b\}:\alpha,\ \{c\}:\neg\beta,\ \{c\}:\neg(\beta \wedge (\beta \rightarrow \alpha)),\ \{d\}:\neg(\beta \wedge (\beta \rightarrow \alpha)) \in \mathsf{arguments}(\Delta)$. Consequently*

*$\{\{a,b\}\} : \alpha$, $\{\{c\},\{d\}\} : \neg(\beta \wedge (\beta \rightarrow \alpha)) \in \Diamond(\Delta)$. The target confirmation is $\{\{a,b\}\} : \alpha$ and the attacking confirmation is $\{\{c\},\{d\}\} : \neg(\beta \wedge (\beta \rightarrow \alpha))$. Let $\mathrm{X} = \{\{a,b\}\}$ and $\mathrm{Z} = \{\{c\},\{d\}\}$. There are no arguments for $\neg\alpha$ and indeed no arguments undercut by $\mathrm{X} : \alpha$, so for completeness it is stated that the unfounded confirmation $\emptyset : \neg\alpha \in \Diamond(\Delta)$ is passed in as the reflector in question. Thus $\langle \mathrm{Z} : \neg(\beta \wedge (\beta \rightarrow \alpha)), \mathrm{X}, \mathrm{X} : \alpha\rangle \in \mathsf{directPreclusiveUndercuts}(\mathrm{X} : \alpha, \emptyset, \emptyset : \neg\alpha, \Delta)$ is a direct preclusive undercut of $\mathrm{X} : \alpha$ as no rebuttal exists, i.e. because $\Delta \nvdash \neg\alpha$.*

### 5.3.1 Properties of Direct Preclusive Undercuts

Given a direct preclusive undercut and the confirmation it is not reflected off, then information about the supports involved is known.

**Proposition 5.3.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $I : \alpha$, $J : \beta$, $K : \gamma \in \mathsf{arguments}(\Delta)$ and let $\mathrm{V} : \alpha, \mathrm{W} : \beta, \mathrm{X} : \alpha, \mathrm{Y} : \beta, \mathrm{Z} : \gamma \in \Diamond(\Delta)$.*

$$\text{If } \langle \mathrm{Z} : \gamma, \mathrm{W}, \mathrm{Y} : \beta\rangle \in \mathsf{directPreclusiveUndercuts}(\mathrm{Y} : \beta, \mathrm{V}, \mathrm{X} : \alpha, \Delta),$$
$$\text{then for all } K \in \mathrm{Z} \text{ there does not exist an } I \in \mathrm{V} \text{ such that } K \subseteq I.$$

*Proof.* Given that $\langle \mathrm{Z} : \gamma, \mathrm{W}, \mathrm{Y} : \beta\rangle \in \mathsf{directPreclusiveUndercuts}(\mathrm{Y} : \beta, \mathrm{V}, \mathrm{X} : \alpha, \Delta)$ then by Definition 5.3.1 it is the case that for every $K \in \mathrm{Z}$ that for all $I \in \mathrm{V}$ and $J \in \mathrm{Y}$ there does not exist a $K : \gamma \in \mathsf{reflectedCanonicalUndercuts}(J : \beta, I : \alpha, \Delta)$. If there had been a reflected canonical undercut then it would have been the case that $K \subseteq I$, so as there is no reflection it is clear that for all $K \in \mathrm{Z}$ and all $I \in \mathrm{V}$ it is the case that $K \not\subseteq I$. Therefore if $\langle \mathrm{Z} : \gamma, \mathrm{W}, \mathrm{Y} : \beta\rangle \in \mathsf{directPreclusiveUndercuts}(\mathrm{Y} : \beta, \mathrm{V}, \mathrm{X} : \alpha, \Delta)$ then for all $K \in \mathrm{Z}$ there does not exist an $I \in \mathrm{V}$ such that $K \subseteq I$. □

Direct preclusive undercuts can be mono-target or multi-target preclusive undercuts. Reflected preclusive undercuts and all canonical undercuts, in comparison, are only ever mono-target undercuts.

**Proposition 5.3.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\mathrm{V} : \alpha, \mathrm{W} : \beta, \mathrm{X} : \alpha, \mathrm{Y} : \beta, \mathrm{Z} : \gamma \in \Diamond(\Delta)$.*

$$\text{If } \langle \mathrm{Z} : \gamma, \mathrm{W}, \mathrm{Y} : \beta\rangle \in \mathsf{directPreclusiveUndercuts}(\mathrm{Y} : \beta, \mathrm{V}, \mathrm{X} : \alpha, \Delta) \text{ then } |\mathrm{W}| \geq 1.$$

*Proof.* By Definition 5.3.1 of direct preclusive undercut and its use of Definition 4.4.1 of reflected canonical undercut there is no further constraint on W, beyond that for ordinary preclusions, see Definition 3.4.2, so the cardinality is the same as for ordinary preclusions. So if $\langle \mathrm{Z} : \gamma, \mathrm{W}, \mathrm{Y} : \beta\rangle \in \mathsf{directPreclusiveUndercuts}(\mathrm{Y} : \beta, \mathrm{V}, \mathrm{X} : \alpha, \Delta)$ then $|\mathrm{W}| \geq 1$. □

Furthermore it is possible to be quite specific about the cardinality of the direct preclusive undercuts function.

**Proposition 5.3.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\mathrm{V} : \alpha, \mathrm{X} : \alpha, \mathrm{Y} : \beta \in \Diamond(\Delta)$.*

$$0 \leq |\mathsf{directPreclusiveUndercuts}(\mathrm{Y} : \beta, \mathrm{V}, \mathrm{X} : \alpha, \Delta)| \leq |\mathsf{preclusiveUndercuts}(\mathrm{Y} : \beta, \Delta)|.$$

*Proof.* Depending upon $\Delta$ there may exist a $Y:\beta$ which simply does not have any undercuts, let alone direct undercuts, so the lower limit is 0. It may also be the case that $X = \emptyset$ when there are no reflecting arguments for $\alpha$ being passed in and thus $V = X = \emptyset$ is an empty reflector and hence there are no reflections. Consequently, at the upper limit, every preclusive undercut of $Y:\beta$ would be a direct preclusive undercut. Therefore $0 \leq |\mathsf{directPreclusiveUndercuts}(Y:\beta, V, X:\alpha, \Delta)| \leq |\mathsf{preclusiveUndercuts}(Y:\beta, \Delta)|$. □

### 5.3.2 Maximal Direct Preclusive Undercut

In the above definition of direct preclusive undercut, only one reflector is considered rather than all possible reflectors. Furthermore no constraint is applied that forces the attacking confirmation of a direct preclusive undercut to be as large a cardinality confirmation as possible. The two advantages of the following maximal cardinality direct preclusive undercuts function are that a) it considers all possible reflectors and b) the attacking confirmation is as large a cardinality as possible. Motivational discussion needed to justify the form of the maximal direct preclusive undercut is aided by Figure 5.2's annotated confirmation graph.

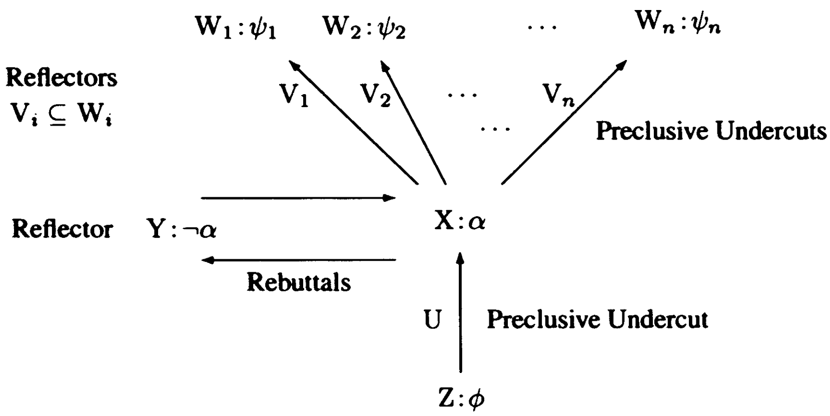


Figure 5.2: Reflectors to Consider for the Maximal Direct Preclusive Undercut $Z:\phi$

One might think that to determine the form of the maximal direct preclusive undercut, $Z:\phi$ in Figure 5.2 above, would involve two types and many occurrences of reflector. The two types of reflector would be rebuttal and preclusive undercut. While there can be only one rebuttal confirmation reflector, $Y:\neg\alpha$, there could be many preclusive undercut reflectors $V_1:\psi_1, \ldots, V_n:\psi_n$. However, following the extension of Base Proposition Two, to be provided shortly as Proposition 5.5.1, it is the case that every one of these preclusive undercut reflectors will be reflected down to the rebuttal $Y:\neg\alpha$ and therefore they do not need independent consideration as reflectors. Secondarily, a reflector can be part of either a direct or a reflected attack. If the reflector is itself part of a reflected attack then it will always exist in an attack chain where at the head of that attack chain is a direct argument. Thus, when using the maximum cardinality confirmation for $\neg\alpha$, i.e. $\mathsf{top}(\Diamond(\neg\alpha, \Delta))$ as the reflector it is not necessary to know which of these rebuttal arguments are direct and which are reflected.

A confirmation graph may be cast into a confirmation tree, following the common approach for casting graphs into trees, see for example (Chalmers, 2003), even if the resultant tree of confirmations is an infinite one. Now assume that these confirmation chains of Figure 5.2 are part of such a confirmation tree and consider the consequences. Thus it will be the case that $\mathrm{X} : \alpha$ has only one preclusive undercut parent, some $\mathrm{W} : \psi$, (unless $\mathrm{X} : \alpha$ is the root in which case it has no parent). Thus out of the many preclusive undercut reflectors, $\mathrm{V}_1 : \psi_1, \ldots, \mathrm{V}_n : \psi_n$, which could act as reflectors, there is only one preclusive undercut reflector, $\mathrm{V} : \psi$, that is active as all the other possible preclusive reflectors will not be part of the tree and thus rendered irrelevant. There is no need to pass in even this one preclusive reflector as its presence is accounted for by the maximum cardinality rebuttal confirmation: $\mathsf{top}(\Diamond(\neg\mathsf{stripConfirmation}(\mathrm{X}:\alpha), \Delta)) = \mathrm{Y} : \neg\alpha$. Of all these possible reflector rebuttal arguments, considering each $J : \neg\alpha$ where $J \in \mathrm{Y}$, some will be part of the tree and others will not be (they are 'pruned out', analogous to the way that a gardener prunes a rose bush or tree by removing undesirable material) because they are reflected off arguments higher in the tree (nearer to the root). Thus to consider them all as reflectors it is not sufficient to just refer to any rebuttal confirmation of $\mathrm{X}:\alpha$, but rather it is necessary to refer to the maximum cardinality confirmation that has the claim of $\neg\mathsf{stripConfirmation}(\mathrm{X}:\alpha)$. Now I formalise this discussion as the definition for the maximal direct preclusive undercut. For comparison, the non-maximal or ordinary direct preclusive undercuts function is Definition 5.3.1 and the attacker helper function is Definition 3.7.1.

**Definition 5.3.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{V}:\alpha,\ \mathrm{X}:\alpha,\ \mathrm{Y}:\neg\alpha,\ \mathrm{Z}:\phi \in \Diamond(\Delta)$. *The* **maximal direct preclusive undercuts** *function, denoted* $\mathsf{maximalDirectPreclusiveUndercuts}(\mathrm{X}:\alpha,\ \Delta)$, *which provides the set of maximal cardinality preclusive undercuts of* $\mathrm{X}:\alpha$ *that are not reflected off any reflector derivable from* $\Delta$, *is such that:*

$$
\begin{aligned}
&\mathsf{maximalDirectPreclusiveUndercuts}(\mathrm{X}:\alpha, \Delta)\\
&= \{\langle \mathrm{Z}:\phi, \mathrm{V}, \mathrm{X}:\alpha\rangle \in \mathsf{directPreclusiveUndercuts}(\mathrm{X}:\alpha, \mathrm{Y}, \mathrm{Y}:\neg\alpha, \Delta) \mid\\
&\qquad \mathrm{Z}:\phi \in \mathsf{top}(\mathsf{attacker}(\mathsf{directPreclusiveUndercuts}(\mathrm{X}:\alpha, \mathrm{Y}, \mathrm{Y}:\neg\alpha, \Delta)))\ \textit{and}\\
&\qquad \mathrm{Y}:\neg\alpha = \mathsf{top}(\Diamond(\neg\mathsf{stripConfirmation}(\mathrm{X}:\alpha), \Delta))\}.
\end{aligned}
$$

The first $\mathsf{top}()$ ensures that $\mathrm{Z}:\phi$ is the maximum cardinality member of its power set, i.e. that the attacker of any other preclusive undercut, $\langle \mathrm{Z}':\phi', \mathrm{V}', \mathrm{X}:\alpha\rangle$, where $\mathrm{V} \neq \mathrm{V}' \subsetneq \mathrm{X}$, that exists that is also not reflected off the same reflectors and that attacks $\mathrm{X}:\alpha$, has a label of lower cardinality than the maximum cardinality $\mathrm{Z} : \phi$. Clearly there can only be one top member of the set of subsets of Z. The defending confirmation $\mathrm{X}:\alpha$ can still have many maximal cardinality direct preclusive undercuts attacking it as X can have many subsets, $\mathrm{V} \subseteq \mathrm{X}$, and thus contain many targets. Each maximal cardinality confirmation attacking $\mathrm{X} : \alpha$ will have a different claim. The target V is unconstrained, apart from $\mathrm{V} \subseteq \mathrm{X}$, so the preamble of Definition 5.3.2 contains the phrase 'the set of' to cover all the targets (not the power set of attackers). I now give an example of a maximal direct preclusive undercut.

**Example 5.3.2. Maximal direct preclusive undercut.** *Let* $\Delta = \{a : \beta, b : \beta \to \alpha, c : \gamma, d : \gamma \to (\neg\beta \vee \neg\pi), e:\pi, f:\neg\pi \vee \alpha, g:\neg\lambda, h:\lambda \vee \neg(\beta \wedge \pi), i:\neg\zeta, j:\alpha \to \zeta\}$. *Let* $I = \{a,b\}, J = \{c,d\}, K = \{e,f\}, L = \{g,h\}, M = \{i,j\}$ *and let* $\mathrm{T} = \{I\}, \mathrm{U} = \{K\}, \mathrm{V} = \{J\}, \mathrm{W} = \{L\}, \mathrm{X} = \{I,K\}, \mathrm{Y} =$

$\{M\}, Z = \{J, L\}$. *So* $I:\alpha, K:\alpha, M:\neg\alpha, J:\neg\beta\vee\neg\pi, L:\neg(\beta\wedge\pi) \in \mathsf{arguments}(\Delta)$. *Thus* $X:\alpha \in \Diamond(\Delta)$ *is the defending confirmation and* $Y:\neg\alpha \in \Diamond(\Delta)$ *the rebutting confirmation and reflector. The attacking confirmations are* $\langle Y:\neg(\beta\wedge(\beta\rightarrow\alpha)), T, X:\alpha\rangle, \langle Y:\neg(\pi\wedge(\neg\pi\vee\alpha)), U, X:\alpha\rangle, \langle V:\neg(\beta\wedge(\beta\rightarrow\alpha)\wedge\pi\wedge(\neg\pi\vee\alpha)), X, X:\alpha\rangle, \langle W:\neg(\beta\wedge(\beta\rightarrow\alpha)\wedge\pi\wedge(\neg\pi\vee\alpha)), X, X:\alpha\rangle, \langle Z:\neg(\beta\wedge(\beta\rightarrow\alpha)\wedge\pi\wedge(\neg\pi\vee\alpha)), X, X:\alpha\rangle \in \mathsf{preclusiveUndercuts}(X:\alpha,\Delta)$. *However* $\langle Y:\neg(\beta\wedge(\beta\rightarrow\alpha)), T, X:\alpha\rangle, \langle Y:\neg(\pi\wedge(\neg\pi\vee\alpha)), U, X:\alpha\rangle \in \mathsf{reflectedPreclusiveUndercuts}(X:\alpha, Y, Y:\neg\alpha,\Delta)$. *Consequently* $\langle V:\neg(\beta\wedge(\beta\rightarrow\alpha)\wedge\pi\wedge(\neg\pi\vee\alpha)), X, X:\alpha\rangle, \langle W:\neg(\beta\wedge(\beta\rightarrow\alpha)\wedge\pi\wedge(\neg\pi\vee\alpha)), X, X:\alpha\rangle, \langle Z:\neg(\beta\wedge(\beta\rightarrow\alpha)\wedge\pi\wedge(\neg\pi\vee\alpha)), X, X:\alpha\rangle \in \mathsf{directPreclusiveUndercuts}(X:\alpha, Y, Y:\neg\alpha,\Delta)$. *However only* $\langle Z:\neg(\beta\wedge(\beta\rightarrow\alpha)\wedge\pi\wedge(\neg\pi\vee\alpha)), X, X:\alpha\rangle \in \mathsf{maximalDirectPreclusiveUndercuts}(X:\alpha,\Delta)$. *It is the case that* $\langle V:\neg(\beta\wedge(\beta\rightarrow\alpha)\wedge\pi\wedge(\neg\pi\vee\alpha)), X, X:\alpha\rangle, \langle W:\neg(\beta\wedge(\beta\rightarrow\alpha)\wedge\pi\wedge(\neg\pi\vee\alpha)), X, X:\alpha\rangle \notin \mathsf{maximalDirectPreclusiveUndercuts}(X:\alpha,\Delta)$ *as they are not maximal. It is also the case that* $\langle Y:\neg(\beta\wedge(\beta\rightarrow\alpha)), T, X:\alpha\rangle, \langle Y:\neg(\pi\wedge(\neg\pi\vee\alpha)), U, X:\alpha\rangle \notin \mathsf{maximalDirectPreclusiveUndercuts}(X:\alpha,\Delta)$ *as they are reflected.*

To conclude this section on direct preclusive undercuts I emphasise their relevance to debate tracking; i.e. to the tracking of all and of only direct arguments. Whilst this and the previous chapter cover both reflected and direct arguments, it is the reflections which are to be avoided or removed and the direct arguments which are of lasting value. This is perhaps akin to woodwork, where the direct arguments are the piece of art or equipment being created and the reflections the waste or scrap wood around the work that need to be cut off and discarded. Maximal direct preclusive undercuts are especially relevant as they are immune to the relative weakness I pointed out in Section 4.5.1 on direct canonical undercuts. In a maximal direct preclusion, by using a maximum cardinality reflector, which stands for all other reflectors, the status of being direct is strengthened from being relative to being absolute.

## 5.4 Mixed Preclusive Undercuts and Invalidation

A mixed preclusive undercut is a preclusive undercut that contains a *mixture* of *direct* and *reflected* preclusive undercuts. All of the arguments in one of these attacking confirmations will have the same claim, but different supports. One might think that this is a mixture of direct and reflected canonical undercuts, however that constraint is unnecessary as the direct arguments may not all be representable as direct canonical undercuts - due to the possibility of multi-target claims. A confirmation is a mixed preclusive undercut iff it is a preclusive undercut and yet not a reflected preclusive undercut nor a direct preclusive undercut, as I now define:

**Definition 5.4.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $V:\alpha,\ X:\alpha,\ Y:\beta, \in \Diamond(\Delta)$. *A* **mixed preclusive undercut**, *denoted* $\mathsf{mixedPreclusiveUndercuts}(Y:\beta, V, X:\alpha,\Delta)$, *is a* *preclusive undercut where some of its arguments are direct and some are reflected, such that:*

$$\begin{aligned}
&\mathsf{mixedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta) \\
&= \;(\mathsf{preclusiveUndercuts}(\mathrm{Y}:\beta, \Delta) \setminus \\
&\qquad \mathsf{directPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)) \setminus \\
&\qquad \mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta).
\end{aligned}$$

The existence of mixed preclusive undercuts, as shown in the following example, illustrates that judges which rely on the counting or existence of arguments need to distinguish between direct and reflected arguments *within a preclusion* – as the two can be mixed together and only direct arguments should be allowed to affect professional debate outcomes.

**Example 5.4.1. Mixed preclusive undercut.** *Let* $\Delta = \{a : \beta, b : \beta \to \alpha, c : \neg\beta, d : \neg(\beta \to \alpha)\}$. *Let* $I = \{a, b\}, J = \{c\}, K = \{d\}$. *So* $I : \alpha, K : \neg\alpha, J : \neg\beta, J : \neg\beta \vee \neg(\beta \to \alpha), J : \neg(\beta \wedge (\beta \to \alpha)), K : \neg(\beta \to \alpha), K : \neg(\beta \to \alpha) \vee \neg\beta, K : \neg(\beta \wedge (\beta \to \alpha)) \in \mathsf{arguments}(\Delta)$. *Consequently* $\{I\} : \alpha, \{K\} : \neg\alpha, \{J, K\} : \neg(\beta \wedge (\beta \to \alpha)) \in \Diamond(\Delta)$. *It follows that* $\langle\{J, K\} : \neg(\beta \wedge (\beta \to \alpha)), \{I\}, \{I\} : \alpha\rangle \in \mathsf{mixedPreclusiveUndercuts}(\{I\} : \alpha, \{K\}, \{K\} : \neg\alpha, \Delta)$ *is a mixed preclusive undercut. The attacking confirmation label of this mixed preclusive undercut is the union of the attacking confirmation label of the direct preclusive undercut* $\langle\{J\} : \neg(\beta \wedge (\beta \to \alpha)), \{I\}, \{I\} : \alpha\rangle \in \mathsf{directPreclusiveUndercuts}(\{I\} : \alpha, \{K\}, \{K\} : \neg\alpha, \Delta)$ *and the attacking confirmation label of the reflected preclusive undercut* $\langle\{K\} : \neg(\beta \wedge (\beta \to \alpha)), \{I\}, \{I\} : \alpha\rangle \in \mathsf{reflectedPreclusiveUndercuts}(\{I\} : \alpha, \{K\}, \{K\} : \neg\alpha, \Delta)$, *i.e.* $\{J, K\} = \{J\} \cup \{K\}$.

A simple observation is that a direct preclusive undercut can never be invalidated by a reflected preclusive undercut. Definition 3.4.2 of preclusive undercut, together with the definitions of terms in the three paragraphs following it, establish that any invalidating preclusive undercut must have the same attacking confirmation label as the invalidated preclusive undercut. If two preclusive undercuts have the same attacking confirmation label they must be either both direct or both reflected. So direct preclusive undercuts can never be invalidated by reflected preclusive undercuts.

Another simple observation is that a reflected preclusive undercut can never be invalidated. If a preclusive undercut is multi-target then it must be direct. Proposition 5.2.2 shows that reflected preclusive undercuts are all mono-target. Proposition 5.3.2 shows that direct preclusive undercuts can be mono-target or multi-target. Therefore a multi-target preclusion cannot be the result of reflection and thus must be direct. Thus, it follows that a reflected preclusive undercut can never be invalidated; not by a direct preclusive undercut, not by a reflected preclusive undercut, indeed not by any preclusive undercut. The reason that reflected preclusive undercuts cannot be invalidated is that they can only ever be mono-target and hence the formulae of the attacking confirmation label can never imply a non-empty shorter claim. Thus if invalidation occurs it must be that a direct preclusive undercut has been invalidated by another direct preclusive undercut.

## 5.5 Type II Reflection – Reflected Confirmation Rebuttals

To recap for context, the previous chapter, in Section 4.7, covered the reflected rebuttals arising from a single pair of arguments (Definition 4.7.1), one attacking the other, analysing it as a Type I Reflection. Now I turn to the situation of reflected rebuttals arising from a pair of confirmations, again with one attacking the other, and establish that this is a new type of reflection which I call Type II Reflection. This pattern is also a parallel of Section 4.4 on reflected canonical undercuts being elaborated upon by Section 5.2 on reflected preclusive undercuts. Now reflected argument rebuttals, Definition 4.7.1, are elaborated upon by reflected confirmation rebuttal, Definition 5.5.1. My focus here in the current section is on confirmations containing multiple arguments, i.e. multi-attack confirmations, as that distinguishes them from self-confirmed single arguments. It is the rebuttals reflected off multi-target preclusive undercuts, in particular, that provide novel behaviour. Note that reflected argument rebuttals between single arguments, addressed earlier with Definition 4.7.1 as a form of Type I Reflection, does not require the $K \subseteq I_1 \cup \ldots \cup I_n$ clause introduced below. Figure 5.3 shows an annotated confirmation graph illustrating the requirements and form of Type II Reflection.

| | | |
|---|---|---|
| | $\mathrm{X}:\alpha$ | |
| Reflector Confirmation is $\mathrm{V}:\alpha$ where $\mathrm{V} = \{I_1, \ldots, I_n\}$, and $\mathrm{V} \subseteq \mathrm{X}$. | $\uparrow$ V | $\langle \mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha \rangle \in \mathsf{preclusiveUndercuts}(\mathrm{X}:\alpha, \Delta)$ |
| | $\mathrm{Y}:\beta$ | |
| | $\uparrow$ Y | $\mathrm{Z}:\neg\beta \in \mathsf{confirmationRebuttals}(\mathrm{Y}:\beta, \Delta)$ |
| Reflected Confirmation | $\mathrm{Z}:\neg\beta$ | For each $K \in \mathrm{Z}$, $K \subseteq I_1 \cup \ldots \cup I_n$. |

Figure 5.3: Type II Reflection is Provided by $\mathsf{reflectedConfirmationRebuttals}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$

In defining the phenomena of reflected confirming rebuttals there could be two situations: a) the reflector is subject to preclusive undercut, or b) the reflector is subject to confirmation rebuttal. Even though including b) would fit better with the abstract perspective of the last chapter and of this chapter so far, it would be trivial. If a confirmation rebuttal is reflected off another rebuttal then the reflector is itself – which does not add any new arguments, nor materially add to the tracking of professional debates. Instead I analyse here the situation a) above, as then the following properties and definitions are more practical and intuitive. I now define the shape or form of the reflected confirmation rebuttals function. While the definition is akin to that for reflected preclusive undercuts, Definition 5.2.1, a number of differences are necessary. This function returns the full power set of confirmations, not just the maximum cardinality one.

**Definition 5.5.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $I_1 : \alpha, \ldots, I_n : \alpha, K : \neg\beta \in \mathsf{arguments}(\Delta)$ and let $\mathrm{V}:\alpha,\ \mathrm{X}:\alpha,\ \mathrm{Y}:\beta,\ \mathrm{Z}:\neg\beta \in \Diamond(\Delta)$. The* **reflected confirmation rebuttals**

*that can be inferred from the existence of the confirmation attack of* $\mathrm{Y}:\beta$ *on defending confirmation* $\mathrm{X}:\alpha$ *with target confirmation* $\mathrm{V}:\alpha$*, denoted* $\mathsf{reflectedConfirmationRebuttals}(\mathrm{Y}:\beta,\mathrm{V},\mathrm{X}:\alpha,\Delta)$*, is a set of confirmations such that:*

$$\begin{aligned}
&\mathsf{reflectedConfirmationRebuttals}(\mathrm{Y}:\beta,\mathrm{V},\mathrm{X}:\alpha,\Delta)\\
&= \{\mathrm{Z}:\neg\beta \in \mathsf{confirmationRebuttals}(\mathrm{Y}:\beta,\Delta) \mid\\
&\qquad \langle \mathrm{Y}:\beta,\mathrm{V},\mathrm{X}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{X}:\alpha,\Delta),\\
&\qquad \textit{for each } K \in \mathrm{Z} \textit{ it is the case that } K \subseteq I_1 \cup \ldots \cup I_n,\\
&\qquad \textit{where } \mathrm{V} = \{I_1,\ldots,I_n\}.
\end{aligned}$$

The fact that a reflected confirmation rebuttal is also a confirmation rebuttal ensures that neither the attacked nor attacking confirmations are labelled with the empty set. I now give an example of a reflected confirmation rebuttal, which uses the '$=$' part of the '$\subseteq$' on the last line of the Definition 5.5.1.

**Example 5.5.1. Reflected confirmation rebuttal.** *Let* $\Delta = \{a:\beta, b:\beta\rightarrow\alpha, c:\zeta, d:\zeta\rightarrow\alpha, e:\neg\eta, f:\neg\alpha\rightarrow\eta, g:\neg\beta\vee\neg\zeta\}$*. Let* $I=\{a,b\}, J=\{c,d\}, K=\{e,f\}, L=\{g\}, M=\{a,b,c,d\}$*. So* $I:\alpha, J:\alpha, K:\alpha, L:\neg(\beta\wedge(\beta\rightarrow\alpha)\wedge\zeta\wedge(\zeta\rightarrow\alpha)), M:\beta\wedge(\beta\rightarrow\alpha)\wedge\zeta\wedge(\zeta\rightarrow\alpha)\in\mathsf{arguments}(\Delta)$*. Let* $\mathrm{X}=\{I,J,K\}, \mathrm{W}=\{I,J\}, Y=\{L\}, \mathrm{Z}=\{M\}$*. Thus* $\mathrm{X}:\alpha, \mathrm{W}:\alpha, \mathrm{Y}:\neg(\beta\wedge(\beta\rightarrow\alpha)\wedge\zeta\wedge(\zeta\rightarrow\alpha)), \mathrm{Z}:\beta\wedge(\beta\rightarrow\alpha)\wedge\zeta\wedge(\zeta\rightarrow\alpha)\in\Diamond(\Delta)$*. Consequently* $\langle\mathrm{Y}:\neg(\beta\wedge(\beta\rightarrow\alpha)\wedge\zeta\wedge(\zeta\rightarrow\alpha)),\mathrm{W},\mathrm{X}:\alpha\rangle\in\mathsf{preclusiveUndercuts}(\mathrm{X}:\alpha,\Delta)$*. It also follows that* $\mathrm{Z}:\beta\wedge(\beta\rightarrow\alpha)\wedge\zeta\wedge(\zeta\rightarrow\alpha)\in\mathsf{confirmationRebuttals}(\mathrm{Y}:\neg(\beta\wedge(\beta\rightarrow\alpha)\wedge\zeta\wedge(\zeta\rightarrow\alpha)),\Delta)$*. Matching with the definition of reflected confirmation rebuttal, the observation can be made that in this example indeed* $M\subseteq I\cup J$*, in fact* $M = I\cup J$*. Thus the conclusion can be drawn that* $\mathrm{Z}:\beta\wedge(\beta\rightarrow\alpha)\wedge\zeta\wedge(\zeta\rightarrow\alpha)\in\mathsf{reflectedConfirmationRebuttals}(\mathrm{Y}:\neg(\beta\wedge(\beta\rightarrow\alpha)\wedge\zeta\wedge(\zeta\rightarrow\alpha)),\mathrm{W},\mathrm{X}:\alpha,\Delta)$.

The above example also illustrates the point that the reflector is always the target confirmation, here $\mathrm{W}:\alpha$, not the larger defending confirmation, here $\mathrm{X}:\alpha$, unless of course $\mathrm{W}=\mathrm{X}$ given that always $\mathrm{W}\subseteq\mathrm{X}$. This point is not a property or separate design decision as it is contained within the Definition 5.5.1 of reflected confirmation rebuttals. The example also brings out that this definition requires not just such a target confirmation, but also that $\mathrm{Y}:\beta$ attacks $\mathrm{X}:\alpha$, in the form of a preclusive undercut.

### 5.5.1 Properties of Reflected Confirmation Rebuttals

Now that the form of reflected confirmation rebuttals is established it is possible to show that given their prerequisites, reflected confirmation rebuttals always exist.

**Proposition 5.5.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{W}:\alpha,\mathrm{X}:\alpha,\mathrm{Y}:\beta\in\Diamond(\Delta)$.

$$\begin{aligned}
&\textit{If } \langle\mathrm{Y}:\beta,\mathrm{W},\mathrm{X}:\alpha\rangle\in\mathsf{preclusiveUndercuts}(\mathrm{X}:\alpha,\Delta)\\
&\quad \textit{then } \mathsf{reflectedConfirmationRebuttals}(\mathrm{Y}:\beta,\mathrm{W},\mathrm{X}:\alpha,\Delta)\neq\emptyset,\\
&\quad \textit{otherwise } \mathsf{reflectedConfirmationRebuttals}(\mathrm{Y}:\beta,\mathrm{W},\mathrm{X}:\alpha,\Delta)=\emptyset.
\end{aligned}$$

*Proof.* Given that $\langle \mathrm{Y} : \beta, \mathrm{W}, \mathrm{X} : \alpha \rangle \in \mathsf{preclusiveUndercuts}(\mathrm{X} : \alpha, \Delta)$ it follows from Definition 3.4.2 of preclusive undercut that $\mathrm{X} \neq \emptyset, \mathrm{Y} \neq \emptyset$ and that there must exist a target confirmation $\mathrm{W} : \alpha$ such that $\emptyset = \mathrm{W} \subseteq \mathrm{X}$. From the same definition, as $\langle \mathrm{Y} : \beta, \mathrm{W}, \mathrm{X} : \alpha \rangle$ is a preclusive undercut of $\mathrm{X} : \alpha$ with target $\mathrm{W} : \alpha$, it follows that $\neg\beta = \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(\mathrm{W} : \alpha), \Delta))$. Now to rebut $\mathrm{Y} : \beta$ just requires one or more argument $K : \neg\beta$, where $K \in \mathrm{Z}$, $\mathrm{Z} : \neg\beta \in \mathsf{reflectedConfirmationRebuttals}(\mathrm{Y} : \beta, \mathrm{W}, \mathrm{X} : \alpha, \Delta)$ and $K$ is any support that infers $\neg\beta$ to ensure a non-empty confirmation label $\mathsf{label}(\mathrm{Z} : \neg\beta)$. Such a K always exists as given these preconditions it follows that $\mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(\mathrm{W} : \alpha), \Delta)) \vdash \neg\beta$, so there will always exist the reflexive argument $K : \neg\beta \in \mathsf{arguments}(\Delta)$ where $K \subseteq I_1 \cup \ldots \cup I_n$ and $\mathrm{W} = I_1 \cup \ldots \cup I_n$. This union $I_1 \cup \ldots \cup I_n$ therefore always implies $\neg\beta$, the negation of the claim of the preclusive undercut attacker $\mathrm{Y} : \beta$. The full union may not be needed however, as $I_1 \cup \ldots \cup I_n \vdash \neg\beta$ may not be a minimal argument. If any of the prerequisites are not met there cannot be a reflection. So if $\langle \mathrm{Y} : \beta, \mathrm{W}, \mathrm{X} : \alpha \rangle \in \mathsf{preclusiveUndercuts}(\mathrm{X} : \alpha, \Delta)$ then $\mathsf{reflectedConfirmationRebuttals}(\mathrm{Y} : \beta, \mathrm{W}, \mathrm{X} : \alpha, \Delta) \neq \emptyset$ otherwise $\mathsf{reflectedConfirmationRebuttals}(\mathrm{Y} : \beta, \mathrm{W}, \mathrm{X} : \alpha, \Delta) = \emptyset$. □

With the reflected argument rebuttal, one reflecting argument gives rise to one and only reflected argument, as shown in Proposition 4.7.5. However with the reflected confirmation rebuttal the number of reflections is not as simple. The above example, Example 5.5.1, shows two reflector arguments in the target confirmation giving rise to one reflected argument. The following example shows four defending arguments with two reflector arguments in the target confirmation giving rise to two reflected arguments. Here there are two subsets (that make up the support of the reflected attacking confirmation) of the union of the supports of the reflector target confirmation.

**Example 5.5.2. Subset.** *Let* $\Delta = \{a : \lambda \wedge \pi, b : \lambda \wedge \pi, c : \lambda \rightarrow \alpha, d : \pi \rightarrow \alpha, e : (\lambda \vee \pi) \wedge \neg\alpha\}$. *Let* $I = \{a, c\}, J = \{a, d\}, K = \{b, c\}, L = \{b, d\}, M = \{e\}, N = \{a, c, d\}, O = \{b, c, d\}$. *So* $I : \alpha, J : \alpha, K : \alpha, L : \alpha, M : (\lambda \wedge \neg\alpha) \vee (\pi \wedge \neg\alpha), M : \neg(\lambda \rightarrow \alpha) \vee \neg(\pi \rightarrow \alpha), M : \neg(\lambda \wedge \pi \wedge (\lambda \rightarrow \alpha) \wedge (\pi \rightarrow \alpha)), N : \lambda \wedge \pi \wedge (\lambda \rightarrow \alpha) \wedge (\pi \rightarrow \alpha), O : \lambda \wedge \pi \wedge (\lambda \rightarrow \alpha) \wedge (\pi \rightarrow \alpha) \in \mathsf{arguments}(\Delta)$. *Let* $\mathrm{X} = \{I, J, K, L\}, \mathrm{W} = \{I, L\}, \mathrm{Y} = \{M\}, \mathrm{Z} = \{N, O\}$. *Thus* $\mathrm{X} : \alpha, \mathrm{W} : \alpha, \mathrm{Y} : \neg(\lambda \wedge \pi \wedge (\lambda \rightarrow \alpha) \wedge (\pi \rightarrow \alpha)), \mathrm{Z} : \lambda \wedge \pi \wedge (\lambda \rightarrow \alpha) \wedge (\pi \rightarrow \alpha) \in \Diamond(\Delta)$. *Consequently* $\langle \mathrm{Y} : \neg(\lambda \wedge \pi \wedge (\lambda \rightarrow \alpha) \wedge (\pi \rightarrow \alpha)), \mathrm{W}, \mathrm{X} : \alpha \rangle \in \mathsf{preclusiveUndercuts}(\mathrm{X} : \alpha, \Delta)$. *The point is that* $\mathrm{Z} : \lambda \wedge \pi \wedge (\lambda \rightarrow \alpha) \wedge (\pi \rightarrow \alpha) \in \mathsf{reflectedConfirmationRebuttals}(\mathrm{Y} : \neg(\lambda \wedge \pi \wedge (\lambda \rightarrow \alpha) \wedge (\pi \rightarrow \alpha)), \mathrm{W}, \mathrm{X} : \alpha, \Delta)$. *Here* $|\mathrm{Z}| = 2$ *and each* $N, O \in \mathrm{Z}$ *is such that* $N \subsetneq (I \cup L), O \subsetneq (I \cup L)$. *At the assumption level observe that* $N = \{a, c, d\} \subsetneq \{a, b, c, d\}$ *and* $O = \{b, c, d\} \subsetneq \{a, b, c, d\}$. *So here is a reflected confirmation rebuttal* $\mathrm{Z} : \lambda \wedge \pi \wedge (\lambda \rightarrow \alpha) \wedge (\pi \rightarrow \alpha)$ *where the defending confirmation* $\mathrm{X} : \alpha$ *contains four arguments, the target or reflector* $\mathrm{W} : \alpha$ *contains two and the reflection two.*

In the above example the target confirmation is uniquely set by the $\mathrm{W} = \{I, L\}$. However, still with all of the following being identical to the example: knowledgebase $\Delta$, defending $\mathrm{X} : \alpha$, attacking $\mathrm{Y} : \neg(\lambda \wedge \pi \wedge (\lambda \rightarrow \alpha) \wedge (\pi \rightarrow \alpha))$ and reflected $\mathrm{Z} : \lambda \wedge \pi \wedge (\lambda \rightarrow \alpha) \wedge (\pi \rightarrow \alpha)$, the target could have been defined differently. A valid target would have to include the assumptions $a, b, c, d$, meaning it could have been any one of $\{I, L\}, \{J, K\}, \{I, J, K\}, \{I, J, L\}, \{I, K, L\}, \{J, K, L\}$ or $\{I, J, K, L\}$.

It can also be the case that the number of reflected arguments in the reflected confirmation rebuttal is greater than the number of reflector arguments in the target of the preclusive undercut at the head of the chain, as I now show.

**Example 5.5.3. Enlarged reflection.** *Let* $\Delta = \{a:\pi, b:\pi, c:\pi \rightarrow \alpha, d:\pi \rightarrow \alpha, e:\neg\pi\}$. *Let* $I = \{a,c\}, J = \{b,d\}, K = \{b,c\}, L = \{a,d\}, M = \{e\}$. *Let* $\mathrm{X} = \{I,J\}, \mathrm{Y} = \{M\}, \mathrm{Z} = \{I,J,K,L\}$. *Thus* $\mathrm{Y}:\neg(\pi \wedge (\pi \rightarrow \alpha)), \mathrm{X}:\alpha, \mathrm{Z}:\alpha, \mathrm{Y}:\neg(\pi \wedge (\pi \rightarrow \alpha) \wedge \pi \wedge (\pi \rightarrow \alpha)), \mathrm{Z}:\pi \wedge (\pi \rightarrow \alpha) \in \Diamond(\Delta)$. *So* $\langle \mathrm{Y}:\neg(\pi \wedge (\pi \rightarrow \alpha)), \mathrm{X}, \mathrm{Z}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Z}:\alpha)$. *The preclusive undercut attack is on the target confirmation label of* X *and hence* X *is the reflector. Hence* $\mathrm{Z}:\pi \wedge (\pi \rightarrow \alpha) \in \mathsf{reflectedConfirmationRebuttals}(\mathrm{Y}:\neg(\pi \wedge (\pi \rightarrow \alpha)), \mathrm{X}, \mathrm{Z}:\alpha, \Delta)$. *Note that* $\langle \mathrm{Y}:\neg(\pi \wedge (\pi \rightarrow \alpha) \wedge \pi \wedge (\pi \rightarrow \alpha)), \mathrm{X}, \mathrm{Z}:\alpha\rangle \notin \mathsf{preclusiveUndercuts}(\mathrm{Z}:\alpha)$ *as it is invalidated.*

Reflected confirmation rebuttals are always confirmed reflexive arguments. See Definition 2.4.7 for reflexive arguments and Definition 2.5.6 for the confirm function as used in the proof below.

**Proposition 5.5.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae, let* $I_j:\neg\beta, I_1:\neg\beta, \ldots, I_n:\neg\beta \in \mathsf{arguments}(\Delta)$ *and let* $\mathrm{V}:\alpha, \mathrm{X}:\alpha, \mathrm{Y}:\beta, \mathrm{Z}:\neg\beta \in \Diamond(\Delta)$.

$$\textit{If } \mathrm{Z}:\neg\beta \in \mathsf{reflectedConfirmationRebuttals}(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta) \textit{ and}$$
$$I_j \in \mathrm{Z} = \{I_1, \ldots, I_n\}$$
$$\textit{then each } I_j:\neg\beta \textit{ is a reflexive argument.}$$

*Proof.* For an argument $I:\alpha$ to be a reflexive argument it is necessary that the claim $\mathsf{stripDeduction}(I:\alpha)$ equals the conjunction of the members of the support $\bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(I:\alpha), \Delta))$. Meeting this condition is a necessary step in the proof of Proposition 5.5.1 which showed that reflected confirmation rebuttals always exist. Thus any reflected confirmation rebuttal $\mathrm{Z}:\neg\beta = \mathsf{confirm}(I_1:\neg\beta, \ldots, I_n:\neg\beta)$ is such that each argument $I_j:\neg\beta$ in it is always a reflexive argument. □

I look more closely at the number of reflected arguments in the next but one section on scaled reflections. That completes my analysis of reflected confirmation rebuttals, which are Type II Reflections, distinct from Type I Reflections analysed above as reflected attacks. Now I turn to the topic of direct confirmation rebuttals.

## 5.6 Direct Confirmation Rebuttals

By symmetry it is apparent that just as there are rebuttals, reflected rebuttals, direct rebuttals, confirmation rebuttals and reflected confirmation rebuttals, so also there must be direct confirmation rebuttals. This is not just a mechanical symmetry, however, as noteworthy properties follow. Now I leverage the definition of confirmation rebuttal, Definition 2.6.8 and reflected confirmation rebuttal, Definition 5.5.1.

**Definition 5.6.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{V}:\neg\beta, \mathrm{W}:\alpha, \mathrm{X}:\alpha, \mathrm{Y}:\beta \in \Diamond(\Delta)$. *The set of* **direct confirmation rebuttals**, *denoted* $\mathsf{directConfirmationRebuttals}(\mathrm{Y}:$

$\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$, *is a power set of confirmation rebuttals of* $\mathrm{Y}:\beta$ *not reflected off* $\mathrm{X}:\alpha$ *such that:*

$$\begin{aligned}
&\mathsf{directConfirmationRebuttals}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta) \\
&= \{\mathrm{Z}:\neg\beta \in \Diamond(\Delta) \mid \langle \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{X}:\alpha, \Delta), \\
&\qquad \mathrm{Z}:\neg\beta \in \mathsf{confirmationRebuttals}(\mathrm{Y}:\beta, \Delta) \textit{ and} \\
&\qquad \textit{there does not exist a } \mathrm{V} \subseteq \mathrm{Z} \textit{ such that} \\
&\qquad \mathrm{V}:\neg\beta \in \mathsf{reflectedConfirmationRebuttals}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)\}.
\end{aligned}$$

As with other reflected and direct confirmations there are a number of power sets involved here. The direct confirmation rebuttal is so defined as to not mandate that any of the particular confirmations are the maximum cardinality members of their respective power sets. What is clear, however, is that all of the support of $\mathrm{Z}:\neg\beta$ is direct as no part of it, i.e. $\mathsf{label}(\mathrm{V}:\neg\beta)$, is reflected off $\mathrm{W}:\alpha$. I now give an example of a direct confirmation rebuttal.

**Example 5.6.1. Direct confirmation rebuttal.** *Let* $\Delta = \{a:\beta, b:\beta \rightarrow \alpha, c:\pi, d:\pi \rightarrow \alpha, e:\neg\beta \vee \neg\pi, f:\sigma \wedge (\pi \rightarrow \alpha), g:\sigma \rightarrow \pi\}$. *Also let* $I = \{a,b\}, J = \{c,d\}, K = \{e\}, L = \{a,b,f,g\}, M = \{f,g\}, \mathrm{X} = \{I,J,M\}, \mathrm{W} = \{I,J\}, \mathrm{Y} = \{K\}, \mathrm{Z} = \{L\}$. *Therefore* $I:\alpha, J:\alpha, K:\neg(\beta \wedge (\beta \rightarrow \alpha) \wedge \pi \wedge (\pi \rightarrow \alpha)), M:\alpha, M:\pi \wedge (\pi \rightarrow \alpha) \in \mathsf{arguments}(\Delta)$. *Furthermore* $\mathrm{X}:\alpha, \mathrm{W}:\alpha, \mathrm{Y}:\neg(\beta \wedge (\beta \rightarrow \alpha) \wedge \pi \wedge (\pi \rightarrow \alpha)), \mathrm{Z}:\beta \wedge (\beta \rightarrow \alpha) \wedge \pi \wedge (\pi \rightarrow \alpha) \in \Diamond(\Delta)$. *It follows that* $\langle \mathrm{Y}:\neg(\beta \wedge (\beta \rightarrow \alpha) \wedge \pi \wedge (\pi \rightarrow \alpha)), \mathrm{W}, \mathrm{X}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{X}:\alpha, \Delta)$. *So there is a direct confirmation rebuttal* $\mathrm{Z}:\beta \wedge (\beta \rightarrow \alpha) \wedge \pi \wedge (\pi \rightarrow \alpha) \in \mathsf{directConfirmationRebuttals}(\mathrm{Y}:\neg(\beta \wedge (\beta \rightarrow \alpha) \wedge \pi \wedge (\pi \rightarrow \alpha)), \mathrm{W}, \mathrm{X}:\alpha, \Delta)$ *because for the* $L \in \mathrm{Z}$ *it is the case that* $L \not\subseteq I \cup J$, *because* $L \not\subseteq \{a,b\} \cup \{c,d\}$, *because* $L \not\subseteq \{a,b,c,d\}$ *because* $\{a,b,f,g\} \not\subseteq \{a,b,c,d\}$.

Direct confirmation rebuttals, such as $\mathrm{Z}:\beta \wedge (\beta \rightarrow \alpha) \wedge \pi \wedge (\pi \rightarrow \alpha)$, are not predictable from any other part of $\Delta$ (i.e. the claim cannot be predicted other than from their own support Z), and thus cannot be predicted from any other part of a tree or graph they might be situated in. So given the prerequisites of two valid confirmations in an attack chain with the attack being a preclusive undercut, it is always the case that a reflected confirmation rebuttal exists, but it only *sometimes* the case that a direct confirmation rebuttal exists. Something that is predictable allows for propositions, but for exactly that reason there is a paucity of propositions about direct arguments and direct confirmations.

### 5.6.1 Maximal Direct Confirmation Rebuttal

The direct confirmation rebuttal does continue the relative weakness identified before - of being relative to just one specific reflector. What is needed is an absolute function that covers all possible reflectors - given its context within a debate. Being maximal there is only one maximal direct confirmation rebuttal, so this rebuttal function is written in the singular. The function is maximal in two ways: i) it covers all possible reflectors and hence yields the complete set of direct rebuttal arguments, and ii) is a maximum cardinality confirmation. Here is the absolute function, the maximal direct confirmation rebuttal, which builds on the relative Definition 5.6.1 for direct confirmation rebuttals:

**Definition 5.6.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{W} : \alpha, \mathrm{X} : \alpha,\ \mathrm{Y} : \beta,\ \mathrm{Z} : \neg\beta \in \Diamond(\Delta)$. *The* **maximal direct confirmation rebuttal** *function, denoted* $\mathsf{maximalDirectConfirmationRebuttal}(\mathrm{Y} : \beta, \mathrm{X} : \alpha, \Delta)$*, which provides the maximal cardinality confirmation rebuttal of* $\mathrm{Y} : \beta$ *that is not reflected off any confirmation for* $\alpha$*, is a single confirmation such that:*

$$\begin{aligned}
&\mathrm{Z}{:}\neg\beta = \mathsf{maximalDirectConfirmationRebuttal}(\mathrm{Y}{:}\beta, \mathrm{X}{:}\alpha, \Delta)\\
&\quad \textit{iff}\ \ \mathrm{Z}{:}\neg\beta = \mathsf{top}(\mathsf{directConfirmationRebuttals}(\\
&\qquad\quad \mathrm{Y}{:}\beta, \mathsf{label}(\mathsf{top}(\Diamond(\mathsf{stripConfirmation}(\mathrm{X}{:}\alpha), \Delta))), \mathsf{top}(\Diamond(\mathsf{stripConfirmation}(\mathrm{X}{:}\alpha), \Delta))\ \textit{and}\\
&\qquad\quad \langle \mathrm{Y}{:}\beta, \mathrm{W}, \mathrm{X}{:}\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{X}{:}\alpha, \Delta).
\end{aligned}$$

The target confirmation label W need not be passed in as it is derived. It can be the case that there are no direct confirmation rebuttals for $\mathrm{Y}{:}\beta$ and hence no maximum cardinality one, so it is possible for $\mathsf{maximalDirectConfirmationRebuttal}(\mathrm{Y}{:}\beta, \mathrm{X}{:}\alpha, \Delta) = \emptyset$.

At this stage in the progression of this thesis, the *basic* analysis of Type I and II reflected arguments and reflected confirmations is complete. However there is another, more advanced, group of properties relating to scaled reflections; as I now establish.

## 5.7 Scaled and Distorted Reflections

This section on scaled and distorted reflections defines and examines the concepts of reduced, exact, enlarged and distorted reflections. The starting point is a function that measures the average scaling factor for a particular reflection. A finding of this section is that there are two mechanisms that cause enlarged reflection, namely the multi-pair enlarged preclusive reflection as distinct from the mono-pair enlarged preclusive reflection (from Section 4.6). I start by defining some helper functions plus the formal terminology for scaled reflections. The term 'targeted attack', used below, was defined in the text immediately after Definition 3.9.4 of confirmation attacks.

**Definition 5.7.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{W} : \alpha, \mathrm{X} : \alpha, \mathrm{Y} : \beta, \mathrm{Z}{:}\phi \in \Diamond(\Delta)$. *The* **reflected confirmation attacks** *function, denoted* $\mathsf{reflectedConfirmationAttacks}(\mathrm{Y} : \beta, \mathrm{W}, \mathrm{X}{:}\alpha, \Delta)$*, is a set of targeted attack tuples such that:*

$$\begin{aligned}
&\mathsf{reflectedConfirmationAttacks}(\mathrm{Y}{:}\beta, \mathrm{W}, \mathrm{X}{:}\alpha, \Delta)\\
&\quad = \ \mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}{:}\beta, \mathrm{W}, \mathrm{X}{:}\alpha, \Delta)\ \cup\\
&\qquad\quad \{\langle \mathrm{Z}{:}\phi, \mathsf{label}(\mathrm{Y}{:}\beta), \mathrm{Y}{:}\beta\rangle \mid\\
&\qquad\quad \mathrm{Z}{:}\phi \in \mathsf{reflectedConfirmationRebuttals}(\mathrm{Y}{:}\beta, \mathrm{W}, \mathrm{X}{:}\alpha, \Delta)\}.
\end{aligned}$$

This function should appear as an unsurprising extension of the work so far, for example because attacks on reflexive arguments are both undercuts and rebuttals at the same time – and, as established, all arguments can be expressed as confirmations. Given this reflected confirmation attacks function, it is possible to define the reflection scaling factor in a way that applies to both reflected preclusive undercuts and reflected confirmation rebuttals.

The first helper function here gives the conjectured or anticipated reflection size as equal to the size of the target confirmation, which is a simple, but incorrect assessment of the actual reflection size. This definition of the intuitively anticipated number of reflected arguments is based on the notion that a reflected argument can act as a proxy for the reflector arguments.

**Definition 5.7.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{W}:\alpha, \mathrm{X}:\alpha, \mathrm{Y}:\beta \in \Diamond(\Delta)$. *The* **anticipated reflection size**, *denoted* $\mathsf{anticipatedReflectionSize}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$, *given the reflection* $\mathsf{reflectedConfirmationAttacks}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta))$, *is:*

$$\mathsf{anticipatedReflectionSize}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta) = |\mathsf{label}(\mathrm{W}:\alpha)|.$$

The second helper function gives the actual reflection size - which is a somewhat more complicated, but correct, assessment of the actual reflection.

**Definition 5.7.3.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{V}_1:\beta, \ldots, \mathrm{V}_n:\beta$, $\mathrm{X}:\alpha, \mathrm{Y}:\beta$, $\mathrm{Z}_1:\gamma_1, \ldots, \mathrm{Z}_l:\gamma_1, \mathrm{Z}_m:\gamma_p, \ldots, \mathrm{Z}_n:\gamma_p \in \Diamond(\Delta)$. *The* **actual number of reflected arguments**, *denoted* $\mathsf{actualReflectionSize}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$, *for the reflection* $\mathsf{reflectedConfirmationAttacks}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)) = \{\langle \mathrm{Z}_1:\gamma_1, \mathrm{V}_1, \mathrm{Y}:\beta\rangle, \ldots, \langle \mathrm{Z}_l:\gamma_1, \mathrm{V}_1, \mathrm{Y}:\beta\rangle, \ldots, \langle \mathrm{Z}_m:\gamma_p, \mathrm{V}_p, \mathrm{Y}:\beta\rangle, \ldots, \langle \mathrm{Z}_n:\gamma_p, \mathrm{V}_p, \mathrm{Y}:\beta\rangle\}$ *is such that:*

$$\begin{aligned}&\mathsf{actualReflectionSize}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)\\ =\ & |\mathsf{label}(\mathsf{top}(\mathsf{attacker}(\langle \mathrm{Z}_1:\gamma_1, \mathrm{V}_1, \mathrm{Y}:\beta\rangle), \ldots, \mathsf{attacker}(\langle \mathrm{Z}_l:\gamma_1, \mathrm{V}_1, \mathrm{Y}:\beta\rangle)))| + \ldots +\\ & |\mathsf{label}(\mathsf{top}(\mathsf{attacker}(\langle \mathrm{Z}_m:\gamma_p, \mathrm{V}_p, \mathrm{Y}:\beta\rangle), \ldots, \mathsf{attacker}(\langle \mathrm{Z}_n:\gamma_p, \mathrm{V}_p, \mathrm{Y}:\beta\rangle)))|).\end{aligned}$$

Clearly $\mathrm{V}_1, \ldots, \mathrm{V}_p \subseteq \mathrm{Y}, \wp(\mathrm{Y})\backslash\emptyset = \{\mathrm{V}_1, \ldots, \mathrm{V}_p\}$, all the arguments that attack target $\mathrm{V}_1$ are in the confirmations $\mathrm{Z}_1:\gamma_1, \ldots, \mathrm{Z}_l:\gamma_1$ and all the arguments that attack target $\mathrm{V}_p$ are in the confirmations $\mathrm{Z}_m:\gamma_p, \ldots, \mathrm{Z}_n:\gamma_p$. Given these two helper functions, the reflection scaling factor follows simply as one divided by the other. I call this the 'down tree' reflection scaling factor to distinguish it from the subsequent 'up tree' one in Section 5.9.2.

**Definition 5.7.4.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{W}:\alpha, \mathrm{X}:\alpha, \mathrm{Y}:\beta \in \Diamond(\Delta)$. *The* **down tree reflection scaling factor** *function, denoted* $\mathsf{reflectionScalingFactorDownTree}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$, *provides a rational number such that:*

$$\mathsf{reflectionScalingFactorDownTree}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta) = \frac{\mathsf{actualReflectionSize}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)}{\mathsf{anticipatedReflectionSize}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)}.$$

The reflector employed to compute the scaling factor is the target confirmation and not the defending confirmation. The reflection scaling factor gives the size ratio between a naive expectation of exact reflection and an actual reflection that may be enlarged or reduced.

**Definition 5.7.5.** *Let* $\Delta$ *be a knowledgebase of labelled assumptions and let* $\mathrm{W}:\alpha, \mathrm{X}:\alpha, \mathrm{Y}:\beta \in \Diamond(\Delta)$.

*If* $\mathsf{reflectionScalingFactorDownTree}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta) < 1$

*then* $\mathsf{reflectedConfirmationAttacks}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$ *is a* **reduced reflection**, *If* $\text{reflectionScalingFactorDownTree}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta) = 1$

*then* $\text{reflectedConfirmationAttacks}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$ *is an* **exact reflection,**

*If* $\text{reflectionScalingFactorDownTree}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta) > 1$

*then* $\text{reflectedConfirmationAttacks}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$ *is an* **enlarged reflection.**

These helper functions can now be used to clarify the distinctions between enlarged, reduced and exact reflections.

## 5.7.1 Reduced Reflection

Reduced reflection occurs for reflected confirmation attacks, and within that it occurs for both reflected preclusive undercuts and for reflected confirmation rebuttals. Support shortening can give rise to the phenomena I call support merging. Support merging for confirmations can in turn give rise to reduced reflections. I hold that this behaviour should influence the structure of debate tracking trees and the examination of arguments, either through counting or existential schemes, in judge functions.

In the case of reduced reflection there are fewer reflected arguments than there are reflector arguments. Many examples in this section include a graphical depiction of the relationships involved; these graphs are contradiction graphs, see Definition 3.9.6, as they show the illustrated behaviour well. I now give an example of reduced preclusive undercut reflection together with its contradiction graph in Figure 5.4.

**Example 5.7.1. Reduced preclusive undercut reflection.** *Let* $\Delta = \{a:\alpha, b:\alpha \to \beta, c:\lambda\wedge\pi\wedge\gamma\wedge\neg\alpha, d:\gamma \to \neg\beta, e:\pi \to \neg\beta, f:\lambda \to \neg\beta\}$. *Let* $I = \{a, b\}, J = \{c, d\}, K = \{c, e\}, L = \{c, f\}, M = \{c\}$. *So* $I:\beta, J:\neg\beta, K:\neg\beta, L:\neg\beta, M:\neg(\alpha\wedge(\alpha\to\beta)), I:\alpha\wedge(\alpha\to\beta) \in \text{arguments}(\Delta)$. *Also* $J:\neg(\alpha\wedge(\alpha\to\beta)), K:\neg(\alpha\wedge(\alpha\to\beta)), L:\neg(\alpha\wedge(\alpha\to\beta)) \in \text{deductions}(\Delta)$, *but they are not members of* $\text{arguments}(\Delta)$ *as they are not minimal. Consequently* $\{I\}:\beta, \{J,K,L\}:\neg\beta, \{M\}:\neg(\alpha\wedge(\alpha\to\beta)), \{I\}:\alpha\wedge(\alpha\to\beta) \in \Diamond(\Delta)$. *It follows that* $\langle\{M\}:\neg(\alpha\wedge(\alpha\to\beta)), \{I\}, \{I\}:\beta\rangle \in \text{preclusiveUndercuts}(\{I\}:\beta, \Delta), \{I\}:\beta \in \text{confirmationRebuttals}(\{J,K,L\}:\neg\beta, \Delta)$' *and* $\langle\{M\}:\neg(\alpha\wedge(\alpha\to\beta)), \{I\}, \{I\}:\beta\rangle \in \text{reflectedPreclusiveUndercuts}(\{I\}:\beta, \{J,K,L\}, \{J,K,L\}:\neg\beta, \Delta)$ *is the reflected attack. So* $\{J,K,L\}:\neg\beta$ *is the reflector confirmation and* $\{M\}:\neg(\alpha\wedge(\alpha\to\beta))$ *is the reflected confirmation. The resultant reflection scaling factor is thus the actual number of reflections* $|\{M\}|$ *divided by the intuitively anticipated number of reflections* $|\{J,K,L\}|$ *, i.e.* $\frac{1}{3}$. *As the reflection scaling factor is less than one this is a reduced reflection.*

$$\begin{tikzcd}
\langle\{I\}, \{J,K,L\}\rangle:\beta \\
\langle\{M\}, \{I\}\rangle:\neg(\alpha\wedge(\alpha\to\beta)) \arrow[u, "\{I\}"']
\end{tikzcd}$$

Figure 5.4: Reduced Preclusive Undercut Reflection Shown in a Contradiction Graph

In the above example there is only one reflected undercut argument, not three (one per rebuttal) as one might expect. Thus this is a reduced reflection.

### 5.7.2 Exact Reflection

Now I come to another behaviour of reflection, which I call the exact reflection. In this case there are exactly the same number of reflected arguments within the reflected confirmation as there are arguments in the reflector. Exact reflection occurs with all three types of confirmation attacks: reflected confirmation attacks, reflected preclusive undercuts and reflected confirmation rebuttals.

**Example 5.7.2. Exact preclusion reflection.** *Let* $\Delta = \{a:\beta, b:\beta \to \alpha, c:\gamma, d:\gamma \to \alpha, e:\neg\pi \wedge \neg\gamma \wedge \sigma \wedge \lambda, f:\sigma \to \neg\alpha, g:\lambda \to \neg\alpha\}$. *Let* $I = \{a,b\}, J = \{c,d\}, K = \{e,f\}, L = \{e,h\}, M = \{e\}$. *Therefore* $I:\alpha, J:\alpha, K:\neg\alpha, L:\neg\alpha, M:\neg(\pi \wedge (\pi \to \alpha)), M:\neg(\gamma \wedge (\gamma \to \alpha)) \in \mathsf{arguments}(\Delta)$. *Clearly* $M \subseteq K, M \subseteq L$. *Consequently* $\langle\{M\}:\neg(\beta \wedge (\beta \to \alpha)), \{I\}, \{I,J\}:\alpha\rangle, \langle\{M\}:\neg(\gamma \wedge (\gamma \to \alpha)), \{J\}, \{I,J\}:\alpha\rangle \in \mathsf{reflectedPreclusiveUndercuts}(\{I,J\}:\alpha, \{K,L\}, \{K,L\}:\neg\alpha, \Delta)$. *So the reflector is* $\{K,L\}:\neg\alpha$ *and the reflected confirmations are* $\{M\}:\neg(\beta \wedge (\beta \to \alpha))$ *and* $\{M\}:\neg(\gamma \wedge (\gamma \to \alpha))$. *Hence the reflection scaling factor* = $|\{K,L\}| \div (|\{M\}| + |\{M\}|) = 2 \div 2 = 1$, *meaning that this is an exact reflection. This example is now shown as a contradiction tree in Figure 5.5.*

$$\begin{tikzcd}
 & \langle\{I,J\},\{K,L\}\rangle:\alpha & \\
\langle\{M\},\{I\}\rangle:\neg(\beta \wedge (\beta \to \alpha)) \arrow[ur, "\{I\}"] & & \langle\{M\},\{J\}\rangle:\neg(\gamma \wedge (\gamma \to \alpha)) \arrow[ul, "\{J\}"']
\end{tikzcd}$$

Figure 5.5: Exact Preclusive Undercut Reflection Shown in a Contradiction Graph

There still may *not* be a one-to-one mapping between individual reflector arguments and reflected arguments, in that, for example, two rebuttals may result in one undercut, while a different rebuttal may result in two undercuts, giving the overall effect of exact reflection. Reflections that lack a one-to-one mapping from reflector to reflected I cover under distorted reflections.

In the case of reflected confirmation rebuttals, the Example 5.5.2 illustrates the existence of an exact reflection that is also a rebuttal. This example of exact confirmation reflection is perhaps surprising as the form of the Type II Reflection function intuitively tends to give reduced reflections – because the reflected support is the union of multiple reflector supports.

### 5.7.3 Enlarged Reflection

Enlarged reflection occurs with reflected confirmation attacks and within that it occurs for reflected preclusive undercuts and for reflected confirmation rebuttals. There exist $\Delta$ where enlarged preclusive reflections occur, as illustrated in the following example.

**Example 5.7.3. Enlarged preclusive reflection.** *Let* $\Delta = \{a:\pi, b:\pi \to \alpha, c:\sigma \wedge \neg\pi, d:(\sigma \to \neg\alpha) \wedge \neg\pi, e:\lambda, f:\lambda \to \alpha, g:\eta, h:\eta \to \neg\alpha\}$, *so* $|\Delta| = 8$. *Let* $I = \{a,b\}, J = \{e,f\}, K = \{c,d\}, L = \{g,h\}, M = \{c\}, N = \{d\}$. *Consequently* $I:\alpha, J:\alpha, K:\neg\alpha, L:\neg\alpha, M:\neg(\pi \wedge (\pi \to \alpha)), N:\neg(\pi \wedge (\pi \to \alpha)), L:\neg(\pi \wedge (\pi \to \alpha)), K:\neg(\lambda \wedge (\lambda \to \alpha)), L:\neg(\lambda \wedge (\lambda \to \alpha)) \in \mathsf{arguments}(\Delta)$. *Importantly although* $K:\neg(\pi \wedge (\pi \to \alpha)) \in \mathsf{deductions}(\Delta)$ *it is the case that* $K:\neg(\pi \wedge (\pi \to \alpha)) \notin$

arguments($\Delta$) *as it is not minimal. Thus* $\{M,N,L\}:\neg(\pi\wedge(\pi\rightarrow\alpha)), \{K,L\}:\neg(\lambda\wedge(\lambda\rightarrow\alpha)), \{I,J\}:\alpha, \{K,L\}:\neg\alpha \in \Diamond(\Delta)$. *There are two preclusive undercuts* $\langle\{M,N,L\}:\neg(\pi\wedge(\pi\rightarrow\alpha)), \{I\}, \{I,J\}:\alpha\rangle, \langle\{K,L\}:\neg(\lambda\wedge(\lambda\rightarrow\alpha)), \{J\}, \{I,J\}:\alpha\rangle \in$ preclusiveUndercuts($\{I,J\}:\alpha,\Delta$), *both of which are reflected, i.e.* $\langle\{M,N,L\}:\neg(\pi\wedge(\pi\rightarrow\alpha)), \{I\}, \{I,J\}:\alpha\rangle, \langle\{K,L\}:\neg(\lambda\wedge(\lambda\rightarrow\alpha)), \{J\}, \{I,J\}:\alpha\rangle \in$ reflectedPreclusiveUndercuts($\{I,J\}:\alpha, \{K,L\}, \{K,L\}:\neg\alpha,\Delta$). *The scaling factor is thus* reflectionScalingFactorDownTree($\{I,J\}:\alpha, \{K,L\}, \{K,L\}:\neg\alpha,\Delta$) $= (|\{M,N,L\}| + |\{K,L\}|) \div |\{K,L\}| = 5 \div 2 = 2.5$ *so* reflectedPreclusiveUndercuts($\{I,J\}:\alpha, \{K,L\}, \{K,L\}:\neg\alpha,\Delta$) *is an enlarged reflection. This enlarged reflection is illustrated as the contradiction graph in Figure 5.6 below.*

$$\begin{tikzcd}
 & \langle\{I,J\},\{K,L\}\rangle:\alpha & \\
\langle\{M,N,L\},\{I\}\rangle:\neg(\pi\wedge(\pi\rightarrow\alpha)) \arrow[ur, "\{I\}"] & & \langle\{K,L\},\{J\}\rangle:\neg(\lambda\wedge(\lambda\rightarrow\alpha)) \arrow[ul, "\{J\}"']
\end{tikzcd}$$

Figure 5.6: Enlarged Preclusive Undercut Reflection Shown in a Contradiction Graph

Enlarged reflected confirmation rebuttals also exist. In contrast, however, enlarged reflected argument rebuttals do not exist, so that non-existence of enlargement is an exception as all of the other forms of reflection analysed in this thesis have enlargement behaviour.

What the definition of enlargement does not relay is *why* a particular preclusion is an enlarged preclusive reflection – it turns out that it could be due to mono-pair enlarged reflection (as defined earlier in Definition 4.6.1) or due to multi-pair preclusive reflection defined immediately below.

### 5.7.4 Multi-Pair Enlarged Preclusive Reflection

This subsection focusses on the situation where preclusive undercuts are reflected off a contradiction. For this situation, I now define a pure multi-pair enlarged preclusive reflections, as the case where the total number of reflected arguments is more than the number of reflector arguments and mono-pair enlarged reflection is not involved. The term 'enlarged reflection' was defined in Definition 5.7.5 and 'mono-pair enlargement' in Definition 4.6.1.

**Definition 5.7.6.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae, let* $I:\alpha, J:\neg\alpha \in$ arguments($\Delta$) *and let* $\mathrm{W}:\neg\alpha, \mathrm{X}:\alpha, \mathrm{Y}:\neg\alpha, \mathrm{Z}:\gamma \in \Diamond(\Delta)$. *A* **multi-pair enlarged preclusive reflection**, *denoted* multiPairEnlargedReflections($\mathrm{Y}:\neg\alpha,\mathrm{X},\mathrm{X}:\alpha,\Delta$), *is an enlarged preclusive reflection where no mono-pair enlargement is involved, such that:*

multiPairEnlargedReflections($\mathrm{Y}:\neg\alpha,\mathrm{X},\mathrm{X}:\alpha,\Delta$)

$=$ $\{\langle \mathrm{Z}:\gamma, \mathrm{W}, \mathrm{Y}:\neg\alpha\rangle \in$ reflectedPreclusiveUndercuts($\mathrm{Y}:\neg\alpha,\mathrm{X},\mathrm{X}:\alpha,\Delta$) $|$

reflectedPreclusiveUndercuts($\mathrm{Y}:\neg\alpha,\mathrm{X},\mathrm{X}:\alpha,\Delta$) *is an enlarged reflection and*

*for each* $I \in \mathrm{X}$ *and* $J \in \mathrm{Y}$ *it is the case that*

monoPairEnlargedReflection($J:\neg\alpha, I:\alpha,\Delta$) $= \emptyset\}$.

Multi-pair enlarged reflections do not always exist. They may exist depending upon the existence of prerequisites. The basic mechanism is that given a contradiction $\langle X, Y\rangle : \alpha$ for every member of the set $X \times Y$ there exists a candidate reflected undercut. Reflected undercuts are always mono-target (be they preclusive or canonical). Given these candidates, however, the final solution may contain a somewhat different set of reflected arguments due to interactions with mono-pair enlargement. Alternatively or additionally, the final solution may contain a reduced number of reflected arguments due to support shortening and hence support merging.

This analysis shows that the above Example 5.7.3 of enlarged reflection contains a mixture of mono and multi-pair enlargements. In that example, it is the reflector support $K$ giving rise to attacking supports $M$ and $N$ that is the mono-pair enlargement. The reflection of attacking $K$ and $L$ with target of $J$ is half of the multi-pair enlargement; the other half is partially obscured by the mono-pair enlargement with only the reflection of attacking support $L$ with target of $I$ remaining directly visible.

The multi-pair enlarged reflection could present a complication or difficulty for notions of judging debates by counting arguments. As reflection can produce reductions and enlargements, even a distorted mixture of enlargements and reductions, it follows that, without proper consideration, reflection can confound the counting of arguments and lead to incorrect judgements. This predicament is akin to trying to make precise measurements in a distorting mirror. In the extreme case, with a variety of related $\Delta$'s, the presence of distortion could reduce a judge that does not consider the effects of reflection, across this variety of $\Delta$'s, to appear to be an arbitrary judge. Arbitrary here is relative to an intuitive assessment based on an appreciation of professional debates.

The number of reflected arguments arising from a mono-pair enlarged reflection was provided in Proposition 4.6.2. The next steps are to establish the number of arguments in i) a multi-pair enlarged reflection in isolation and then ii) with mono-pair and multi-pair together.

**Proposition 5.7.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let* $W : \alpha, X : \alpha,\ Y : \beta \in \Diamond(\Delta)$. *For the reflection* $\mathsf{multiPairEnlargedReflections}(Y : \beta, W, X : \alpha, \Delta)$ *it is the case that:*

$$2 \leq \mathsf{actualReflectionSize}(Y : \beta, W, X : \alpha, \Delta) \leq |\mathsf{label}(X : \alpha)| \cdot |\mathsf{label}(Y : \beta)|.$$

*Proof.* If the prerequisites are not met, e.g. $\langle Y : \beta, W, X : \alpha\rangle \notin \mathsf{confirmationAttacks}(X : \alpha, \Delta)$ for any $W \subseteq X$, then there is no reflection. However, as it is a given that there will exist a $\langle Z_i : \gamma_i, V, Y : \beta\rangle \in \mathsf{multiPairEnlargedReflections}(Y : \beta, W, X : \alpha, \Delta)$, where $V : \beta \in \Diamond(\Delta)$ then it follows that a) the prerequisites have been met and b) a multi-pair enlarged reflection has occurred. A multi-pair enlarged reflection is, by Definition 5.7.6, a special case of an enlarged reflection and hence has $\mathsf{reflectionScalingFactorDownTree}(Y : \beta, W, X : \alpha, \Delta) > 1$. To guarantee such a $\mathsf{reflectionScalingFactorDownTree}(Y : \beta, W, X : \alpha, \Delta)$ requires that at least $2 \leq |\mathsf{multiPairEnlargedReflections}(Y : \beta, W, X : \alpha, \Delta)|$ for the low end case of $|X| = 1$. Now, recall the mechanism for how reflected canonical undercuts arise from a pair $J : \beta, I : \alpha$ where $J : \beta \in \mathsf{argumentAttacks}(I : \alpha, \Delta)$. In either the case $J : \beta \in \mathsf{rebuttals}(I : \alpha, \Delta)$ or $J : \beta \in \mathsf{canonicalUndercuts}(I : \alpha, \Delta)$ the number of reflected canonical undercuts is the same. In the absence of mono-pair enlargement, a single argument $J:\beta$ combines with a single reflector $I:\alpha$ to yield one $K:\gamma$ = reflectedCanonicalUndercuts($J:\beta,\ I:\alpha,\ \Delta$). The word 'combines' here means the five steps of i) expanding the claim of $J:\beta$ [to $J:\neg\alpha$ for rebuttal or to $J:\neg(\psi_1 \wedge \ldots \wedge \psi_n)$ for undercut, where $I = \{\psi_1,\ldots,\psi_n\}$], ii) expanding the support of $J:\beta$ [to $(\phi_1 \wedge \ldots \wedge \phi_p) \rightarrow \neg\alpha$, or $(\phi_1 \wedge \ldots \wedge \phi_p) \rightarrow (\psi_1 \wedge \ldots \wedge \psi_n)$, where $J = \{\phi_1,\ldots,\phi_p\}$ ], iii) taking the contrapositive of what was $J:\beta$ [to $\alpha \rightarrow \neg(\phi_1 \wedge \ldots \wedge \phi_p)$ or $(\psi_1 \wedge \ldots \wedge \psi_n) \rightarrow \neg(\phi_1 \wedge \ldots \wedge \phi_p)$] and then iv) joining with the reflector $I:\alpha$ [either by hypothetical syllogism or straight substitution to give $I:\neg(\phi_1 \wedge \ldots \wedge \phi_p)$] and v) remembering that the subset $K \subsetneq I$ may be involved for minimality $K:\neg(\phi_1 \wedge \ldots \wedge \phi_p)$. Thus there can be one multi-pair enlarged reflected argument for each member of the cross product label($X:\alpha$) $\times$ label($Y:\beta$). The number of members of this cross product comes from multiplying |label($X:\alpha$)| by |label($Y:\beta$)|. Hence the maximum number of multi-pair enlarged reflections is |label($X:\alpha$)| $\cdot$ |label($Y:\beta$)|. So for the reflection multiPairEnlargedReflections($Y:\beta, W, X:\alpha, \Delta$) it is the case that $2 \leq$ actualReflectionSize($Y:\beta, W, X:\alpha, \Delta$) $\leq$ |label($X:\alpha$)| $\cdot$ |label($Y:\beta$)|. □

Finally here is the maximum and minimum size of an enlarged reflection where both multi-pair and mono-pair enlargement are involved.

**Proposition 5.7.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $J_i:\beta, J_1:\beta,\ldots,J_{|Y|}:\beta \in$ arguments($\Delta$) and let $X:\alpha,\ \ Y:\beta \in \Diamond(\Delta)$. For an enlarged reflection* reflectedPreclusiveUndercuts($Y:\beta, X, X:\alpha, \Delta$)*, where $J_i \in Y$, it is the case that:*

$$2 \leq \text{actualReflectionSize}(Y:\beta, X, X:\alpha, \Delta) \leq |\text{label}(X:\alpha)| \cdot \text{midLattice}(J_i).$$

*Proof.* This proof combines that for mono-pair and multi-pair enlarged reflected preclusive undercuts, i.e. Proposition 4.6.2 and Propositions 5.7.2. The claims reveal that if $X:\alpha, Y:\beta \in \Diamond(\Delta)$ then for every $I \in X$ and $J \in Y$ it is the case that $I:\alpha \in$ argumentAttacks($J:\beta, \Delta$). The midLattice($J_i$) in the above is the maximum number of subsets of $J_i$ that are not subsets of each other. Suppose midLattice($J_i$) $= 1$, for the case of no mono-pair enlargement with support $J_i$, then $(\sum_{i=1}^{i=|Y|}$ |label($X:\alpha$)|$\cdot 1)$ is equal to |label($X:\alpha$)| $\cdot$ |label($Y:\beta$)|. Hence if there is no mono-pair enlargement the result is the same as for multi-pair enlargement in isolation. However, as both mono-pair and multi-pair enlargement can occur for the same attack chain $\langle Y:\beta, W, X:\alpha\rangle \in$ confirmationAttacks($X:\alpha, \Delta$) the two cardinalities must be multiplied. Hence for an enlarged reflection reflectedPreclusiveUndercuts($Y:\beta, X, X:\alpha, \Delta$), where $J_i \in Y$ it is the case that $2 \leq$ actualReflectionSize($Y:\beta, X, X:\alpha, \Delta$) $\leq$ |label($X:\alpha$)| $\cdot$ midLattice($J_i$). □

As this is an enlargement there exist more reflected canonical undercuts than there are rebuttal arguments in the reflector, as I now demonstrate.

**Example 5.7.4. Multi-pair enlarged reflection.** *Let $\Delta = \{a:\alpha, b:\alpha \rightarrow \beta, c:\gamma, d:\gamma \rightarrow \beta, e:\lambda, f:\lambda \rightarrow \beta, g:\zeta, h:\zeta \rightarrow \beta\}$. Let $I = \{a,b\}, J = \{c,d\}, K = \{e,f\}, L = \{g,h\}, V = \{I\}, W = \{J\}, Y = \{I,J\}, X = \{K,L\} = Z$. So there are clearly two defending arguments and two rebuttal arguments of $\beta$, namely $I:\beta, J:\beta, K:\neg\beta, L:\neg\beta \in$* arguments($\Delta$)*. There must also be four canonical undercuts, namely $K:\neg(\alpha \wedge (\alpha \rightarrow \beta)), L:\neg(\alpha \wedge (\alpha \rightarrow \beta)) \in$* canonicalUndercuts($I:\beta, \Delta$) *and*

$K:\neg(\gamma \wedge (\gamma \to \beta)), L:\neg(\gamma \wedge (\gamma \to \beta)) \in \mathsf{canonicalUndercuts}(J:\beta, \Delta)$. *These canonical undercuts are reflected and can be represented as* $\langle \mathrm{Z}:\neg(\alpha \wedge (\alpha \to \beta)), \mathrm{V}, \mathrm{Y}:\beta\rangle, \langle \mathrm{Z}:\neg(\gamma \wedge (\gamma \to \beta)), \mathrm{W}, \mathrm{Y}:\beta\rangle \in \mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{X}, \mathrm{X}:\neg\beta, \Delta)$. *There are no mono-pair enlargements present, consequently there exists a multi-pair enlarged reflection.* $\{\mathrm{Z}:\neg(\alpha \wedge (\alpha \to \beta)), \mathrm{Z}:\neg(\gamma \wedge (\gamma \to \beta))\} = \mathsf{multiPairEnlargedReflections}(\mathrm{Y}:\beta, \mathrm{X}, \mathrm{X}:\neg\beta, \Delta)$. *This multi-pair enlarged reflection is illustrated as the contradiction graph of Figure 5.7 below.*

$$\begin{tikzcd}
& \langle\{I,J\},\{K,L\}\rangle:\beta & \\
\langle\{K,L\},\{I\}\rangle:\neg(\alpha\wedge(\alpha\to\beta)) \arrow[ur, "\{I\}"'] & & \langle\{K,L\},\{J\}\rangle:\neg(\gamma\wedge(\gamma\to\alpha)) \arrow[ul, "\{J\}"]
\end{tikzcd}$$

Figure 5.7: Multi-Pair Enlarged Reflection Shown in a Contradiction Graph

This example shows that if one nominates canonical undercuts to stand for the rebuttals that they are derived from, then one must be aware that it is not a one-for-one swap. In (Besnard & Hunter, 2001) the argument is made that a canonical undercut can represent a rebuttal. They use this idea as a motivation for the way that their argument trees exclude all rebuttals, except at the root, and only include root arguments plus their recursive canonical undercuts. My analysis indicates that counting or existential based judging is only accurate for the situation of direct arguments and exact reflections. I revisit this issue in greater detail in the next chapter.

### 5.7.5 Distorted Reflections

Distorted reflections may be enlarged, exact or reduced and may involve any combination of those three situations within their reflections. Within the distortion, if there is enlargement, the cause may be mono-pair enlarged reflection, multi-pair enlarged reflection or a combination of the two. If reduction is involved, however, there is only one mechanism. It is the existence of distorted reflections that prevents or blocks the approach of scaling the number of undercuts to match the number of rebuttals.

**Definition 5.7.7.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{T}:\alpha, \mathrm{U}:\alpha, \mathrm{W}:\alpha, \mathrm{X}:\alpha, \mathrm{Y}:\beta \in \Diamond(\Delta)$.

$\mathsf{reflectedPreclusiveUndercuts}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$ *is a* **distorted refection** *of* $\mathrm{W}:\alpha$

*iff there exists a* $\mathrm{T} \subseteq \mathrm{W}$ *and a* $\mathrm{U} \subseteq \mathrm{W}$ *such that* $\mathrm{T} \cap \mathrm{U} = \emptyset$ *and*

$\mathsf{reflectionScalingFactorDownTree}(\mathrm{Y}:\beta, \mathrm{T}, \mathrm{X}:\alpha, \Delta) \neq$

$\mathsf{reflectionScalingFactorDownTree}(\mathrm{Y}:\beta, \mathrm{U}, \mathrm{X}:\alpha, \Delta)$.

A distorted reflection does not necessarily need to have one part enlarged and another reduced; it just needs to have different scaling factors for the two parts. Furthermore there could be more than two parts to such a distortion by applying the definition repeatedly. I now give an example.

**Example 5.7.5. Distorted reflection.** *Let* $\Delta = \{a:\beta, b:\beta \to \alpha, c:\gamma, d:\gamma \to \alpha, e:\pi \wedge \neg\beta \wedge \eta \wedge \lambda, f:\pi \to \neg\alpha, g:\lambda \to \neg\alpha, h:\eta \to \neg\alpha, i:\sigma \wedge \neg\beta \wedge \neg\gamma, j:\sigma \to \neg\alpha\}$. *Let*

*$I = \{a,b\}, J = \{c,d\}, K = \{e,f\}, L = \{e,g\}, M = \{e,h\}, N = \{i,j\}, O = \{e\}, P = \{i\}$. Thus $I:\alpha, J:\alpha, K:\neg\alpha, L:\neg\alpha, M:\neg\alpha, N:\neg\alpha, P:\neg(\beta \wedge (\beta \rightarrow \alpha)), O:\neg(\beta \wedge (\beta \rightarrow \alpha)), P:\neg(\gamma \wedge (\gamma \rightarrow \alpha)) \in \mathsf{arguments}(\Delta)$. Therefore $\{I,J\}:\alpha, \{K,L,M,N\}:\neg\alpha, \{P\}:\neg(\gamma \wedge (\gamma \rightarrow \alpha)), \{O,P\}:\neg(\beta \wedge (\beta \rightarrow \alpha)) \in \Diamond(\Delta)$. Clearly $P \subseteq N, O \subseteq K, O \subseteq L, O \subseteq M$ so these are reflected preclusive undercuts $\langle\{P\}:\neg(\gamma \wedge (\gamma \rightarrow \alpha)), \{J\}, \{I,J\}:\alpha\rangle, \langle\{O,P\}:\neg(\beta \wedge (\beta \rightarrow \alpha)), \{I\}, \{I,J\}:\alpha\rangle \in \mathsf{reflectedPreclusiveUndercuts}(\{I,J\}:\alpha, \{K,L,M,N\}, \{K,L,M,N\}:\neg\alpha, \Delta)$. Let $\mathrm{T}$, $\mathrm{U}$ and $\mathrm{W}$ from the definition of distortion, Definition 5.7.7, take the values of $\mathrm{T} = \{K,L,M\}, \mathrm{U} = \{N\}$ and $\mathrm{W} = \{K,L,M,N\}$. Consequently $\mathrm{T} \subseteq \mathrm{W}$ and $\mathrm{U} \subseteq \mathrm{W}$ such that $\mathrm{T} \cap \mathrm{U} = \emptyset$. Looking at $\mathrm{T}$, the three candidate reflections from the three reflecting arguments $K:\neg\alpha, L:\neg\alpha, M:\neg\alpha$ merge to become only one actual reflected argument $O:\neg(\beta \wedge (\beta \rightarrow \alpha))$, which is a reduced reflection with a scaling factor of $\frac{1}{3} = 0.33$. Turning now to the subset $\mathrm{U}$, one reflecting arguments $N:\neg\alpha$ has been reflected as two reflected argument $P:\neg(\beta \wedge (\beta \rightarrow \alpha))$ and $P:\neg(\gamma \wedge (\gamma \rightarrow \alpha))$, which is an enlarged reflection with a scaling factor of 2.0. As $0.33 \neq 2.0$ it follows that $\mathsf{reflectionScalingFactorDownTree}(\{I,J\}:\alpha, \mathrm{T}, \mathrm{W}:\neg\alpha, \Delta) \neq \mathsf{reflectionScalingFactorDownTree}(\{I,J\}:\alpha, \mathrm{U}, \mathrm{W}:\neg\alpha, \Delta)$. The fact that $\mathsf{reflectionScalingFactorDownTree}(\{I,J\}:\alpha, \mathrm{T}, \mathrm{W}:\neg\alpha, \Delta) \neq \mathsf{reflectionScalingFactorDownTree}(\{I,J\}:\alpha, \mathrm{U}, \mathrm{W}:\neg\alpha, \Delta)$ means that this is a distorted reflection. The three arguments of $\mathrm{T} = \{K,L,M\}$ give rise to a reduced reflection of just $O:\neg(\beta \wedge (\beta \rightarrow \alpha))$. In contrast, the single argument of $\mathrm{U} = \{N\}$ gives rise to the enlarged reflection of $P:\neg(\beta \wedge (\beta \rightarrow \alpha))$ plus $P:\neg(\gamma \wedge (\gamma \rightarrow \alpha))$. This distorted reflection is shown as a contradiction graph in Figure 5.8 below.*

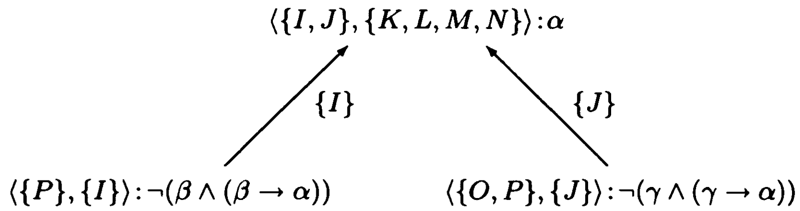


Figure 5.8: Distorted Reflection Shown in a Contradiction Graph

My view is that the existence of distorted reflections should effect the tracking of debates. Suppose for a moment that reflected canonical undercuts were exact reflections – that would form a useful basis to propose that direct rebuttals could be excluded from debate tracking – because the reflected canonical undercuts would act as their representatives. Such a notion however is refuted by the existence of enlarged and reduced reflections. This proposal could be refined by arguing that a scaling factor be applied to compensate for enlargement or reduction – hence it could be imagined that appropriately scaled reflections could still stand in for the excluded direct rebuttals. Such scaling factor usage would supposedly correct any counting errors coming from the scaled reflections. The problem with that refinement is that the reflections are non-linear. The above example clearly establishes the existence of such non-linear or distorted reflections – blocking the hope of using linear scale factors. These properties are further discussed in the next chapter in Section 6.4.

## 5.8 Type III Reflection – Reflected Confirming Arguments

I now introduce another type of reflection, distinct from those discussed so far. I call it Type III Reflection or 'reflected confirming arguments'. The annotated argument graph of Figure 5.9 below, where the additional circle represents the confirmation $\{I, L\} : \alpha$, illustrates the following commentary and definition.

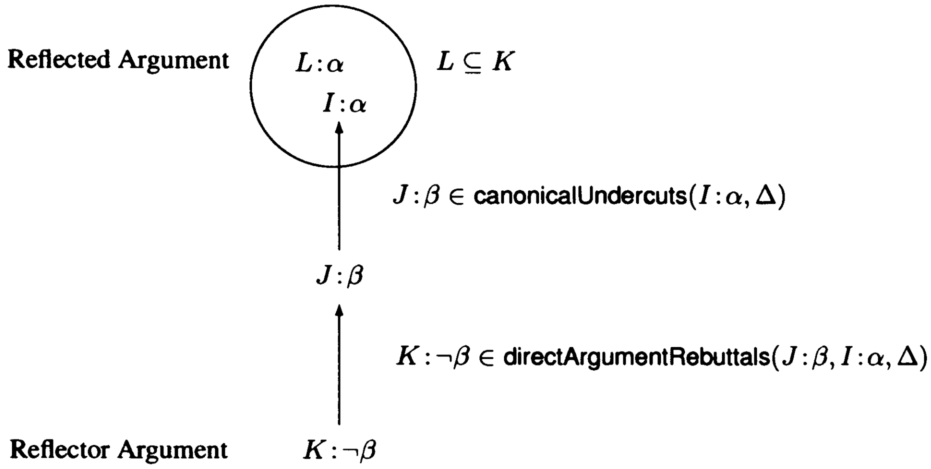


Figure 5.9: Type III Reflection is Provided by reflectedConfirmingArguments$(K:\neg\beta, J:\beta, I:\alpha, \Delta)$

This reflection is from the reflector at the *tail* of the chain to the reflection at the head of the chain. Recall that, in contrast, Type I and II Reflections have the reflector at the head of the chain and the reflection at the tail.

Informally, this reflection is coming from a direct rebuttal and goes up the chain of arguments, via an undercut, appearing as a reflected confirming argument. It is a reflection akin to, but not the same as, my reflectedAttacks() function – so it is a third type of reflection, which I call Type III Reflection. It is a reflection as it is the predictable existence of an argument in a debate given certain other arguments and attacks. This reflection arises from a pattern or interaction between *three* arguments, giving rise to a fourth or more, whereas the reflected attacks function, reflectedAttacks(), arises from a pattern or interaction between *two* arguments, giving rise to a third or more. The number of attacks involved, i.e. two, however, is the same for all of these types of reflection.

The Type III Reflection function's input of a chain of three arguments has the first argument attacked by the middle argument and the middle argument attacked by the third argument. The definition and properties that follow from it are amongst the central findings of this thesis and are relied upon in the next chapter. Although the following definition is in terms of individual arguments, I discuss it in this chapter on confirmations as the reflected argument is in the same confirmation as the reflector.

**Definition 5.8.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $I:\alpha$, $J:\beta$, $K:\neg\beta$ $\in$ arguments$(\Delta)$. *The set of* **reflected confirming arguments**, *denoted* reflectedConfirmingArguments$(K:\neg\beta, J:\beta, I:\alpha, \Delta)$, *arising from the canonical undercut of* $I:\alpha$ *by*

*$J:\beta$ and the direct rebuttal of $J:\beta$ by $K:\neg\beta$ is such that:*

$$\begin{aligned}&\text{reflectedConfirmingArguments}(K:\neg\beta, J:\beta, I:\alpha, \Delta)\\ &= \{L:\alpha \in \text{arguments}(\Delta) \mid J:\beta \in \text{canonicalUndercuts}(I:\alpha,\Delta),\\ &\qquad K:\neg\beta \in \text{directArgumentRebuttals}(J:\beta, I:\alpha, \Delta) \textit{ and } L \subseteq K\}.\end{aligned}$$

In this definition $I:\alpha, J:\beta, K:\gamma \in \text{arguments}(\Delta)$ might have been placed within the conditional part of the definition after the $|$, however it is clear that if they participate in the two attacks $J:\beta \in \text{argumentAttacks}(I:\alpha,\Delta)$ and $K:\gamma \in \text{argumentAttacks}(J:\beta,\Delta)$ where attack was defined earlier in Definition 3.9.2, then they are surely arguments. The same cannot be said for $L:\alpha \in \text{arguments}(\Delta)$ and so it is an explicit condition.

I now give an example of a Type III Reflection, i.e. of a reflected confirming argument.

**Example 5.8.1. Type III Reflection.** *Let $\Delta = \{a:\pi, b:\pi \to \alpha, c:\lambda, d:\lambda \to \neg\pi, e:\mu, f:\mu \to \pi\}$. Let $I = \{a,b\}, J = \{c,d\}, K = \{e,f,b\}$. Thus $I:\alpha, J:\neg(\pi \wedge (\pi \to \alpha)), I:\pi \wedge (\pi \to \alpha), K:\pi \wedge (\pi \to \alpha), K:\alpha \in \text{arguments}(\Delta)$. Therefore $J:\neg(\pi \wedge (\pi \to \alpha)) \in \text{canonicalUndercuts}(I:\alpha,\Delta)$ and $I:\pi \wedge (\pi \to \alpha) \in \text{reflectedArgumentRebuttal}(J:\neg(\pi \wedge (\pi \to \alpha)), I:\alpha, \Delta)$. However, $K:\pi \wedge (\pi \to \alpha) \notin \text{reflectedArgumentRebuttal}(J:\neg(\pi \wedge (\pi \to \alpha)), I:\alpha,\Delta)$, because $K \not\subseteq I$, because $\{b,e,f\} \not\subseteq \{a,b\}$. Additionally $K:\pi \wedge (\pi \to \alpha) \notin \text{reflectedArgumentRebuttal}(J:\neg(\pi \wedge (\pi \to \alpha)), K:\alpha,\Delta)$, because $J:\neg(\pi \wedge (\pi \to \alpha)) \notin \text{canonicalUndercuts}(K:\alpha,\Delta)$. Thus it is the case that $K:\pi \wedge (\pi \to \alpha) \in \text{directArgumentRebuttals}(J:\neg(\pi \wedge (\pi \to \alpha)), I:\alpha,\Delta)$. Additionally $K:\pi \wedge (\pi \to \alpha) \in \text{directArgumentRebuttals}(J:\neg(\pi \wedge (\pi \to \alpha)), K:\alpha,\Delta)$. This allows the conclusion that $K:\alpha \in \text{reflectedConfirmingArguments}(K:\pi \wedge (\pi \to \alpha), J:\neg(\pi \wedge (\pi \to \alpha)), I:\alpha,\Delta)$.*

To provide clarity on the mechanism at work here the above example also includes a reflected argument rebuttal which always exists but does not partake in the reflected confirming argument. The example makes clear the direct argument rebuttals and hence the reflected confirming argument. This example uses the $=$ part of $\subseteq$ from the last line of Definition 5.8.1. Here one argument is reflected off one argument, so this can be called an exact reflection.

Differences between Type I and Type III Reflections are as follows. Type I has an input of two arguments in a chain, while Type III has an input of three. Type I results in a new argument at the tail of the chain, whereas the Type III results in a new argument at the head of the chain. Thus Type I can be said to flow down the chain (in the direction from head to tail), while the Type III flows up the chain (from tail to head). Type I yields an attacking argument, while Type III yields a confirming argument.

## 5.8.1 Properties of Reflected Confirming Arguments

Given their prerequisites reflected confirming arguments always exist, however absent any one prerequisite and they don't exist. This observation is addressed by the following two propositions.

**Proposition 5.8.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\beta, K:$*

$\neg\beta \in \mathsf{arguments}(\Delta)$.

$$\textit{If } J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha,\Delta) \textit{ and } K:\neg\beta \in \mathsf{rebuttals}(J:\beta,\Delta)$$
$$\textit{then } \mathsf{reflectedConfirmingArguments}(K:\neg\beta, J:\beta, I:\alpha, \Delta) \neq \emptyset.$$

*Proof.* The three argument attack chain of canonical undercut, $J : \beta \in \mathsf{canonicalUndercuts}(I : \alpha, \Delta)$, followed by rebuttal, $K : \neg\beta \in \mathsf{rebuttals}(J : \beta, \Delta)$, always yields a reflected confirming argument, $L : \alpha$, as follows. Given $J : \beta \in \mathsf{canonicalUndercuts}(I : \alpha, \Delta)$ it is known that $\neg\beta = \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(I : \alpha), \Delta))$. Clearly $\mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(I : \alpha), \Delta)) \vdash \alpha$. Given $K:\neg\beta \in \mathsf{arguments}(\Delta)$ it is a simple fact that $\mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(K : \neg\beta), \Delta)) \vdash \neg\beta$, so given $K : \neg\beta \in \mathsf{rebuttals}(J : \beta, \Delta)$ it follows that $\mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(K : \neg\beta), \Delta)) \vdash \neg\neg \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(I : \alpha), \Delta))$. Thus $\mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(K : \neg\beta), \Delta)) \vdash \alpha$, i.e. that given these relationships $\{\neg\beta\} \vdash \alpha$, therefore $K : \alpha \in \mathsf{deductions}(\Delta)$. $K$ may not be minimal, so let $L \subseteq K$, hence there exists minimal $L:\alpha \in \mathsf{arguments}(\Delta)$. Therefore there exits $L:\alpha \in \mathsf{reflectedConfirmingArguments}(K : \neg\beta, J : \beta, I : \alpha, \Delta)$. Note, regardless of whether $K : \neg\beta \in \mathsf{directArgumentRebuttals}(J : \beta, \Delta)$ or $K : \neg\beta \in \mathsf{reflectedArgumentRebuttal}(J : \beta, \Delta)$, it follows that $K : \neg\beta \in \mathsf{rebuttals}(J : \beta, \Delta)$. If this rebuttal is reflected then the reflected confirming argument $L : \alpha$ is the same as $I : \alpha$ – still non-empty. So given the necessary prerequisites $\mathsf{reflectedConfirmingArguments}(K:\neg\beta, J:\beta, I:\alpha, \Delta) \neq \emptyset$. □

Thus if the prerequisites are not met then reflected confirming arguments do not exist. The predictable nature of reflected confirming arguments establishes that they are not just a minor novelty, but rather that they should have a major impact on the design of debate-tracking trees. It is not that they sometimes exist and are therefore tolerable – rather it is that they always exist and therefore should always be considered in professional debate. I thus argue that these reflected arguments need to be pruned out of or removed from any debate tracking tree (however at this stage of the thesis I have not shown how to prune them out).

Now I return to the detailed behaviour: if the second argument in the attack chain is a reflected argument rebuttal then no new argument is produced. There is a reflection, but it contributes nothing new to the debate. There is hence no redundancy.

**Proposition 5.8.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I : \alpha, J : \beta, K : \neg\beta \in \mathsf{arguments}(\Delta)$.*

$$\begin{aligned} &\{\mathsf{reflectedConfirmingArguments}(K:\neg\beta, J:\beta, I:\alpha, \Delta) \mid \\ &\qquad J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta), \\ &\qquad K:\neg\beta \in \mathsf{reflectedArgumentRebuttal}(J:\beta, I:\alpha, \Delta)\} \\ = &\quad \{I:\alpha\}. \end{aligned}$$

*Proof.* Because $K : \neg\beta$ is reflected it must be the case that $K \subseteq I$ and specifically because $K : \neg\beta \in \mathsf{reflectedArgumentRebuttals}(J : \beta, I : \alpha, \Delta)$ it follows that $K = I$. From Definition 5.8.1 of reflected confirming argument, $L \subseteq K$, so $L \subseteq I$. Given that $I:\alpha \in \text{arguments}(\Delta)$ there cannot exist an $L \subsetneq I$ such that $L:\alpha \in \text{arguments}(\Delta)$ otherwise $I:\alpha$ would not be minimal. So it is not the case that $L \subsetneq I$ thus $L = I$. Therefore the whole set equals $I:\alpha$. □

Referring back to Figure 5.9, one might think that each reflected confirming argument, $L:\alpha$, would then act as a reflector that would transform the direct argument rebuttal $K:\neg\beta$ at the tail of the attack chain into a reflected argument rebuttal. This is generally not the case as the middle of the attack chain does not usually attack the reflected confirming argument, $L:\alpha$, except for the one situation when the assumption formulae are identical. Thus it is always the case that the existence of a reflected confirming argument, given that its assumption formulae are not identical to the reflector argument, does not turn a direct argument rebuttal into a reflected argument rebuttal.

**Proposition 5.8.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $I:\alpha, J:\beta, K:\neg\beta, L:\alpha \in \text{arguments}(\Delta)$.*

$$\begin{aligned}&\textit{If } L:\alpha \in \mathsf{reflectedConfirmingArguments}(K:\neg\beta, J:\beta, I:\alpha, \Delta)\\ &\quad \textit{and } \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(I:\alpha), \Delta)) \neq\\ &\quad\quad \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(L:\alpha), \Delta))\\ &\textit{then } K:\neg\beta \notin \mathsf{reflectedArgumentRebuttal}(J:\beta, L:\alpha, \Delta).\end{aligned}$$

*Proof.* Clearly for $L:\alpha \in \mathsf{reflectedConfirmingArguments}(K:\neg\beta, J:\beta, I:\alpha, \Delta)$ it must be the case that $J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta)$ and that $K:\neg\beta \in \mathsf{directArgumentRebuttals}(J:\beta, I:\alpha, \Delta)$. Thus $K \not\subseteq I$ and $\neg\beta = \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(I:\alpha), \Delta))$. Also because $L:\alpha \in \mathsf{reflectedConfirmingArguments}(K:\neg\beta, J:\beta, I:\alpha, \Delta)$ it follows that $L \subseteq K$. Therefore $L \subseteq K \not\subseteq I$ so $L \neq I$. Furthermore if constraint $\bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(I:\alpha), \Delta)) \neq \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(L:\alpha), \Delta))$ is met then $\neg\beta \neq \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(L:\alpha), \Delta))$ therefore in general $J:\beta \in \mathsf{canonicalUndercuts}(L:\alpha, \Delta)$. The only exception to the general case is when $\bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(I:\alpha), \Delta)) = \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(L:\alpha), \Delta))$. Thus if $L:\alpha \in \mathsf{reflectedConfirmingArguments}(K:\neg\beta, J:\beta, I:\alpha, \Delta)$ and $\bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(I:\alpha), \Delta)) \neq \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(L:\alpha), \Delta))$ then it is always the case that $K:\neg\beta \notin \mathsf{reflectedArgumentRebuttal}(J:\beta, L:\alpha, \Delta)$. □

If it were the case that $\bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(I:\alpha), \Delta)) = \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(L:\alpha), \Delta))$ then in addition to $J:\beta \in \mathsf{canonicalUndercuts}(I:\alpha, \Delta)$ it would also be the case that $J:\beta \in \mathsf{canonicalUndercuts}(L:\alpha, \Delta)$. So if it were not for the use of labels to allow two experts to state the same formulae, the proposition would be simplified to just that 'the existence of a reflected confirming argument does not transform a direct argument rebuttal into a reflected argument rebuttal.'

I use the term 'mutual destruction' to describe this rare situation where what would normally be a reflected confirming argument is not because it turns the direct argument rebuttal in the attack chain into a reflected confirming argument. Mutual destruction occurs when two arguments $I:\alpha, J:\alpha$ are such that

$I, J \in \mathrm{X}\!:\!\alpha$ and stripAssumptions(formulae(label($I : \alpha$), $\Delta$)) = stripAssumptions(formulae(label($J : \alpha$), $\Delta$)) and the other conditions are met.

## 5.9 Type IV Reflection – Reflected Confirming Confirmations

I now extend the reflectedConfirmingArguments() function from the level of individual arguments to that of confirmations. I classify reflected confirming confirmations as a fourth type of reflection, which I call Type IV Reflection, as the following definition and properties show them to materially differ from the earlier reflection Types I to III. The confirmation graph in Figure 5.10 below provides an overview of Type IV Reflection. The additional circle indicates a confirmation for $\alpha$ with confirmation label of $\mathrm{V} \cup \mathrm{X}$. Definitions 3.4.2 of preclusive undercut and 5.6.1 of direct confirmation rebuttal are reused here.

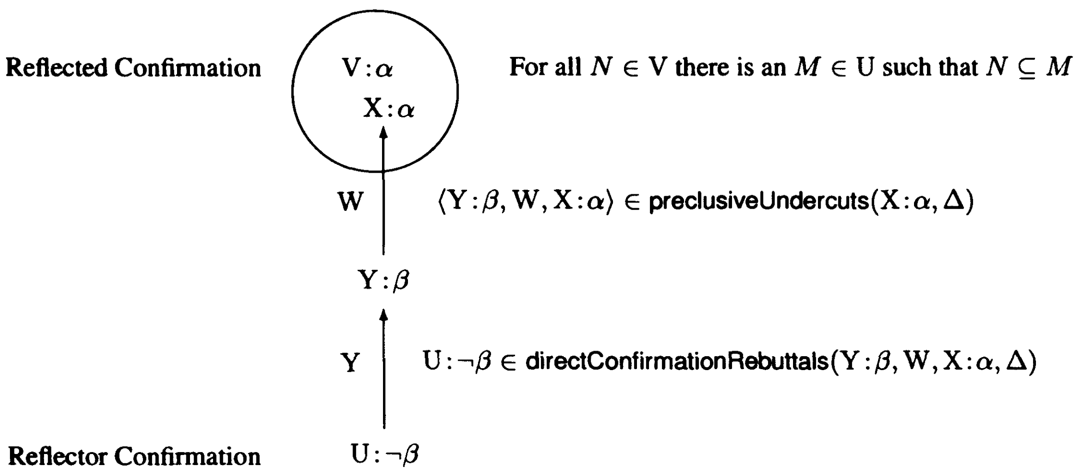


Figure 5.10: Type IV Reflection is reflectedConfirmingConfirmations($\mathrm{U}\!:\!\neg\beta, \mathrm{Y}\!:\!\beta, \mathrm{W}, \mathrm{X}\!:\!\alpha, \Delta$)

**Definition 5.9.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $M : \neg\beta, N : \alpha \in$ arguments($\Delta$) and let $\mathrm{U}\!:\!\neg\beta, \mathrm{W}\!:\!\alpha, \mathrm{X}\!:\!\alpha, \mathrm{Y}\!:\!\beta \in \Diamond(\Delta)$. The* **reflected confirming confirmations**, *denoted* reflectedConfirmingConfirmations($\mathrm{U}\!:\!\neg\beta, \mathrm{Y}\!:\!\beta, \mathrm{W}, \mathrm{X}\!:\!\alpha, \Delta$), *arising from the preclusive undercut of $\mathrm{X}\!:\!\alpha$ by $\mathrm{Y}\!:\!\beta$ and the direct rebuttal of $\mathrm{Y}\!:\!\beta$ by $\mathrm{U}\!:\!\neg\beta$ is a set of confirmations such that:*

$$\begin{aligned}
&\text{reflectedConfirmingConfirmations}(\mathrm{U}\!:\!\neg\beta, \mathrm{Y}\!:\!\beta, \mathrm{W}, \mathrm{X}\!:\!\alpha, \Delta) \\
&= \{\mathrm{V}\!:\!\alpha \in \Diamond(\Delta) \mid \langle \mathrm{Y}\!:\!\beta, \mathrm{W}, \mathrm{X}\!:\!\alpha\rangle \in \text{preclusiveUndercuts}(\mathrm{X}\!:\!\alpha, \Delta), \\
&\qquad \mathrm{U}\!:\!\neg\beta \in \text{directConfirmationRebuttals}(\mathrm{Y}\!:\!\beta, \mathrm{W}, \mathrm{X}\!:\!\alpha, \Delta) \textit{ and} \\
&\qquad \textit{for all } N \in \mathrm{V} \textit{ there exists an } M \in \mathrm{U} \textit{ such that } N \subseteq M\}.
\end{aligned}$$

I call $\mathrm{U} : \neg\beta$ the **reflector** confirmation and $\mathrm{V} : \alpha$ the **reflected** confirmation. Because preclusive undercuts can be multi-target it is not possible to define reflected confirming confirmations solely in terms of reflected confirming argument, which is Definition 5.8.1, hence the above form is needed. Thus we have here and require a fourth type of reflection, Type IV Reflection.

### 5.9.1 Properties of Reflected Confirming Confirmations

One implication of X being relatively unconstrained in this definition is that none, some or all of the reflected arguments $N:\alpha$ are such that $N \in \mathrm{X}$ for the head of the attack chain $\mathrm{X}:\alpha$. Thus it can be the case that $\mathrm{V}:\alpha$ is such that $\mathrm{V} \subseteq \mathrm{X}$ or $\mathrm{V} \cap \mathrm{X} \neq \emptyset$. It depends on the inputs to the reflected confirming confirmations function. So supports $N$, i.e. the argument labels $\mathsf{label}(N:\alpha)$, can be both part of the input and the output to this function. Likewise none, some or all of $N \in \mathrm{W}$. The above definition allows $\mathrm{X}:\alpha \neq \mathsf{top}(\Diamond(\alpha,\Delta))$. Furthermore $\mathrm{V}:\alpha$ is allowed to be a power set of confirmations and not just a single confirmation. The function is defined so that the output is $\{\mathrm{V}:\alpha \subsetneq \Diamond(\alpha,\Delta)\}$, the reason being that $\emptyset \notin \mathrm{V}$. There is no stipulation in the definition that $\mathrm{V}:\alpha \cup \mathrm{W}:\alpha = \mathsf{top}(\Diamond(\alpha,\Delta))$, nor that $\mathrm{V}:\alpha \cup \mathrm{X}:\alpha = \mathsf{top}(\Diamond(\alpha,\Delta))$. None of the confirmations in the attack chain can be labelled with the emptyset as that is prohibited by the definitions reused here.

**Example 5.9.1. Reflected confirming confirmation.** *Let* $\Delta = \{a:\gamma, b:\gamma \to \alpha, c:\pi, d:\pi \to \alpha, e: \neg\pi \vee \neg\gamma, f:\pi \wedge \alpha\}$. *Also Let* $I = \{a,b\}, J = \{c,d\}, K = \{e\}, L = \{a,b,c,d\}, M = \{b,d,f\}, N = \{b,f\}, O = \{d,f\}$. *Consequently* $I:\alpha, J:\alpha, N:\alpha, O:\alpha, K:\neg(\gamma \wedge (\gamma \to \alpha) \wedge \pi \wedge (\pi \to \alpha)), L: \gamma \wedge (\gamma \to \alpha) \wedge \pi \wedge (\pi \to \alpha), M:\gamma \wedge (\gamma \to \alpha) \wedge \pi \wedge (\pi \to \alpha) \in \mathsf{arguments}(\Delta)$. *Additionally let* $\mathrm{T} = \{L\}, \mathrm{U} = \{M\}, \mathrm{V} = \{N,O\}, \mathrm{W} = \{I,J\}, \mathrm{X} = \{I,J,N,O\}, \mathrm{Y} = \{K\}$. *So* $\mathrm{V}:\alpha, \mathrm{W}:\alpha, \mathrm{X}:\alpha, \mathrm{Y}: \neg(\gamma \wedge (\gamma \to \alpha) \wedge \pi \wedge (\pi \to \alpha)), \mathrm{T}:\gamma \wedge (\gamma \to \alpha) \wedge \pi \wedge (\pi \to \alpha), \mathrm{U}:\gamma \wedge (\gamma \to \alpha) \wedge \pi \wedge (\pi \to \alpha) \in \Diamond(\Delta)$. *It thus follows that* $\langle \mathrm{Y}:\neg(\gamma \wedge (\gamma \to \alpha) \wedge \pi \wedge (\pi \to \alpha)), \mathrm{W}, \mathrm{X}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{X}:\alpha,\Delta)$. *Also* $\mathrm{U}:\gamma \wedge (\gamma \to \alpha) \wedge \pi \wedge (\pi \to \alpha) \in \mathsf{directConfirmationRebuttals}(\mathrm{Y}:\neg(\gamma \wedge (\gamma \to \alpha) \wedge \pi \wedge (\pi \to \alpha)), \mathrm{W}, \mathrm{X}:\alpha, \Delta)$. *As an aside, there is also a Type II Reflection* $\mathrm{T}:\gamma \wedge (\gamma \to \alpha) \wedge \pi \wedge (\pi \to \alpha) \in \mathsf{reflectedConfirmationRebuttals}(\mathrm{Y}:\neg(\gamma \wedge (\gamma \to \alpha) \wedge \pi \wedge (\pi \to \alpha)), \mathrm{W}, \mathrm{X}:\alpha, \Delta)$ *as* $L \subseteq I \cup J$. *The main point of this example is that* $V:\alpha \in \mathsf{reflectedConfirmingConfirmations}(\mathrm{U}:\gamma \wedge (\gamma \to \alpha) \wedge \pi \wedge (\pi \to \alpha), \mathrm{Y}:\neg(\gamma \wedge (\gamma \to \alpha) \wedge \pi \wedge (\pi \to \alpha)), \mathrm{W}, \mathrm{X}:\alpha, \Delta)$ *because* $N \subseteq M$ *and* $O \subseteq M$. *So looking ahead to tracking a professional debate, the two confirmations in this example that would have to be pruned out are the reflected confirmation rebuttal* $\mathrm{T}:\gamma \wedge (\gamma \to \alpha) \wedge \pi \wedge (\pi \to \alpha)$ *and the reflected confirming confirmation* $V:\alpha$.

Given the prerequisites of an appropriate attack chain, i.e. a direct confirmation rebuttal attacking a preclusive undercut, where all three vertices in the chain are valid confirmations, reflected confirming confirmations always exist.

**Proposition 5.9.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\mathrm{U}:\neg\beta, \mathrm{W}:\alpha, \mathrm{X}:\alpha, \mathrm{Y}:\beta \in \Diamond(\Delta)$.

*If* $\langle \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{X}:\alpha,\Delta)$ *and*

$\mathrm{U}:\neg\beta \in \mathsf{directConfirmationRebuttals}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$

*then* $\mathsf{reflectedConfirmingConfirmations}(\mathrm{U}:\neg\beta, \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta) \neq \emptyset$.

*Proof.* Given the preclusive undercut $\langle \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{X}:\alpha,\Delta)$ it is known that $\beta = \neg \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(\mathrm{W}:\alpha),\Delta))$, where $\mathrm{W}:\alpha$ is the target confirmation.

Also for each $M \in \mathrm{U}$ it must be the case that $\mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(M : \neg\beta), \Delta)) \vdash \neg\beta$. Therefore for each $M \in \mathrm{U}$ it follows that $\mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(M : \neg\beta), \Delta)) \vdash \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(\mathrm{W}:\alpha), \Delta))$ and therefore that $\mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(M : \neg\beta), \Delta)) \vdash \alpha$. That $M$ may not be minimal, so suppose an $N \subseteq M$ and it follows that given the above attack chain, for each $M \in \mathrm{U}$ there exists a valid argument $N:\alpha \in \mathsf{arguments}(\Delta)$. All of these resultant $N:\alpha, N \subseteq M$ form a confirmation $\mathrm{V}:\alpha$, where $N \in \mathrm{V}$. Therefore $\mathrm{V} \neq \emptyset$. Thus if $\langle \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha\rangle \in \mathsf{preclusiveUndercuts}(\mathrm{X}:\alpha, \Delta)$ and $\mathrm{U}:\neg\beta \in \mathsf{directConfirmationRebuttals}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$ then $\mathsf{reflectedConfirmingConfirmations}(\mathrm{U}:\neg\beta, \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta) \neq \emptyset$. □

It is not quite so simple to say that otherwise $\mathsf{reflectedConfirmingConfirmations}(\mathrm{U}:\neg\beta, \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta) = \emptyset$ because if the second attack in the attack chain is a reflected confirmation rebuttal (rather than a direct one) then the reflected confirmation is the same as the reflector confirmation and hence not the emptyset. If the preclusion is absent, however, the result is the empty set.

Given that these reflections always exist, given their prerequisites, the next question is how many arguments are there in the reflection.

## 5.9.2 Scaling and Reflected Confirming Confirmations

I now introduce a variation of the reflection scaling factor of Definition 5.7.4 to extend the concept of that function to also cover reflected confirming confirmations. I append the function name with the words 'up tree' to avoid ambiguity.

**Definition 5.9.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let* $\mathrm{U}:\neg\beta, \mathrm{W}:\alpha, \mathrm{X}:\alpha, \mathrm{Y}:\beta \in \Diamond(\Delta)$. *The* **up tree reflection scaling factor**, *denoted* $\mathsf{reflectionScalingFactorUpTree}(\mathrm{U}:\neg\beta, \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$, *is a rational number such that:*

$$\begin{aligned}&\mathsf{reflectionScalingFactorUpTree}(\mathrm{U}:\neg\beta, \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)\\ =\;& \frac{|\mathsf{label}(\mathsf{top}(\mathsf{reflectedConfirmingConfirmations}(\mathrm{U}:\neg\beta, \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)))|}{|\mathsf{label}(\mathrm{U}:\neg\beta)|}.\end{aligned}$$

The definitions of the key words for the amount of scaling also need to apply to the different domain.

**Definition 5.9.3.** *Let $\Delta$ be a knowledgebase of labelled assumptions and* $\mathrm{U}:\neg\beta, \mathrm{W}:\alpha, \mathrm{X}:\alpha, \mathrm{Y}:\beta \in \Diamond(\Delta)$.

*If* $\mathsf{reflectionScalingFactorUpTree}(\mathrm{U}:\neg\beta, \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta) < 1$
*then* $\mathsf{reflectedConfirmationConfirmationsU}:\neg\beta, \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$ *is a* **reduced reflection**.

*If* $\mathsf{reflectionScalingFactorUpTree}(\mathrm{U}:\neg\beta, \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta) = 1$
*then* $\mathsf{reflectedConfirmationConfirmations}(\mathrm{U}:\neg\beta, \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$ *an* **exact reflection**.

*If* $\mathsf{reflectionScalingFactorUpTree}(\mathrm{U}:\neg\beta, \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta) > 1$
*then* $\mathsf{reflectedConfirmationConfirmations}(\mathrm{U}:\neg\beta, \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$ *an* **enlarged reflection**.

Now a slightly modified example to illustrate why the earlier definition of reflected confirming arguments uses $L \subseteq K$ rather than $L = K$ (which would have yielded only one reflected argument of $K:\alpha$). What this example shows is enlargement, exactly as with Type I Reflections. Here the $\subsetneq$ is active.

**Example 5.9.2. Enlarged reflected confirming confirmation.** *Let* $\Delta = \{a:\alpha, b:\gamma \rightarrow \alpha, c:\lambda, d:\lambda \rightarrow \neg\gamma, e:\pi \wedge \alpha, f:\alpha \wedge (\pi \rightarrow \alpha)\}$. *Let* $I = \{a,b\}, J = \{c,d\}, K = \{e.f\}, L = \{e\}, M = \{f\}$. *Thus* $I:\alpha, J:\neg\gamma, J:\neg(\gamma \wedge (\neg\gamma \rightarrow \alpha)), I:\gamma \wedge (\gamma \rightarrow \alpha), K:\gamma \wedge (\gamma \rightarrow \alpha), L:\gamma \rightarrow \alpha, L:\alpha, M:\alpha \in \mathsf{arguments}(\Delta)$. *Thus* $J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)) \in \mathsf{canonicalUndercuts}(I:\alpha,\Delta)$. *Also* $I:\gamma \wedge (\gamma \rightarrow \alpha), K:\gamma \wedge (\gamma \rightarrow \alpha) \in \mathsf{rebuttals}(J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)),\Delta)$. *It is the case that* $I:\gamma \wedge (\gamma \rightarrow \alpha) \in \mathsf{reflectedArgumentRebuttal}(J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)), I:\alpha,\Delta)$*, because obviously* $I \subseteq I$*. It is also the case that* $K:\gamma \wedge (\gamma \rightarrow \alpha) \in \mathsf{directArgumentRebuttals}(J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)), I:\alpha,\Delta)$*, because* $K \not\subseteq I$*. Given this attack chain of three arguments, it follows that there are two reflected confirming arguments, namely* $\mathsf{reflectedConfirmingArguments}(K:\gamma \wedge (\gamma \rightarrow \alpha), J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)), I:\alpha,\Delta) = \{L:\alpha, M:\alpha\}$ *where* $L \subsetneq K, M \subsetneq K$*. Now two checks to ensure the safety of this result. Firstly* $K:\gamma \wedge (\gamma \rightarrow \alpha) \notin \mathsf{reflectedArgumentRebuttal}(J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)), L:\alpha,\Delta)$*, because* $J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)) \notin \mathsf{canonicalUndercuts}(L:\alpha,\Delta)$*. Secondarily* $K:\gamma \wedge (\gamma \rightarrow \alpha) \notin \mathsf{reflectedArgumentRebuttal}(J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)), M:\alpha,\Delta)$*, because* $J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)) \notin \mathsf{canonicalUndercuts}(M:\alpha,\Delta)$*. So the reflection off the one argument* $K:\gamma \wedge (\gamma \rightarrow \alpha)$ *has yielded two arguments* $L:\alpha$ *and* $M:\alpha$*, giving an enlarged reflection. Thus* $\mathsf{reflectionScalingFactorUpTree}(\{K\}:\gamma \wedge (\gamma \rightarrow \alpha), \{J\}:\neg(\gamma \wedge (\gamma \rightarrow \alpha)), \{I\}, \{I\}:\alpha,\Delta) = |\{L,M\}| \div |\{K\}| = 2 \div 1 = 2$.

The above proves by example that Type III and IV Reflections are not always exact reflections. They can be enlarged. This example suggests problematic behaviour would arise if reflected confirming arguments were used to stand in for the direct rebuttals they are reflected from.

A further implication is that the existence of reduced reflections is also inevitable. Just as shown before for preclusive undercuts, the $\subsetneq$ part of $\subseteq$ allows support shortening (e.g. a reflector with support of $\{a,b,c\}$ giving rise to reflections with supports such as $\{b,c\}$). Consequently two different shortened subsets (e.g. reflectors $\{a,b,c\}, \{b,c,d\}$ yielding reflections $\{b,c\}, \{b,c\}$) which then go on to merge (to be just $\{b,c\}$). Hence a reduced reflection. Here is an example of one.

**Example 5.9.3. Reduced reflected confirming confirmation.** *Let* $\Delta = \{a:\gamma, b:\gamma \rightarrow \alpha, c:\lambda, d:\lambda \rightarrow \neg\gamma, e:\pi \wedge \mu \wedge \alpha, f:\pi \rightarrow \gamma, g:\mu \rightarrow \gamma\}$. *Let* $I = \{a,b\}, J = \{c,d\}, K = \{e,f\}, L = \{e,g\}, M = \{e\}$. *Therefore* $I:\alpha, J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)), I:\gamma \wedge (\gamma \rightarrow \alpha), M:\gamma \rightarrow \alpha, K:\gamma \wedge (\gamma \rightarrow \alpha), L:\gamma \wedge (\gamma \rightarrow \alpha) \in \mathsf{arguments}(\Delta)$. *So* $J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)) \in \mathsf{canonicalUndercuts}(I:\alpha,\Delta)$ *and* $K:\gamma \wedge (\gamma \rightarrow \alpha), L:\gamma \wedge (\gamma \rightarrow \alpha) \in \mathsf{directArgumentRebuttals}(J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)), I:\alpha,\Delta)$ *– these two arguments are direct because* $K \not\subseteq I$ *and* $L \not\subseteq I$*. Naturally there is also* $I:\gamma \wedge (\gamma \rightarrow \alpha) \in \mathsf{directArgumentRebuttals}(J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)), I:\alpha,\Delta)$*. Now for the point of this example: firstly* $M:\alpha \in \mathsf{reflectedConfirmingArguments}(K:\gamma \wedge (\gamma \rightarrow \alpha), J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)), I, I:\alpha,\Delta)$ *because* $M \subsetneq K$ *and secondly* $M:\alpha \in \mathsf{reflectedConfirmingArguments}(L:\gamma \wedge (\gamma \rightarrow \alpha), J:\neg(\gamma \wedge (\gamma \rightarrow \alpha)), I:\alpha,\Delta)$ *because* $M \subsetneq L$*. So although there are two reflectors at the tail of the argument chain there is only one reflected argument at the head. Because* $|\{M\}| < |\{K,L\}|$ *this a reduced reflection.*

*Consequently* reflectionScalingFactorUpTree$(\{L\} : \gamma \wedge (\gamma \rightarrow \alpha), \{J\} : \neg(\gamma \wedge (\gamma \rightarrow \alpha)), \{I\}, \{I\} : \alpha, \Delta) = |\{M\}| \div |\{K, L\}| = 1 \div 2 = 0.5$.

As enlarged and reduced reflected confirming confirmations have been shown, it follows that a combination of the two mechanisms can be employed to yield exact reflections.

### 5.9.3 Direct Confirming Confirmations

By symmetry, the next and final step in this pattern of reflected and direct functions is the topic of direct confirming arguments and direct confirming confirmations. If small steps were used the steps would be direct confirming argument, then direct confirming confirmation leading to a maximal direct confirming confirmation. Similar ground has been covered several times already so it is practical to move immediately to maximal direct confirming confirmations, which for short I call direct confirmations. This definition leverages Definition 5.9.1 of reflected confirming confirmations.

**Definition 5.9.4.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $L : \alpha \in$ arguments$(\Delta)$ and let $\mathrm{U} : \neg\beta, \mathrm{V} : \alpha, \mathrm{W} : \alpha, \mathrm{X} : \alpha, \mathrm{Y} : \beta \in \Diamond(\Delta)$. The set of* **direct confirmations***, denoted* directConfirmations$(\mathrm{X} : \alpha, \Delta)$*, which is devoid of reflected confirming confirmations from any reflector is the set of confirmations such that:*

$$\begin{aligned}
&\text{directConfirmations}(\mathrm{X}:\alpha, \Delta) \\
&= \{\mathrm{T}:\alpha \in \Diamond(\Delta) \mid \textit{for every } L \in \mathrm{T} \textit{ there does not exist an } L \in \mathrm{V} \textit{ such that} \\
&\qquad \mathrm{V}:\alpha = \text{top}(\text{reflectedConfirmingConfirmations}(\mathrm{U}:\neg\beta, \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)), \\
&\qquad \langle \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha \rangle \in \text{preclusiveUndercuts}(\mathrm{X}:\alpha, \Delta) \textit{ and} \\
&\qquad \mathrm{U}:\neg\beta \in \text{directConfirmationRebuttals}(\mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)\}.
\end{aligned}$$

The set of direct confirmations for $\alpha$ is a power set of confirmations directConfirmations$(\mathrm{X}:\alpha, \Delta) \subseteq \Diamond(\alpha, \Delta)$. It is a filter or pruning function because $\mathrm{T} \subseteq \mathrm{X}$. In the definition W is such that $\mathrm{W} \subseteq \mathrm{X}$.

A direct confirmation for $\alpha$ is not necessarily the maximum cardinality confirmation for $\alpha$, but it is maximal in the sense that it takes into account all possible reflectors. The directConfirmations$(\mathrm{X}:\alpha, \Delta)$ does require the input parameter of $\Delta$ as it ensures that there do not exist any reflectors that could give rise to reflected arguments in the output.

To look ahead for a moment, to be free of reflections, the root node of a debate tracking tree therefore intuitively has to be the contradiction $\langle \mathrm{X}, \mathrm{Y} \rangle : \alpha$ where :

$$\mathrm{X}:\alpha = \text{top}(\text{directConfirmations}(\text{top}(\Diamond(\alpha, \Delta), \Delta).$$

$$\mathrm{Y}:\neg\alpha = \text{top}(\text{directConfirmations}(\text{top}(\Diamond(\neg\alpha, \Delta), \Delta)).$$

## 5.10 Conclusion for Reflected and Direct Confirmations

The ten principal functions from this chapter, plus three more included for context, are summarised in Table 5.1. This table is a close parallel to the one at the end of Chapter 4, with the exclusion of abstract reflections and the inclusion of maximal reflections. It also includes the up-chain reflections arising from the combination of rebuttal and undercut. The key difference with the previous summary table, Table 4.3, is of course that the one from the last chapter is at the level of individual arguments whilst the table below is at the level of sets of arguments, i.e. confirmations. This analysis of reflection on the level of confirmations uncovers a number of properties that bear on the automation of debate.

The two maximal direct functions and the direct confirmation are the ones used in the next chapter - on debate tracking trees. All of the functions in the above table return power sets of confirmations, with the maximalDirectConfirmationRebuttal() being a special case as it always returns a singleton set – hence its singular name.

| Attack | Reflection | Confirmation Attacks Function | Def. |
|---|---|---|---|
| Rebuttal | Reflected ∪ Direct | confirmationRebuttals$(\mathrm{X}:\alpha, \Delta)$ | 2.6.8 |
| Undercut | Reflected ∪ Direct | preclusiveUndercuts$(\mathrm{X}:\alpha, \Delta)$ | 3.4.2 |
| Attacks | Reflected ∪ Direct | confirmationAttacks$(\mathrm{X}:\alpha, \Delta)$ | 3.9.4 |
| Rebuttal | Reflected Only | reflectedConfirmationRebuttals$(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$ | 5.5.1 |
| Undercut | Reflected Only | reflectedPreclusiveUndercuts$(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$ | 5.2.1 |
| Attacks | Reflected Only | reflectedConfirmationAttacks$(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$ | 5.7.1 |
| Rebuttal | Direct Only | directConfirmationRebuttals$(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$ | 5.6.1 |
| Undercut | Direct Only | directPreclusiveUndercuts$(\mathrm{Y}:\beta, \mathrm{V}, \mathrm{X}:\alpha, \Delta)$ | 5.3.1 |
| Rebuttal | Direct, Maximal | maximalDirectConfirmationRebuttal$(\mathrm{Y}:\beta, \mathrm{X}:\alpha, \Delta)$ | 5.6.2 |
| Undercut | Direct, Maximal | maximalDirectPreclusiveUndercuts$(\mathrm{Y}:\beta, \Delta)$ | 5.3.2 |
| Undercut then Rebuttal | Reflected Only | reflectedConfirmingArguments$(K:\neg\beta, J:\beta, I:\alpha, \Delta)$ | 5.8.1 |
| Undercut then Rebuttal | Reflected Only | reflectedConfirmingConfirmations$(\mathrm{U}:\neg\beta, \mathrm{Y}:\beta, \mathrm{W}, \mathrm{X}:\alpha, \Delta)$ | 5.9.1 |
| Undercut then Rebuttal | Direct, Maximal | directConfirmations$(\mathrm{X}:\alpha, \Delta)$ | 5.9.4 |

Table 5.1: Summary of Reflected and Direct Confirmation Attacking Functions

Reflections can flow both up and down a chain of arguments, with the reflected undercuts and rebuttals all being down chain. The reflected confirming arguments, however, flow up the chain from leaf to root. The reflection scaling factors which occur with confirmations undercutting and rebutting each other are shown in Table 5.2 below.

This chapter has uncovered the principle properties of reflections amongst confirmations, many of which might not be intuitively expected and all of which bear on the automation of professional debate. Further research may uncover further forms of reflection and certainly further properties of those forms defined here. A key observation from this and the last chapter is that reflections are subject to scaling

| Reflected | Enlarged | Exact | Reduced |
|---|---|---|---|
| Preclusive Undercut: | Exist<br>Example 5.7.3<br>Multi-Pair, Section 5.7.4<br>Mono-Pair, Definition 4.6.1 | Exist<br>Example 5.7.2 | Exist<br>Example 5.7.1 |
| Confirmation Rebuttal: | Exist<br>Example 5.5.3<br>(relies on labels) | Exist<br>Example 5.5.2<br>(relies on labels) | Exist<br>Example 5.5.1 |
| Confirming Confirmation: | Exist<br>Example 5.9.2 | Exist<br>Section 5.9.2 | Exist<br>Example 5.9.3 |

Table 5.2: Summary of Scaling Factors Found With Confirmation Reflections

and thus can result in enlarged, reduced, exact or even distorted reflections – except, as shown in the previous chapter, for reflected argument rebuttals which are only ever exact. An area for further research is to examine more closely the role of labels in scaling and see which scaling situations in Table 5.2 would not occur if labels were removed.

There are two mechanisms that can cause enlarged preclusive reflections (mono-pair and multi-pair enlarged reflection), but only one one mechanism for reduced reflection (support merging). Scaling in general, and distortion in particular, show the notion that arguments in a debate can be counted by counting their reflections to be impractical or unworkable. The targets of all reflected preclusive undercuts are always mono-attack.

One possible area for future research would be to investigate more fully the properties of an abstract confirmation attacks functions, including reflected confirmation attacks, direct confirmation attacks and maximal direct confirmation attacks. Such an approach might build on the work of Dung, working at at abstract level of 'attacks', but employing confirmations rather than individual arguments as the graph nodes.

This chapter lays the foundations for constructing debate tracking trees in the next chapter and therefore to judging debates in the chapter after that.

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# Chapter 6

# How to Track a Debate

This chapter establishes that an effective way to track a debate, the whole debate and nothing but the debate is to use a tree of contradictions with all of the reflected arguments pruned out of the tree, leaving just the direct arguments. This proposal is based on a critical analysis of trees in the literature, with a specific analysis of the tree in (Besnard & Hunter, 2001) and a general analysis that includes the trees of (Prakken & Sartor, 1997; Amgoud & Cayrol, 2002; García & Simari, 2004; Dung et al., 2006). My examination shows that the question of how to handle reflection in argument graphs is a general problem in the literature not limited to any one particular publication.

## 6.1 Overview of the Chapter

I develop this chapter's evaluation of how to track a debate in seven stages. It moves from showing issues with approaches in the literature to proposing a new approach that responds to these issues.

**Debate Tracking in the Literature** Section 6.2 describes various approaches found in the literature and then picks one for detailed analysis of the effects of reflection.

**Argument Trees are Incomplete** Section 6.3 shows several distinct mechanisms that lead to argument trees lacking necessary arguments making them incomplete.

**Argument Trees are Inordinate** Section 6.4 shows that there are additional mechanisms which result in argument trees containing excessive, i.e. unnecessary or redundant, arguments.

**Further Argument Tree Behaviours** Section 6.5 shows i) the proxy arguments that argument trees contain do not accurately represent the arguments that are excluded, ii) that argument trees are sensitive to the way the knowledgebase is written down and iii) they can be asymmetrical.

**Rationale for Matt Opaque Contradiction Trees** Section 6.6 provides a reasoned path from the problems I identify with debate tracking in the literature to the justification of the new approach of the subsequent section.

**Definition of Debate Trees** Section 6.7 starts with a series of helper functions to succinctly define this new debate tracking structure so that it is free from reflection. I call this matt opaque contradiction tree a $\mathsf{debateTree}(\alpha, \Delta)$ or just $\blacklozenge^*(\alpha, \Delta)$ for short (see Definition 6.7.7).

**Properties of Debate Trees** Section 6.8 rounds out the chapter by examining the behaviour and establishing a number of properties of these new trees.

I propose that my debate tree is a practical solution to the problem of how to track a professional debate. For any particular debate a necessary prerequisite to correctly judging the outcome is thus, I would argue, to populate the debate tree data structure.

## 6.2 Debate Tracking in the Literature

The examples of debate tracking covered so far in this thesis, notably in Section 2.7.2 on simple illustrative judges, take the elementary approach of relying on a single contradiction. Works in this area include (Benferhat et al., 1993; Krause et al., 1995; Elvang-Gøransson & Hunter, 1995). For further use of a single contradiction to track a debate in the informal literature see (Willcox & Bridgewater, 1977) and descriptions in (Krause et al., 1998) and (Fox & McBurney, 2002). The limitation of tracking with a single contradiction is that undercut cannot be taken into account and as shown in Section 3.2.2 undercuts can change debate outcomes, so including them is prudent. Consequently a more sophisticated data structure is warranted and there is an established consensus in the literature that some form of graph, tree or forest is appropriate. To create a graph to track a debate, it is necessary to make a choice for what the graph vertices represent and another choice for the graph edges.

**Vertex representation.** While graphs in the literature tend to use arguments for the vertices, my finding and recommendation is that *sets* of arguments, namely confirmations, or even better contradictions, are better suited as the nodes in a debate-tracking graph. Each contradiction then isolates a point of classical inconsistency. Much of this thesis is a justification for mapping vertices to contradictions. The thesis touches upon five different argument definitions (see Table 6.1), which are the main five to be found in the literature (excluding the related field of priorities or preferences). My focus is on classical logic labelled mincons as I find them well suited to tracking contributions in debates. These vertex representations are summarised as Table 6.1. The citation is the earliest formal definition I have located.

| | **Argument** | **Earliest Use** | **Reference** |
|---|---|---|---|
| 1. | Abstract argument | (Dung, 1993) | Definition 3.9.1 |
| 2. | Semi-abstract | (Bondarenko et al., 1997) | Definition 4.3.1 |
| 3. | Defeasible rule | (Pollock, 1992) | Not defined in thesis |
| 4. | Mincon | (Doyle, 1979) | Definition 2.4.2 |
| 5. | Labelled mincon | (Gabbay, 1991) | Definition 2.4.2 |
| 6. | Contradiction | this thesis | Definition 2.6.1 |

Table 6.1: Candidates Forms for Vertices in Debate Tracking Graphs

**Edge representation.** Now to turn to the question of what the inter-nodal links should be. There is a general consensus in the non-bipolar majority of the literature that the edges should be some form of attack, but what kind of attack is not agreed upon. In this thesis, I analyse at least seven different definitions that could be seen as attack, most of which are used in debate tracking structures in the literature, as summarised in Table 6.2.

| | Attack | Citation | Reference |
|---|---|---|---|
| 1. | Abstract attack | (Dung, 1993) | Definition 3.9.1 |
| 2. | Rebuttal only | (Benferhat et al., 1993) | Section 2.7 |
| 3. | Undercut only | (Pollock, 1992) | Definition 3.2.2 |
| 4. | Rebuttal or undercut | (Amgoud & Cayrol, 1998) | Definition 3.9.2 |
| 5. | Direct rebuttal | This thesis | Definition 4.8.1 |
| 6. | Direct canonical undercut | This thesis | Definition 4.5.1 |
| 7. | Direct preclusive undercut | This thesis | Definition 5.3.1 |

Table 6.2: Candidates Forms for Edges in Debate Tracking Graphs

Given the lists of possible vertices and edges the question arises: in what ways have these been combined and explored in the literature? Prior to doing further detailed research, can it be said which ones may contain reflection and are there any where reflection can definitely be ruled out?

| | Publication | Vertices | Edges | Name |
|---|---|---|---|---|
| 1. | (Pollock, 1992) | Defeasible rule | Undercut or rebuttal | Defeat graph |
| 2. | (Dung, 1993) | Abstract argument | Abstract attack | Dung graph |
| 3. | (Prakken & Sartor, 1997) | Defeasible rule | Undercut or rebuttal | Dialogue tree |
| 4. | (Besnard & Hunter, 2001) | Mincon | Canonical undercut | Argument tree |
| 5. | (Amgoud & Cayrol, 2002) | Abstract argument | Abstract attack | Dialogue tree |
| 6. | (García & Simari, 2004) | Defeasible rule | Undercut or rebuttal | Dialectical tree |
| 7. | (Dung et al., 2006) | Semi-abstract | Undercut only | Dispute tree |
| 8. | This thesis | Contradiction | Direct preclusive undercut | Debate tree |

Table 6.3: Summary of Debate Tracking Graphs or Trees

Table 6.3 provides a representative sampling of debate tracking structures found in the literature and additionally in this thesis. I now comment on each of these leading debate tracking forms in the literature before making some more general observations about reflection in these various trees of arguments.

**Defeat graph of (Pollock, 1992).** The defeat graph from (Pollock, 1992) is based on defeasible logic with the modus ponens proof rule dominating in examples. What is not clear is exactly what proof rules are available and thus what forms of reflection are present. It would appear that contraposition is probably not allowed, however that does not rule out reflection as several other forms of reflection, e.g. Type III, are not dependent on contraposition.

**Dung graph of (Dung, 1993).** While the writings of Dung do not major heavily on graphical visualisations of arguments, those commenting on his works, such as(Prakken & Vreeswijk, 2002) do make extensive use of such graphs. The vertices are single abstract arguments and the edges denote are one argument attacking one argument with an abstract attack of unspecified form. Clearly out of scope are vertices covering more than one argument and attacks attacking more than one argument. The Dung graphs are quite different from the other lines in this table as a) they are the only one that is a graph not a tree and b) the vertices and edges are abstract rather than concrete.

**Dialectic tree of (Prakken & Sartor, 1997).** This is another defeasible logic tree with an existential judge building on the work of Pollock. Priorities are included together with a novel feature for debating priorities. They make a distinction between i) deciding whether an individual argument is justified with a 'proof theory' which I would call a debate tracking structure plus judge and ii) a 'semantics' which identifies sets of justified arguments in the style of Dung. Their undercuts are similar but not identical to Wigmore undercuts, and do not involve priorities, whereas their rebuttals do. This landmark paper has clearly provided some inspiration to at least (Amgoud & Cayrol, 2002; García & Simari, 2004).

**Argument tree of (Besnard & Hunter, 2001).** The existence of reflection is acknowledged in this paper and the tree features a no-recycling constraint (NRC) and an edge choice that appear to have been influenced by this understanding. The removal of reflection is partial. Another strength of this paper is that it is explicit in stating how the outcomes of multiple trees are combined - an area I would suggest is underdeveloped in the literature. It uses the full classical logic , setting it apart from the body of papers which limit themselves to a smaller set of proof rules.

**Dialogue tree of (Amgoud & Cayrol, 2002).** The dialogue tree of (Amgoud & Cayrol, 2002) is presented as a proof procedure to establish if a particular argument A is acceptable. The argument A is placed at the root of the tree and each attack on A, and each attack on those attackers etc., analysed. While this paper contains a discussion of mincon arguments and Wigmore undercuts, the vertices of the dialogue tree are defined as abstract arguments based on (Dung, 1993), augmented with priorities in the style of (Amgoud & Cayrol, 1998). The edges are thus abstract attacks meaning that it is not possible to say precisely whether the tree does or does not contain reflection. The implications of following through with the mincon and Wigmore undercut specifics for dialogue tree and its judgements would be an interesting area for further research.

**Dialectical tree of (García & Simari, 2004).** This is another defeasible tree employing priorities and an existential judge (gaining at least some inspiration from Prakken). The proof rules are limited and contraposition is explicitly excluded. As with Prakken the undercut-like attack is not one analysed in my earlier Chapter 3, but unlike (Besnard & Hunter, 2001) and perhaps (Dung et al., 2006), the attack is arguably more general as it includes rebuttal. The root of each tree is still a single argument yielding a forest of trees for a claim.

**Dispute tree of (Dung et al., 2006).** I established in Section 4.3 that the semi-abstract ABA framework allows reflection, which would suggest that reflection exists in the trees of (Dung et al., 2006), as these trees are based on the same semi-abstract approach. This 2006 paper builds on the approach of (Prakken & Sartor, 1997; Amgoud & Cayrol, 2002) and others in treating the tree as the recording of a dialogue between proponent and opponent. In contrast, the approach of (Elvang-Gøransson & Hunter, 1995; Besnard & Hunter, 2001; Fox & McBurney, 2002), others and this thesis is to build the tree from a single knowledgebase, where all allowed arguments and attacks are computed and automatically included as part of the debate. While edges of the tree are specifically undercuts only, the authors present an argument as to why they believe rebuttals are also catered for. The theme of (Dung et al., 2006), I would argue, is probably one of the most promising area for future research in argumentation - where Dung-style semantics are integrated with concrete assumption-based argumentation.

**Debate tree** of this thesis. . Later on in this chapter I define the debate tree as a debate tracking structure which overcomes the shortcomings I identify with trees in the literature. The vertices of my debate tree are contradictions and the edges are preclusive undercuts between contradictions. The tree is so defined as to be devoid of reflected arguments. All of the arguments in one debate can be tracked in a single tree, rather than in a forest of trees, as is the situation with for example with (Besnard & Hunter, 2001; García & Simari, 2004; Dung et al., 2006).

More recent note-worthy existing research on trees of arguments includes the further analysis of defeasible trees by (Prakken et al., 2005) and classical trees by (Amgoud et al., 2004). The differences between these two approaches, defeasible and classical, are relatively small, when contrasted with the abstract Dung and semi-abstract ABA approaches.

**Observation on reflections in trees.** One might think that various mechanisms in these trees would ensure that reflections were absent, however each such conjecture is not robust:

1. **Defeasible logic**. In many defeasible logics the set of proof rules is more restrictive than in classical logic and in some contraposition, $\frac{\beta\rightarrow\alpha}{\neg\alpha\rightarrow\neg\beta}$, is specifically excluded, e.g. (García & Simari, 2004). It might be thought that without contraposition reflection would not be possible, however there are two responses to that idea.

   - Although my proof of Base Proposition One (Proposition 4.2.1) uses contraposition, other the predictable existence of forms of reflection, including Base Proposition Two (Proposition 4.7.2), Type III (Proposition 5.8.1) and Type IV do (Proposition 5.9.1) not rely on contraposition. Hence, ruling out the contraposition proof rule does not rule out reflection.

   - Furthermore, the same outcome can be achieved by using a combination of different proof rules, e.g. $\frac{\beta\rightarrow\alpha}{\neg\beta\vee\alpha}$, $\frac{\neg\beta\vee\alpha}{\alpha\vee\neg\beta}$, $\frac{\alpha\vee\neg\beta}{\neg\alpha\rightarrow\neg\beta}$ yields the contrapositive, but doesn't use contraposition. Some would argue, however, that this sequence of proof rules is still contraposition.

2. **Wigmore undercut**. While the bulk of my analysis of reflection uses canonical undercuts, many of these trees employ a form of Wigmore undercut instead, e.g. (Amgoud & Cayrol, 2002; Dung et al., 2006). It might be thought that that switch would prohibit reflection, however Proposition 4.3.1 shows reflection always existing even with Wigmore undercut.

3. **Excluding rebuttal**. The trees of (Besnard & Hunter, 2001; García & Simari, 2004; Dung et al., 2006) have already excluded all or most rebuttals and hence it would appear plausible that reflection (off rebuttals to undercuts) was already catered for. The counter to this suggestion is provided in Base Proposition Four showing that even with undercuts alone reflection occurs.

4. **No-recycling constraint** (NRC). Some of these trees employ a device to keep them finite, which is commonly called a no-recycling constraint. Some NRCs are simple in that they prevent repetition of an argument in a path from leaf to root (e.g. (Prakken & Sartor, 1997; Amgoud & Cayrol, 2002), while others NRCs, such as that in (Besnard & Hunter, 2001), are more sophisticated in preventing the reuse of supports. Given the absence of support reuse it might be thought that reflection would be excluded. The counter is that even by excluding undercut to undercut reflection there are other kinds, such as rebuttal to undercut and up-tree, that are not excluded.

Furthermore, none of the trees in the literature appear to contain any explicit device to exclude all reflection. Some appreciation of reflection appears to have affected the forms in (Besnard & Hunter, 2001) and seems to have influenced the intuitions of (Dung et al., 2006). The conclusion has to be drawn that all of the trees of arguments in the literature probably do contain reflection and that a body of further research is needed to confirm or refute that opinion.

As all of the approaches in the literature appear to lack a full appreciation of reflection so all of them are likely to yield arbitrary judgements for some knowledgebases. Of these debate tracking approaches, the classical forest approach of (Besnard & Hunter, 2001) is arguably the most promising as it has at least some consideration of reflection, however it too also exhibits a number of properties which could be problematic in mapping to professional debate. To perform a detailed analysis of all of the forms of reflection in all of the trees of arguments in the literature would be too broad a topic for this thesis so I pick one and document that detailed analysis as follows.

### 6.2.1 Argument Trees and Argument Structures – Classical Logic Forests

The argument tree was defined by (Besnard & Hunter, 2001). The same definition is also to be found in several follow on papers, notably (Cayrol & Lagasquie-Schiex, 2002; Hunter, 2004a; Hunter, 2004b). I now recap the standard definition of an argument tree, first using the notation of the original:

**Definition 6.2.1.** *(Besnard & Hunter, 2001)'s Definition 6.1. An* **argument tree** *for $\alpha$ is a tree where the nodes are arguments such that:*

1. *The root is an argument for $\alpha$,*
2. *For no node $\langle\Phi, \beta\rangle$ with ancestor nodes $\langle\Phi_1, \beta_1\rangle, \ldots, \langle\Phi_n, \beta_n\rangle$ is $\Phi$ a subset of $\Phi_1 \cup \ldots \cup \Phi_n$,*
3. *The children nodes of a node $N$ consist of all canonical undercuts for $N$ that obey 2.*

Clause 2. is called the **no-recycling constraint**. When a graph is cast into a tree it is the NRC that ensures the resultant tree is finite. The next step, as in the original, is to allow argument trees to be collected together to form argument structures or forests with the aim of tracking the whole debate.

**Definition 6.2.2.** *(Besnard & Hunter, 2001)'s Definition 8.1. An* **argument structure** *for a formula $\alpha$ is a pair of sets $\langle\mathcal{P}, \mathcal{C}\rangle$ where $\mathcal{P}$ is the set of argument trees for $\alpha$ and $\mathcal{C}$ is the set of argument trees for $\neg\alpha$.*

The above two definitions build on the same standard classical logic mincon argument and canonical undercut definitions as I employ, except they do not use a labelled deductive system (LDS). A key to understanding the two definitions above is Besnard and Hunter's argument that canonical undercuts can stand in for, or represent, rebuttals. Hence they track no rebuttals (apart from at the root) and only track undercuts.

A minor challenge with the above definition of argument tree is that the vertices do not form a set. In standard graph theory, a graph comprises a set of vertices and a set of edges. In the above, however, it is allowed that a particular argument may appear more than once in the tree and hence as more than one vertex. (Dung et al., 2006) observe the same issue in their tree and use a mapping function somewhat akin to the one I describe below (in that they refer to labelling a node with an argument, rather than the node being the argument), but (García & Simari, 2004) follow the same approach as (Besnard & Hunter, 2001). Here is an example of this minor challenge:

**Example 6.2.1. Repeated vertices in an argument tree.** *Let* $\Delta = \{a{:}\beta, b{:}\beta \to \alpha, c{:}\sigma, d{:}\sigma \to \neg\beta, e{:}\gamma, f{:}\gamma \to \neg\beta, g{:}\neg\gamma \wedge \neg\sigma, h{:}\gamma \vee \sigma\}$. *Let* $I = \{a, b\}, J = \{c, d\}, K = \{e, f\}, L = \{g\}, M = \{h\}$. *Members of* $\mathsf{arguments}(\Delta)$ *appear as nodes in the resultant argument tree for* $\alpha$ *below. The argument* $M : \neg(\neg\gamma \wedge \neg\sigma)$ *appears twice as two separate nodes in this argument tree showing vertex repetition. Figure 6.1 shows this repetition using the form of an argument graph (Definition 3.9.3).*

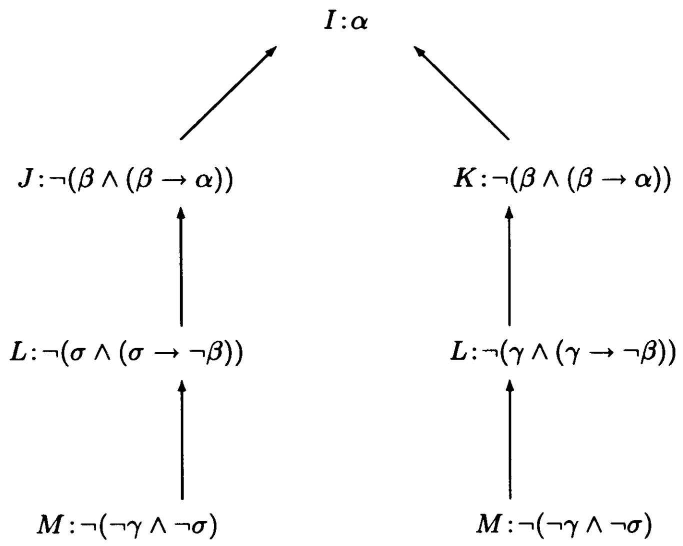



Figure 6.1: Repeated Vertices in an Argument Tree

A minor nomenclature improvement would thus be to have a set that represents vertices, a further set that represents arguments and then use a third device, a surjective function to map many vertices to single arguments, so that each vertex has attached to it a specific argument.

To facilitate subsequent definitions and overcome the above minor challenge I now rewrite the definition of argument tree in my notation to use such a mapping function. Certainly by doing so I add the possibility of additional behaviour due to labels, but that is not my motivation, and certainly not the cause of the properties established below. So here is the argument tree adapted to have unique vertices and edges.

**Definition 6.2.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae. An* **argument tree** *for an argument for $\alpha$, $I:\alpha$, denoted* $\text{argumentTree}(I:\alpha,\Delta) = \langle \mathcal{V}, \mathcal{E}, \text{arguments}(\Delta), f\rangle$, *where $\mathcal{V}$ is a set of vertices, $\mathcal{E}$ is a set of edges and $f:\mathcal{V}\mapsto \text{arguments}(\Delta)$, is such that:*

1. *The root $v_{root} \in \mathcal{V}$ of a tree for $I:\alpha$ is such that $f(v_{root}) \in \text{arguments}(\alpha,\Delta)$,*
2. *$v_{child} \in \mathcal{V}$ is a child of $v \in \mathcal{V}$*

   *iff* $(f(v_{child}) \in \text{canonicalUndercuts}(f(v),\Delta)$ *and*

   $\text{label}(f(v_{child})) \not\subseteq \text{label}(f(v_1)) \cup \ldots \cup \text{label}(f(v_n))$,

   *where $v_1,\ldots,v_n \in \mathcal{V}$ are the ancestors of $v_{child}$).*

The last two lines together comprise the **no-recycling constraint** which is a mechanism to ensure that the tree is finite and free from cycles, i.e. closed walks, of repeated arguments.

I use the above definition when documenting my analysis of reflection in argument trees below. In this definition every $v \in \mathcal{V}$ has one and only one $f(v) \in \text{arguments}(\Delta)$, that is, every node maps to an argument, rather than being an argument. Also any edge, representing a canonical undercut, is $\langle v_{child}, v\rangle \in \mathcal{E}$. To avoid repeating Definition 6.2.2, I now state that forests of argument trees $\text{argumentTree}(I:\alpha,\Delta)$ can also be held in the argument structure of Definition 6.2.2.

In the following three sections I establish that argument trees have several behaviours that do not map well to professional debate, namely that they are incomplete, inordinate and inchoate. Thus argument trees exist which lack necessary arguments, which contain unnecessary arguments and where the arguments they do contain do not accurately represent the ones that are absent. Consequently there will exist judge functions for these argument trees that will yield debate outcomes that are different from those commonly or intuitively expected by professional debate.

The definitions of inordinate and incomplete address establishing both the existence and the right number of arguments. If the right number are present, but they do not behave as they should then that is inchoate tree behaviour. I employ a graphical notation in the following examples where on the left I place an argument graph (Definition 3.9.3) portraying an argument structure (Definition 6.2.2) and on the right the intuition of a robust professional debate portrayed as a contradiction graph (see Definition 3.9.6). The right side trees are, in fact, debate trees as defined later, but for now they should be viewed as informal structures that build motivation for the later formalism.

A slightly different graphical form is used in (Besnard & Hunter, 2001) for the visual presentation of argument trees. In contrast to my argument graphs they show less information about the claim and more about the support. My view is that these mathematical presentational differences are not fundamental as i) they have no impact on the underlying formalisms, ii) that the undercuts' claims can be inferred from the attacked supports and vice versa, iii) that the point a debater is making with an argument is of more primary or immediate importance than the set of assumptions they use to backup their point and iv) to help someone appreciate and genuinely understand the parts of a debate I would suggest that a considerably more sophisticated and adjustable graphical user interface (GUI) is required - see for example the illustrations in the book *Visualizing Argumentation* by (Kirschner et al., 2003).

## 6.3 Argument Trees are Incomplete

The first of these properties of argument trees is that they are incomplete. I use the term incomplete to mean that necessary arguments are missing. An argument tree is incomplete if the $\Delta$ is such that one of the four following situations occurs.

### 6.3.1 Type A Incomplete - Non-root Rebuttals Excluded

Argument trees, by definition, *exclude all rebuttals* below the root, and hence exclude all direct rebuttals below the root. I hold a contrary position, that direct rebuttals are a legitimate and necessary part of professional debate. The following example highlights this difference of approach.

**Example 6.3.1. Type A Incomplete.** *This example illustrates Type A Incompleteness by providing a graphical representation of the earlier Example 4.8.1 of a direct argument rebuttal, see Figure 6.2.*

$$\begin{tikzcd} I:\alpha \quad \emptyset \\ J:\neg(\pi \wedge (\pi \rightarrow \alpha)) \arrow[u] \end{tikzcd} \qquad\qquad \begin{tikzcd} \langle\{I\},\emptyset\rangle:\alpha \\ \langle\{J\},\{L\}\rangle:\neg(\pi \wedge (\pi \rightarrow \alpha)) \arrow[u, "\{I\}"'] \end{tikzcd}$$

Incomplete Argument Structure — Complete Professional Debate

Figure 6.2: Type A Incomplete - Non-root Rebuttals Excluded

The direct argument rebuttal $L:\pi \wedge (\pi \rightarrow \alpha)$ would be part of a complete professional debate but would be excluded from an argument tree. Here the argument tree has fewer arguments than I deem is necessary.

The argument of (Besnard & Hunter, 2001) is that their exclusion of rebuttals from argument trees is legitimate because these absent rebuttals are represented by their reflected canonical undercuts. My counter-argument to that defence is that the resultant Type B Incompleteness (and the inordinate behaviours discussed in the next section) bring more disadvantages than advantages.

### 6.3.2 Type B Incomplete - Reduced Reflections from Excluded Direct Rebuttals

Reflected canonical undercuts arising from direct rebuttals are not good representatives of those rebuttals due to changes in the scaling factor of the reflection. Type B Incompleteness arises when there is a *reduced reflection* of reflected canonical undercuts, reflected off two or more direct argument rebuttals. In this situation the debate tracking, as recorded in an argument tree, has fewer arguments than I deem necessary.

To illustrate Type B Incompleteness I revisit Example 5.7.1. Even though it uses preclusive notation it is still about mono-target attacks and hence about canonical undercuts. The use of confirmation and preclusion notations makes it easier to describe and see the incomplete behaviour.

**Example 6.3.2. Type B Incomplete.** *Let* $\Delta = \{a:\beta, b:\beta \rightarrow \alpha, c:\gamma, d:\gamma \rightarrow \neg\beta, e:\neg\gamma\wedge\pi\wedge\lambda\wedge\sigma\wedge\beta, f: \pi \rightarrow (\beta \rightarrow \alpha), g:\lambda \rightarrow (\beta \rightarrow \alpha), h:\sigma \rightarrow (\beta \rightarrow \alpha)\}$. *Let* $I = \{a,b\}, J = \{c,d\}, K = \{e,f\}, L =$

$\{e,g\}, M = \{e,h\}, N = \{e\}$. *So* $I:\alpha, J:\neg\beta \vee \neg(\beta \to \alpha), J:\neg(\beta \wedge (\beta \to \alpha)), K:\beta \wedge (\beta \to \alpha), L: \beta \wedge (\beta \to \alpha), M:\beta \wedge (\beta \to \alpha), N:\neg\gamma \vee \neg(\gamma \to \neg\beta), N:\neg(\gamma \wedge \neg(\gamma \to \neg\beta)) \in \mathsf{arguments}(\Delta)$. *The root of this argument tree for* $\alpha$ *is* $I:\alpha$. *The root has one child which is a direct canonical undercut, namely* $J:\neg(\beta \wedge (\beta \to \alpha)) \in \mathsf{directCanonicalUndercuts}(I:\alpha, \emptyset, \Delta)$. *This direct canonical undercut has three direct argument rebuttals, i.e.* $K:\beta \wedge (\beta \to \alpha), L:\beta \wedge (\beta \to \alpha), M:\beta \wedge (\beta \to \alpha) \in \mathsf{directArgumentRebuttals}(J:\neg(\beta \wedge (\beta \to \alpha)), I:\alpha, \Delta)$. *These three direct rebuttals are not part of the argument tree, but they would be part of a professional debate. If it is the case that these direct argument rebuttals can be represented by their reflected canonical undercuts, then that turns out to be just* $N:\neg(\gamma \wedge \neg(\gamma \to \neg\beta)) \in \mathsf{reflectedCanonicalUndercuts}(J:\neg(\beta \wedge \neg(\beta \to \alpha)), K:\beta \wedge (\beta \to \alpha), \Delta)$. *If the representation was 1:1 there should be three reflected canonical undercuts, but in fact there is only one for all three reflectors – because also* $N:\neg(\gamma \wedge \neg(\gamma \to \neg\beta)) \in \mathsf{reflectedCanonicalUndercuts}(J: \neg(\beta \wedge (\beta \to \alpha)), L:\beta \wedge (\beta \to \alpha), \Delta)$ *and* $N:\neg(\gamma \wedge \neg(\gamma \to \neg\beta)) \in \mathsf{reflectedCanonicalUndercuts}(J: \neg(\beta \wedge (\beta \to \alpha)), M:\beta \wedge (\beta \to \alpha), \Delta)$. *This* **reduced reflection** *occurs as* $N \subsetneq M, N \subsetneq L, N \subsetneq K$. *So the only child of* $J:\neg(\beta \wedge (\beta \to \alpha))$ *in the argument tree is* $N:\neg(\gamma \wedge \neg(\gamma \to \neg\beta))$. *Thus, as shown in Figure 6.3, this argument tree for* $\alpha$*, quantitatively has two arguments missing, relative to professional debate, and is thus an incomplete tree.*

$$\begin{tikzcd}
I:\alpha \\
J:\neg(\beta \wedge (\beta \to \alpha)) \arrow[u] \\
N:\neg(\gamma \wedge (\gamma \to \neg\beta)) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle\{I\},\emptyset\rangle:\alpha \\
\langle\{J\},\{K,L,M\}\rangle:\neg(\beta \wedge (\beta \to \alpha)) \arrow[u, "\{I\}"']
\end{tikzcd}$$

Incomplete Argument Tree | Complete Professional Debate

Figure 6.3: Type B Incomplete - Reduced Reflections from Excluded Direct Rebuttals

A counter argument to my position is that reflections with support shortening are good as they are a more economical or more conservative way of expressing the same information. This counter argument is present in (Besnard & Hunter, 2001) via the definition of a 'more conservative' argument meaning one having a subset of the support and inferring the same conclusion as another argument. To match ordinary English, I would suggest the (Besnard & Hunter, 2001) definition of more conservative would be better phrased as 'as conservative as or more conservative than'. Thus by being reflected does not automatically make an argument more conservative. In practice, and as can be seen from the examples of this thesis, it is much more common place that the support of the reflection is equal to the full support of the reflector, so reflections are rarely 'more conservative' than their reflectors. In any case, I would argue that the relative benefit of a slightly shorter support is out weighed by the effect reflection has on the outcome of judgement (be that existential or quantitative). This favouring an economy of support as more important than other considerations is a key rationale for defending argument trees against a variety of the criticisms made in this chapter.

Whilst the above two forms of incompleteness have nothing to do with the NRC, the next form is entirely due to Besnard and Hunter's choice of NRC.

### 6.3.3 Type C Incomplete - Premise Reuse Prohibition

In a professional debate it is acceptable to reuse the premises of a previous argument as long as a different claim (or point) is being made. A barrister might be cautioned 'are you repeating material that you covered earlier?', to which he or she replies 'no, I am making a new point'. The following example illustrates what I would argue is the legitimate reuse of premises, something which is not allowed with argument trees. The kind of premise reuse I focus on here is within a chain of arguments from one specific leaf to the root; a second kind of premise reuse called 'divergent arguments' is discussed in (Walton, 1989) and refers to reusing a premise from anywhere in the tree. Within argument trees, divergent arguments drawing on premises from other paths are allowed. In debate trees all divergent arguments are allowed.

**Example 6.3.3. Type C Incomplete.** *Let* $\Delta = \{a : \beta \wedge \gamma, b : (\beta \rightarrow \alpha) \wedge (\gamma \rightarrow \neg\lambda), c : \pi, d : \pi \rightarrow \neg\beta, e : \lambda, f : \lambda \rightarrow \neg\pi\}$. *Let* $I = \{a, b\}, J = \{c, d\}, K = \{e, f\}$. *The arguments, confirmations and contradictions are as shown in Figure 6.4. Here the support I is legitimately reused to make the point* $\neg(\lambda \wedge (\lambda \rightarrow \neg\pi))$ *as the previous use of I was in support of a different claim, i.e.* $\alpha$.

$$
\begin{tikzcd}
I : \alpha \\
J : \neg(\beta \wedge \gamma \wedge (\beta \rightarrow \alpha) \wedge (\gamma \rightarrow \neg\lambda)) \arrow[u] \\
K : \neg(\pi \wedge (\pi \rightarrow \neg\beta)) \arrow[u]
\end{tikzcd}
$$

Incomplete Argument Tree

$$
\begin{tikzcd}
\langle \{I\}, \emptyset \rangle : \alpha \\
\langle \{J\}, \emptyset \rangle : \neg(\beta \wedge \gamma \wedge (\beta \rightarrow \alpha) \wedge (\gamma \rightarrow \neg\lambda)) \arrow[u, "\{I\}"'] \\
\langle \{K\}, \emptyset \rangle : \neg(\pi \wedge (\pi \rightarrow \neg\beta)) \arrow[u, "\{J\}"'] \\
\langle \{I\}, \emptyset \rangle : \neg(\lambda \wedge (\lambda \rightarrow \neg\pi)) \arrow[u, "\{K\}"']
\end{tikzcd}
$$

Complete Professional Debate

Figure 6.4: Type C Incomplete - Premise Reuse Prohibition

In any of these trees, an argument can never reuse the full support of its immediate parent because if it did then the support would have to be inconsistent; and hence not would not be a valid argument. My analysis shows that in any tree an argument should not be allowed to reuse even a subset of the support of its grandparent, with no other premises added, because if it did it would be reflected. Additionally I argue that the support of a grandparent's grandparent should not be allowed to be reused (even a subset thereof) because of reflection. This grandparent's grandparent constraint can be legitimately applied repeatedly all the way up to the root (or child of the root). However, as shown in the example, I hold that it is perfectly acceptable in professional debate to reuse the support of a great grandparent. For longer chains each legitimately reusable vertex's grandparent is also legitimate. While these reuse properties could be formally established and do constitute an area for further research, my point is that the (Besnard & Hunter, 2001) NRC does not accurately mirror professional debate. My point is made by the above Example 6.3.3; these additional observations are only made to reinforce that point.

This completes the identification of types of incompleteness in argument trees, allowing me to now move on to types of inordinateness.

## 6.4 Argument Trees are Inordinate

The second accuracy issue with argument trees is that they are inordinate. By inordinate I mean that unnecessary arguments are present. The word inordinate has a dictionary definition of 'unusually or disproportionately large; excessive', for example 'He placed an inordinate amount of butter on his bread'. I use the term inordinate to mean specifically that the tree contains more arguments than necessary to track the debate and hence contains unwanted redundancy.

### 6.4.1 Type A Inordinate - Root Reflected Canonical Undercuts

A reflected canonical undercut which is an immediate child of the root, reflected off other contradicting roots within the argument structure, see Example 6.4.1 is Type A Inordinate. The argument is counted once as a rebuttal of the root (existing as the root of another tree in the argument structure) and counted a second time as a reflected canonical undercut.

**Example 6.4.1. Type A Inordinate.** *Let* $\Delta = \{a{:}\beta, b{:}\beta \rightarrow \alpha, c{:}\gamma, d{:}\gamma \rightarrow \neg\alpha\}$. *Let* $I = \{a, b\}, J = \{c, d\}$. *Consequently* $I : \alpha, J : \neg\alpha, J : \neg(\beta \wedge (\beta \rightarrow \alpha)), I : \neg(\gamma \wedge (\gamma \rightarrow \neg\alpha)) \in \mathsf{arguments}(\Delta)$. *The root of the tree for* $\alpha$ *is* $I : \alpha$. *Likewise the tree for* $\neg\alpha$ *has a root of* $J : \neg\alpha$. *It follows that* $J : \neg(\beta \wedge (\beta \rightarrow \alpha)) \in \mathsf{reflectedCanonicalUndercuts}(I : \alpha, J : \neg\alpha, \Delta)$ *and* $I : \neg(\gamma \wedge (\gamma \rightarrow \neg\alpha)) \in \mathsf{reflectedCanonicalUndercuts}(J : \neg\alpha, I : \alpha, \Delta)$. *As these two immediate children of the roots are utterly predictable arguments these are inordinate trees. See Figure 6.5.*

$$\begin{array}{cccc} I{:}\alpha & J{:}\neg\alpha & \quad & \langle\{I\},\{J\}\rangle{:}\alpha \\ \uparrow & \uparrow & & \\ J{:}\neg(\beta \wedge (\beta \rightarrow \alpha) & I{:}\neg(\gamma \wedge (\gamma \rightarrow \neg\alpha) & & \end{array}$$

Inordinate Argument Trees — Ordinate Professional Debate

Figure 6.5: Type A Inordinate - Root Reflected Canonical Undercuts

Any argument structure $\langle \mathcal{P}, \mathcal{C} \rangle$ with a non-empty $\mathcal{P}$ and a non-empty $\mathcal{C}$ will contain reflected canonical undercuts. Here I refer back to my Definition 6.2.2 of Argument Structure, which is adapted from (Besnard & Hunter, 2001).

**Proposition 6.4.1.** *Let $\langle \mathcal{P}, \mathcal{C} \rangle$ be an argument structure, with nodes $u, v, w$ where $f(u), f(v), f(w) \in$ $\mathsf{arguments}(\Delta)$ where $\Delta$ is a knowledgebase of labelled assumption formulae.*

*If $\mathcal{P} \neq \emptyset$ and $\mathcal{C} \neq \emptyset$*

*then there exists $w$ child of $u$*

*such that $f(w) \in \mathsf{reflectedCanonicalUndercuts}(f(u), f(v), \Delta)$,*

*where $u$ is root of a tree in $\mathcal{P}$ and $v$ is root of a tree in $\mathcal{C}$.*

*Proof.* Let the root of an argument tree in $\mathcal{P}$ be $I : \alpha = f(v)$, likewise for $\mathcal{C}$ let $J : \neg\alpha = f(u)$, where $I:\alpha, J:\neg\alpha \in \mathsf{arguments}(\Delta)$ and $\Delta$ is a knowledgebase of labelled assumption formulae. Consequently there exists $K:\phi \in \mathsf{reflectedCanonicalUndercuts}(I:\alpha, J:\neg\alpha, \Delta)$. It follows that $K:\phi = f(w)$ always exists as $I:\alpha, J:\neg\alpha \in \mathsf{arguments}(\Delta)$ and $I:\alpha \in \mathsf{attacks}(J:\neg\alpha, \Delta)$. □

By symmetry, swapping around $v$ and $u$, it can be seen that just as their exists $x$ child of $v$, so also there will also exist $y$ child of $u$ such that $f(y) \in \mathsf{reflectedCanonicalUndercuts}(f(u), f(v), \Delta)$.

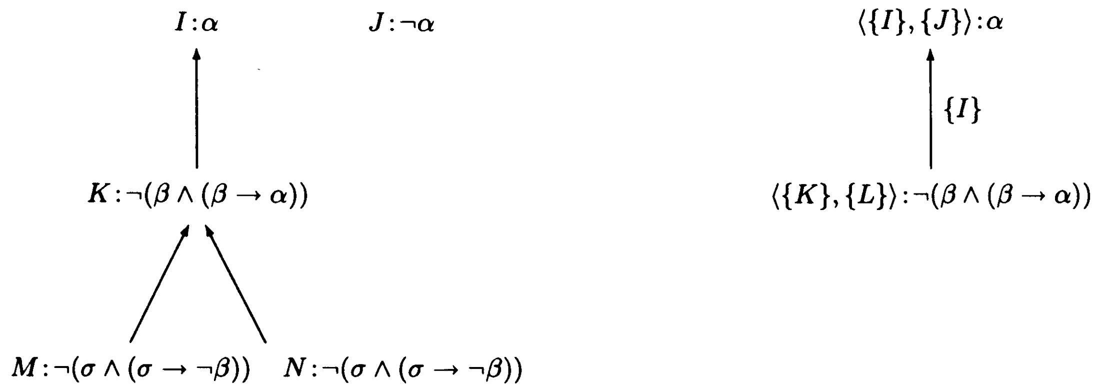
Another form of redundancy in an argument tree is from an expanded reflection of canonical undercuts off one or more off-tree direct argument rebuttals. By 'off-tree' I am referring to arguments which are in $\mathsf{arguments}(\Delta)$ but not in the vertices of $\mathsf{argumentTree}(I:\alpha, \Delta)$ or $\mathsf{debateTree}(\alpha, \Delta)$ (see Definition 6.7.7), depending upon context. Such 'off-tree' arguments are not part of the tree due to some part of the tree definition: they may be excluded by the NRC, or excluded as they are reflected, or simply because the definition does not deem them to be part of the debate. Such off-tree arguments can still have an influence on a debate, notably where they act as reflectors, relaying the effects to on-tree arguments across the tree. Expanded reflections of canonical undercuts off rebuttals may occur from a mixture of mono-pair and multi-pair enlargement, so I describe this as two mechanisms.

### 6.4.2 Type B Inordinate - Mono-Pair Enlarged Canonical Reflection

An enlarged reflection of canonical undercuts from an off-tree direct argument rebuttal, caused by a mono-pair enlarged reflection, see Example 4.6.2, is Type B Inordinate. The number of reflected canonical undercuts is greater than the number of off-tree rebuttals, hence each off-tree rebuttal is counted more than once. The following example illustrates Type B Inordinate behaviour by providing a graphical representation of a mono-pair enlarged reflection. The following example in focussing on the mono-pair enlargement does not diagram every argument in the trees.

**Example 6.4.2. Type B Inordinate.** *Let $\Delta = \{a : \beta, b : \beta \rightarrow \alpha, c : \pi, d : \pi \rightarrow \neg\alpha, e : \sigma, f : \sigma \rightarrow \neg\beta, g : \beta \wedge \neg\sigma, h : (\beta \rightarrow \alpha) \wedge \neg\sigma\}$. Let $I = \{a, b\}, J = \{c, d\}, K = \{e, f\}, L = \{g, h\}, M = \{g\}, N = \{h\}$. Therefore $I : \alpha, J : \neg\alpha, K : \neg\beta, K : \neg(\beta \wedge (\beta \rightarrow \alpha)), L : \beta \wedge (\beta \rightarrow \alpha), M : \neg(\sigma \wedge (\sigma \rightarrow \neg\beta)), N : \neg(\sigma \wedge (\sigma \rightarrow \neg\beta)) \in$ $\mathsf{arguments}(\Delta)$. It follows that $\{M, N\} : \neg(\sigma \wedge (\sigma \rightarrow \neg\beta)) \in$ $\mathsf{monoPairEnlargedReflection}(J : \neg(\beta \wedge (\beta \rightarrow \alpha)), L : \beta \wedge (\beta \rightarrow \alpha), \Delta)$. It might be thought that*

*$L:\neg(\sigma \wedge (\sigma \rightarrow \neg\beta)) \in$ reflectedCanonicalUndercuts$(K:\neg(\beta \wedge (\beta \rightarrow \alpha), L:\beta \wedge (\beta \rightarrow \alpha), \Delta)$, but that is not the case as $L:\neg(\sigma \wedge (\sigma \rightarrow \neg\beta))$ is not even an argument as its not minimal. Also note that $L:\beta \wedge (\beta \rightarrow \alpha)$ is not part of the argument tree for $\alpha$ and hence is 'off-tree'. See Figure 6.6.*




Inordinate Argument Trees　　Ordinate Professional Debate

Figure 6.6: Type B Inordinate - Mono-Pair Enlarged Canonical Reflection

In professional debate neither of the reflected undercuts, $M:\neg(\sigma \wedge (\sigma \rightarrow \neg\beta))$ and $N:\neg(\sigma \wedge (\sigma \rightarrow \neg\beta))$ would be allowed, but the direct rebuttal $L:\beta \wedge (\beta \rightarrow \alpha)$ would be. Mono-pair enlarged canonical undercut reflection is thus one of the several mechanisms that leads argument trees to contain superfluous arguments. It also endorses rejection of the notion that 'reflection is good because it leads to fewer arguments and is therefore more efficient or conservative' as in Type B Incompleteness; rejection because reflection can just as easily give rise to more arguments than less.

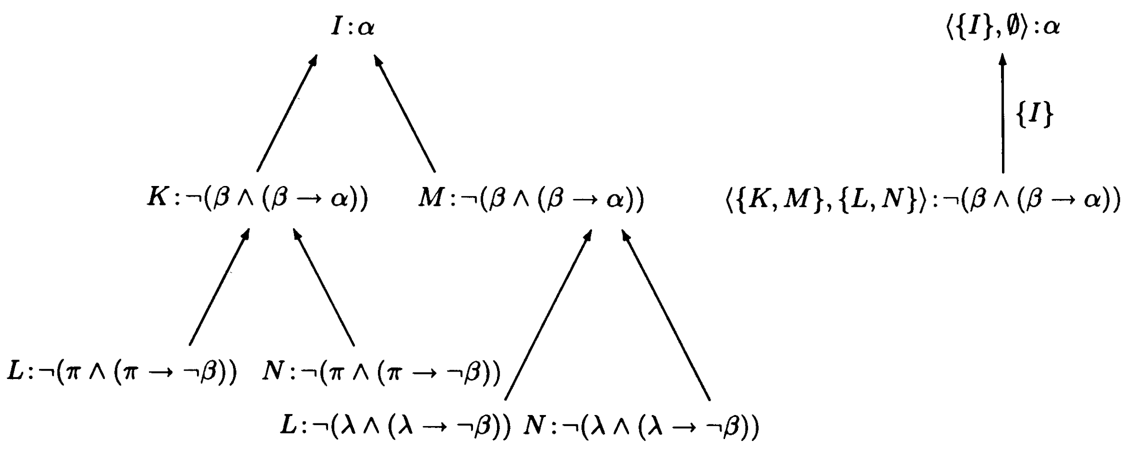
### 6.4.3 Type C Inordinate - Multi-Pair Enlarged Canonical Reflection

An enlarged reflection of canonical undercuts from off-tree direct argument rebuttals, caused by a multi-pair enlarged reflection, see Example 5.7.4, is Type C Inordinate. Again, the number of reflected canonical undercuts is greater than the number of off-tree rebuttals, leading again to over counting of the effect of the off-tree rebuttals.

**Example 6.4.3. Type C inordinate.** *Let $\Delta = \{a:\beta, b:\beta \rightarrow \alpha, c:\pi, d:\pi \rightarrow \neg\beta, e:\beta, f:\beta \rightarrow \alpha, g:\lambda, h:\lambda \rightarrow \neg\beta, i:\beta \rightarrow \alpha\}$. Also let $I = \{a,b\}, K = \{c,d\}, L = \{e,f\}, M = \{g,h\}, N = \{e,i\}$. Therefore $I:\alpha, K:\neg(\beta \wedge (\beta \rightarrow \alpha)), M:\neg(\beta \wedge (\beta \rightarrow \alpha)), L:\beta \wedge (\beta \rightarrow \alpha), N:\beta \wedge (\beta \rightarrow \alpha), L:\neg(\lambda \wedge (\lambda \rightarrow \neg\beta)), N:\neg(\lambda \wedge (\lambda \rightarrow \neg\beta)), L:\neg(\pi \wedge (\pi \rightarrow \neg\beta)), N:\neg(\pi \wedge (\pi \rightarrow \neg\beta)) \in$ arguments$(\Delta)$. The arguments $L:\beta \wedge (\beta \rightarrow \alpha)$ and $L:\beta \wedge (\beta \rightarrow \alpha)$ are direct rebuttals; for an argument tree they are 'off tree'. See Figure 6.7.*

### 6.4.4 Type D Inordinate - Reflected Confirming Argument

Another form of redundancy is when an exact reflected confirming argument and a reflected canonical undercut both attempt to stand for a single direct argument rebuttal off-tree reflector. The reflected confirming argument can be viewed as a reflection flowing up the tree (i.e. the reflection is nearer to the




Inordinate Argument Tree Ordinate Professional Debate

Figure 6.7: Type C Inordinate - Multi-Pair Enlarged Canonical Reflection

root than the reflector), whilst the reflected canonical undercut is a reflection flowing down the tree (i.e. the reflection is further from the root than the reflector).

This taxonomy of kinds of incomplete and inordinate behaviour is somewhat arbitrary as if increasingly subtle distinctions were made further kinds of undesirable behaviour could be delineated. For example, an enlarged reflected confirming argument where there is no reflected canonical undercut from the direct argument rebuttal due to the NRC would be another kind of inordinate behaviour. Many reflected confirming arguments would be attempting to stand for a single off-tree direct argument rebuttal.

## 6.5 Three Further Behaviours

Argument trees are inchoate (as defined below), are sensitive to the way the knowledgebase is written and are asymmetrical.

### 6.5.1 Argument Trees are Inchoate

The third undesirable property of argument trees is that they are inchoate. I define inchoate as meaning that those arguments present in the tree do not accurately represent those arguments missing from the tree.

Just suppose that all reflections were exact reflections and thus that the counting of reflected arguments would accurately stand for the counting of direct arguments. Or imagine that some corrective lens could be constructed so that enlarged and reduced reflected arguments were again transformed back into an exact mirror of the originals – at least to satisfy the purposes of existence and counting. In either of these situations, judging of debates would still be arbitrary – because the reflected arguments can behave differently from their reflectors. This I call inchoate behaviour.

The dictionary definition of inchoate means 'not fully formed or developed' and the term definitely does not extend to mean chaotic, confused or incoherent. I use the term to clarify the situation where an argument A is proposed as a proxy, representative or substitute for an argument B. However B can only proxy for A if B is a fully formed, fully developed version of A. If A does not behave in the same way as B then A is an inchoate form of B and thus not a robust proxy for B.

I now give an example to show reflected arguments that behave differently from their reflectors. Here a reflected canonical undercut behaves differently from the off-tree direct argument rebuttal it was reflected off. The terms 'reinstated' and 'undefeated' are used here informally; the latter part of this example is revisited more formally in Section 7.4.

**Example 6.5.1. Inchoate.** *Let* $\Delta = \{a:\beta, b:\beta \to \alpha, c:\gamma, d:\gamma \to \neg\beta, e:\beta \wedge \pi \wedge \neg\gamma, f:\pi \to (\beta \to \alpha), g:\neg\beta \vee \neg\pi \vee \gamma \vee \neg(\pi \to (\beta \to \alpha))\}$. *Let* $I = \{a, b\}, J = \{c, d\}, K = \{e, f\}, L = \{e\}, M = \{g\}$. *Therefore* $I:\alpha, J:\neg(\beta \wedge (\beta \to \alpha)), K:\beta \wedge (\beta \to \alpha), L:\neg(\gamma \wedge (\gamma \to \neg\beta)), M:\neg(\beta \wedge \pi \wedge \neg\gamma \wedge (\pi \to (\beta \to \alpha))) \in \mathsf{arguments}(\Delta)$. *Consequently* $J:\neg(\beta \wedge (\beta \to \alpha)) \in \mathsf{directCanonicalUndercuts}(I:\alpha, \emptyset, \Delta)$ *and* $K : \beta \wedge (\beta \to \alpha) \in \mathsf{directArgumentRebuttals}(J : \neg(\beta \wedge (\beta \to \alpha)), I : \alpha, \Delta)$. *Furthermore* $L : \neg(\gamma \wedge (\gamma \to \neg\beta)) \in \mathsf{reflectedCanonicalUndercuts}(J : \neg(\beta \wedge (\beta \to \alpha)), K : \beta \wedge (\beta \to \alpha), \Delta)$. *Additionally* $M:\neg(\beta \wedge \pi \wedge \neg\gamma \wedge (\pi \to (\beta \to \alpha))) \in \mathsf{directCanonicalUndercuts}(K:\beta \wedge (\beta \to \alpha), J:\neg(\beta \wedge (\beta \to \alpha)), \Delta)$. *However* $\mathsf{canonicalUndercuts}(L:\neg(\gamma \wedge (\gamma \to \neg\beta)), \Delta) = \emptyset$. *So for the argument tree the attack chain is* $L:\neg(\gamma \wedge (\gamma \to \neg\beta))$ *attacks* $J:\neg(\beta \wedge (\beta \to \alpha))$ *attacks* $I:\alpha$. *The chain contains three arguments. The attack on* $I:\alpha$ *is direct, the other reflected. For the professional debate, the chain is* $M:\neg(\beta \wedge \pi \wedge \neg\gamma \wedge (\pi \to (\beta \to \alpha)))$ *attacks* $K:\beta \wedge (\beta \to \alpha)$ *attacks* $J:\neg(\beta \wedge (\beta \to \alpha))$ *attacks* $I:\alpha$. *Thus this chain contains four arguments and all attacks are direct. The attack of* $K:\beta \wedge (\beta \to \alpha)$ *on* $J:\neg(\beta \wedge (\beta \to \alpha))$ *in the debate tree is a direct rebuttal, whereas all the other attacks for both trees in this example are canonical undercuts. Imagine a simple judge function that takes i) the existence of an attack to mean that the attacked argument is defeated and ii) an attack on an attack as reinstatement and thus not defeat (i.e. concordant with the intuition of (Dung, 1995). So in the argument tree* $I:\alpha$ *is not defeated, whereas in the professional debate* $I:\alpha$ *is defeated - exactly the opposite outcome. Therefore argument trees are inchoate, in that they do not behave as intuitively expected. See Figure 6.8.*

$$\begin{tikzcd} I:\alpha \\ J:\neg(\beta \wedge (\beta \to \alpha)) \arrow[u] \\ L:\neg(\gamma \wedge (\gamma \to \neg\beta)) \arrow[u] \end{tikzcd}$$

Inchoate Argument Tree

$$\begin{tikzcd} \langle\{I\}, \emptyset\rangle:\alpha \\ \langle\{J\}, \{K\}\rangle:\neg(\beta \wedge (\beta \to \alpha)) \arrow[u, "\{I\}"'] \\ \langle\{M\}, \emptyset\rangle:\neg(\beta \wedge \pi \wedge \neg\gamma \wedge (\pi \to (\beta \to \alpha))) \arrow[u, "\{K\}"'] \end{tikzcd}$$

Choate Professional Debate

Figure 6.8: Inchoate - Present Arguments Do Not Represent Absent Ones

This example, in particular its idea of a simple judge function, is revisited and formalised in Chapter 8 in Examples 7.5.1 and 7.5.2.

I now give a proposition that examines the notion that further down the tree one could conjecture the existence of reflected canonical undercuts where the reflector is the grandparent canonical undercut. However such a conjecture does not hold as the NRC of Definition 6.2.1 rules them out.

**Proposition 6.5.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae, let* $\mathsf{argumentTree}(I\!:\!\alpha, \Delta) = \langle \mathcal{V}, \mathcal{E}, f, \mathsf{arguments}(\Delta) \rangle$ *be an argument tree and let* $u, v \in \mathcal{V}$.

$$\textit{If } f(u) \in \mathsf{canonicalUndercuts}(f(v)) \textit{ and } \langle v, u \rangle \in \mathcal{E}$$
$$\textit{then } \mathsf{reflectedCanonicalUndercuts}(f(u), f(v), \Delta) = \emptyset.$$

*Proof.* The NRC used in the argument tree definition clause 2. is 'For no node $\langle \Phi, \beta \rangle$ with ancestor nodes $\langle \Phi_1, \beta_1 \rangle, \ldots, \langle \Phi_n, \beta_n \rangle$ is $\Phi$ a subset of $\Phi_1 \cup \ldots \cup \Phi_n$'. Now suppose $\mathsf{reflectedCanonicalUndercuts}(K\!: \phi, L\!:\!\gamma, \Delta) \neq \emptyset$. However, for such a $M : \psi \in \mathsf{reflectedCanonicalUndercuts}(K : \phi, L : \gamma, \Delta)$ it would have to be the case that $M \subseteq L$ as that is a requirement for reflection. Having $M \subseteq L$ is the same as having a node with label M having an ancestor with label L, which is forbidden by the NRC. So the supposition is false and thus if $K\!:\!\phi \in \mathsf{canonicalUndercuts}(L\!:\!\gamma) \in \mathcal{E}, f(u) = K\!:\!\phi, f(v) = L\!:\!\gamma$ and $\langle u, v \rangle \in \mathcal{E}$ then it is always the case that $\mathsf{reflectedCanonicalUndercuts}(K\!:\!\phi, L\!:\!\gamma, \Delta) = \emptyset$. □

So the form of NRC in the definition of argument tree does not just prevent repetition of arguments it also rules out reflected canonical undercuts that have a canonical undercut as their reflector. What it does not rule out is reflected canonical undercuts having a rebuttal as their reflector, i.e. Type A Inordinance. As there are no rebuttals in the tree below the root then the only direct argument rebuttals in the tree are at the root.

### 6.5.2 Sensitivity To The Way the Knowledgebase is Written

The fourth undesirable behaviour of argument trees, beyond incomplete, inordinate and inchoate, is that they are sensitive to the way that the knowledgebase is written. This sensitivity that argument trees have to the way that the knowledgebase is written down means that two logically equivalent knowledgebases tracked as argument trees can give entirely different results. For example $\{a\!:\!\alpha \wedge \beta\} \subseteq \Delta$, will give a different debate outcome from the logically equivalent $\{a\!:\!\alpha, b\!:\!\beta\} \subseteq \Delta$.

Consider the situation where two police clerks take the same verbal statement from a witness. One clerk encodes the statement as $\Delta = \{a : \alpha, b : \beta, c : \gamma, \ldots\}$ whereas the second uses $\wedge$I to write the logically equivalent $\Delta = \{a\!:\!\alpha \wedge \beta \wedge \gamma \wedge \ldots\}$ . In the second case the witness would not be allowed to defend themselves against any attack as the argument tree NRC would prohibit it. In a professional debate, however, assumptions can be reused as long as a new claim is being made. So an argument tree can result in arbitrary judgements because it is sensitive to the way the knowledgebase is written down.

In defence of argument trees here, it should be pointed out that all mincon systems display syntax sensitivity - as this class of behaviour is commonly called. The framework of this thesis too is not immune from the effects of syntax sensitivity, so the open question, meriting further research, is the matter of degree.

### 6.5.3 Pro and Con Reflections Do Not Cancel

The final point is not an undesirable behaviour, but rather the analysis of a possible (but incorrect) intuition about reflection in these trees. It might be thought that even though reflections exist in argument structures that the reflections would be symmetrical and thus that those for $\alpha$ balance out and cancel those for $\neg\alpha$ giving no net effect. The analysis of Section 4.6.1, however, shows that while reflection has a symmetry of existence it does not have a symmetry of cardinality.

Relating this asymmetry back to the current theme of analysis, I now draw Example 4.6.3, see below in Figure 6.9, as a pair of argument trees, also known as an argument structure. Study of the arguments derivable from that example's $\Delta$ (with $|\Delta| = 4$) shows that there are no other arguments for $\beta$ or $\neg\beta$ and hence no other argument trees that belong in this argument structure.

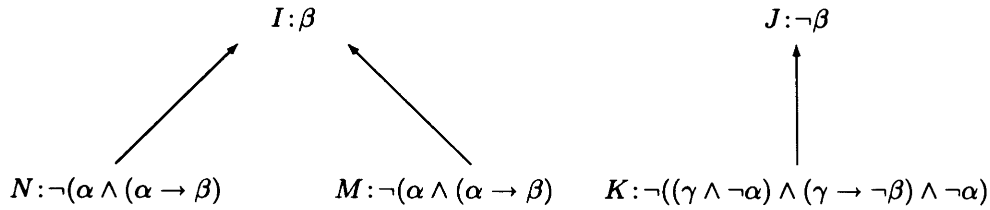



Figure 6.9: Asymmetry - of Reflected Cardinality in an Argument Structure

This diagram shows that the reflected canonical undercuts to be found in a pair of argument trees can be asymmetrical. Thus any suggestion that the reflected canonical undercuts cancel each other out is not well founded.

That concludes the comparative analysis of argument trees and professional debate trees. I now move on to the formalisation of the debate tracking tree as a form that better meets the criteria (the debate, the whole debate and nothing but the debate) outlined so far.

## 6.6 Rationale for Matt Opaque Contradiction Trees

I use the term 'matt' tree or graph to mean one devoid of Type I and II Reflection and 'opaque' to mean free of Type III and IV Reflection. The structures I have mentioned so far, notably including defeat graphs (Section 6.2), defeasible trees (Section 6.2), argument trees (Section 6.2.1) and argument structures (Section 6.2.1), would appear to not be matt or opaque. The goal of this chapter is to define robustly a debate tracking tree that is both matt and opaque (see Section 6.7). The word 'graphs' would have been more general in the title of this chapter, with the understanding that trees are a kind of graph, however to prevent the infinite cycles that can arise in graph traversal I will focus exclusively on trees from here on, as does the bulk of the literature.

I start by summarising the key points of my critique of the argument trees and argument structures approach, which may be extended to critique other argument aggregation trees in the literature, notably (Pollock, 1992; Simari et al., 1994; Prakken & Sartor, 1997; Amgoud & Cayrol, 2002; García & Simari, 2004; Dung et al., 2006). This informal analysis, running through the rest of this section, moves from critiquing the literature to the introduction of a new tree.

I have presented examples and discussion showing that argument trees contain reflections (see Examples 6.4.1, 6.4.2 and 6.4.3) and thus that they are not matt opaque trees. The reflections can thus give rise to 'double counting' in the argument-aggregation functions of (Besnard & Hunter, 2001) (which they call categoriser and accumulator functions). This double, or even multiple counting, will be of an argument and its reflections. My examples also show that even though their NRC removes some reflections it by no means removes all.

My analysis shows that the argument tree NRC attempts to do two jobs, namely to prevent cycles (so as to make the tree finite) and to prevent reflection. I would argue that it over achieves at the first job and underachieves at the second, suggesting that using a single tool to do two jobs here yields sub-optimal results on both. Consequently I define, shortly, a separate (and different) NRC, which is not burdened with the job of removing any reflections and is thus a) more akin to professional debate and b) able to make the tree finite. The NRC of (Besnard & Hunter, 2001) introduces somewhat puzzling behaviour (see my Example 6.3.3), however none of these NRCs has a strong theoretical basis so from that perspective they are equally valid. The (Besnard & Hunter, 2001) NRC is fully effective at removing grandparent reflections. Reflections from rebuttals coming from the roots of other trees, however, are not removed by the NRC.

Base Proposition Four, also called the grandparent relationship, shows that in a tree of arguments reflections can occur from any node to the level of its grandchildren. Prior to Base Proposition Four, with only Base Propositions One and Two at hand one might have conjectured that all reflections were off-tree rebuttals and thus somehow not necessary to include in the debate. Base Proposition Four shows such a conjecture to be less than the full story as these off-tree rebuttals cause reflections back into the main body of the tree, regardless whether the off-tree rebuttals are formally part of the tree or not. The immediate children of the root of an argument tree or will also be subject to Base Proposition One reflections.

Reflections can be enlarged, reduced or distorted. Any attempt to rationalise the counting of reflected arguments with scaling factors is problematic as the number of reflections can be a smaller or larger number than the original. Even for one confirmation, parts of it can be enlarged and parts reduced creating distorted reflections, ensuring that *scaling-based approaches to judgement would be complex and problematic*. I am not claiming that such scaling-based approaches are impossible, but rather that any such definition would be long, inelegant and hard to understand by professional debaters.

Argument tree edges are a mix of direct and reflected canonical undercuts. The reflected ones are repetitions of other attacks in the tree. Therefore it is sensible to exclude all reflected canonical undercuts from any such tree. The remaining edges would thus all be direct canonical undercuts.

My use of a different undercut definition is, I argue, helpful. I hold that (Besnard & Hunter, 2001)'s canonical undercuts do not tell the full story of human debating, however they do provide a robust undercutting mechanism that has the advantage of being simpler than mine while still covering the majority of cases in professional debate.

Argument trees exclude direct argument rebuttals - apart from at the root. Direct rebuttals exist and do not arise from reflection. In evaluating or judging a debate, direct rebuttals have or should have a definite effect on the outcome. They should be included. The reasoning underpinning (Besnard & Hunter, 2001) would appear to be that they do address argument rebuttals by means of reflection. However if one accepts the earlier point about distorted reflections rendering reflection-unaware judging schemes impractical, then this suggestion of leveraging Type I Reflections to represent rebuttals is also likely to be impractical.

Direct argument rebuttals are also reflectors for Type III and IV Reflections. These reflections are subject to distortion, rendering them too as unhelpful to represent rebuttals. If no distortion was present Type III and IV Reflections could have been sound representatives of the direct rebuttals; however that is not the case. If distortion was absent direct rebuttals might be assesed indirectly by looking at the arguments that reflect off them. Both Type I and II Reflections are not limited to exact reflections and thus proscribe indirect counting or assessment.

Therefore I conclude that *it is necessary to include all direct argument rebuttals in the tree*. Consequently, to build on the discussion of Section 3.9, it is clear that the most practical attack assumption is the union of direct rebuttal and direct undercut. This inclusion of direct rebuttal is necessary to make counting-based judges viable and, it would appear, to prevent all judges from giving incorrect answers. Even existential judges, which do not use integers to count arguments, are sensitive to the existence of arguments involved in attacks and thus sensitive to reflection. Note that although the edges of the trees of (Pollock, 1992; Prakken & Sartor, 1997; García & Simari, 2004) encompass both rebuttal and undercut they also include both direct and reflected arguments making them somewhat different from my proposal.

There are two ways that this inclusion of direct argument rebuttals could be achieved, one enhancing the vertex definition, the other enhancing the edges. If vertex enhancement is adopted then the tree nodes need to be contradictions. Such an approach is arguably more compact and succinct, yielding a simpler notation. It could be argued, however, that vertex enhancement is making an excessive or perhaps artificial distinction between rebuttal and undercut. If the vertices are contradictions then the edges would need to be direct undercuts.

If edge enhancement is used then the edges would be *direct attacks*, where a direct attack is defined as either a direct argument rebuttal or a direct canonical undercut. Edge enhancement has appeal as it is closer to an intuitive view of attack. This approach has a weaknesses in that when the edges include preclusive undercuts then what the edge labels for the rebuttal arcs should be becomes unclear. One could just use blank for rebuttal edge labels, but even that is unintuitive. It could also be argued that there is a weakness in that the two nodes and one edge representing two confirmations and their direct rebuttals between the confirmations could be more succinctly represented by the functionally equivalent and more compact notation of the contradiction - of Definition 2.6.1.

A point that applies to either approach is that any judge function traversing any of these refined trees would have to know the difference between undercut and rebuttal because multi-target preclusions need counting in a different way from mono-target preclusions. In either case there is no need for argument structures, or any kind of pairing of trees or forests (i.e. one or more for $\alpha$ plus one or more for $\neg\alpha$). A tree including rebuttal (using either the edge or vertex approach) would be just a single tree for the entire debate, which is a strong advantage for visualisation, comprehension and simplicity of judgement. Regarding the edges in an edge enhanced tree, these direct attacks could arise from undercut or rebuttal. The undercuts are preclusive undercuts and similarly the rebuttals are confirmation rebuttals. Thus the attacks are direct confirmation attacks. Direct confirmation attack removes all Type I and Type II Reflections. It does not however remove any Type III Reflections - which require a different approach for their removal. In either case (vertex or edge enhancement) it is necessary to exclude Type III and IV Reflections (that is the up tree reflections coming off direct rebuttals and appearing as confirming arguments).

Given this motivation it is now possible to progress to a robust definition of a reflection-free tree where direct rebuttals are included at every level of the debate (not just the root) thus adopting the vertex enhanced approach.

## 6.7 Definition of Debate Trees

I select the approach of vertex-enhanced trees as superior to edge-enhanced trees for tracking debates, based on the rationale of the last section. Thus I leave edge-enhanced trees as an area for further research. My vertex-enhanced trees have each vertex map to a contradiction. Thus some of the attacks are intra-node, whilst others are inter-node. The intra-node attacks are rebuttals, while the inter-node attacks are undercuts.

Six helper functions are required to succinctly define the debate tree. These retrace the steps of the last three chapters, but manipulate contradictions instead of confirmations and build on Definition 6.7.1 of preclusive undercut between contradictions. Just as the analysis of reflection and undercut moved from the level of individual arguments to confirmations, it is necessary now, for the sake of complete and ordinate debate tracking, to move on again to the level of contradictions. The six helper functions that lead to the debate tree, Definition 6.7.7, are:

1. the preclusive undercuts between a pair of contradictions, Definition 6.7.1,
2. the matt function that removes or prunes out Type I and Type II Reflections, Definition 6.7.2,
3. the opaque function that prunes out Type III and IV Reflections, Definition 6.7.3,
4. a defending path function that aids in identifying the ancestors of a vertex, Definition 6.7.4,
5. an ancestors function required to support the definition of the NRC of the debate tree, Definition 6.7.5 and finally
6. an ancestor arguments function which provides the set of all of the arguments in the ancestor vertices, Definition 6.7.6.

The naming of the opaque and matt functions continues the optical metaphor used so far with reflection. For an arbitrary vertex, $v$, in a tree, Type I and II Reflections allow one to partially observe vertices nearer to the root than $v$. The analogy here is that if $v$ was painted with matt paint then it would not give rise to such reflections. The situation in an argument tree thus is akin to gloss paint which gives rise to reflections of things above the painted layer, closer to the observer. Type III and IV Reflections allow one to see something of vertices further from the root – consequently if $v$ was painted with opaque paint it would not allow one to see through to the descendants of $v$. For argument trees the situation is thus akin to transparent paint where things below the painted layer, further from the observer, become visible. Thus to avoid redundancy in the tracked debate the paint to use in this metaphor is opaque matt paint.

The first helper function I now define is a variation of the original preclusive undercuts function, which was Definition 3.4.2, this time between contradictions rather than between confirmations.

**Definition 6.7.1.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae, let* $\langle \mathrm{V}, \mathrm{W} \rangle : \phi, \langle \mathrm{X}, \mathrm{Y} \rangle : \alpha \in \blacklozenge(\Delta)$ *and let* $\mathrm{Z} \in \mathcal{C}$. *The set of* **preclusive undercuts between contradictions** *attacking* $\langle \mathrm{X}, \mathrm{Y} \rangle : \alpha$ *derivable from* $\Delta$, *denoted* $\mathsf{preclusiveUndercutsBetweenContradictions}(\langle \mathrm{X}, \mathrm{Y} \rangle : \alpha, \Delta)$ *is such that:*

$$
\begin{aligned}
&\mathsf{preclusiveUndercutsBetweenContradictions}(\langle \mathrm{X}, \mathrm{Y} \rangle : \alpha, \Delta) \\
&= \{\langle\, \langle \mathrm{V}, \mathrm{W} \rangle : \phi,\ \mathrm{Z},\ \langle \mathrm{X}, \mathrm{Y} \rangle : \alpha \,\rangle \mid \\
&\qquad \textit{If } \mathrm{Z} \subseteq \mathrm{X} \textit{ then } \langle \mathrm{V} : \phi,\ \mathrm{Z},\ \mathrm{X} : \alpha \rangle \in \mathsf{preclusiveUndercuts}(\mathrm{X} : \alpha, \Delta), \\
&\qquad\qquad \textit{else } \langle \mathrm{V} : \phi,\ \mathrm{Z},\ \mathrm{Y} : \neg\alpha \rangle \in \mathsf{preclusiveUndercuts}(\mathrm{Y} : \neg\alpha, \Delta), \\
&\qquad \textit{and } \mathrm{W} : \neg\phi = \mathsf{top}(\mathsf{confirmationRebuttals}(\mathrm{V} : \phi, \Delta))\}.
\end{aligned}
$$

For a preclusive undercut between two contradictions, which is of the form of a 3-tuple $\langle\, \langle \mathrm{V}, \mathrm{W} \rangle : \phi,\ \mathrm{Z},\ \langle \mathrm{X}, \mathrm{Y} \rangle : \alpha \,\rangle$, I call $\langle \mathrm{V}, \mathrm{W} \rangle : \phi$ the **attacking contradiction**, I call $\mathrm{V} : \phi$ the **attacking confirmation**, I call Z the **target confirmation label** and I call $\langle \mathrm{X}, \mathrm{Y} \rangle : \alpha$ the **defending contradiction**. If $\mathrm{Z} \subseteq \mathrm{X}$ then I call $\mathrm{X} : \alpha$ the **defending confirmation**, else $\mathrm{Z} \subseteq \mathrm{Y}$ and I call $\mathrm{Y} : \alpha$ the **defending confirmation**. I call the contradiction $\langle \mathrm{V}, \mathrm{W} \rangle : \phi$ a **preclusive undercut** of a contradiction $\langle \mathrm{X}, \mathrm{Y} \rangle : \alpha$ iff $\langle \langle \mathrm{V}, \mathrm{W} \rangle : \phi, \mathrm{Z}, \langle \mathrm{X}, \mathrm{Y} \rangle : \alpha \rangle \in \mathsf{preclusiveUndercutsBetweenContradictions}(\langle \mathrm{X}, \mathrm{Y} \rangle : \alpha, \Delta)$.

This definition is not essentially a new kind of undercut. Rather, it is exactly the same preclusive undercut, or process of attack, from earlier, now with more information on the context of the attacker and the attacked.

One feature of the definition of preclusive undercut between contradiction is that attacks on reflexive arguments, Definition 2.4.7, appear twice – as illustrated in the following example.

**Example 6.7.1. Preclusive undercut between contradictions.** *Let* $\Delta = \{a : \alpha, b : \gamma, c : \gamma \to \neg\alpha\}$ *and* $I = \{a\}, J = \{b, c\}$. *Then* $I : \alpha, J : \neg\alpha \in \mathsf{arguments}(\Delta)$ *where* $I : \alpha$ *is a reflexive argument. Furthermore* $\{I\} : \alpha, \{J\} : \neg\alpha \in \lozenge(\Delta)$. *Thus* $\langle \{I\}, \{J\} \rangle : \alpha, \langle \{J\}, \{I\} \rangle : \neg\alpha \in \blacklozenge(\Delta)$. *It can be seen that* $J : \neg\alpha \in \mathsf{canonicalUndercuts}(I : \alpha, \Delta)$ *and also* $J : \neg\alpha \in \mathsf{rebuttals}(I : \alpha, \Delta)$. *So* $\langle \{J\} : \neg\alpha, \{I\}, \{I\} : \alpha \rangle \in \mathsf{preclusiveUndercuts}(\{I\} : \alpha, \Delta)$ *and also* $\{J\} : \neg\alpha \in \mathsf{confirmationRebuttals}(\{I\} : \alpha, \Delta)$. *So now, using Definition 6.7.1, it is clear that* $\langle\ \langle \{J\}, \{I\} \rangle : \neg\alpha,\ \{I\}\ , \langle \{I\}, \{J\} \rangle : \alpha\ \rangle \in \mathsf{preclusiveUndercutsBetweenContradictions}(\langle \{I\}, \{J\} \rangle : \alpha, \Delta)$. *The reflexive argument* $\{I\} : \alpha$ *is attacked once by* $\{J\} : \neg\alpha$ *visible in the defending contradiction, where the attack is a rebuttal, and again* *by* $\{J\} : \neg\alpha$ *in the attacking contradiction where the attack is an undercut. There is also a further reflection in this example, not due to reflexivity, where* $I : \alpha$ *appears a second time in the attacking contradiction.*

This feature of reflexive arguments appearing twice is not a problem as removing reflections removes the second (i.e. if traversing from root to leaf, the undercutting) occurrence. The instance of $J:\neg\alpha$ that undercuts $I:\alpha$ is a reflection of the instance of $J:\neg\alpha$ that is rebutted by and rebuts $I:\alpha$, so $J:\neg\alpha \in \mathsf{reflectedAttacks}(I:\alpha, J:\neg\alpha, \Delta)$.

The following matt function, thus, not only removes reflected arguments, but also double attacks arising from reflexivity. Given a child node and the context of its parent (it will have only one parent, being in a tree), it is possible to filter that child node to yield direct arguments only and no Type I or II Reflections. The definitions reused here are Definition 5.3.2 for maximal direct preclusive undercuts and Definition 5.6.2 for maximal direct confirmation rebuttal.

**Definition 6.7.2.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae and let* $\langle\, \langle \mathrm{V}, \mathrm{W}\rangle : \phi,\ \mathrm{Z},\ \langle \mathrm{X}, \mathrm{Y}\rangle : \alpha \,\rangle$, $\langle\, \langle \mathrm{T}, \mathrm{U}\rangle : \phi,\ \mathrm{Z},\ \langle \mathrm{X}, \mathrm{Y}\rangle : \alpha \,\rangle \in \mathsf{preclusiveUndercutsBetweenContradictions}(\langle \mathrm{X}, \mathrm{Y}\rangle : \alpha,\ \Delta)$. *The* **matt** *function, denoted* $\mathsf{matt}(\,\langle\, \langle \mathrm{V}, \mathrm{W}\rangle : \phi,\ \mathrm{Z},\ \langle \mathrm{X}, \mathrm{Y}\rangle : \alpha \,\rangle, \Delta)$, *given a preclusive undercut between two contradictions, removes Type I and II Reflections from the attacking contradiction, such that:*

$$
\begin{aligned}
&\mathsf{matt}(\langle\, \langle \mathrm{V}, \mathrm{W}\rangle : \phi,\ \mathrm{Z},\ \langle \mathrm{X}, \mathrm{Y}\rangle : \alpha \,\rangle, \Delta) \\
&\quad = \{\langle \mathrm{T}, \mathrm{U}\rangle : \phi \in \blacklozenge(\Delta) \mid \\
&\qquad\qquad \textit{if } \mathrm{Z} \subseteq \mathrm{X} \\
&\qquad\qquad\qquad \textit{then } \langle \mathrm{T} : \phi, \mathrm{Z}, \mathrm{X} : \alpha\rangle \in \mathsf{maximalDirectPreclusiveUndercuts}(\mathrm{X} : \alpha, \Delta), \\
&\qquad\qquad\qquad\qquad \mathrm{U} : \neg\phi = \mathsf{maximalDirectConfirmationRebuttal}(\mathrm{V} : \phi, \mathrm{X} : \alpha, \Delta)), \\
&\qquad\qquad\qquad \textit{else } \langle \mathrm{T} : \phi, \mathrm{Z}, \mathrm{Y} : \neg\alpha\rangle \in \mathsf{maximalDirectPreclusiveUndercuts}(\mathrm{Y} : \neg\alpha, \Delta)), \\
&\qquad\qquad\qquad\qquad \mathrm{U} : \neg\phi = \mathsf{maximalDirectConfirmationRebuttal}(\mathrm{V} : \phi, \mathrm{Y} : \neg\alpha, \Delta)\}.
\end{aligned}
$$

Thus the input to the matt function is a single preclusive undercut between contradictions, $\langle\ \langle \mathrm{V}, \mathrm{W}\rangle : \phi,\ \mathrm{Z},\ \langle \mathrm{X}, \mathrm{Y}\rangle : \alpha$, (plus the knowledgebase $\Delta$) and the output is a single contradiction $\langle \mathrm{T}, \mathrm{U}\rangle : \phi$, such that $\mathrm{T} \subseteq \mathrm{V}$ and $\mathrm{U} \subseteq \mathrm{W}$. If $\mathrm{T} \subsetneq \mathrm{V}$ then I say V has been **pruned** to yield T. Likewise if $\mathrm{U} \subsetneq \mathrm{W}$ then I say W has been **pruned** to yield U. This process of pruning removes Type I and Type II Reflections from the input $\langle \mathrm{V}, \mathrm{W}\rangle : \phi$, thereby ensuring that all of the arguments in the output $\langle \mathrm{T}, \mathrm{U}\rangle : \phi$ are direct. I say that the members of $\mathrm{V}\backslash\mathrm{T}$ and $\mathrm{W}\backslash\mathrm{U}$ have been pruned out. If $\mathrm{T} = \mathrm{V}$ and $\mathrm{U} = \mathrm{W}$ then no pruning is needed.

The function is defined to consider a chain of two contradictions, where the attacking contradiction is $\langle \mathrm{V}, \mathrm{W}\rangle : \phi$ and the attacked contradiction is $\langle \mathrm{X}, \mathrm{Y}\rangle : \alpha$. The definition does not explicitly state or require that these contradictions are in a tree, however visualising them as such can help understanding. Thus, the matt function can be viewed as considering an arbitrary vertex $\langle \mathrm{X}, \mathrm{Y}\rangle : \alpha$ in a tree of contradictions, along with one of its children $\langle \mathrm{V}, \mathrm{W}\rangle : \phi$. The input tree contains reflection. The function then describes how to remove some of the arguments, i.e. the reflected arguments, from the child vertex $\langle \mathrm{V}, \mathrm{W} \rangle : \phi$ to yield the equivalent direct child vertex $\langle \mathrm{T}, \mathrm{U} \rangle : \phi$. Applying this definition repeatedly, in a recursive fashion, prunes out all of the Type I and II Reflections from a tree containing reflection, yielding a matt tree.

The definition has been written with a certain economy of style, whereby although there are two defending confirmations $\mathrm{X} : \alpha$, $\mathrm{Y} : \neg\alpha$ in $\langle \mathrm{X}, \mathrm{Y} \rangle : \alpha$ there is only one target confirmation label Z. For both of the input attacks, $\langle \langle \mathrm{V}, \mathrm{W} \rangle : \phi, \mathrm{Z}, \langle \mathrm{X}, \mathrm{Y} \rangle : \alpha \rangle \in$ preclusiveUndercutsBetweenContradictions$(\langle \mathrm{X}, \mathrm{Y} \rangle : \alpha, \Delta)$ and the output attack $\langle \langle \mathrm{T}, \mathrm{U} \rangle : \phi, \mathrm{Z}, \langle \mathrm{X}, \mathrm{Y} \rangle : \alpha \rangle \in$ preclusiveUndercutsBetweenContradictions$(\langle \mathrm{X}, \mathrm{Y} \rangle : \alpha, \Delta)$ it is thus always the case that for the 'then' clause, above, $\mathrm{Z} \subseteq \mathrm{X}$ and for the 'else' clause, $\mathrm{Z} \subseteq \mathrm{Y}$.

So far this commentary has focussed on the default case where pruning occurs, however if a reflector is empty then some reflections will not occur and will thus be absent. So if the attack is on X, it follows that Y is the reflector; thus if $\mathrm{Y} = \emptyset$ then if follows that $\mathrm{T} = \mathrm{V}$ and no pruning of V occurs. Likewise if the attack is on Y, it follows that X is the reflector; so if $\mathrm{X} = \emptyset$ it follows that $\mathrm{U} = \mathrm{W}$ and no pruning of W occurs. An additional step of analysis is needed for reflected rebuttals; one might think if X were subject to preclusive undercut and $\mathrm{X} = \emptyset$ then there would be no reflected rebuttal arguments in $\mathrm{W} : \neg\phi$, however a preclusive undercut cannot attack an empty target and hence this conjectured case cannot occur. One might also conjecture that a further way for reflection to be absent is if no attack has occurred, however that too cannot be the case as a prerequisite for this function is a valid preclusive undercut. Hence given a non-empty attacked reflector, then $\mathrm{T} \subsetneq \mathrm{V}$, $\mathrm{U} \subsetneq \mathrm{W}$ and pruning is definite. Thus summing up these various situations, it is the case that $\mathrm{T} \subseteq \mathrm{V}$ and $\mathrm{U} \subseteq \mathrm{W}$, i.e. that the matt confirmation labels will always be a subset or equal to the confirmation labels that include reflection.

Furthermore, the matt function always returns a unique top, or maximum cardinality contradiction, containing only direct arguments. Suppose $\mathrm{V} = \{I_1, \ldots, I_n\}$, then each $I_i \in \mathrm{V}$ is either direct or reflected, (here the reflector is fixed as $\langle \mathrm{X}, \mathrm{Y} \rangle : \alpha$). Hence the direct subset of V, i.e. $\mathrm{T} = \{I_l, \ldots, I_m\} \subseteq \mathrm{V} = \{I_1, \ldots, I_n\}$ is unique, likewise for $\mathrm{U} \subseteq \mathrm{W}$. This maximum cardinality behaviour follows from the use of the maximal versions of the direct confirmation rebuttal and direct preclusive undercut functions.

Now that Type I and II Reflections are removed it is necessary to move on to Type III and IV Reflections. The opaque function has an input of a contradiction. Its output is again a single contradiction, but now devoid of Type III and IV Reflections. I reuse the direct confirmation function, Definition 5.9.4, as it removes Type III and IV Reflections from a confirmation.

**Definition 6.7.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\langle \mathrm{X}, \mathrm{Y} \rangle : \alpha \in \blacklozenge(\Delta)$. The* **opaque** *function, denoted* opaque$(\langle \mathrm{X}, \mathrm{Y} \rangle : \alpha, \Delta)$*, given a contradiction for $\alpha$, delivers the maximum* *cardinality contradiction for $\alpha$ that is free from reflected confirming confirmations such that:*

$$\begin{aligned}
&\mathsf{opaque}(\langle \mathrm{X}, \mathrm{Y}\rangle : \alpha, \Delta) \\
&= \{\langle \mathrm{V}, \mathrm{W}\rangle : \alpha \in \blacklozenge(\Delta) \mid \\
&\qquad \mathrm{V} : \alpha = \mathsf{top}(\mathsf{directConfirmations}(\mathrm{X} : \alpha, \Delta)), \\
&\qquad \mathrm{W} : \neg\alpha = \mathsf{top}(\mathsf{directConfirmations}(\mathrm{Y} : \neg\alpha, \Delta))\}.
\end{aligned}$$

The opaque function cannot add arguments to X or Y, in that it only prunes or removes arguments. It will always be the case that $\mathrm{V} \subseteq \mathrm{X}, \mathrm{W} \subseteq \mathrm{Y}$ which is why I call opaque() a pruning function. If $\mathrm{X} : \alpha$ is not subject to any preclusive undercuts then $\mathrm{V} = \mathrm{X}$, likewise if $\mathrm{Y} : \neg\alpha$ is not subject to any preclusive undercuts then $\mathrm{W} = \mathrm{Y}$.

The order in which the matt and opaque functions are applied, in principle, makes no difference, as set difference is commutative: $A \setminus B \setminus C = A \setminus C \setminus B$. My specific definitions, however, require different input parameters to provide the correct reflectors and thus it is not straightforward to show this commutativity as a property.

Finally I come to the last three helper functions needed for the debate tree which are all graph theory related:

**Definition 6.7.4.** *Let $\langle \mathcal{V}, \mathcal{E}\rangle$ be a tree where $\mathcal{V}$ is a set of vertices and $\mathcal{E}$ a set of edges. Let $v, v_1, v_2, v_n, v_{root} \in \mathcal{V}$ and $\langle v, v_1\rangle, \langle v_1, v_2\rangle, \langle v_n, v_{root}\rangle \in \mathcal{E}$. The* **defending path**, *denoted* defendingPath$(v, \mathcal{V}, \mathcal{E})$, *is the set of continuously connected edges from $v$ to the root $v_{root}$ such that:*

$$\mathsf{defendingPath}(v, \mathcal{V}, \mathcal{E}) = \{\langle v, v_1\rangle, \langle v_1, v_2\rangle, \ldots, \langle v_n, v_{root}\rangle\}.$$

Given this set of edges the next step is to extract the vertices from the edges, giving a set of vertices.

**Definition 6.7.5.** *Let $\langle \mathcal{V}, \mathcal{E}\rangle$ be a tree where $\mathcal{V}$ is a set of vertices and $\mathcal{E}$ a set of edges. Let $v, v_1, v_2, \ldots, v_n, v_{root} \in \mathcal{V}$. The* **ancestors** *function, for a vertex $v \in \mathcal{V}$ in the tree $\langle \mathcal{V}, \mathcal{E}\rangle$, is the set of vertices between $v$ and the root, including the root, denoted* ancestors$(v, \mathcal{V}, \mathcal{E})$ *such that:*

$$\mathsf{ancestors}(v, \mathcal{V}, \mathcal{E}) = \{v_1, v_2, \ldots, v_n, v_{root} \mid \mathsf{defendingPath}(v, \mathcal{V}, \mathcal{E})\}.$$

Clearly $v_1$ is the parent of $v$. Observe that the vertex $v$ is not included within its ancestors. While the above two definitions could apply to any tree, the next one only applies to trees where the vertices are mapped to contradictions.

**Definition 6.7.6.** *Let $\Delta$ be a knowledgebase of labelled assumptions. Let $\langle \mathcal{V}, \mathcal{E}, \blacklozenge(\Delta), f\rangle$ be a tree where $\mathcal{V}$ is a set of vertices, $\mathcal{E}$ is a set of edges and $f : \mathcal{V} \mapsto \blacklozenge(\Delta)$. The* **ancestor arguments** *function, denoted* ancestorArguments$(v, \langle \mathcal{V}, \mathcal{E}, \blacklozenge(\Delta), f\rangle)$, *provides the set of all the arguments present in the vertices above v to the root in the tree $\langle \mathcal{V}, \mathcal{E}, \blacklozenge(\Delta), f\rangle$, such that:*

$$\begin{aligned}
&\mathsf{ancestorArguments}(v, \langle \mathcal{V}, \mathcal{E}, \blacklozenge(\Delta), f\rangle) \\
&= \{I_k^1 : \alpha_k, \ldots, I_k^n : \alpha_k, J_k^1 : \neg\alpha_k, \ldots, J_k^m : \neg\alpha_k \mid \\
&\qquad \textit{for each } v_k \in \mathsf{ancestors}(v, \mathcal{V}, \mathcal{E}) \\
&\qquad \textit{it is the case that } f(v_k) = \langle \{I_k^1, \ldots, I_k^n\}, \{J_k^1, \ldots, J_k^m\}\rangle : \alpha_k\}.
\end{aligned}$$

The new matt, opaque and preclusive undercut for contradictions definitions, plus the other tree-related helper functions, can now be used in defining the debate tree, which is free of all reflections.

Key definitions reused here are: contradiction Definition 2.6.1, my shorthands for contradiction Definitions 2.6.2 and 2.6.3, maximum cardinality contradiction Definition 2.6.6, plus the six helper functions just defined.

The assertion $f : \mathcal{V} \mapsto \blacklozenge(\Delta)$ establishes that each vertex in the tree is mapped onto a contradiction. Thus the arbitrary vertex $v \in \mathcal{V}$ is such that $f(v) = \langle \mathrm{U}, \mathrm{V} \rangle : \psi \in \blacklozenge(\Delta)$, where $\psi$ is any formula. It would be untrue to say that each vertex *is* a contradiction as the vertices of a tree form a set, any given contradiction may appear more than once in a tree and repeated members are not allowed in sets. Thus for each vertex $v \in \mathcal{V}$ there exists one and only one contradiction $\langle \mathrm{U}, \mathrm{V} \rangle : \psi \in \blacklozenge(\Delta)$. However for each contradiction in $\blacklozenge(\Delta)$ there may or may not exist a corresponding vertex or vertices $v \in \mathcal{V}$.

The edges in an edge-labelled tree are three element tuples $\langle u, \mathrm{Z}, v \rangle$, where $u$ is the vertex the edge is incident from, $v$ the vertex the edge is incident to and Z the edge label.

**Definition 6.7.7.** *Let* $\Delta$ *be a knowledgebase of labelled assumption formulae, let* $\langle \mathrm{V}, \mathrm{W} \rangle : \phi$, $\langle \{I_1, \ldots, I_n\}, \{J_1, \ldots, J_m\} \rangle : \phi \in \blacklozenge(\Delta)$ *and* $\mathrm{Z} \in \mathcal{C}$. *The* **debate tree** *for* $\alpha$ *derivable from* $\Delta$, *denoted* $\mathsf{debateTree}\ (\alpha, \Delta) = \langle \mathcal{V}, \mathcal{E}, \blacklozenge(\Delta), f \rangle$ *where* $\mathcal{V}$ *is a set of vertices,* $\mathcal{E}$ *is a set of labelled edges and* $f : \mathcal{V} \mapsto \blacklozenge(\Delta)$, *is an edge-labelled contradiction tree devoid of reflection such that:*

1. *the root* $v_{root} \in \mathcal{V}$ *has* $f(v_{root}) = \mathsf{opaque}(\mathsf{top}(\blacklozenge(\alpha, \Delta)), \Delta)$,
2. *for each* $v \in \mathcal{V} \setminus \{v_{root}\}$,

    *for each* $\langle \langle \mathrm{V}, \mathrm{W} \rangle : \phi, \mathrm{Z}, f(v) \rangle \in \mathsf{preclusiveUndercutsBetweenContradictions}(f(v), \Delta)$,

        *where* $\langle \mathrm{V}, \mathrm{W} \rangle : \phi = \mathsf{top}(\blacklozenge(\phi, \Delta))$,

    *if there exists* $\langle \{I_1, \ldots, I_n\}, \{J_1, \ldots, J_m\} \rangle : \phi$ *such that*

        $\mathsf{opaque}(\mathsf{matt}(\langle \langle \mathrm{V}, \mathrm{W} \rangle : \phi, \mathrm{Z}, f(v) \rangle, \Delta), \Delta) = \langle \{I_1, \ldots, I_n\}, \{J_1, \ldots, J_m\} \rangle : \phi$

        *and* $I_1 : \alpha, \ldots, I_n : \alpha, J_1 : \neg\alpha, \ldots, J_m : \neg\alpha \notin \mathsf{ancestorArguments}(v, \langle \mathcal{V}, \mathcal{E}, \blacklozenge(\Delta), f \rangle)$,

    *then there exists a child vertex* $v_{child} \in \mathcal{V}$ *of* $v$ *such that*

        $f(v_{child}) = \langle \{I_1, \ldots, I_n\}, \{J_1, \ldots, J_m\} \rangle : \phi$.

I use the shorthand of $\blacklozenge^*(\alpha, \Delta) = \mathsf{debateTree}\ (\alpha, \Delta) = \langle \mathcal{V}, \mathcal{E}, \blacklozenge(\Delta), f \rangle$ as an option for shortening subsequent notation. Each edge is annotated with an additional piece of information that is the target confirmation label, however that is not needed to define the tree. So for the labelled tree $v_{child}$ and $v$ are connected by edge $\langle v_{child}, \mathrm{Z}, v \rangle \in \mathcal{E}$. Now the debate tree has been defined it is possible to state which earlier examples were in fact debate trees – they are Examples 6.3.1, 6.3.2, 6.3.3, 6.4.1, 6.4.2, 6.4.3 and 6.5.1 . The Examples 5.7.2, 5.7.3, 5.7.4 and 5.7.5, while showing trees of contradiction vertices with preclusion edges are not debate trees as they contain reflection.

The NRC, which is the sixth line of clause 2. above, disallows reusing an argument that appears anywhere in the parent contradictions from the vertex in question up to the root. An area for further research would be to explore variations on this NRC where only arguments in the targets between the vertex and the root are counted towards no-recycling. I would defend the approach taken here as being somewhat conservative and robust; it is clearly intolerant of repeated arguments.

While $\mathsf{arguments}(\Delta)$ can hold many attacks of many types on a reflexive arguments, the debate tree only holds one type of attack because the undercutting arguments are pruned out as a reflection of each rebutting argument.

This tree tracks the debate on the level of contradictions, thereby ensuring that direct rebuttals are included in the tracking. The tree can be understood on two levels: that of contradiction and that of confirmation. On the level of contradiction, the central feature is that $\langle \mathrm{V}, \mathrm{W} \rangle : \phi$ undercuts $\langle \mathrm{X}, \mathrm{Y} \rangle : \psi$. On the level of confirmation there are two possible undercuts: $\mathrm{V} : \phi$ undercuts either the defending confirmation $\mathrm{X} : \psi$ with a target confirmation of $\mathrm{Z} : \psi$; or the defending confirmation $\mathrm{Y} : \neg\psi$ with target confirmation of $\mathrm{Z} : \neg\psi$. The definition of preclusive undercut ensures $\mathrm{Z} \subseteq \mathrm{X}$ or $\mathrm{Z} \subseteq \mathrm{Y}$.

Reflections can arise off any argument, be that argument a direct argument or a reflected argument. Thus a reflector may itself be direct or reflected and that distinction makes no difference to its ability to act as a reflector. Thus to make sure that the direct argument is devoid of reflections it is necessary to include reflections in the reflector confirmation. Consequently two different confirmations are needed for these two different purposes. The confirmation that is a part of the tree has to be the direct confirmation. The reflector confirmation used to compute the direct confirmations has to be a maximum cardinality confirmation.

The approach to building a debate tree is the same for immediate children of the root as it is for vertices further down the tree. One might think that the notion of grandparent reflections, which could affect further down the tree, would not occur for the immediate children of the root. Such immediate children of the root have no grandparents. In reality, however, the behaviour is the same at any level of the tree below the root as grandparent reflections (appearing as a chain of two undercuts) are always intermediated by a rebuttal.

## 6.8 Properties of Debate Trees

I now examine the properties of debate trees, starting with some introductory observations most of which can be established directly from the tree's definition and then move on to formal properties with more complicated proofs. The key property I build the rationale for is that debate trees are finite.

### 6.8.1 Introductory Observations

The early portion of this chapter criticised trees from the literature for not adhering to a number of properties deemed to be desirable. Each of these cases was proven by example, with each example giving not just an argument tree, but also the corresponding a debate tree. These examples can be re-examined to show that debate trees are:

- Debate trees are not Type A Incomplete as non-root rebuttals are included - given that they are direct, rather than reflected rebuttals.
- Debate trees are not Type B Incomplete as they contain no reflected arguments and hence no reduced reflections.

- Debate trees are not Type C Incomplete as premise reuse is allowed within a path from leaf to root (and also between paths).

- Debate trees are not Type A Inordinate as they contain no reflected arguments and hence no root reflected canonical undercuts.

- Debate trees are not Type B Inordinate as they contain no reflected arguments and hence no mono-pair enlargement.

- Debate trees are not Type C Inordinate as they contain no reflected arguments and hence no mono-pair enlargement.

- Debate trees are not Inchoate as all necessary arguments are included and all unnecessary arguments excluded, so there is no possibility of proxy arguments misrepresenting the arguments they proxy for.

- Debate trees are considerably less sensitive to the way the knowledgebase is written, relative to argument trees.

Before establishing properties requiring proofs there are a number of observations that can be made simply by pointing them out in the definitions.

The debate tree is a single tree, in contrast to the argument tree and argument structure approach which requires a forest to track a debate. The debate tree could be viewed as taking the definition of a (Besnard & Hunter, 2001) argument structure and applying it uniformly to every vertex in the tree rather than just at the root. Nowhere in the literature can I find anyone tracking a complete debate as a single tree.

**Proposition 6.8.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\blacklozenge^*(\alpha, \Delta)$ be a debate tree.*

$$|\{\blacklozenge^*(\alpha, \Delta)\}| = 1.$$

*Proof.* The root of the tree maps to a maximum cardinality contradiction and building on Proposition 2.5.9, it follows that there is only ever one maximum cardinality contradiction for a given formula $\alpha$. Given the root $\alpha$ and the knowledgebase $\Delta$ the rest of the tree is uniquely determined. Therefore $[|\{\blacklozenge^*(\alpha, \Delta)\}| = 1$. □

The debate tree is devoid of reflections, regardless of whether the reflection is a Type I, II, III or IV Reflection. Hence it is ordinate, complete and choate. Thus the possibility of judges counting an argument twice is avoided.

Because the debate is modelled by a tree not a graph, it follows that there are no parallel edges. In a conjectured graph, if a vertex $v_{child}$ attacks two targets, V, W in parent $v$, then the result would be parallel edges $\langle v_{child}, \mathrm{V}, v\rangle, \langle v_{child}, \mathrm{W}, v\rangle \in \mathcal{E}$. In a debate tree, however, $v$ would have two children $v_{child1} \neq v_{child2}$ where $f(v_{child1}) = f(v_{child2})$ and $\langle v_{child1}, \mathrm{V}, v\rangle, \langle v_{child2}, \mathrm{W}, v\rangle \in \mathcal{E}$.

Clearly any graph can contain infinite cycles, while a finite tree, by definition, cannot contain infinite cycles. Additionally the NRC ensures that there are no repeated arguments and thus no repeated chains of arguments in the tree. A third feature of some graphs is loops, that is arguments that attack themselves (as seen with Dung). Loops cannot occur with mincons, but can with abstract arguments. Hence the debate tree clearly has no loops, cycles nor parallel edges.

Debate trees include preclusive undercuts - unlike the debate tracking structures in the literature. Additionally debate trees track which expert contributed which knowledge (due to the use of a Labelled Deductive System).

There is no dependable mapping between argument trees and debate trees. Sometimes it is possible to derive one from the other, but not always. Specifically, the examples of incomplete argument trees are ones where there is not enough information to construct a debate tree. Furthermore, the examples of inordinate argument trees provide ones which cannot be constructed (at least not in a straightforward mapping fashion) from the debate tree for the same $\alpha$ and $\Delta$.

I now look at repetition within a debate tree in more detail. In any formally defined tree, including debate trees, a vertex cannot be repeated as the vertices form a set and hence are unique. My $f(v) = \langle \mathrm{X}, \mathrm{Y} \rangle : \phi$ mapping construct ensures this vertex uniqueness. Similarly in a well formed tree, including debate trees, edges cannot be repeated.

A contradiction, however, can be repeated in a debate tree and that is acceptable and normal in professional debate. As a contradiction can be repeated in the tree, so also can its parts: confirmations, arguments, supports, claims and assumptions. On a path from leaf to root, however, the NRC ensures that an argument cannot be repeated, and hence on such a path no confirmation or contradiction can be repeated either.

Within a path from leaf to root it is still allowable to repeat a support, a consequent or assumption, as long as no entire argument is repeated. Now to touch on the empty set.

**Example 6.8.1.** *Let $\Delta = \emptyset$. The debate tree $\blacklozenge^*(\alpha, \Delta)$ is $\langle \emptyset, \emptyset \rangle : \alpha$. The set of arguments for $\alpha$ is the empty set. Note that $\{\emptyset\} : \alpha$ would mean that $\alpha$ was a tautology, which it is not the case here. Here $\mathcal{V} = \langle \emptyset, \emptyset \rangle : \alpha$ and $\mathcal{E} = \emptyset$.*

### 6.8.2 Vertex Size

The number of arguments in an arbitrary vertex in any debate tree is finite.

**Proposition 6.8.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\blacklozenge^*(\alpha, \Delta) = \langle \mathcal{V}, \mathcal{E}, \blacklozenge(\Delta), f \rangle$ be the debate tree for $\alpha$ derivable from $\Delta$.*

$$0 \leq (\ |\mathrm{X}| + |\mathrm{Y}|\ ) \leq \mathsf{midLattice}(\Delta), \ \textit{where } v \in \mathcal{V}, f(v) = \langle \mathrm{X}, \mathrm{Y} \rangle : \phi.$$

*Proof.* If the number of arguments for $\phi$ derivable from $\Delta$ is zero, then $|\mathrm{X}| = 0$. Likewise if there are no arguments for $\neg\phi$ then $|\mathrm{Y}| = 0$, hence the lower limit is zero. The support for one argument for $\phi$ can never be a subset of the support for another argument for $\phi$. Thus the maximum number of arguments for $\phi$ is $\mathsf{midLattice}(\Delta)$. Any argument can be for $\phi$ or for $\neg\phi$, but never for both. Thus regardless of the distribution of arguments between $\phi$ or $\neg\phi$ the total maximum is still always $\mathsf{midLattice}(\Delta)$. Hence the upper limit for the number of arguments in any vertex of a debate tree is $\mathsf{midLattice}(\Delta)$. □

If some of the arguments for $\phi$ or for $\neg\phi$ are reflected then the upper limit will not be reached. Thus although this property is for a debate tree, it is essentially about the size of contradictions.

### 6.8.3 Branching Factor

The number of edges incident upon an arbitrary vertex $v$ in any debate tree $\blacklozenge^*(\alpha, \Delta)$ is finite.

**Proposition 6.8.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\blacklozenge^*(\alpha, \Delta) = \langle \mathcal{V}, \mathcal{E}, \blacklozenge(\Delta), f\rangle$ be the debate tree for $\alpha$ derivable from $\Delta$.*

$$0 \leq |\{\langle u, W, v\rangle \in \mathcal{E} \mid u, v \in \mathcal{V}, W \in \mathcal{C}, f(v) = \langle \mathrm{X}, \mathrm{Y}\rangle : \psi\}| \leq 2^{|\mathrm{X}|} + 2^{|\mathrm{Y}|} - 2.$$

*Proof.* A leaf vertex will have no edges incident on it, so the lower bound is 0. Any edge incident on a vertex $v$ represents a preclusive undercut with defending confirmation $\mathrm{X} : \psi$ or $\mathrm{Y} : \neg\psi$ where $f(v) = \langle \mathrm{X}, \mathrm{Y}\rangle : \psi$. Each edge will have a unique target confirmation label $\mathrm{W} \subseteq \mathrm{X}$ or $\mathrm{W} \subseteq \mathrm{Y}$. Any subset of X or Y can be a valid target confirmation label, except for the emptyset. Here $u \in \mathrm{V}$ is any child vertex of $v$ and W any valid target confirmation label. Hence the upper limit for the number of targets is $2^{|\mathrm{X}|} - 1$, so the upper limit for the contradiction taken as a whole is $2^{|\mathrm{X}|} + 2^{|\mathrm{Y}|} - 2$. □

This proposition is the same as asking how many preclusive undercuts can be incident upon a confirmation and then doubling it to get the answer for a contradiction.

### 6.8.4 Path Length

To look at the path length from vertex $v$ to the root, where $v$ and root are part of the path, it is helpful to define odd and even levels in the tree. Thus the root is odd, given that its path length, $x$, is 1.

**Definition 6.8.1.** *Let $\langle \mathcal{V}, \mathcal{E}\rangle$ be a tree where $\mathcal{V}$ is its set of vertices and $\mathcal{E}$ its set of edges, $v \in \mathcal{V}$.*

$$v \textit{ is even if } \left(\frac{x}{2} - \mathsf{trunc}(\frac{x}{2})\right) = 0 \textit{ else } v \textit{ is odd, where } x = |\mathsf{ancestors}(v, \mathcal{V}, \mathcal{E})| + 1.$$

While the following proposition could be stated more formally it is unambiguous and avoids the introduction of further helper functions.

**Proposition 6.8.4.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\blacklozenge^*(\alpha, \Delta) = \langle \mathcal{V}, \mathcal{E}, \blacklozenge(\Delta), f\rangle$ be the debate tree for $\alpha$ derivable from $\Delta$. Let $v \in \mathcal{V}$ and $I : \phi, J : \psi \in$* $\mathsf{arguments}(\Delta)$.

*If $I : \phi$ exists at an odd [even] level in the tree,*

*then no $J : \psi$, $J \subseteq I$ can exist on any other odd [even] level*

*of the same path further from the root.*

*Proof.* If such a $J : \psi$, $J \subseteq I$ did exist it would be pruned out by the $\mathsf{matt}()$ function. The same applies to even levels. □

The number of ancestors for a given vertex is finite.

**Proposition 6.8.5.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $\blacklozenge^*(\alpha, \Delta) = \langle \mathcal{V}, \mathcal{E}, \blacklozenge(\Delta), f \rangle$ be the debate tree for $\alpha$ derivable from $\Delta$.*

$$0 \leq \mid \mathsf{ancestors}(v, \mathcal{V}, \mathcal{E}) \mid \leq (2^{|\Delta|} - 1)^2 - 2^{|\Delta|}.$$

*Proof.* The NRC states that any argument added to the tail of an argument chain must not already exist in its ancestors. Hence the longest chain will use up as few arguments as possible in the ancestors and therefore have one argument only per vertex which will be mono-attack. For two arguments $I : \phi, J : \psi$ to be different, either support $I \neq J$, or claim $\phi \neq \psi$ or both support and claim differ. If an argument $I : \phi$ is child of an argument $J : \psi$ then, from the definition of preclusive undercut, it is known that $\phi = \neg \bigwedge \mathsf{stripAssumptions}(\mathsf{formulae}(\mathsf{label}(J : \psi)))$. The number of arguments derivable from $\Delta$ with different supports is $2^{|\Delta|}$, however $\emptyset$ as a support cannot attack or be attacked, so the practical number for creating chains is $2^{|\Delta|} - 1$. Hence any of $2^{|\Delta|} - 1$ supports can combine with any of $2^{|\Delta|} - 1$ claims. Furthermore loops are disallowed so a support $\langle \{I\} : \phi, \{I\}, \{I\} : \phi \rangle \notin \mathsf{preclusiveUndercuts}(\{I\} : \phi, \Delta)$. So the upper limit is $(2^{|\Delta|} - 1)^2 - 2^{|\Delta|}$. □

One might think that the odd / even behaviour of Proposition 6.8.4 would constrain the maximum path length, but it does not. The reason there is no constraint is that it is possible to have a sequence of subsets of $\Delta$ that starts with the shortest support length (for the root) and proceeds to longer supports whereby none are subsets of those nearer the root.

### 6.8.5 Debate Trees are Finite

**Proposition 6.8.6.** *Debate trees are finite.*

*Proof.* There are just three ways that a debate tree could be conjectured to be infinite: 1) the number of arguments in a vertex could be unbounded; 2) the branching factor could be unbounded; 3) the path length could be unbounded. However, it is established 1) that vertex contents are finite by Proposition 6.8.2, 2) that the branching factor is finite by Proposition 6.8.3 and 3) that path lengths are finite by Proposition 6.8.5. Therefore debate trees are finite. □

That completes the analysis of the key properties of debate trees. Further research is warranted to deepen this picture of how debate trees behave and also how well and in what ways they do and do not map to professional debate.

## 6.9 Conclusion for How To Track a Debate

The main conclusion of this chapter, and indeed of the thesis, is that to track a debate, the whole debate and nothing but the debate requires a debate tree, or something similar with the same properties of being complete and choate. The debate tree, denoted $\blacklozenge^*(\alpha, \Delta)$, has each vertex mapped to a contradiction and each edge representing a preclusive undercut. The two main differences between argument trees and debate trees are that a) debate trees include all necessary arguments (e.g. direct rebuttals below the root) and b) exclude all unnecessary arguments (e.g. reflected arguments). The NRC of the debate tree is simpler than that for argument trees as it is only doing one job (preventing repetition of arguments) rather than two (preventing repetition and some reflection).

The following table summarises areas where argument trees and argument structures do not map well to professional debate. In each of these areas, debate trees provide a more precise mapping to professional debate. Even so there are several areas where debate trees are lacking; these are discussed in Chapter 8.

| Behaviour | Type | Confirmation Attacks Function | Section |
|---|---|---|---|
| Incomplete | A | Non-root rebuttals excluded | 6.3.1 |
| Incomplete | B | Reduced reflections from direct rebuttals | 6.3.2 |
| Incomplete | C | Premise reuse prohibition | 6.3.3 |
| Inordinate | A | Root reflected canonical undercuts | 6.4.1 |
| Inordinate | B | Mono-pair enlarged canonical reflection | 6.4.2 |
| Inordinate | C | Multi-pair enlarged canonical reflection | 6.4.3 |
| Inordinate | D | Reflected confirming argument | 6.4.4 |
| Inchoate | | Present arguments do not represent missing ones | 6.5.1 |
| Additional | | Sensitivity to way knowledgebase is written | 6.5.2 |
| Additional | | Asymmetry in reflections do not cancel | 6.5.3 |

Table 6.4: Summary of Comparison of Argument Trees with Professional Debate

The definition of debate trees is preceded by six helper functions, notably preclusiveUndercutsBetweenContradictions() that is still fundamentally between confirmations, matt() which excludes Type I and II Reflections, opaque() which excludes Type III and IV Reflections and three more helper functions to facilitate defining the NRC.

The properties of debate trees are mostly self evident from examining the definitions. Namely they include undercuts, specifically preclusive undercuts, and exclude all reflected arguments. As they are devoid of reflection they thus exclude unnecessary arguments. Non-root direct rebuttals are included, which is part of the way that they include necessary arguments. The key property established is that debate trees are finite. It is conceivable that further research could identify additional forms of reflection in which case the definition of debate tree would have to be refined.

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## Chapter 7

# How To Judge a Debate

The chapter is less formal than those that precede it, so many of the terms and concepts discussed or introduced here are not formally defined. Formalisation of these concepts is suggested as one area for future research.

## 7.1 Overview of Chapter

The focus of this chapter is on how to judge a debate and draws out a preliminary result indicating that reflection skews or modifies judge outcomes. Judgement was defined informally in Chapter 1 and formally in Chapter 2, with the existential and quantitative distinction covered in both places. The body of this chapter comprises six sections:

**Judge Selection** Section 7.2 raises the need for a theoretical basis for the design of judge functions and selection of a particular judge for a specific application.

**Judge Properties** Section 7.3 provides an informal but fairly comprehensive discussion of judge properties including such matters as awareness of undercuts, reflection and confirmation.

**Reflection and Judge Outcomes** Section 7.4 provides a motivational and reasoned discussion of the question of whether reflection alters judge outcomes. As not removing any reflection leaves an infinite and not sensibly judgeable tree the choice becomes between removing some and removing all reflection.

**Reflection and Existential Judge Outcomes** Section 7.5 shows that, given certain assumptions, reflection can change debate outcomes when judged with an existential judge, and therefore that reflection should be considered in the deliberations.

**Judging a Debate Tree** Section 7.6 introduces a new judge function that is designed to judge the reflection free debate tree while also taking into account preclusive undercut and the filtering of noise.

**Reflection and Quantitative Judge Outcomes** Section 7.7 shows the indication that quantitative judges, like qualitative or existential judges, can be skewed by reflection.

A question at the core of this thesis is: does reflection make a difference to the outcome of debates? The preliminary result of the four key examples of this chapter (namely Examples 7.5.1, 7.5.2, 7.7.1 and 7.7.2 in Sections 7.4 through 7.7) suggest that the answer is yes, reflection does alter debate outcomes. It would appear that the outcomes from both existential and quantitative judging schemes are affected by reflection. Therefore, in my view, reflection ought to be taken into account by judge functions, regardless of whether they are existential or counting judges.

## 7.2 Judge Selection

Some facets of judge selection have been begun to be explored in the literature, while other facets, I would suggest, are less well developed. A topic which has received attention is that of semantics in the abstract argumentation literature, and to some degree in the semi-abstract literature. There now exists a a broad and growing range of semantics from Dung and those building on this work. How to select the optimum semantics for a particular application is, however, I would suggest, under developed, not least because abstract argumentation is a theoretical rather than an applied discipline. The need to develop this area I call judge selection has been recognised by (Caminada & Amgoud, 2007). Certain general guidelines do exist for the most well known half dozen abstract semantics regarding which semantics are good for what jobs, but formal analysis is not so developed. Just by casting a graph of arguments into a tree, thereby removing attack cycles will always cause agreement between grounded, preferred and stable semantics (Dung, 1995). An important and recent classification of semantics is (Baroni & Giacomin, 2008) which shows that for seven well known semantics, (grounded, preferred, stable, complete, CF2, semi-stable and ideal) that there are just 14 equivalence classes, or situations, out of a possible 120 where the semantics give the same result as each other.

Judges based on concrete argumentation, in contrast, have in the main lacked an underlying theoretical framework and thus selecting one over another has been somewhat arbitrary. The comparison and selection of particular concrete judges for particular applications is not well developed in the literature. Questions such as 'for what particular applications would a Benferhat judge be better than a Franklin judge?' are yet to be thoroughly explored.

The literature on argumentation semantics provides functions where the output is a set (or a set of sets) of arguments that is a subset (or are subsets) of the input set of arguments. My judge functions, in contrast, seek an output that is a) a proposition rather than an argument and b) a single item not a set of items, nor a set of sets of items.

In the ABA framework, attack is constrained to Wigmore undercut and by definition excludes rebuttal. Hence in ABA, due to the absence of an explicit rebuttal, it is impossible to attack a claim formula in isolation. On page 7 of (Dung et al., 2006) the authors provide their perspective of how undercut can be said to imply rebuttal. That view in turn hinges its justification on (Kowalski & Toni, 1996) and illustrates the notion with an example. The view of (Kowalski & Toni, 1996) is also argued on the basis of an example. I would suggest that more research is warranted to investigate how these examples, which rely on different logics from my classical mincon analysis, compare and contrast to reflection and reflexivity as presented herein.

Both the ABA and (Besnard & Hunter, 2001) frameworks limit attack in this way and thus create one tree per argument for $\alpha$ or $\neg\alpha$. Additionally (Besnard & Hunter, 2001) offer a way to aggregate all of these trees together, by means of the argument structure. There are some indications of more recent ABA research venturing into this territory of judging propositions rather than arguments. Because the ABA framework appears to be built on certain unstated assumptions about reflection, it becomes more challenging to map its semantics to professional debate. Professional debate requires judgement of a motion not of individual arguments, and also requires recognition of rebuttal as a valid form of argumentation, rendering most of the work on semantics, I would suggest, to be not as helpful as it might initially appear to be for clarifying judge selection.

Clearly, there are many possible concrete judges, which may be categorised in various possible ways. Common to them all is the same issue as to why to choose a particular judge for a particular application. As different judges can give different outcomes for the same debate (as demonstrated, for example in Sections 2.8.6 and 3.2.2) some suspicion about automated judgement is natural.

One approach to tackling this lack of basis is by means of what I call the **properties approach**, another by means of what I all the **inner workings approach**; I now describe these two approaches. The properties approach has five steps.

1. Formally define desirable or possible judge properties.

2. Formally define judge functions.

3. Use logical proof to establish the precise properties for each judge.

4. Then for a particular application decide what judge properties, picked from the list immediately above, are required for that application.

5. Hence, select the best judge for the job, accepting that compromise is often inevitable.

The inner workings approach instead seeks to define the safest judge possible, whereby the way that it reaches conclusions is as impeccable as it can be, i.e. as free from the prospect of criticism as possible. A feature of courts of law is the appeal (see Section 1.1 point 5), and if in the appeal some flaw or weakness can be established that brings into question the *way* that the conclusion was reached then that can cause the original judgement to be 'thrown out' or rejected. Therefore if the goal is to achieve safety in cases of appeal then it is prudent to focus the design on impeccability of judge inner workings. However, safety on appeal, i.e. safety of inner workings, is arguably just another judge property so even here, I would say, the theoretical basis is not robust. A focus on impeccability rather than on resolving inconsistency may suggest the use of an existential judge. A focus on resolving inconsistency, thereby reducing the frequency of stalemates, suggests counting judges.

## 7.3 Judge Properties

This section informally introduces a number of desirable judge properties leaving their formal definition as an area for further research. More formal discussions of judge properties can be found in (Manor &

Rescher, 1970; Benferhat et al., 1993; Elvang-Gøransson & Hunter, 1995; Cayrol & Lagasquie-Schiex, 1995). This literature is strong for judge properties linked to the parts of propositional logic covered in discrete mathematics text books, but not, I would argue, in entertaining the notions of confirmation, preclusion and reflection. Additionally there is a strong literature on the semantics for selecting subsets of arguments from a larger set of given arguments, which in my view is not so pertinent to the current discussion because professional debates have a formula as the motion, not a set of arguments. The nine properties I now outline are rebuttal awareness, existential judging, quantitative judging, undercut awareness, reflection awareness, confirmation awareness, cumulativity, bipolarity and metaknowledge awareness.

### 7.3.1 Rebuttal Aware

Arguably the most basic of judge properties is that a judge should consider rebuttal arguments and thus take them into account when reaching a conclusion. The rebuttals (or pros and cons, or arguments for and against) in a debate can be tracked using a single contradiction. Judges 1.0 and 2.0 of Chapter 2 are rebuttal aware judges, each relying on a single contradiction as the basis for their judgement. Rebuttal awareness appears to be the most widely accepted and prevalent judge property in the literature, but is not universal (see (Dung et al., 2006) for example).

### 7.3.2 Existential

The simplest, and arguably the most robust, way to reach a conclusion is to look at only the presence or absence of arguments - and nothing else. Existential judges thus never count arguments, nor perform any subtler evaluations such as weighing or considering probabilities or priorities. The Benferhat judge, judge 1.0, of Chapter 2 is an example of an existential judge, as also is the judge of (García & Simari, 2004), which is defined shortly as judge 3.0 in Definition 7.5.1. Existential judges are strong on appeal, but their ability or power to resolve inconsistencies or stalemates is weak. They can also deliver a lot of null results (i.e. the empty set rather than $\alpha$ or $\neg\alpha$) making them unsuitable for some applications. Section 7.4 looks at the effect of reflection on existential judges.

### 7.3.3 Quantitative

A quantitative judge is sensitive to the number of arguments and takes these counts into consideration when reaching conclusions. The Franklin judge, judge 2.0, of Chapter 2 is an example of a quantitative judge. Another quantitative judge is the example judge in (Besnard & Hunter, 2001) to be found on their pages 222 and 223 and provided below as Definition 7.6.1. A benefit of counting arguments is that the judge's ability to resolve inconsistencies, which I call its resolving power, is considerably enhanced. A downside of quantitative judges is that they make more assumptions about the nature of argumentation than existential judges and are hence weaker on appeal. Other complaints are that different arguments can have different weights (or degrees of worth in influencing a conclusion) and that the meaning of counts is unclear. Generally speaking a judge is either existential or quantitative. Section 7.7 touches on the effect of reflection on quantitative judges.

I would argue that existential judges are particularly robust for applications that only ever have zero, one or perhaps occasionally two arguments for a claim and hence where inconsistency and consequently stalemate are rare. For applications with larger and more complex knowledgebases producing a large number of arguments for claims then there may be more of a value in employing counting schemes; more value as inconsistencies can be resolved and do not automatically result in stalemates.

The following quotation provides the suggestion that quantitative judges, if properly formulated, could be of considerable value, but also acknowledges the challenges that appear to be inherent in their delineation.

> *'The notion of confirmation plays a central role in scientific practice, and is therefore a most important task for philosophers of science to clarify and perhaps systematise and improve this notion, or, in other words to construct a theory of confirmation. Unfortunately it has proved surprisingly difficult to develop confirmation theory. It is not even clear how we should set about this task, or what a successful confirmation theory would look like.'* (Gillies, 1998).

A further point I would like to make on quantitative judges is that they would appear to offer an approach to living with, resolving and taking advantage of inconsistency. Existential judges, in comparison have a tendency, after identifying the existence of inconsistency and avoiding ex falso quodlibet, to deliver the empty set, which is almost as unhelpful. One of the most frequently cited papers in the field of inconsistency raises exactly this issue of how to live with inconsistency, rather than just attempting to delete or avoid it (Gabbay & Hunter, 1991).

Finally, quantitative judges that deliver not just set membership judge outputs, but also numeric strength of the net support for the motion, are useful for deciding between different motions. In the Soft Systems Methodology it is common to examine a challenging problem, formulate a small number of possible solutions and then debate the merits of each one. Quantitative judges delivering numeric output could be of assistance in conducting such trade-off debates.

### 7.3.4 Undercut Aware

For a judge to be undercut aware requires that a consideration of undercut to be taken into account in its workings to reach a conclusion. Both judges 1.0 and 2.0 introduced in Chapter 2 are *not* undercut aware. However judges 3.0 onwards, introduced below, are undercut aware. Section 3.2.2 showed that undercuts can change debate outcomes, and therefore that including them in deliberations is prudent to obtain what would generally be agreed to be accuracy or the avoidance of errors.

I use the term 'undercut aware' as a general term without tying it specifically to Wigmore, Pollock, canonical, preclusive or some other kind of undercut. A judge that relies on a single contradiction to track a debate cannot be undercut aware. On the contrary, undercut awareness requires that the debate tracking structure is some kind of a tree (or possibly a graph). Undercut aware judges in the literature are almost all also rebuttal aware, but less so vice versa.

### 7.3.5 Reflection Aware

For a judge to be reflection aware it must include an understanding of reflection in its deliberations. If a judge is rebuttal aware, but not undercut aware, as are many in the literature - for example (Fox & Das, 2000), then there is no need or option to be reflection aware. Undercut aware judges, however, should, in my view, all be made to be reflection aware. There is no current known proof or sound theoretical basis that would make reflection awareness a mandatory requirement for a judge to be a valid judge. Judge 6.0 is reflection aware, while the judges in this thesis prior to 6.0 are not.

### 7.3.6 Confirmation Aware

Another judge property developed in this thesis in that of confirmation - in that professional debates are often so structured as to only consider confirmed arguments, such as $\{I, J\} : \alpha$. Any claim for which there is only one argument, i.e. where $\mathsf{top}(\Diamond(\alpha, \Delta)) = \{I\} : \alpha$, will be removed from consideration in such a debate. Journalists are taught to check their facts through confirmation and not to submit for publication unconfirmed, i.e. unsubstantiated, stories. Surgeons also commonly seek a 'second opinion' from another surgeon before conducting a major operation to ensure that the procedure is properly warranted. In some professions and professional practices the number of arguments required for a claim increases beyond two to larger numbers. This number, or variable threshold (documented as threshold in judge 6.0), for the required level of proof or evidence is another input to a debate and hence best treated as an item of metaknowledge (see Section 7.3.9). The workings of juries, parliaments and senates can be understood in this light of confirmation thresholds. The benefit of using such a threshold is that it filters out noise, i.e claims for which there is a paucity of compelling evidence. Noise is not the same as inconsistency.

### 7.3.7 Cumulativity

The ability of a judge to determine the outcomes of multiple debates without the set of conclusions taken together implying inconsistency is called cumulativity and was discussed in the light of Definition 2.7.2. Almost all of the judges in the literature lack this property of cumulativity, however in professional debate this property is deemed to be important. See Example 2.7.2 and the whole of Section 2.7 for further discussion of cumulativity. I would suggest that cumulativity is likely to be a fruitful yet challenging area for future research. The judges of (Manor & Rescher, 1970) and of (Dung, 1993) share the feature of judging more than one claim or arguments at a time, but neither do it in a way that appears to be directly transferrable to make non-cumulative judges cumulative.

### 7.3.8 Bipolar

The term 'bipolar' in the argumentation literature (Amgoud et al., 2004; Cayrol & Lagasquie-Schiex, 2005) has become associated with the notion of not just having an attacks or negative relationship between arguments but also having an assisting, supporting or positive relationship. Typically there is seen to be a kind of inverse symmetry between attacking and assisting. I introduced this notion in Section 1.3 where I pointed out that my confirmation function, $\Diamond(\alpha, \Delta)$, can be viewed as a form of positive relationship between arguments. I revisited the notion in Section 2.4.1 and will touch on it again in Section 8.4.

While some papers deal with this symmetry on the level of abstract arguments (Cayrol & Lagasquie-Schiex, 2005), my examination is at a concrete level. Therefore, in addition to my confirmations being a form of bipolar positive argument, I formally propose, in Section 8.4, a second kind of positive relationship, which I call an assisting argument. With confirmations the positive relationship is to an argument's claim; in assists it is to the support. For a judge to be a bipolar judge it must take a consideration of both positive and negative argument relationships into account. Confirmation and assisting arguments can thus be seen as two aspects of bipolarity.

### 7.3.9 Metaknowledge Aware

John Locke is credited with the view that when drawing conclusions on a subject it is prudent not just to use the logical knowledge we know about that subject, but also to draw into consideration everything else we know about the matter (Krause & Clark, 1993). Benjamin Franklin also argued that not just the number of arguments for and against but also their weights (he called them weights) was of importance. Accordingly, a dominant use of metaknowledge is to hold information about weights of arguments, be that using some from of priorities, possibilities or probabilities. Of these various weighing approaches that of priorities, e.g. see (Cayrol et al., 1992; Prakken & Sartor, 1997; Amgoud & Cayrol, 1998; Modgil, 2007), is the one gaining most attention in the literature and I would suggest as likely to be the most fruitful for further research as it assumes less about a domain than other approaches. Other kinds of metaknowledge include thresholds and domain specific ontologies, see (Williams & Hunter, 2007). For a judge to be a metaknowledge aware judge it must take a consideration of at least some metaknowledge into account in its deliberations.

Beyond these judge properties touched on here there are naturally many others meriting further research, as cited earlier.

## 7.4 Reflection and Judge Outcomes

A fundamental question raised by this thesis is 'Does reflection change judge outcomes?', be they existential or quantitatively judged. I now share some observations stemming from this question in a narrative style akin to that of the earlier Section 6.6, where in both cases I provide motivation by arguing for a specific interpretation and approach.

This question can be paraphrased as 'Does there exist a knowledgebase $\Delta$, a motion $\alpha$, metaknowledge $\Theta$ and a pair of judges $\text{judge}_{\text{with reflection}}(\alpha, \Delta, \Theta)$ and $\text{judge}_{\text{without reflection}}(\alpha, \Delta, \Theta)$ such that $\text{judge}_{\text{with reflection}}(\alpha, \Delta, \Theta) \neq \text{judge}_{\text{without reflection}}(\alpha, \Delta, \Theta)$?' As this is a question of existence it can be addressed with proof by example. The example requires two judge functions to be contrasted for the same $\Delta, \alpha$ and $\Theta$. I will stop mentioning $\Theta$ now in this section as it is typically the empty set for simple judges and is not material to my main argument. Recall that it was established in Section 2.8.6 that two different judges can give different outcomes for the same $\Delta$ and $\alpha$, and this distinction was prior to any consideration of reflection. One challenge is that if the two judges are different then how can there be confidence that a difference in outcome is due to reflection and not just due to differences in the approach to judgement? My analysis, outlined in this section and detailed in Section 7.6.2, aims to address this challenge. Putting aside, for the moment, the issues of defining the two judges, the main path to addressing the question is to provide examples with and without reflection, both with the same $\Delta, \alpha$ and $\Theta$.

The situation with reflection is not easy to assess as for any of the many worked examples I have done I keep discovering more and more reflections with increasingly complex logic involved in their derivation. Much proliferation still occurs, in my experience, even with constraints that are intended to limit the number of reflections.

An immediate observation is that if a computer is used to generate a tree of arguments and display it for a user or users then that tree will be cluttered, complicated and difficult to understand if *any* reflection is involved or allowed. The kinds of automation I have in mind here is tools such as (Efstathiou & Hunter, 2008) for computing arguments and Araucaria (Reed & Rowe, 2004) or Argumed (Verheij, 2003) for displaying them. I thus draw the conclusion that even just for the purpose of clarity and understandability in argumentation visualisation it is necessary to remove or prevent reflections. Even if the tool only validates arguments, or even only records arguments, it will still become cluttered and unmanageable if reflected arguments are introduced, i.e. are allowed to or can be introduced, by one or more debaters. Of course the human facilitator could provide a filter to reject reflected arguments, according to some 'common sense' definition of reflection. In truth I believe that is what happens, to quite a degree, in practice with computer-assisted debates - that it is the facilitator intervening to provide the intelligence. Clearly it is modelling this human intelligence to apply 'common sense' that is at the heart of my thesis.

In earlier sections I have proven that given their prerequisites reflections always exist. Proposition 4.4.1 shows that every attack is subject to a reflected undercut. Propositions 4.7.2 shows that every attack is also subject to a reflected rebuttal, which is trivial if the attack is a rebuttal, but not if the attack is an undercut. Furthermore, Proposition 5.8.1 shows the existence of additional up-tree reflections. The proofs just referenced show that the attack chains are not just long, they are infinite (which is not disputing the proof in (Besnard & Hunter, 2001) that argument trees are finite). In doing examples by hand to identify all of the reflections even just two or three attacks away from the root, I have found the exercise to be considerably harder than one might expect: there are so many reflections. There also may be other forms of reflection involved that I have not discovered. More research is needed to identify if any other types of reflection exist and ideally to prove completeness, i.e. that all possible types have been discovered.

Infinite attack chains cannot be easily judged, certainly not, I would suggest, by any sensible, meaning fitting with a easily understandable motivational intuition, deterministic existential judge. (I do note at this point that an admissible semantics, such as the grounded semantics (Dung, 1995), can handle infinite attack chains but is non-deterministic.) In an existential judge alternating levels of the tree, (in terms of number of vertices between a vertex and the root), are alternately defeated or undefeated. So the question arises: are the attack chains even or odd. Is infinity an even or an odd number? (It is neither). Therefore for existential judges to have any credibility some no-recycling constraint (NRC) must be used to make judged trees finite. While it is not the case that an NRC removes all reflection, it is the case that the existence of a judge with decidable outcomes means that some form of NRC must have been used to make the infinite finite. Thus the choice of NRC is intimately tied up with the question of how to handle reflection. (I still argue that pruning out reflection and the choice of the NRC are separable concepts, even though I acknowledge they are related).

Given the points so far, I am now able to move to another main point regarding how much reflection to remove. Logically there are just three possible situations relating to the inclusion or participation of reflection in judgement and debate outcome: none, all or some. Option A, *none* means that the tracking and judgement are such that *all* reflections are removed and thus that *no* reflections are considered. This is the aim of this thesis, this is the matt-opaque situation. The second option, option B is that the tracking and judgement is such that *no* reflections are removed and thus that *all* reflections are considered. This option I have shown to be impossible as infinite structures cannot be sensibly judged. Or alternatively option C has the tracking and judgement being such that *some* reflections are removed and thus that *some* reflections remain and are considered by the judge. It is this 'some' situation that I believe is the only logical alternative to all, and which I have shown, by analysis in the last chapter, to exist in the literature.

A further key conclusion is that the question here thus becomes 'What are the merits of removing *some* reflections versus removing *all*'. Simply disregarding reflection is not an option as it leads to infinite structures and undecideability. This suggestion of undecideability deserves a proof, which is a matter for further research. Consequently the issue of NRC is fundamental to the question of does reflection alter judge outcomes. The choice of NRC is somewhat confounded by the approach I outline here to address the core question of this section. I do not know of a superior approach, however it remains that there are difficulties in the way the NRCs of (García & Simari, 2004; Besnard & Hunter, 2001) and of this thesis interact with reflection in the approach outlined. To sum up my position so far: my debate tree is free from all forms of reflection known to me, is finite and thus is judgeable. Any tree with all reflections present is simply not sensibly judgeable. Thus to be judgeable at least some reflections need to be removed, which begs the question as to which reflections and how many should be removed. Where does one stop?

Returning now to my core question from the first paragraph of this section (does reflection change debate outcomes), the challenges are that a) with my tree there is no reflection, so with what to compare it to be able to say that the removal of reflection makes a difference? My analysis has clearly selected the argument tree as the reflection-containing debate tracking structure of focus. So how to judge the argument tree? The (García & Simari, 2004) existential judge would appear to be an obvious choice for the judge. But then the question arises about how to handle the differences between the trees of (García & Simari, 2004) and (Besnard & Hunter, 2001). The (García & Simari, 2004) tree includes rebuttals as edges, which I accept as a robust approach, however the (Besnard & Hunter, 2001) tree does not include rebuttals as edges - except at the root where rebuttals do appear, by courtesy of their argument structure (of Definition 6.2.2). The (García & Simari, 2004) tree uses defeasible logic and the set of proof rules available is defined as limited, so the resultant reflections would appear to be still extant, but not as prolific as in other examples. In contrast, the paper (Besnard & Hunter, 2001) and its successors do not appear to provide an existential judge function. Thus this comparison is still somewhat unsatisfactory: my set of casting and connecting devices seems at best to be complex and at worst unnecessarily complex and implausible. What I describe can thus only be described as inspired by (García & Simari, 2004) and not as a map or accurate portrayal of their work. Clearly, further research is warranted to understand the effects of reflection on judgement. What I can offer is early results that suggest the answer to the question is that reflection can alter debate outcomes.

### 7.4.1 Steps for Examining Debate Outcomes With and Without Reflection

| Judge Type $\rightarrow$<br><br>Debate Tracking Structure $\downarrow$ | Existential Judge<br>$\mathsf{judge}_{3.0}()$ is Definition 7.5.2<br>from (García & Simari, 2004) | Quantitative Judge<br>$\mathsf{judge}_{6.0}()$ is Definition 7.6.4<br>from this thesis |
|---|---|---|
| With Reflection | argument structure - Definition 6.2.2<br>of argument trees - Definition 6.2.3.<br><br>Example 7.5.1 with its Figure 7.1<br>is judged by $\mathsf{judge}_{3.0}()$.<br>Outcome: $\alpha$ | argument structure cast<br>to be a tree of contradictions<br>by cast two.<br>Example 7.7.1 with its Figure 7.3<br>$\mathsf{judge}_{6.0}()+$ cast two $= \mathsf{judge}_{7.0}()$<br>Outcome: $\alpha$ |
| Without Reflection | debate tree cast<br>to be a tree of arguments<br>by cast one - Definition 7.5.3.<br>Example 7.5.2 with its Figure 7.2<br>$\mathsf{judge}_{3.0}()+$ cast one $= \mathsf{judge}_{4.0}()$<br>Outcome: $\emptyset$ | debate tree - Definition 6.7.7<br>(same situation as<br>implementation of Section 8.5).<br>Example 7.7.2 with its Figure 7.4<br>is judged by $\mathsf{judge}_{6.0}()$.<br>Outcome: $\emptyset$ |

Table 7.1: Analysis Showing that Reflection Changes Debate Outcomes

Given this preamble I now examine the outcome of debates with and without a consideration of reflection. The current section is thus directly parallel to Section 3.2.2 'Undercuts can change debate outcomes'. The question of whether refection can change outcomes has two aspects, i.e. the two kinds of judge: existential (considered in the immediately following section) and quantitative (considered in Section 7.7). Table 7.1 summarises my analytical approach and preliminary findings. The example I have selected to focus on for this 'proof by example' is Example 6.5.1 with its Figure 6.8.

The data structures of the argument tree, argument structure and debate tree have already been defined in the last chapter and provide debate tracking with and without reflection. I will subsequently define two tree-to-tree casting functions, which I call cast one and cast two, to allow like-to-like comparisons whereby the same judge can be applied to virtually the same tree with and without reflection.

## 7.5 Reflection and Existential Judge Outcomes

The following existential judge function for a tree of arguments is adapted from (García & Simari, 2004). While the tree of (García & Simari, 2004) uses defeasible logic, their judge appears to be more widely applicable. I say 'appears to be' as it is not fully apparent exactly how it is to be applied to argument trees and further research is warranted. This interpretation of (García & Simari, 2004) to obtain an existential judge for trees from (Besnard & Hunter, 2001) is somewhat speculative and not as robust as the main body of this thesis.

I now restate (García & Simari, 2004)'s Procedure 5.1 and follow it with an existential judge that seeks to turn their Procedure 5.1 into a judge function.

**Definition 7.5.1.** *Let $\mathcal{T}_{\langle A,\alpha\rangle}$ be a tree, where the vertices are arguments derivable from $\Delta$, the edges are attacks and the root vertex argument is $\langle A,\alpha\rangle$ having support $A$ and claim $\alpha$. Let $\mathcal{T}^*_{\langle A,\alpha\rangle}$ be the same tree, but with each vertex being able to be marked as 'D' for defeated or 'U' for undefeated. The process to mark the tree is such that:*

1. *All leaves in $\mathcal{T}_{\langle A,\alpha\rangle}$ are marked as 'U's" in $\mathcal{T}^*_{\langle A,\alpha\rangle}$.*
2. *Let $\langle B,q\rangle$ be a non-leaf vertex of $\mathcal{T}_{\langle A,\alpha\rangle}$.*
   *$\langle B,q\rangle$ will be marked as 'U' in $\mathcal{T}^*_{\langle A,\alpha\rangle}$*
   *iff every child of $\langle B,q\rangle$ is marked as 'D' in $\mathcal{T}^*_{\langle A,\alpha\rangle}$.*
   *$\langle B,q\rangle$ will be marked as 'D' in $\mathcal{T}^*_{\langle A,\alpha\rangle}$*
   *iff it has at least one child marked 'U' in $\mathcal{T}^*_{\langle A,\alpha\rangle}$.*

In reusing the above for other trees, how to make the jump from marking one García and Simari tree that has a root argument for $\alpha$, and edges of rebuttal and undercut, to judging all trees for $\alpha$ and $\neg\alpha$ where edges are just undercut requires some interpretation. A quotation from (García & Simari, 2004) is that '*In DeLP a literal $h$ will be warranted if there exists a non-defeated argument structure $\langle A,h\rangle$*.' This quotation is then defined as their Definition 5.2 on their page 119 as follows '*Definition 5.2 (Warranted literals) Let $\langle A,\alpha\rangle$ be an argument structure and $\mathcal{T}^*_{\langle A,h\rangle}$ its associated marked dialectical tree. The literal $h$ is warranted iff the root of $\mathcal{T}^*_{\langle A,h\rangle}$ is marked as 'U'. We will say that $A$ is a warrant for $h$.*' I interpret that quotation to mean 'In DeLP, a literal $\alpha$ will be warranted if there exists a non-defeated argument $\langle A,\alpha\rangle$ at the root of a tree $\mathcal{T}^*_{\langle A,\alpha\rangle}$.'

The García and Simari root is only one argument for $\alpha$, so given that a professional debate may have multiple arguments for $\alpha$ and $\neg\alpha$ there will be two forests of trees, one forest for $\alpha$ and one forest for $\neg\alpha$. Any one of the García Simari trees for $\alpha$ will take into account all of the trees for $\neg\alpha$ as the roots of those trees for $\neg\alpha$ will appear as rebuttal attacks on the root of the tree for $\alpha$. How one (García & Simari, 2004) tree for $\alpha$ should stand for all the other trees for $\alpha$ is I suggest an open question. In lieu of further research I follow the quote above so that if there exists an undefeated tree for $\alpha$ then $\alpha$ is warranted, regardless of the state of any other trees for $\alpha$ or $\neg\alpha$.

**Definition 7.5.2.** *Let $\Delta$ be a knowledgebase of labelled or unlabelled assumption formulae. The* $\mathsf{judge}_{3.0}(\alpha, \Delta, \emptyset)$ *procedure for judging $\mathcal{T}_{\langle A,\alpha\rangle}$ is such that:*

*If the root of at least one tree for $\alpha$ is 'U' then* $\mathsf{judge}_{3.0}(\alpha, \Delta, \emptyset) = \alpha$.

An implication of Definition 7.5.2 is that if i) the roots of all trees for $\alpha$ are 'D' or there are no trees for $\alpha$ and ii) the roots of all trees for $\neg\alpha$ are 'D' or there are no trees for $\neg\alpha$ then given i) and ii) $\mathsf{judge}_{3.0}(\alpha, \Delta, \emptyset) = \emptyset$. However, we know that if the root of at least one tree for $\alpha$ is 'U' then $\mathsf{judge}_{3.0}(\alpha, \Delta, \emptyset) = \alpha$. Therefore if the marking for one root is 'U' the markings of other roots become irrelevant (courtesy of the inclusion of rebuttals as edges in the García Simari tree). Also aggregating together the markings for many roots is probably not necessary. I use this $\mathsf{judge}_{3.0}(\alpha, \Delta, \emptyset)$ in my later analysis, even though I am not entirely comfortable with its application to argument trees, as they do not include rebuttals as edges.

### 7.5.1 Situation with Reflection and Existential Judgement

I now revisit my earlier Example 6.5.1, which showed that argument trees are inchoate, and judge it with judge 3.0. This is a manageable example to use as it has no arguments for $\neg\alpha$ and thus has considerably less reflection than if that were the case.

**Example 7.5.1.** *This example reuses the knowledgebase of Example 6.5.1. See the left side of Figure 6.8 for the resultant unmarked argument tree. This example then judges that tree with the existential García and Simari-like judge 3.0. The resultant marked tree is shown as Figure 7.1 below. As the root is undefeated the judge outcome is that the motion is undefeated so the marked tree shows that* $\mathsf{judge}_{3.0}(\alpha, \Delta, \emptyset) = \alpha$.

$$\begin{array}{c}
\{I : \alpha\}^U \\
\uparrow \\
\{J : \neg(\beta \wedge (\beta \rightarrow \alpha))\}^D \\
\uparrow \\
\{L : \neg(\gamma \wedge (\gamma \rightarrow \neg\beta))\}^U
\end{array}$$

Figure 7.1: Marked Argument Tree - Contains Reflection - Outcome $\alpha$

To see if reflection affects the outcome it is now necessary to look at the same knowledgebase but without reflection.

### 7.5.2 Situation without Reflection and Existential Judgement

In order to apply judge 3.0 to reflection free debate trees in a way that permits direct comparison to its application above to argument trees I propose the approach of i) not changing the judge itself, ii) casting the debate tree into a form suitable for judge 3.0 and then iii) applying judge 3.0 directly to that cast tree.

Hence the comparison should be like-with-like. To cast a debate tree into a tree of arguments that is in a form suitable for judge 3.0 the following procedure is used. First I establish some basic nomenclature to label the parts of a tree of contradictions and parts of a tree of arguments. The nomenclature shown in Table 7.2 below is used in subsequent definitions to refer to different parts of a contradiction tree. The nomenclature and analysis focusses on an arbitrary vertex $v$ in the tree.

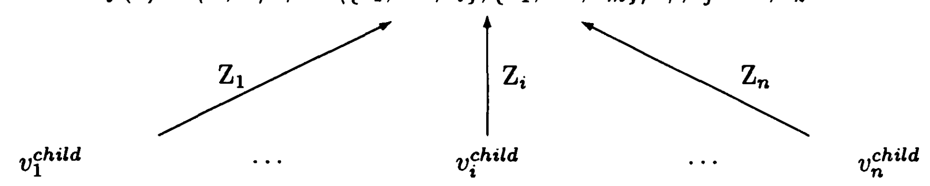
| | |
|---|---|
| 1. | Let $v \in \mathcal{V}$ be an arbitrary vertex in a contradiction tree $\blacklozenge^*(\alpha, \Delta) = \langle \mathcal{V}, \mathcal{E}, f, \blacklozenge(\Delta) \rangle$. |
| 2. | Let $f(v) = \langle \mathrm{X}, \mathrm{Y} \rangle : \phi \in \blacklozenge(\Delta)$, where $I_j \in \mathrm{X} = \{I_1, \ldots, I_l\} \in \mathcal{C}$, and where $J_k \in \mathrm{Y} = \{J_1, \ldots, J_m\} \in \mathcal{C}$. |
| 3. | Let the children vertices of $v$ be $v_1^{child}, \ldots, v_n^{child}$, where $v_i^{child} \in \{v_1^{child}, \ldots, v_n^{child}\}$, with each child connected to $v$ by a labelled edge $\langle v_i^{child}, \mathrm{Z}_i, v \rangle$ equating to an intra-contradiction preclusive undercut $\langle f(v_i^{child}), \mathrm{Z}_i, f(v) \rangle$. |
| 4. | Showing these nomenclature points diagrammatically as part of a contradiction tree gives: Arbitrary vertex $v$, where $f(v) = \langle \mathrm{X}, \mathrm{Y} \rangle : \phi = \langle \{I_1, \ldots, I_l\}, \{J_1, \ldots, J_m\} \rangle : \phi, I_j \in \mathrm{X}, J_k \in \mathrm{Y}$  |

Table 7.2: Tree Nomenclature: An Arbitrary Vertex $v$ in a Debate Tree

**Definition 7.5.3.** *Let $\blacklozenge^*(\alpha, \Delta)$ be a debate tree. The* **cast one** *function casts a debate tree into a forest of trees of arguments as follows:*

1. *Starting with the root of the debate tree apply the following procedure for an arbitrary vertex thereby creating one cast tree for each argument $I_j : \phi$ or $J_k : \neg\phi$ in that debate tree root.*

2. *Procedure for an arbitrary vertex. For each argument $I_j : \phi$ or $J_k : \neg\phi$ in the arbitrary vertex v of the debate tree, for each attack on that argument:*

   *(a) If the attack is a multi-target preclusive undercut then do not create any child argument in the cast tree for that attack,*

   *(b) If the attack is a mono-target preclusive undercut then for each argument $J : \psi$ in the attacking confirmation $\mathrm{Z} : \psi$, $J \in \mathrm{Z}$ create a child vertex mapped to the argument $J : \psi$ in the cast tree (the connecting edge is a canonical undercut),*

   *(c) If the attack is a rebuttal then for each argument attack create a child vertex mapped to the reflected canonical undercut from the reflecting rebutting argument.*

3. *Repeat for each vertex in the debate tree working down from the root towards the leaves, using the same steps as defined above for the arbitrary vertex, until each and every debate tree leaf is*

*reached. Apply the no recycling constraint of the argument tree as each new prospective child node is identified.*

I use the identifier $\mathsf{judge}_{4.0}(\alpha, \Delta, \Theta)$ for the combination of cast one with $\mathsf{judge}_{3.0}(\alpha, \Delta, \Theta)$, and give no separate definition. As this fourth judge reuses the third judge the two can be directly compared and contrasted to see the effects of reflection. Note that all attacks in the cast tree are mono-target and thus edge labels do not add information and are therefore excluded. I now show the marked cast debate tree after being judged by judge 4.0. As the root is defeated the outcome is that $\mathsf{judge}_{4.0}(\alpha, \Delta, \Theta) = \emptyset$.

**Example 7.5.2.** *This example reuses the knowledgebase of Example 6.5.1 and the resultant debate tree seen on the right of Figure 6.8. This example then uses the cast one function to cast that debate tree into a forest of trees of arguments, which in this case is a single tree, and then judges that tree with the existential García and Simari-like judge. The combination of cast one and judge 3.0 is judge 4.0. The resultant marked tree is shown as Figure 7.2. As the root is defeated the judge outcome is that the motion is defeated so the marked tree shows that* $\mathsf{judge}_{4.0}(\alpha, \Delta, \emptyset) = \emptyset$.

$$
\begin{array}{c}
\{I : \alpha\}^D \\
\uparrow \\
\{J : \neg(\beta \land (\beta \to \alpha))\}^U \\
\uparrow \\
\{K : \beta \land (\beta \to \alpha)\}^D \\
\uparrow \\
\{M : \neg(\beta \land \pi \land \neg\gamma \land (\pi \to (\beta \to \alpha)))\}^U
\end{array}
$$

Figure 7.2: Marked Argument Tree - Without Reflection - Outcome $\emptyset$

Thus the implication is that reflection can change debate outcomes. The question remains is this modification of outcome by reflection unique to existential judges or does it also occur with quantitative judges.

## 7.6 Judging a Debate Tree

Having covered existential judges I now move on to quantitative judges. My purpose is to enquire if quantitative judges, like existential judges are also sensitive to reflection. Before introducing my own reflection-aware judge function I start by providing some context from the literature. This published function is relevant as it provides some of the inspiration for the structure of my judge 6.0.

### 7.6.1 Undercut Aware Quantitative Judgement in the Literature

This quantitative discussion from the literature has the property of being undercut aware. The categoriser function below delivers a normalised real number in the range 0 to 1. The Besnard Hunter judge 5.0 is defined in four steps using the accumulator and categoriser functions of Definitions 8.10 and 8.11 in (Besnard & Hunter, 2001).

**Definition 7.6.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae. The* **Besnard Hunter quantitative** *judge, denoted* $\mathsf{judge}_{5.0}(\alpha, \Delta, \emptyset)$*, is a judge such that:*

1. *Given a tree of arguments, with root $R$ being an argument for $\alpha$, the* **h-categoriser** *function, denoted h, returns a real number $h(R) \in \mathbb{R}$ defined recursively by:*

$$h(N) = \frac{1}{1 + h(N_1) + \ldots + h(N_l)},$$

*where $N_1, \ldots, N_l$ are the children vertices for N, and*

*if $l = 0$ then $h(N_1) + \ldots + h(N_l) = 0$.*

2. *Applying the h-categoriser to each argument tree in an argument structure provides a* **categorisation** *for $\alpha$, which is a pair of multisets of real numbers, written*

$$\langle [X_1, \ldots, X_n], [Y_1, \ldots, Y_n] \rangle, \text{ where } X_1, \ldots, X_n, Y_1, \ldots, Y_n \in \mathbb{R}$$

3. *Given the categorisation for $\alpha$, the* **log accumulator** *function, denoted $l(\langle X, Y \rangle)$ returns two real numbers $\alpha^+, \alpha^- \in \mathbb{R}$ such that:*

$$l(\langle X, Y \rangle) = (log(1 + X_1 + \ldots + X_n), log(1 + Y_1 + \ldots + Y_m)) = \{\alpha^+, \alpha^-\},$$

$$\text{where } X = [X_1, \ldots, X_n] \text{ and } Y = [Y_1, \ldots, Y_n].$$

4. *The two real numbers determine the membership of* $\mathsf{judge}_{5.0}(\alpha, \Delta, \emptyset)$*, such that:*

$$\text{If } \alpha^+ > \alpha^- \text{ then } \mathsf{judge}_{5.0}(\alpha, \Delta, \emptyset) = \alpha.$$

$$\text{If } \alpha^+ < \alpha^- \text{ then } \mathsf{judge}_{5.0}(\alpha, \Delta, \emptyset) = \neg\alpha.$$

$$\text{If } \alpha^+ = \alpha^- \text{ then } \mathsf{judge}_{5.0}(\alpha, \Delta, \emptyset) = \emptyset.$$

A feature of this judge is that finite chains of defeaters never fully defeat the argument at the head of the chain. Furthermore there is no binary distinction between odd and even length chains, so this judge presents a contrast to judges 3.0 and 6.0, which do provide such a binary distinction.

### 7.6.2 Quantitative Judge Aware of Confirmation, Preclusion and Reflection

I now define a judge function that processes the debate tree. It thus automatically excludes reflected arguments from its deliberations. It also includes consideration of preclusive undercut and applies a threshold-based filter to exclude uncorroborated points or noise. This tree judge is an extension of or elaboration upon the Franklin judge described earlier coupled with the intuition that in essence an attack by an undercut has the same force as an attack by a rebuttal. It is a recursive function defined in two steps, firstly to determine the argumentative strength of a vertex in the tree and then secondly to use that within a standard judge function. First a simple helper function to set negative numbers to zero and some further nomenclature for trees of contradictions.

**Definition 7.6.2.** *Let $r \in \mathbb{R}$. The positive function, denoted* $\mathsf{pos}(r)$, *takes a real number of any value and returns a non-negative real number such that if* $r < 0$ *then* $\mathsf{pos}(r) = 0$ *else* $\mathsf{pos}(r) = r$.

The arguments in $\langle X, Y \rangle : \phi = f(v)$ are subject to attacks from the children of $v$, however not every child attacks every argument in the parent. The arguments that are attacked are as followed:

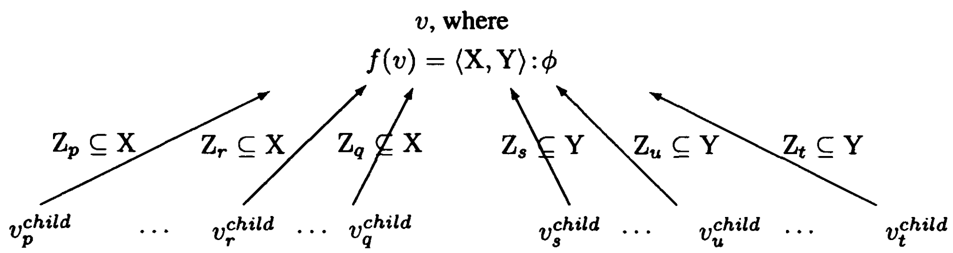
| | |
|---|---|
| 1. | Let argument $I_j$ be undercut by each $v_r^{child} \in \{\{v_p^{child}, \ldots, v_q^{child}\} \subseteq \{v_1^{child}, \ldots, v_n^{child}\} \mid I_j \subseteq Z_r \subseteq X\}$. |
| 2. | Let argument $J_k$ be undercut by each $v_u^{child} \in \{\{v_s^{child}, \ldots, v_t^{child}\} \subseteq \{v_1^{child}, \ldots, v_n^{child}\} \mid I_j \subseteq Z_u \subseteq Y\}$. |
| 3. | These distinctions of which child vertices attack which arguments area clarified by the graph:  |



Table 7.3: Tree Nomenclature: Attacks on an Arbitrary Vertex $v$ in a Debate Tree

Given this nomenclature is it now possible to succinctly define a judge function for the debate tree. The two steps of definition are, firstly to define the strength of a vertex (which is the same as the strength of the claim of the stripped contradiction of the vertex) and then secondly to use that function in judge 6.0.

**Definition 7.6.3.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae and let $v \in \mathcal{V}$ be a vertex in a contradiction tree $\blacklozenge^*(\alpha, \Delta) = \langle \mathcal{V}, \mathcal{E}, f, \blacklozenge(\Delta)\rangle$. The* **strength of the proposition** $\mathsf{stripContradiction}(f(v))$*, denoted* $\mathsf{strength}(v, \blacklozenge^*(\alpha, \Delta))$*, is a real number showing the argumentative strength of* $\phi = \mathsf{stripContradiction}(f(v))$ *in its context in the tree such that:*

$$
\begin{aligned}
&\mathsf{strength}(v, \blacklozenge^*(\alpha, \Delta)) \\
&\quad = \sum_{j=1}^{j=l} \mathsf{pos}(1 - \sum_{r=p}^{r=q} \mathsf{pos}(\mathsf{strength}(v_r^{child}, \blacklozenge^*(\alpha, \Delta))) \div |\mathrm{Z}_r|) \\
&\quad - \sum_{k=1}^{k=m} \mathsf{pos}(1 - \sum_{u=s}^{u=t} \mathsf{pos}(\mathsf{strength}(v_u^{child}, \blacklozenge^*(\alpha, \Delta))) \div |\mathrm{Z}_u|).
\end{aligned}
$$

The strength of an argument in a leaf vertex $v$ is always 1 (prior to the consideration of rebuttals) as it is never weakened by any undercuts. The strength of a leaf vertex, however, is an integer that can be either positive, zero or negative depending on rebuttals, i.e. on the cardinalities of X and Y where $f(v) = \langle \mathrm{X}, \mathrm{Y}\rangle : \phi$. Thus the strength of a leaf vertex in isolation is a direct parallel to the earlier Franklin judge.

I now extend the signature of the judge function to include an item of metaknowledge as a third parameter that can be passed in. Metaknowledge refers to any additional knowledge available about the situation which is not contained in $\Delta$. Keeping $\Theta$ as separate from $\Delta$ allows $\Delta$ to be the same for all judge functions, while the nature of $\Theta$ is allowed vary depending on the subscript $j$. In the following judge, where j is 6.0, the metaknowledge is a threshold which is a real number controlling the filtering of noise.

**Definition 7.6.4.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, let $v_{root} \in \mathcal{V}$ be the root vertex in a contradiction tree $\blacklozenge^*(\alpha, \Delta) = \langle \mathcal{V}, \mathcal{E}, f, \blacklozenge(\Delta)\rangle$ and let* $\mathsf{threshold} \in \mathbb{R}$*. The* **debate tree judge**, *denoted* $\mathsf{judge}_{6.0}(\alpha, \Delta, \mathsf{threshold})$*, is a judge function such that:*

$$
\begin{aligned}
\alpha \in \mathsf{judge}_{6.0}(\alpha, \Delta, \mathsf{threshold}) \quad &\textit{iff} \quad \mathsf{strength}(v_{root}, \blacklozenge^*(\alpha, \Delta)) > \mathsf{threshold}, \\
\neg\alpha \in \mathsf{judge}_{6.0}(\alpha, \Delta, \mathsf{threshold}) \quad &\textit{iff} \quad \mathsf{strength}(v_{root}, \blacklozenge^*(\neg\alpha, \Delta)) > \mathsf{threshold}.
\end{aligned}
$$

It follows that if $\mathsf{strength}(v_{root}, \blacklozenge^*(\alpha, \Delta)) \leq \mathsf{threshold}$ then $\alpha \notin \mathsf{judge}_{6.0}(\alpha, \Delta, \mathsf{threshold})$. Likewise if $\mathsf{strength}(v_{root}, \blacklozenge^*(\neg\alpha, \Delta)) \leq \mathsf{threshold}$ then $\neg\alpha \notin \mathsf{judge}_{6.0}(\alpha, \Delta, \mathsf{threshold})$, so the judge can never imply inconsistency. The judge threshold thus acts as a filter to remove uncorroborated evidence.

One criticism of this judge, and its underlying strength function, is that it could be said to read more into the information relayed by a preclusive undercut than is perhaps strictly valid. A preclusion says A or B may be false, which I interpret as meaning that A and B are equally weakened. However when more information becomes available it could be that B is flawless and A is false. I would argue that my scheme allows for that new information to affect the debate tracking and outcome, but that I make best use of the available information.

It would also be valuable to extend this judge to be cumulative. In judging a debate tree a non-cumulative judge will look at only one motion $\alpha$ and may produce an outcome that is inconsistent with other debates run on the same knowledgebase. A cumulative judge would have to be more sophisticated in examining many motions and hence is left as an area for future research.

## 7.7 Reflection and Quantitative Judge Outcomes

I approach this question of 'Are quantitative judges skewed by reflection?' in two steps, firstly by looking at the situation with reflection and then without reflection. While I focus on Example 6.5.1 to show the impact of reflection, further examples of reflection changing judge outcomes when judged with a quantitative judge involve applying my judge to Examples 6.3.1 and 6.3.3. In Example 6.3.1 the outcome with reflection, following the approach of an argument tree cast into a tree of contradictions by cast two and judged with my judge is the empty set, but once reflection is removed the outcome changes from $\emptyset$ to $\alpha$ - effectively the chain of two arguments becomes a chain of three arguments. In Example 6.3.3 the outcome changes from $\alpha$ with reflection to $\emptyset$ without.

### 7.7.1 Situation with Reflection and Quantitative Judgement

Rather than give a formal definition of cast two the following example should be simple enough for the mapping to be clear - and as proof by example is being used a general cast two function is not required. If there is a future need for such a cast two function in the further research of reflection then clearly a general definition would be necessary. Judge 7.0 is defined as the combination of cast two and judge 6.0.

**Example 7.7.1.** *This example reuses the knowledgebase of Example 6.5.1 and the resultant argument tree on the left of Figure 6.8. Thus* $\Delta = \{a\!:\!\beta, b\!:\!\beta \rightarrow \alpha, c\!:\!\gamma, d\!:\!\gamma \rightarrow \neg\beta, e\!:\!\beta \wedge \pi \wedge \neg\gamma, f\!:\!\pi \rightarrow (\beta \rightarrow \alpha), g\!:\!\neg\beta \vee \neg\pi \vee \gamma \vee \neg(\pi \rightarrow (\beta \rightarrow \alpha))\}$ *and* $I = \{a, b\}, J = \{c, d\}, K = \{e, f\}, L = \{e\}, M = \{g\}$. *This example then casts that argument structure into a tree of contradictions using the cast two function. It is not a debate tree as it contains reflection. The casting in this case produces a single tree. The example then judges that tree with the quantitative judge 6.0 of Definition 7.6.4. The resultant tree, annotated with vertex strengths, is shown as Figure 7.3. As the root strength is 1 the judge outcome is that the motion is carried so the annotated tree shows that* $\mathsf{judge}_{7.0}(\alpha, \Delta, \emptyset) = \alpha$.

$$\begin{array}{ll}
v_{root}, f(v_{root}) = \langle\{I\}, \emptyset\rangle\!:\!\alpha & \mathsf{strength}(v_{root}, \blacklozenge^*(\alpha, \Delta)) = 1 \\
\quad\uparrow & \\
v_1, f(v_1) = \langle\{J\}, \emptyset\rangle\!:\!\neg(\beta \wedge (\beta \rightarrow \alpha)) & \mathsf{strength}(v_1, \blacklozenge^*(\alpha, \Delta)) = 0 \\
\quad\uparrow & \\
v_2, f(v_2) = \langle\{L\}, \emptyset\}\!:\!\neg(\gamma \wedge (\gamma \rightarrow \neg\beta)) & \mathsf{strength}(v_2, \blacklozenge^*(\alpha, \Delta)) = 1
\end{array}$$

Figure 7.3: Argument Structure cast by Cast Two - Contains Reflection - Outcome $\alpha$

### 7.7.2 Situation without Reflection and Quantitative Judgement

Now I provide the final example in this set of four.

**Example 7.7.2.** *This example reuses the knowledgebase of Example 6.5.1 and the resultant debate tree seen on the right of Figure 6.8. Thus* $\Delta = \{a\!:\!\beta, b\!:\!\beta \rightarrow \alpha, c\!:\!\gamma, d\!:\!\gamma \rightarrow \neg\beta, e\!:\!\beta\wedge\pi\wedge\neg\gamma, f\!:\!\pi \rightarrow (\beta \rightarrow \alpha), g\!:\!\neg\beta \vee \neg\pi \vee \gamma \vee \neg(\pi \rightarrow (\beta \rightarrow \alpha))\}$ *and* $I = \{a, b\}, J = \{c, d\}, K = \{e, f\}, L = \{e\}, M = \{g\}$. *This example covers the normal situation that I recommend for professional debate and also as is used for my design of Section 8.5. It is my standard debate tree judged by my judge. No casting is involved.*

$$
\begin{tikzcd}
v_{root}, f(v_{root}) = \langle\{I\}, \emptyset\rangle : \alpha & \mathsf{strength}(v_{root}, \blacklozenge^*(\alpha, \Delta)) = 0 \\
v_1, f(v_1) = \langle\{J\}, \{K\}\rangle : \neg(\beta \wedge (\beta \rightarrow \alpha)) \arrow[u, "\{I\}"'] & \mathsf{strength}(v_1, \blacklozenge^*(\alpha, \Delta)) = 1 \\
v_2, f(v_2) = \langle\{M\}, \emptyset\rangle : \neg(\beta \wedge \pi \wedge \neg\gamma \wedge (\pi \rightarrow (\beta \rightarrow \alpha))) \arrow[u, "\{K\}"'] & \mathsf{strength}(v_2, \blacklozenge^*(\alpha, \Delta)) = 1
\end{tikzcd}
$$

Figure 7.4: Debate Tree - Without Reflection - Outcome $\emptyset$

These two examples show quantitative judge outcomes of $\alpha$ with reflection and $\emptyset$ without reflection, establishing that quantitative judgement too is affected by reflection.

That concludes this outlining of the need for further research into the effects of reflection. The next chapter will touch on another possible source of reflection, that of extending an argumentation framework to have a 'bipolar' nature.

## 7.8 Closing Remarks and Conclusion for How To Judge a Debate

This chapter provides the preliminary result, given certain assumptions, that reflection can change debate outcomes - as illustrated by reanalysis of earlier Examples 6.3.1, 6.3.3 and 6.5.1. The existence of changes to debate outcomes appears to be the case regardless of whether quantitative or existential judges are used. More research is warranted to go more deeply into this point. Even in the simplest of debates the nature and number of all the reflections becomes a complex matter that is hard to compute manually in a way that provides confidence of completeness. A computer is thus required even for relatively simple debates.

It is my 'matt opaque contradiction tree', which for short I call a 'debate tree' that I propose as a practical tool for tracking all of the arguments in a debate whilst avoiding redundancy. The debate tree is the basis of the mature judgement of Section 7.6 and the tool design of Section 8.5.

The above list summarises the judge functions defined and discussed in this thesis. Another judge of importance in the literature and not in the above list is called 'net support'. Net support is documented

| Judge Function | Definition | Reference | Nature |
|---|---|---|---|
| $\mathsf{judge}_{1.0}(\alpha, \Delta, \emptyset)$ | Definition 2.7.4 | (Benferhat et al., 1993) | Existential |
| $\mathsf{judge}_{2.0}(\alpha, \Delta, \emptyset)$ | Definition 2.7.5 | Franklin (Willcox & Bridgewater, 1977) | Quantitative |
| $\mathsf{judge}_{2.1}(\alpha, \Delta, \emptyset)$ | not formally defined | Franklin plus undercut | Quantitative |
| $\mathsf{judge}_{3.0}(\alpha, \Delta, \emptyset)$ | Definition 7.5.2 | (García & Simari, 2004) | Existential |
| $\mathsf{judge}_{4.0}(\alpha, \Delta, \emptyset)$ | cast one + $\mathsf{judge}_{3.0}()$ | Extends (García & Simari, 2004) | Existential |
| $\mathsf{judge}_{5.0}(\alpha, \Delta, \emptyset)$ | Definition 7.6.1 | (Besnard & Hunter, 2001) | Quantitative |
| $\mathsf{judge}_{6.0}(\alpha, \Delta, \mathsf{threshold})$ | Definition 7.6.4 | This thesis | Quantitative |
| $\mathsf{judge}_{7.0}(\alpha, \Delta, \emptyset)$ | cast two + $\mathsf{judge}_{6.0}()$ | Extends Definition 7.6.4 | Quantitative |

Table 7.4: Judge Functions in Thesis; 1.0 and 2.0 are Undercut Unaware

in (Fox & Das, 2000) and mentioned in several earlier papers by Fox et al. Net support is an enhanced from of the Franklin judge, $\mathsf{judge}_{2.0}(\alpha, \Delta, \emptyset)$. While this judge is not undercut aware it has been used in production argumentation applications for the UK's National Health Service.

Finally I would suggest that identifying and proving potential applications for the quantitative argumentation-based debate tracking and judging functions described in this thesis would be a area of future research warranting attention. This quantitative approach has the key benefit of being better able to resolve stalemates, i.e. inconsistencies, than approaches which ignores the number of arguments and are only sensitive to the existence of arguments. Clearly more work is needed on the theoretical underpinnings of the quantitative approach, however the use of quantitative methods in practical debates is a good motivation for further investigation and application.

---

# Chapter 8

# Conclusion and Discussion

This first part of this chapter summarises areas for further research and dwells on the framework refinement of assisting arguments drawn from the bipolar approach. The chapter then touches on the design of an implementation. The latter part of the chapter goes on to draw conclusions about the contributions made herein. This is a discussion chapter so any new concepts introduced are not particularly formal, thereby leaving their formalisation as an area of further research.

## 8.1 Overview of Chapter

The body of this chapter is made up of six sections:

**Further Research** Section 8.2 summarises, with a little discussion, areas for further research drawing from the many earlier suggestions throughout this thesis.

**Short List** Section 8.3 provides a short list of areas, taken from the previous section, that are likely to be particularly interesting and fruitful for research. These topics could potentially provide valuable contributions to the field.

**Assisting Arguments** Section 8.4 describes a possible and desirable framework refinement to cover an assisting relationship between arguments to compliment the more established attacking relationship.

**Implementation** Section 8.5 outlines the design of a tool that tracks and judges debates. It is intended to be operated by a skilled facilitator in support of a group of professionals engaged in a professional debate.

**Design Rationale** Section 8.6 suggests the utility of this tool for the areas of decision rationale, design rationale, decision trade-off and design trade-off. A worked example is given of decision rational taken from the field of corporate acquisition.

**Contributions and Limitations** Section 8.7 summarises the various contributions made by this thesis and examines their strengths and weaknesses. A useful angle for describing the weaknesses has been to identify the limitations of contributions, which in turn folds back to the areas of further research.

Of these areas I am most drawn to developing an implementation and using it in real professional debates, with the aim of then clarifying requirements and obtaining insight into which areas for future research that would be most useful.

## 8.2 Areas for Further Research

I now summarise the areas for further research identified in earlier chapters. I then pick out a few which I feel are particularly important. The sequence of presentation here is the same as the sequence of chapters in the thesis, and is not sorted by gravity or import.

**Introduction - Chapter 1.** The introduction made clear that several topics are outside the scope of investigation of this thesis. They can therefore be seen as major topics for further research.

**Computational Complexity.** There are a number of facets to the computations presented herein that would merit investigation. Finding all of the arguments for a claim is crucial, however if there is also a need to find all of the confirmations or all of the contradictions for a particular knowledgebase then these too would merit quantification. A starting place likely to yield results would be to map the work of Dunne and Bench-Capon onto that of this thesis.

**Algorithms and Heuristics.** Algorithms and heuristics is a promising argumentation topic as it appears to be possible to transform apparently intractable problems into ones that are manageable. Work here could commence by investigating the research by Efstathiou and Hunter, integrating it in with that of this thesis.

**Dialectical Analysis.** The idea of optimising the winning of debates by controlling the choice and sequence of arguments presented merits further research. Further work could either follow the legal path, building on writings of Prakken and Sartor or it could take a purer form following in the wake of such authors as Dung, Kowalski and Toni.

**Priorities or Preferences.** Again a huge area for future research in argumentation, priorities could be worked into the approach of this thesis. Leaders in this field to draw from would be Modgil, Amgoud and Cayrol.

**Probabilities or Possibilities.** Casual observers from outside the academic discipline of argumentation often comment that its greatest weakness is not being able to handle different weights of arguments. The idea that one argument can be stronger than another, and to provide some numeric ratio of their strengths is not well developed in the formal argumentation literature. Clearly my thesis touches a little on this topic by determining the weight of a proposition, however, observers talk of adding weights to the knowledgebase and also to the arguments derived from the knowledgebase, not just to the conclusions of debates. This facet of argumentation is particularly under-developed, with only elementary pointers for future direction having been provided by such authors as McBurney, Chesñevar, Simari, Godo, Haenni, Kohlas and Lehmann.

**Basic Framework - Chapter 2.** The most useful area for further research in this chapter is probably the notion of extending the basic framework to other logics, notably first order predicate logic. Others include the following:

- Explore the algebra of labels within this labelled deductive system, especially for explanation and provenance.
- Subsume other published argumentation frameworks.
- Extend the framework to cover first order predicate logic. This may be a challenging piece of work, however it could yield valuable results.
- Extend the framework to cover other logics, such as defeasible, temporal and modal logics. This would be likely to be a large piece of work which could yield interesting results.
- Use a part of each assumption label to track which agent or expert originated that assumption.
- Relate confirmation to the concept of 'extensions to a theory' in abstract argumentation. The basic insight here for this item of research is minor, however any insights into how to better link abstract and concrete argumentation are likely to be welcomed by the field.
- Identify a more succinct way to define confirmations and related aggregations. While improving the basic definitions here is a good idea this point is unlikely to yield profound results.
- Exploring differences, similarities and a possible integration between a) judge functions as proposed herein and b) Dung semantics would be a particularly useful area for further research.

**Undercut Framework - Chapter 3.** It is surprising to me that the literature contains so little consensus on precisely what constitutes an undercut. I would be keen to know just how useful or constraining the various approaches to undercut documented in this thesis are when applied to professional debates with human debaters.

- More research could be done to further compare and contrasting Wigmore, Pollock, canonical and preclusive undercuts.
- Review the literature further to see if there are any other published forms of undercut.
- Analyse bridging assumptions between concrete and abstract argumentation approaches. This examination could focus on bridges to and from the mincon classical form of concrete argumentation. The initial investigation of bridging assumptions could be without a consideration of reflection.
- How abstract argumentation could be extended or refined to incorporate the practical reality of preclusion is an interesting question. I call this idea 'abstract preclusion'.

**Reflection Between Arguments - Chapter 4.** I find it surprising that no one else has documented reflection before, albeit that there are pointers in this direction coming from Besnard, Hunter, Dung, Kowalski and Toni. It is the implications of reflection for judging debates or evaluating semantics that I find most fascinating. Specific points identified earlier for further research are:

- Questions around the existence or possible existence of reflection in other argumentation frameworks from the literature are likely to be quite important. The ABA framework is a valuable place to start as this thesis examines only one instance of ABA. Note that the bulk of this thesis, except for Section 4.3, is not precisely an instance of ABA due to differences in the definition to attack.
  - What other forms of reflection exist within the ABA framework?
  - What are the minimum sets of proof rules needed for each form of reflection?
  - What alternative minimum sets of proof rules can be used to show each reflection?
  - Which of the non-monotonic logics that are special cases of the ABA framework permit which forms of reflection?
  - Which semantics are affected by the observation that given the right prerequisites each attacked argument is automatically subject to counterattack and thus reinstatement?
  - Which of the semantics referred to in ABA papers are affected by the existence of which forms of reflection and what is the effect?
- Defeasible logics may also contain reflection and therefore also warrant such evaluation.
- An area for further research would be to generalise all of the reflection functions in this thesis to include Wigmore and Pollock undercuts as well as canonical and preclusive undercuts. While this would make the topic tidier from an intellectual perspective it could, however, make it even more complicated and harder to understand.
- It would be worth deepening the understanding of bridging assumptions in the light of reflection between arguments.

**Reflection Between Confirmations - Chapter 5.** It should be emphasised that what gives reflection merit for further research is the behaviour of reflections between confirmations.

- Of key interest is the question of whether there are further forms of reflection beyond those described herein.
- Proving that all forms of reflection have been identified would be of particular value.
- Deepening the understanding of the various forms of reflection, both identified herein and possible others, is important.
- One specific question is to examine more closely the role of labels in scaling and see which scaling situations would not occur if labels were removed.
- A broader question is to return to the issue of scaling and see if there are any robust ways of catering for it while still allowing reflected arguments to exist within the debate tracking. I have to admit that I feel this is unlikely theoretically, and unlikely to be helpful in applications

as a) filtering out reflections is fundamentally easy and b) including them clutters debates making them hard to follow.

- One possible area for future research would be to investigate more fully the properties of an abstract confirmation attacks functions, including reflected confirmation attacks, direct confirmation attacks and maximal direct confirmation attacks. Such an approach might build on the work of Dung, working at at abstract level of attacks, but employing confirmations rather than individual arguments as the graph nodes. This point would build on that of abstract preclusion mentioned earlier in this section.

**Tracking a Debate - Chapter 6.** From the perspective of the application of argumentation to professional debate, the question of how best to track a debate is particularly important. The survey (Kirschner et al., 2003) shows a variety of ideas in this field, and comparison of that survey with the more formal research, for example at COMMA '08, shows quite a contrast between informal and formal approaches.

- Study the debate tracking trees and graphs in the literature and establish definitively which contain which types of reflection and which, if any, are free of such reflections. I flag this item as important.
- This analysis should clearly include any further forms of reflection not described herein.
- A more robust approach, with a stronger theoretical basis, is needed to select the form of the vertices and edges of the optimum debate tracking tree.
- As is common in AI, one form of tree and graph may not be ideal for all applications, so it is worthy to qualify under what conditions which form is most suited.
- The properties of the debate tree should be better understood.
- One specific focus is to investigate the dialogue tree of (Amgoud & Cayrol, 2002) and see if the abstract proof procedures presented there can be more closely combined with the concrete argumentation and attack forms discussed in the same paper.
- The question of what no recycling constraint (NRC) is best to keep a debate tracking tree finite merits work.
  - Should the NRC be distinct from the mechanism that filters out reflection or is overlap acceptable?
  - Is one form of NRC good for all applications or are different NRCs best for different situations?
  - How profoundly and in what specific ways does the choice of NRC affect the behaviour and properties of debate tracking trees?
- The topic of syntax sensitivity merits further research: to what degree and in what ways does the way the knowledgebase is written affect the form of the debate tracking tree or graph?

**Judging a Debate - Chapter 7.** My own view is that the definition of judge functions and the way to select the optimum judge function for a particular application is surprisingly underdeveloped in the literature. I suggest that the role of reflection in the process of judgement is a fundamental step to get right, before other subtleties of judgement can be properly evaluated.

- Does reflection affect debate outcomes?
  - When judged with a quantitative judge?
  - How to best structure an existential judge for a tree or forest?
  - When judged with an existential judge?
  - How to better cast between different forms of debate tracking tree?
  - How to better cast between debate tracking graphs and trees?
- How can these questions be proved definitively for all cases and not just with an example of a single case?
- Assuming there is a provable effect, can the situations and boundaries where the effect occurs and doesn't occur be better understood?
- Are there ways to better motivate, formalise and place on a stronger theoretical under-footing the area of judgement, judge properties and metaknowledge.
- Identify and formalise further judge properties, both desirable and undesirable.
- Deepen the understanding of cumulativity.

## 8.3 Further Research Shortlist

Although a number of these items suggest building stronger bridges with the abstract and semi-abstract forms of argumentation I would suggest the particularly interesting areas are where the work of this thesis takes a radically different direction from abstract and semi-abstract work. This thesis brings to awareness and even to question several basic assumptions which the abstract and semi-abstract work appears to be making. These apparent abstract assumptions which I ponder over include:

**Abstract preclusion.** In abstract argumentation an argument can only attack one argument, not more than one argument. My preclusive undercuts, however, attack more than one argument.

**Confirmation graphs and contradiction graphs.** Vertices in a debate tracking graph should be single arguments, not confirmations or contradictions. The debate tree and related motivational work on preclusions brings this assumption into question.

**Graded defeat.** An argument is either fully defeated or fully undefeated. This rigid binary simplicity does not seem to fit well with the intuitions of quantitative judges both herein and in the informal and formal literature.

**Judgement of claims.** Judging arguments is considered to be important in the abstract literature, whereas judging claims is not. Professional debates and the judges discussed focus on the claim or motion not on any particular argument or set of arguments.

There appears to have grown up an acceptance of the 'semantics' approach with its 'extensions to a theory' as the way to judge debates. In contrast, theoretical work on judging motions in professional debates is considerably less developed. I agree that the work on semantics has a profound mathematical elegance, however, from the needs of applications I would suggest more be done to create robust formal underpinnings for quantitative judging with its graded defeat and preclusions.

My view is that papers on the semantics approach are perhaps a bit too ready to accept stalemates yielding the empty set and not so well equipped in resolving stalemates and hence in resolving inconsistency. If there are many arguments both for and against a claim then I would suggest that the full defeat of a subset of individual arguments via undercut should not be the only mechanism available to decide the overall status of that claim.

In concluding this shortlist, I flag this area that questions the underpinnings of abstract argumentation and its semantics as probably the most valuable one for future research highlighted by this thesis.

## 8.4 Assisting Arguments

The topic of catenate inferences and what I call assisting arguments, touched upon in Sections 2.4.1 and 7.3.8, merits further development not least because it is quite prevalent in the informal argumentation literature. Wigmore appears to have been the first to introduce the distinction between 'affirmatory and negatory' forces of arguments within a debate (Wigmore, 1937; Twining, 1985; Reed & Rowe, 2006). Also, see the more recent (Kirschner et al., 2003) for an argumentation visualisation survey with many examples of works using informal assisting arguments. This notion has also begun to be explored in the formal argumentation literature (Amgoud et al., 2004; Cayrol & Lagasquie-Schiex, 2005) where the topic is now commonly referred to as 'bipolarity'. As outlined in Section 7.3.8 my conceptual approach to 'bipolarity' is to view confirmation as one form of positive argument interaction and assist as a second form of positive argument interaction, as summarised in Table 8.1 below.

| **Bipolar Relationship →**<br>**Argument Connection ↓** | **Positive** | **Negative** |
|---|---|---|
| Claim-to-Premises | Assist | Undercut |
| Claim-to-Claim | Confirmation | Rebuttal |

Table 8.1: Summary of Bipolar Functions

Thus rebuttal can be viewed as a claim-to-claim negative relationship and confirmation as a claim-to-claim positive relationship. Likewise undercut is a claim-to-premises negative relationship and assist a claim-to-premises positive relationship. The following brief but formal examination shows these assisting arguments to have some perhaps counter-intuitive properties that make them not quite as straightforward as might initially appear. My findings reinforce the general controversy in the literature over the wisdom of including bipolarity into an argumentation framework. My suggestion is that although, from a user requirements perspective, they are highly desirable (see (Kirschner et al., 2003) for support for this point of view), more work is required to create a concrete bipolar argumentation framework where assisting arguments have intuitive behaviour.

The avenue I suggest pursuing in this formalisation is that of 'assisting arguments' as follows: given an argument $\{\beta, \beta \to \alpha\} \vdash \alpha$ then the argument $\{\gamma, \gamma \to \beta\} \vdash \beta$ is an assisting argument. It is an assisting argument because it reinforces one of the assumptions of the original argument.

Assisting argument definitions could be formalised to match and extend any or each of the undercut definitions covered in Chapter 3. The Wigmore assist would be a variant on the Wigmore undercut, see Definition 3.2.1, the Pollock assist a variation of the Pollock Undercut, Definition 3.2.2, the canonical assist would build on the canonical undercut of Definition 3.3.2 and the preclusive assist would build on preclusive undercut Definition 3.4.2. To transform one of these undercut definitions into an assist definition would primarily be a matter of removing the negation symbol. This simple approach in isolation might not be that satisfactory for canonical and preclusive assists as the assisting argument would need a consequent of the conjunction of all the premise formulae of the assisted argument, which would be a challenging requirement to meet in real world debates. Hence the subset or equals sign approach of the Pollock undercut would seem to be more appropriate than the conjunction of the full support of the suggested canonical and preclusive assists. Hence:

**Definition 8.4.1.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae, and let $I : \alpha, J : \phi_1 \wedge \ldots \wedge \phi_n \in \mathsf{arguments}(\Delta)$.*

$J : \phi_1 \wedge \ldots \wedge \phi_n$ *is a* **Pollock assist** *of* $I : \alpha$ *iff* $\{\phi_1, \ldots, \phi_n\} \subseteq \mathsf{stripAssumptions}(\mathsf{formulae}(I, \Delta))$.

I call $J : \phi_1 \wedge \ldots \wedge \phi_n$ the **assisting argument** and $I : \alpha$ the **assisted argument**. I say that this pair $I : \alpha, J : \phi_1 \wedge \ldots \wedge \phi_n$ form an **assisting chain** of arguments. Given this intuitive definition of assist, which would appear to match the informal usage in (Kirschner et al., 2003) for example in its Villawood 'one person one job' example, it can be seen that several issues arise. These issues would need to be addressed before such assists could become part of the framework and part of some yet to be defined 'bipolar debate tree'. The first issue is the potential of inconsistency in the chain of assisting arguments as illustrated in Example 2.4.5, which I call **catenate inconsistency**. The second issue is that of a form of reflection that could be called a Type V Reflection, which I define as follows:

**Definition 8.4.2.** *Let $\Delta$ be a knowledgebase of labelled assumption formulae. The set of* **reflected arguments of the fifth type** *arising from the assist of $J : \beta$ on $I : \alpha$, denoted $\mathsf{reflect}_5(J : \beta, I : \alpha, \Delta)$, is such that:*

$$
\begin{aligned}
& \mathsf{reflect}_5(J : \beta, I : \alpha, \Delta) \\
= \;& \{K : \alpha \in \mathsf{arguments}(\Delta) \mid \\
& \quad I : \alpha, J : \beta \in \mathsf{arguments}(\Delta), \\
& \quad J : \beta \textit{ is a Pollock assist of } I : \alpha, \\
& \quad K \subseteq (I \cup J) \textit{ and } K \neq I\}.
\end{aligned}
$$

In this definition it can be seen that the reflected argument $K : \alpha$ and the argument at the head of the chain $I : \alpha$ have the same claim, $\alpha$. Thus there are at least two members, $\{I\} : \alpha, \{K\} : \alpha$, of $\Diamond(\alpha, \Delta)$

making it appear that there is more evidence for $\alpha$ than there actually is - which is an issue. Here is an example:

**Example 8.4.1.** *Let* $\Delta = \{a:\beta, b:\beta \to \alpha, c:\gamma, d:\gamma \to \beta\}$. *Given the assisted argument* $\{a,b\}:\alpha$ *and the assisting argument* $\{c,d\}:\beta$ *there exists the Type V reflected argument* $\{b,c,d\}:\alpha$. *So in this example* $\beta$ *is both an assumption of one argument* $\{a,b\}:\alpha$ *which is the assisted argument and a claim of another* $\{c,d\}:\beta$ *which is the assisting argument. Furthermore it can be seen that* $\{\{b,c,d\}\}:\alpha, \{\{a,b\}\}:\alpha \in \Diamond(\alpha,\Delta)$ *thereby suggesting that there is more evidence for* $\alpha$ *than might be intuitively thought.*

As shown earlier, any such redundancy can have an effect on the conclusions of existential or quantitative judges. If the chain of assisting arguments is longer than two members the number of Type V Reflections permutations increases rapidly.

Further research is needed to analyse this and other possible kinds of reflection that may occur with assists and to prove that they always exist, given their prerequisites. Further research is also needed on the optimal nomenclature to show assisting arguments. A useful destination for this further research would be to provide a bipolar debate tree, together with any supporting definitions needed to make it robust in practical use.

## 8.5 Implementation

I now propose a software design to implement my theory as a tool to support professional debate. My tool's goals are, relative to manual methods, to reach conclusions that are a) better-thought through and b) less error prone, with c) a documented audit trail in d) a more efficient fashion. The emphasis is not on either side trying to win, but rather on ensuring that all involved are well informed and on deepening the collective understanding of the problem at hand.

There exist several implementations of automated argumentation tools, for example Oscar by Pollock, (Pollock, 1992; Pollock, 2000), Verheij's Argue! (Verheij, 1998) and ArguMed, (Verheij, 1999), Reed's Araucaria (Reed & Rowe, 2004), Gaertner and Toni's CaSAPI (Gaertner & Toni, 2007), Krause, Bryant and Vreeswijk Argue tuProlog (Bryant et al., 2006) and the Aspic software from www.argumentation.org. All of these tools have applicability to debate tracking and judging, however none of them appear to incorporate reflection as analysed herein.

The only absolute distinction between my design and others that have been developed in this area is thus that this one is aware of reflection. Relative distinctions are that my design is more automatic than most (users only have to provide assumptions) while at the same time having more attention to argument visualisation and ease of use. Consequently this tool should have a greater ability to resolve inconsistency and filter out uncorroborated evidence, making it, arguably, closer to professional debate in practice.

Of the broadly comparable tools in the literature, some emphasise the user interface and provide less automatic underpinnings (Zeno by (Gordon & Karacapilidis, 1997), Argue! and ArguMed by (Verheij, 1998; Verheij, 1999; Verheij, 2003) and Araucaria by (Reed & Rowe, 2004)) than my proposed tool. In others emphasis is placed on interoperability (Rahwan et al., 2007), which is a worthy but separate goal. Yet other tools in the literature have strong automation but less apparent emphasis on ease of use via argumentation visualisation (e.g. CaSAPI (Gaertner & Toni, 2007)), or additionally having the automation cover the full classical propositional logic rather than the subset of defeasible reasoning (e.g. Oscar (Pollock, 1992; Pollock, 2000)). An area of further research is to obtain copies of these and comparable tools and further study their strengths and weaknesses. Two areas (Prakken & Sartor, 2002) not tackled by the proposed tool are those of a) automating the procedural control over the flow of the debate and b) helping optimise the choice of arguments presented using rhetoric and other means to maximise persuasion. The design aim of my tool is also to provide a superior visualisation of the debate, both because a) my theory allows a single tree to cover a full debate arguably more richly (by including preclusion) and simply (by excluding reflection) than others and b) leveraging recent developments in the theory and user interface tooling for argumentation visualisation and information visualisation.

In the proposed tool, the computer does the work of building the tree from assumptions, notably computing arguments, confirmations, contradictions (and hence rebuttals), preclusive undercuts, building the debate tree vertices and edges and also deciding the outcome of the debate. In contrast to some approaches in the literature the tool does not require the distinction of proponent and opponent or prosecution and defence. Instead all of the users are debaters, recognising that many professional debates involve a large number of stakeholder parties whose needs often form a relatively complex web of complimentary and contradictory points. The tool has no procedural controls thereby avoiding the constraints of dialectical approaches with their use of alternating moves by two players.

I now describe system design in two stages, describing i) the system exterior and ii) the system interior. The system exterior section describes the functionality delivered by the tool to its users. I use the Unified Modelling Language (UML) approach of use cases to document the system functions. The system interior is documented as the components or modules that are needed to deliver the desired function. I employ the UML approach of the package diagram to show the system interior. These two provide the high level system design, that as a next step must be complimented with a detailed low level design, prior to building the system itself.

### 8.5.1 System Exterior - Use Cases

Each use case is a task or job performed by a user with assistance from the computer - it is a unit of functionality documented here at a fairly high level of abstraction. I start with a use case diagram showing the actors, system boundary, use cases and functional flows that occur outside the system.

**Maintain Knowledgebase**. A key job of the facilitator is to maintain the knowledgebase. He or she does this by listening to proposed assumptions from the debaters and then mapping the assumptions into the language of the knowledgebase. Within this use case there are several aspects including maintaining the list of debaters, each with the debater's full name and initials. Each assumption in the knowledgebase is associated with the initials of the debater who proposed it and a unique assumption label number. An assumption is built from a combination of one or more 'atomic formulae' connected by zero or more logical operators to make a 'compound formula'. All atomic formulae are also valid compound formulae. The logical operators are 'and', 'or', 'negation' and 'implication'. Brackets are also available to show

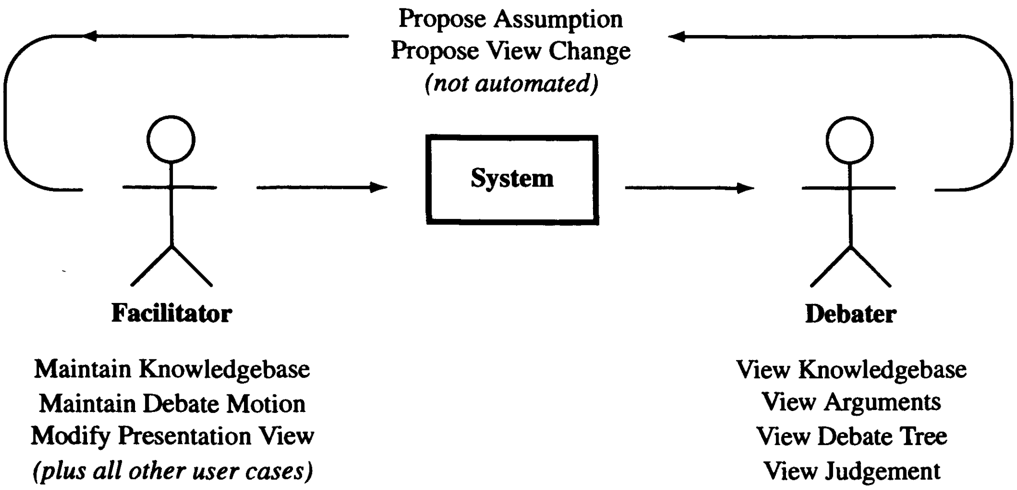



Figure 8.1: Use Case Diagram of a Debate Facilitation Tool

precedence explicitly. Each atom has a symbol plus a linguistic gloss, where the gloss is a phrase or sentence of ordinary English. Thus each assumption also has a gloss built up from the gloss of the atoms and logical operators it is comprised of. The word 'maintain' in the title of this use case is used here to cover adding, editing and deleting assumptions.

**Maintain Debate Motion.** Each and every debate has to have a motion. The motion is an atomic or compound formula, which is separate from the knowledgebase and from the set of arguments as the motion is neither an assumption nor a deduction. The default situation is to have one motion, however it is also possible to have more than one motion under debate at a time, with the given that all motions under debate are underpinned by the same knowledgebase. It is typically only at the outset of the debate that the facilitator selects the motion. One way to select a motion is to enter one or more assumptions into the knowledgebase, allow the system to create the desired argument from those assumptions and then use the resultant claim as the motion of the debate. Hence selecting the motion is not a time consuming part of the facilitators job.

**Modify Presentation View.** Another key part of the facilitator's job is to modify the presentation view seen by all users, i.e. the single view that is shared by the debaters and facilitator. I envision that a projector, or more specifically a bank of three projectors, be used to show the debate tracking on walls or large screens so all see the same information. Controlling the view assists in controlling the focus and flow of the debate. The presentation view is modified i) by changing which window is at the front and therefore not obscured and ii) by panning (i.e. scrolling) and zooming. When a three dimensional view of the debate tree is employed then it is also possible to move the position of the viewer and to modify the direction the viewer is looking in.

The view has the following windows or visible items: 1) list of debaters (with short name and full name), 2) list of atomic formulae (with symbol and gloss), 3) knowledgebase 4) arguments (with assumptions and claim), 5) confirmations built from arguments with the same claim, showing maximum cardinality confirmations, 6) contradictions showing rebuttals between confirmations and hence rebuttals between arguments, 7) preclusive undercuts and most importantly 8) the debate tree.

The facilitator also has access to all of the other user cases in the system and may give verbal instructions to the debaters to pause or continue debating while he or she catches up with the data entry and view adjustment.

**View Knowledgebase.** The knowledgebase is a list of assumptions represented as compound formulate built from atomic formulae and logical connectives (each having proposer, compound logic symbols and computed gloss). The knowledgebase is displayed as a table and a choice may be made to see a more textual or a more mathematical view. A fully textual view would require further work in the area of natural language, but a more mathematical view might confuse some users. One workaround would be to allow the facilitator to edit the textual gloss of assumptions and arguments to make them into more everyday English.

**View Arguments.** Given the set of assumptions, the system determines what arguments follow. The list of arguments can be filtered to either show a) by default, just those direct arguments that actually appear in the debate tree or trees under debate or b) those in a) plus all of the reflected arguments that have been pruned out of debate tree or trees under debate or c) all practical arguments that are derivable from the knowledgebase. More research will be needed to define the term 'practical'; here I mean for example that if $I:\alpha \in \mathsf{arguments}(\Delta)$ then $I:\alpha \vee \beta$, $I:\alpha \vee \beta \vee \gamma$ and $I:\alpha \vee \gamma \vee \beta$ may not be 'practical'. As the automated derivation of arguments is time consuming any of these lists may not necessarily be complete. New arguments are added to the displayed list and may affect the tree and other windows in real time as they are computed.

**View Debate Tree.** Central to the abilities of the system is the display of the debate tree. Different levels of granularity of display are available with the ability to focus on contradictions, confirmations, arguments, assumptions, linguistic gloss, claims, assumption labels, deduction labels, confirmation labels, rebuttals or preclusive undercuts. The strength of each proposition is shown on the display as computed in real time by the judge function. I envision that brightness, seen by the eye as rear illumination by a small light bulb, is used to represent the strength of an argument and strength of a vertex within the tool's visual language.

**View Judgement.** The most important feature of the system is that it shows, in real time, who is winning the debate. Whenever a new argument is added to the debate tree then the judgement is re-evaluated and re-displayed. The judge outcome can be that the motion is carried, defeated or that the there is no outcome. No outcome may be due to a stalemate, the absence of compelling evidence (where threshold has done its noise filtering work) or just the absence of evidence.

Debaters have no direct hands-on way to control the debating tool, hence they must provide their inputs to the facilitator who then enters them into the tool. Proposing assumptions for inclusion in the debate is the main way that debaters are allowed to contribute. The system does not allow an argument to be entered directly as it derives them from the assumptions given. Debaters are encouraged not to propose complete arguments, but if they do it is the facilitators job to break these arguments down into assumptions. Debaters may also request the view to be changed. If contributions are occurring more rapidly than the facilitator can absorb then the facilitator must request the debaters to pause and be quiet until he or she has caught up; a visual cue might assist this pausing.

### 8.5.2 System Interior - Packages

Now that the system exterior has been described I turn to the system interior and document its high level design as a series of packages. The UML approach of the package diagram is akin a modular or component design diagram. My design bridges a) an emphasis on a rich argumentation visualisation user interface with b) the automated discovery of arguments and population of the debate tree.

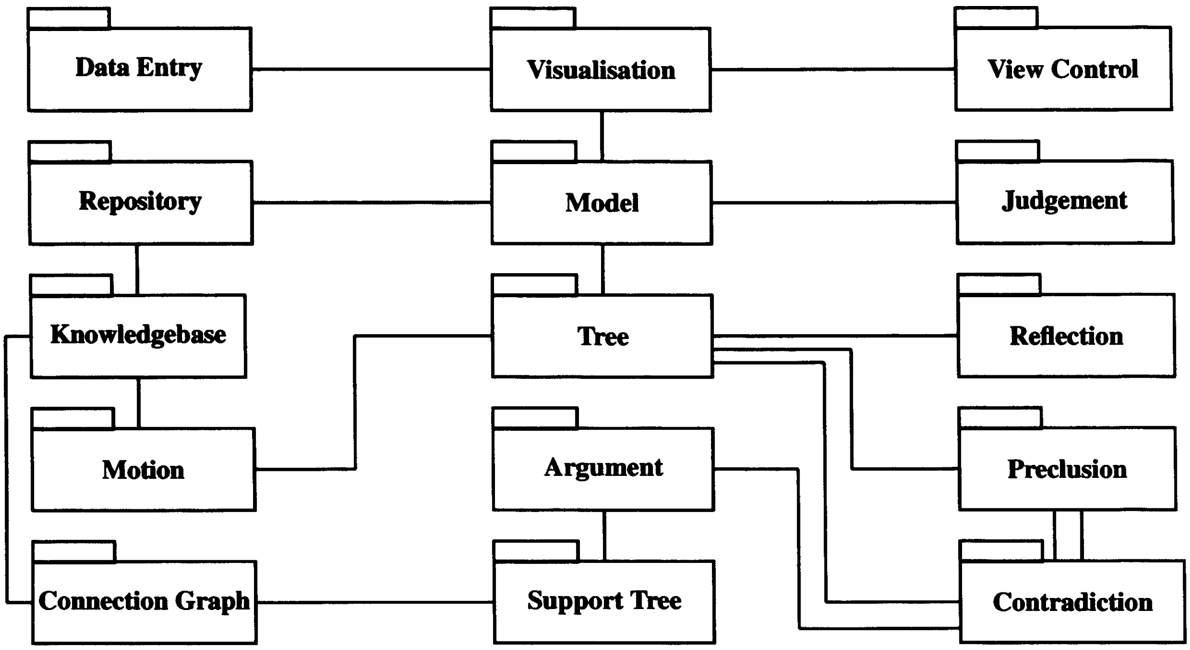



Figure 8.2: Package Diagram of a Debate Facilitation Tool

In describing the system interior I relate it to use cases so that the sequence of messaging from package to package is explained. Package names here begin with a capital letter. The design within the packages is assumed to be an object-oriented class structure. The package to package lines in the diagram are somewhat informal as some represent 'contains', some the flow of data and others the flow of control. Also the direction of flow and whether it is a one way or bidirectional flow is not shown. The diagram also does not show the concurrent nature of multiple processes, with requests passing one way and various argument aggregations passing the other along process pipelines.

The tool starts with an empty Knowledgebase and an empty Repository. The Repository contains the Knowledgebase and also holds all of the other information about the debate, most of which can be seen listed as other package names.

**Motion**. The debate motion is initially empty and has to be specified. The motion is held within the Repository, but not in the Knowledgebase as that holds only the set of labelled assumption formulae. Adding a motion starts with the Data Entry package, and ends with Visualisation displaying the debate's motion. The Data Entry package then talks to the Model, with that communication mediated by Visualisation. Model then puts the motion into the Repository for the sake of persistence. Once the motion is in Model then the view aspect of Visualisation picks it up and the users see it. At this stage the argument generating engine has no assumptions to work on and is thus dormant. The motion does however prime the argument generating engine by putting requests in to Argument for sets of assumptions that prove or disprove the motion. Throughout this design the passing of events and requests will be handled by the observer software design pattern, with listeners registering their interest in classes of event when that listener is instantiated.

**Model - View - Controller (MVC) Pattern.** The MVC pattern is an established way of addressing the underpinnings for a rich user interface. The controller, here split between View Control and Data Entry, listens to events, namely input events from user input devices, such as mouse and keyboard and interprets that input using a state machine that tells it what to do with that input. The Model package contains the data to be displayed in a format that is readily accessible and closely matching that needed for display. On the one hand the Model can contain everything in the Repository, but it also enhances and unpacks that information particularly populating a representation of the virtual three dimensional space visible to the users. The view of MVC manages the display and in this tool takes the form of the Visualisation package. The other event driving the controller is when the model changes due to argumentation computations. A candidate tool for underpinning the MVC is the Struts package from the Apache Software Foundation. MVCs event management will also use the observer pattern.

**Visualisation.** The visualisation package is built on the metaphors of argumentation visualisation and information visualisation to provide an easy to understand graphical user interface. Information visualisation, see for example (Chalmers, 2003), is a collection of user interface design techniques that deliver an information-rich display of many inter-related series of data at one time. Thus there are multiple dimensions of data shown concurrently, some of which are visually modelled as the three dimensions of space. Such visualisation allows users to more quickly gain an intuitive grasp of complex data sets. The information involved in tracking and judging a debate is a good example of such a complex data set. Argumentation visualisation can be seen as specialisation within information visualisation focussed on how to best display the debate tracking and judging information so that it is a) usable, i.e. easy to understand and b) useful, i.e. that productivity gains are achieved from the system. The landmark book (Kirschner et al., 2003) provides several user interface metaphors; I particularly like the displays of Chapter 5 on the Villawood example, especially the one on page 113, as one way of seeing the debate tree. Much of the argument tool literature focusses on the level of showing isolated or individual arguments, but an important exception is (Reed & Rowe, 2006) that advocates using Wigmore's approach to tracking a debate in a diagram.

The ArgDF documentation (Rahwan et al., 2007) suggests a rich three-dimensional way of showing a tree of arguments is to have the root at the centre of a sphere and the other vertices positioned to be spread neatly across the surface of a series of concentric spheres. I propose adopting visualisation that builds on the sphere style and within that the simplicity of the Villawood approach. OpenGL is a candidate graphics library to support these two and three dimensional display metaphors, because of its support for lighting and perspective. OpenGL provides an application programming interface (API) for a rich library of graphics primitives and has bindings to the prevalent languages. Another candidate graphics library is the more recent evolution on top of OpenGL contained in the Apple Macintosh Leopard operating system, known as Core Animation.

**Adding Assumptions to Knowledgebase.** The next step in the debate is for assumptions to be added. The MVC takes care of assumption entry along the same lines as motion entry, with the distinction that assumptions are stored in the Knowledgebase. There is merit in using an extensible mark-up language (XML) format for arguments, notably Argument Markup Language (AML) originating from SRI International, as used in Araucaria and employed by Douglas Walton, the Open University and others. While the main benefit is interoperability, the use of XML also makes for easy to understand and evolve persistence. The Repository, and its sub-components Knowledgebase and Motion, are therefore sensibly to be implemented with XML. I am minded to use an XML database, noting however that the compute intensive, more real-time, elements of the system probably should not be making frequent accesses to XML. A candidate XML database is the open source Xindice from the Apache Software Foundation. As assumptions are added to the Knowledgebase the argument generating engine is able to move from dormant to active.

**Connection Graph.** The construction of arguments begins with the Connection Graph package, passes through Support Tree and delivers its result as instances of Argument. The pattern employed here is factory. The connection graph concept was introduced by (Kowalski, 1975). The main purpose of the Connection Graph package is to transform the knowledgebase into a structure which is more amenable to searching for the basis of arguments. A basic feature of classical logic is that any formula, however complex, can be represented as an equivalent disjunction of literals, known as a clause, where a literal is an atomic formula or the negation of an atomic formula. The support of any argument can then be represented as a conjunction of these disjunctive clauses; this standardisation or normalisation of formulae is commonly known as conjunctive normal form (CNF). Thus the first step of processing in the Connection Graph package is to use a small set of standards proof rules to render a copy of the knowledgebase into CNF, and to maintain a map between the elements in knowledgebase and CNF items.

The second step of processing in this package is to build a graph, called the connection graph, see (Kowalski, 1975), with the CNF formulae as its vertices. The graph edges are provided by a preattacks function, which means that there is a potential logical relationship between the two vertices. If, for example, one vertex only referred to $\pi$ and $\rho$ and a second vertex only referred to $\lambda$ and $\gamma$ then there would be no edge between them. Having a preattacks relationship is necessary but not a sufficient condition for participation in a valid argument. While the paper (Efstathiou & Hunter, 2008) describes a number of graphs and trees involved in the production of arguments, the special feature of the Connection Graph is that it is the only one that is common to all arguments and all claims. Thus maintaining this data structure as separate from the claim specific structures has merit.

Whenever the knowledgebase is changed the contents of Connection Graph have to be automatically updated so that the two remain synchronised. Change here could mean the addition or deletion of a formula or the revision of an existing formula. The Connection Graph is persisted over the life of the debate. In a typical debate the assumption formulae can be expected to arrive in an asynchronous fashion with many seconds or minutes between each change. Hence the computational task is not to build a large graph from scratch, but rather to incrementally grow and revise an existing graph, and hence the compute times for revision can be expected to be slightly better than the complete computation times given in (Efstathiou & Hunter, 2008); I would suggest that benchmarking is warranted, particularly for the range $100 > |\Delta| > 30$.

**Support Tree**. The Support Tree package has inputs of i) the connection graph and ii) the claim of arguments to be found and an output of valid mincon arguments for that claim. A feature of this package is that it is able to find all of the arguments for a given claim derivable from the knowledgebase. The output can of course can be no arguments. The package is able to tell when it has the complete set of arguments for a claim. The package is also able to compute arguments in a considerably more efficient fashion than brute force examination of every subset of the knowledgebase.

Whenever the knowledgebase changes this package recomputes the set of arguments for each claim. Usually that recomputation yields additional arguments, but if formulae have been deleted or revised in the knowledgebase then this recomputation can result in the removal of previously valid arguments. A second aspect of this computation is that for large knowledgebases (with say in excess of 50 formulae) it can take a long time so additional arguments for a claim can be produced at virtually any time during a debate. A benchmark presented in (Efstathiou & Hunter, 2008) is that the computation time of the work I have described for the Connection Graph and Support Tree packages moves from the human perception of instantaneous processing time to that of perceivable processing time at around 30 formulae on a modest computer. My own evaluations of professional debate shows that 40 formulae is a more typical average size for a modest but practical debate, so the process of use of this system will have to tolerate the late arrival of some arguments.

Within Support Tree the processing of the connection graph is performed in a series of steps, as described in (Efstathiou & Hunter, 2008). These steps, if viewed as a Venn diagram, consist of a set of concentric circles (with the odd exceptions) with the outer circle being the connection graph. Each step thus computes a subset of the output of the previous step. In order of increasing specificity the steps are called 1) connection graph, 2) attack graph, 3) closed graph, 4) focal graph, 5) query graph, 6) presupport tree, 7) consistent presupport tree, 8) minimal presupport tree, 9) support tree. Consistent support tree and minimal support tree are both subgraphs of presupport tree. A support tree is a presupport tree that is both consistent and minimal.

One branch of a support tree yields the support for a valid argument for the given claim. Recalling Definition 2.4.2 of the mincon argument, consistency of the support is ensured by step 7) above, minimality by step 8) and implication of the necessary claim by step 6). The chain of responsibility pattern is employed in the Support Tree package to underpin this pipeline.

**Argument**. The first way to understand the Argument package is as a structural element recording or holding the form of part of a debate tree. The overall structural form is that one debate tree (in Tree) starts with a single formula as its motion (in Motion). The debate tree, recalling Definition 6.7.7, has contradictions mapped to vertices (in Contradiction) and preclusive undercuts as edges (in Preclusion). The intra-package links are thus part of the form of the debate tree, showing the relationships between structural elements. The next level of detail is that one contradiction contains one to many arguments (in Argument). Each of these structural element packages also contains processing functions, but they relate to getting, setting, mapping and the like rather than to the compute intensive tasks. Thus the debate tree itself is formed by Tree, Motion, Contradiction, Preclusion and Argument.

The second aspect of the Argument package is that it marshals the creation of arguments, their filtering by Reflection and the process of growing the tree. Each argument is aware of its status as either having all of its confirmations, undercuts and rebuttals identified or not. If it has more of these yet to be identified it issues requests to Support Tree to find and instantiate them. Argument is able to create many parallel threads, one per claim for arguments sought. Further recent theoretical work in speeding and approximating the computation of arguments is also noted, e.g. (Besnard & Hunter, 2006).

**Reflection**. The Reflection package examines the tree looking for reflections and then flags each argument to show which kinds of reflection it is free from. Only arguments that are free from all kinds of reflection from all reflectors are allowed in the tree that is displayed and judged. Thus as Reflection examines Tree it interacts with two trees, one that contains reflection and one that is matt-opaque. Consequently Reflection is able to not only examine each part of the reflection-containing tree, but also to issue commands to flag each argument with its reflected or direct status, by each type of reflection. These flags then control the instantiation of the matt opaque tree. Reflection runs as another asynchronous process moving data along one of the pipeline steps from the Knowledgebase to the Visualisation.

**Tree**. The implemented debate tree corresponds exactly to my mathematical debate tree, with its vertices mapped to Contradictions and its edges existing as Preclusions. Edges in the tree are labelled to show which target confirmation is attacked within which defending confirmation. All of these links between items are achieved with object-oriented references between instantiated objects and also are persisted within the XML of the Repository. The recursive composite pattern is used to hold the tree itself and thus act as a springboard for recursive programming in population of the tree. The tree also delivers the no-recycling constraint checking which is at least a fifth check on argument validity, beyond the mincon three and reflection, for validity in a given tree. The constraints to be valid confirmation, rebuttal or undercut argument could be seen as further integrity checks. As mentioned under reflection, within the Tree packages are two instantiations of contradiction trees, one without reflection (i.e. matt opaque) and one with reflection.

**Preclusion**. The main feature of the Preclusion package is that each instance creates a pair of links to two contradictions, showing the attacker and the defender. A preclusive undercut is also linked to the various confirmations of importance: for attacker, target and defending confirmations. Thus edge labels are accomplished by means of links. There are separate objects for the three confirmations involved, attacker, target and defender, each with a hierarchy of sub-objects for their confirmation, argument and assumption labels.

**Contradiction**. One Contradiction comprises many Arguments. Within Contradiction is confirmation (not shown in the package diagram). A feature of the Contradiction package is the generation of requests to the argument factory as to what claims are being sought. These claims if they result in viable arguments will deliver i) confirmations of existing arguments ii) rebuttals of existing arguments and iii) preclusive undercuts of existing arguments. Thus each Contradiction $\langle X, Y \rangle : \phi$, exploiting its status as part of the tree, has its own ability to increase the cardinalities of X and Y. Each Contradiction also has the ability to create new children for itself and thus link those child Contradictions into the Tree. Once a Preclusion is validated as being an appropriate part of the tree the established Argument transforms itself into a confirmation and then that confirmation transforms itself into a Contradiction. As touched on under Reflection, each Argument within each Contradiction has contextual flags to indicate its reflection status, so while these flags are manipulated by Reflection they are part of the Tree data structure. Note that a particular Contradiction can appear more than once is a Tree and may thus be instantiated more than once.

**Judgement**. The Judge package reads the tree in a parallel way to Reflection, however it is only able to see the matt-opaque tree not the one containing reflection. Judge is another asynchronous process that recomputes its outcome each time the tree changes, again triggered by the observer pattern. Not all of the lines are shown on the package diagram. The main point is that the judgement is not stored as part of the Tree, but rather resides in the Model that contains the Tree. Some obvious but unobstructive way of showing the current state of the Judgement is necessary within the Visualisation.

### 8.5.3 Implementation - Next Steps

Interesting follow-on questions are: i) Can the tool and facilitator keep the data entry of assumptions in synchronisation with such a team of professionals without too much frustrating pausing? - i.e. do the debaters have to wait for data entry? ii) Can the tool give an indicative debate outcome in near-real time as the debate develops? - i.e. do the debaters have to wait for the computation of the debate tree? iii) What is the probability that the outcome is right after four hours? iv) What is the probability that the outcome is right after seven full days of computation? v) What is the probability that the outcome will change purely as a function of the computation from debate end to a week or a fortnight later? Other key questions are: vi) What exactly is the best user interface metaphor to show the debate tree and its parts? vii) Do users find the tool-facilitator combination easy to work with? viii) Do users find the tool-facilitator combination as useful overall? - i.e. does it increase productivity and improve the quality of analysis? Is it perceived as useful ix) during the debate?, x) only at the end of the debate, xi) at the end of say a week when more computation has been done and the debate has been written up and xii) some time later when the debate tracking and judgement are required for audit or reuse?

These questions have the character of research rather than development, and require answers before widespread application of such debate facilitation tools is likely.

## 8.6 Decision and Design Rationale and Trade-off

I suggest the areas of decision and design rationale and trade-off as ones likely to be fruitful for application of this tool and for further research. Decision rationale can be seen as establishing and documenting the reasons or arguments as to why a particular decision takes the form that it does. Likewise design rationale documents the justification for a particular design. Design trade-off, in contrast, involves the design rationale for more than one design with the aim of finding the optimum design solution for a given problem. Decision trade-off is a new part of the more established field of decision support.

Design rationale and trade-off are aspects of systems engineering which appear to be under-developed parts of the literature and extant tooling. Existing work (Buckingham-Shum, 1994) suggests good potential utility for a tool such as that proposed here. By documenting the reasons for a design or decision several benefits may be achieved, notably:

1. improved efficiency of the design or decision team in their work as their mental processes become explicit, recorded, sharable and more quickly appreciated by other team members,
2. improved speed and quality of design or decision modification in the face of changed requirements,
3. easier reuse and adaptation of similar designs or decisions and
4. faster and smoother communication to independent auditors, assessors or advisors as to the nature of a design or decision and its justification.

I now give a small example of such a decision or design rationale exercise, focussed on the area of corporate acquisition, commonly known as mergers and acquisitions or M&A. This example comes from an actual occurrence and has not been fabricated; only the company name has been changed. The motion of this small debate is $\alpha$ with a linguistic gloss of 'Buy WarmHouse', i.e. that we should buy the company WarmHouse Corporation. The following Table 8.2 shows the atomic formulae involved together with their equivalent linguistic glosses. Appreciate that Table 8.2 is not the knowledgebase, but rather just ingredients that the knowledgebase may be built from. The knowledgebase is provided in three steps, $\Delta_1, \Delta_2$ and $\Delta_3$, below.

While the approach of this thesis is monological in nature, I present this example as a series of steps that could be taken as having a dialectical flavour. This step by step presentation does not change the debate outcome versus starting with the full knowledgebase as no arguments are held back.

An initial business analysis of the situation yielded the initial knowledgebase, $\Delta_1$, showing what was known about the situation at that early stage.

$\Delta_1 = \{a\!:\!\beta, b\!:\!\gamma, c\!:\!\delta \wedge \epsilon, d\!:\!(\beta \wedge \gamma \wedge \delta) \rightarrow \alpha, e\!:\!\zeta, f\!:\!\eta, g\!:\!\theta, h\!:\!\iota, i\!:\!\kappa, j\!:\!(\zeta \wedge \eta \wedge \theta \wedge \iota \wedge \kappa) \rightarrow \lambda, k\!:\!\mu, l\!:\!(\lambda \wedge \mu) \rightarrow \nu, m\!:\!\nu \rightarrow \psi, n\!:\!(\lambda \wedge \mu \wedge \psi) \rightarrow \alpha\}$.

At first sight, with $\Delta_1$, the purchase of WarmHouse appeared to be attractive. There are two arguments, see Table 8.3, in favour of purchasing the company and thus these form a confirmation of the motion $\alpha$.

Further due diligence analysis yielded additional basic information for the knowledgebase giving $\Delta_2$ which is $\Delta_2 = \Delta_1 \cup$ some additional assumptions.

$\Delta_2 = \Delta_1 \cup \{o\!:\!\xi, p\!:\!\xi \rightarrow \pi, q\!:\!\xi \rightarrow \rho, r\!:\!(\pi \wedge \rho) \rightarrow \neg\zeta, s\!:\!\sigma, t\!:\!\tau, u\!:\!\tau \rightarrow \upsilon, v\!:\!(\sigma \wedge \upsilon) \rightarrow \neg\eta\}$.

These additional assumptions provide two undercuts, again see Table 8.3, that attack one of the confirming arguments in favour of the motion. These undercuts are shown in Table 8.3 first as Wigmore

| Used In | Atomic Formula | Linguistic Gloss |
|---|---|---|
| Motion and d,n,y,cc | $\alpha$ | Buy the company WarmHouse Corporation |
| a,d,x | $\beta$ | I want to get into the energy conservation business |
| b,d | $\gamma$ | A way to get into the energy business is to buy an existing firm, e.g. a heating firm |
| c,d | $\delta$ | A local heating firm, WarmHouse, is available |
| c,x | $\epsilon$ | My brother is an employee at WarmHouse |
| e,j,r,z | $\zeta$ | WarmHouse has a valuable asset of a vehicle |
| f,j,v,z | $\eta$ | WarmHouse has a valuable asset of existing service contracts bringing in £2,000 a month by direct debit |
| g,j,z | $\theta$ | WarmHouse has a valuable asset of a customer database going back fifteen years |
| h,j,z | $\iota$ | WarmHouse has a valuable asset of many stickers and fridge magnets in the field with their phone number on |
| i,j,z | $\kappa$ | WarmHouse has a valuable asset of accumulated goodwill from its customers and prospects |
| j,l,n,z,aa,cc | $\lambda$ | The assets of WarmHouse are worth over £100,000 |
| k,l,n,aa,cc | $\mu$ | WarmHouse is for sale for £100,000 |
| l,m | $\nu$ | A company X's assets are worth more than its sale price |
| m,n,bb,cc | $\psi$ | Buy that company X |
| o,p,q | $\xi$ | The vehicle is old |
| p,r | $\pi$ | The vehicle is only worth a few thousand |
| q,r | $\rho$ | The vehicle could breakdown on the way to a service call |
| s,v | $\sigma$ | The service contracts incur considerable routine cost for each visit |
| t,u | $\tau$ | Many of the boilers under contract are old |
| u,v | $\upsilon$ | The service contracts could soon require many of the boilers to be replaced |
| aa,bb | $\phi$ | A company X's assets plus projected revenues over 10 years are worth less than its sales price |
| w,x | $\chi$ | My brother knows the home heating business very well |
| y,x | $\omega$ | Start one's own company from scratch |

Table 8.2: Atomic Formulae and their Linguistic Equivalents for WarmHouse

undercuts then as canonical undercuts. The tool should probably only show end users the Wigmore undercuts, keeping by default the canonical undercuts undisplayed and for internal tracking of the debate. At this point there is still an undefeated argument in favour of the motion and no arguments against, so the motion $\alpha$ still carries.

Additional research into company valuation and startup strategy added further assumptions yielding $\Delta_3$. I've adopted the common legal practice of having $aa, bb, cc, \ldots$ follow on from $a, b, c, \ldots$ after z is reached and the alphabet is exhausted. When listening to a speaker, $aa$ is easier to distinguish from earlier set members than the alternative approach of $ab, ac, \ldots$. Here is the final knowledgebase:

$\Delta_3 = \Delta_2 \cup \{w\!:\!\chi, x\!:\!(\beta \wedge \epsilon \wedge \chi) \rightarrow \omega, y\!:\!\omega \rightarrow \neg\alpha, z\!:\!(\neg\zeta \wedge \neg\eta \wedge \theta \wedge \iota \wedge \kappa) \rightarrow \neg\lambda, aa\!:\!(\neg\lambda \wedge \mu) \rightarrow \phi, bb\!:\!\phi \rightarrow \neg\psi, cc\!:\!(\neg\lambda \wedge \mu \wedge \neg\psi) \rightarrow \neg\alpha\}$.

These final assumptions allowed a pair rebuttals to be formed, see Table 8.3. Observe that there is some somewhat informal logic in the two areas where the quantifiers of first order predicate calculus would have been useful, notably around $\nu, \psi$ and $\phi$. Let $I = \{a, b, c, d\}, J = \{e, f, g, h, i, j, k, l, m, n\}, K = \{a, c, w, x, y\}, L = \{g, h, i, k, o, p, q, r, s, t, u, v, aa, bb, cc\}, M = \{o, p, q, r\}, N = \{s, t, u, v\}$.

Some of the reflected arguments in the debate have also been shown in Table 8.3. Here is the resultant debate tree, which by definition does not include any reflections.

$$\begin{array}{c} \langle\{I, J\}, \{K, L\}\rangle : \alpha \\ \Big\uparrow \{J\} \\ \langle\{M, N\}, \emptyset\rangle : \neg\varpi \end{array}$$

Figure 8.3: Debate Tree of the Buying WarmHouse Debate

It can be seen that there are six arguments of merit in the debate. Depending on the judge style used one of the undercuts does not play an active role. Table 8.3 shows twelve reflected arguments, and that is only a sampling of the total - so the number of reflected arguments outnumbers the direct arguments here by at least a factor of three to one. Separate from the question of skewing of the debate outcome by reflection, it is clear here that if reflected arguments were included in Figure 8.3 then it would be much harder to see and understand the essence of the debate.

At the conclusion of the debate there are two undefeated arguments for not buying WarmHouse and one undefeated argument for buying WarmHouse. Thus using either an existential or a quantitative judge the conclusion is not to buy WarmHouse. This argumentation example therefore crisply documents the decision rational for not buying the company. If a similar situation to this presented itself in the future this documentation would allow the earlier work to be adapted more effectively and easily to the new situation. Even if no similar situation arose the documentation provides an audit trail for shareholders, tax auditors, accountants and others.

| No. | Argument | Comment |
|---|---|---|
| 1. | $I:\alpha = \{a,b,c,d\}:\alpha \in \mathsf{arguments}(\Delta_1)$<br>i.e. $\{a:\beta, b:\gamma, c:\delta \wedge \epsilon, d:(\beta \wedge \gamma \wedge \delta) \rightarrow \alpha\} \vdash \alpha$ | Argument for motion. |
| 2. | $J:\alpha = \{e,f,g,h,i,j,k,l,m,n\}:\alpha \in \mathsf{arguments}(\Delta_1)$<br>i.e. $\{e:\zeta, f:\eta, g:\theta, h:\iota, i:\kappa, j:(\zeta \wedge \eta \wedge \theta \wedge \iota \wedge \kappa) \rightarrow \lambda,$<br>$k:\mu, l:(\lambda \wedge \mu) \rightarrow \nu, m:\nu \rightarrow \psi, n:(\lambda \wedge \mu \wedge \psi) \rightarrow \alpha\} \vdash \alpha$ | Confirming argument for motion. |
| 3. | $M:\neg\zeta = \{o,p,q,r\}:\neg\zeta \in \mathsf{arguments}(\Delta_2)$<br>i.e. $\{o:\xi, p:\xi \rightarrow \pi, q:\xi \rightarrow \rho, r:(\pi \wedge \rho) \rightarrow \neg\zeta\} \vdash \neg\zeta$ | 1st Wigmore undercut<br>of argument 2. |
| 4. | $N:\neg\eta = \{s,t,u,v\}:\neg\eta \in \mathsf{arguments}(\Delta_2)$<br>i.e. $\{s:\sigma, t:\tau, u:\tau \rightarrow \upsilon, v:(\sigma \wedge \upsilon) \rightarrow \neg\eta\} \vdash \neg\eta$ | 2nd Wigmore undercut<br>of argument 2. |
| 5. | Let $\varpi = \zeta \wedge \eta \wedge \theta \wedge \iota \wedge \kappa \wedge ((\zeta \wedge \eta \wedge \theta \wedge \iota \wedge \kappa) \rightarrow \lambda)$<br>$\wedge \mu \wedge ((\lambda \wedge \mu) \rightarrow \nu) \wedge ((\lambda \wedge \mu \wedge \nu) \rightarrow \alpha)$<br>$M:\neg\varpi = \{o,p,q,r\}:\neg\varpi \in \mathsf{arguments}(\Delta_2)$<br>i.e. $\{o:\xi, p:\xi \rightarrow \pi, q:\xi \rightarrow \rho, r:(\pi \wedge \rho) \rightarrow \neg\zeta\} \vdash \neg\varpi$ | 1st canonical undercut<br>of argument 2. |
| 6. | $N:\neg\varpi = \{s,t,u,v\}:\neg\varpi \in \mathsf{arguments}(\Delta_2)$<br>i.e. $\{s:\sigma, t:\tau, u:\tau \rightarrow \upsilon, v:(\sigma \wedge \upsilon) \rightarrow \neg\eta\} \vdash \neg\varpi$ | 2nd canonical undercut<br>of argument 2. |
| 7. | $K:\neg\alpha = \{a,c,w,x,y\}:\neg\alpha \in \mathsf{arguments}(\Delta_3)$<br>i.e. $\{a:\beta, c:\delta \wedge \epsilon, w:\chi, x:(\beta \wedge \epsilon \wedge \chi) \rightarrow \omega, y:\omega \rightarrow \neg\alpha\} \vdash \neg\alpha$ | 1st rebuttal<br>of arguments 1 and 2. |
| 8. | $L:\neg\alpha = \{g,h,i,k,o,p,q,r,s,t,u,v,aa,bb,cc\}:\neg\alpha$<br>$\in \mathsf{arguments}(\Delta_3)$<br>i.e. $\{g:\theta, h:\iota, i:\kappa, k:\mu, o:\xi, p:\xi \rightarrow \pi, q:\xi \rightarrow \rho,$<br>$r:(\pi \wedge \rho) \rightarrow \neg\zeta, s:\sigma, t:\tau, u:\tau \rightarrow \upsilon, v:(\sigma \wedge \upsilon) \rightarrow \neg\eta,$<br>$aa:(\neg\lambda \wedge \mu) \rightarrow \phi, bb:\phi \rightarrow \neg\psi,$<br>$cc:(\neg\lambda \wedge \mu \wedge \neg\psi) \rightarrow \neg\alpha\} \vdash \neg\alpha$ | 2nd rebuttal<br>of arguments 1 and 2. |
| 9. | $I:\varpi,$ | Reflection, rebuts $\{M,N\}:\neg\varpi$. |
| 10. | $J:\varpi,$ | Reflection, rebuts $\{M,N\}:\neg\varpi$. |
| 11. | $K:\neg(\beta \wedge \gamma \wedge \delta \wedge \epsilon \wedge ((\beta \wedge \gamma \wedge \delta) \rightarrow \alpha),$ | Reflection, undercuts $I:\alpha$. |
| 12. | $L:\neg(\beta \wedge \gamma \wedge \delta \wedge \epsilon \wedge ((\beta \wedge \gamma \wedge \delta) \rightarrow \alpha),$ | Reflection, undercuts $I:\alpha$. |
| 13. | $K:\neg\varpi,$ | Reflection, undercuts $J:\alpha$. |
| 14. | $L:\neg\varpi,$ | Reflection, undercuts $J:\alpha$. |
| 15. | $I:\neg(\beta \wedge \delta \wedge \epsilon \wedge \chi \wedge ((\beta \wedge \epsilon \wedge \chi) \rightarrow \omega) \wedge (\omega \rightarrow \neg\alpha)),$ | Reflection, undercuts $K:\neg\alpha$. |
| 16. | $J:\neg(\beta \wedge \delta \wedge \epsilon \wedge \chi \wedge ((\beta \wedge \epsilon \wedge \chi) \rightarrow \omega) \wedge (\omega \rightarrow \neg\alpha)),$ | Reflection, undercuts $K:\neg\alpha$. |
| 17. | $I:\neg(\theta \wedge \iota \wedge \kappa \wedge \mu \wedge \xi, p:\xi \rightarrow \pi, q:\xi \rightarrow \rho \wedge$<br>$((\pi \wedge \rho) \rightarrow \neg\zeta) \wedge \sigma \wedge \tau \wedge (\tau \rightarrow \upsilon) \wedge ((\sigma \wedge \upsilon) \rightarrow \neg\eta) \wedge$<br>$(\neg\lambda \wedge \mu) \rightarrow \phi) \wedge (\phi \rightarrow \neg\psi) \wedge ((\neg\lambda \wedge \mu \wedge \neg\psi) \rightarrow \neg\alpha),$ | Reflection, undercuts $L:\neg\alpha$. |
| 18. | $J:\neg(\theta \wedge \iota \wedge \kappa \wedge \mu \wedge \xi, p:\xi \rightarrow \pi, q:\xi \rightarrow \rho \wedge$<br>$((\pi \wedge \rho) \rightarrow \neg\zeta) \wedge \sigma \wedge \tau \wedge (\tau \rightarrow \upsilon) \wedge ((\sigma \wedge \upsilon) \rightarrow \neg\eta) \wedge$<br>$(\neg\lambda \wedge \mu) \rightarrow \phi) \wedge (\phi \rightarrow \neg\psi) \wedge$<br>$((\neg\lambda \wedge \mu \wedge \neg\psi) \rightarrow \neg\alpha) \in \mathsf{arguments}(\Delta_3)$ | Reflection, undercuts $L:\neg\alpha$. |

Table 8.3: Arguments Derived from Knowledgebase for WarmHouse

### 8.6.1 Design Rationale within Private Finance Initiatives (PFIs)

In the UK and elsewhere today many advanced systems are being developed under the Private Finance Initiative (PFI) funding rules and these mandate that an 'Independent Technical Advisor' (ITA) is involved in checking all the major decision rationales, design rationales and design trade-offs for the project (see HM Treasury guidelines *Standardisation of PFI Contracts* (SoPC4)). In today's large PFI projects the ITA work can take two years and thus there is room for greater efficiency if the right tooling could be found. PFI contracts require that before payments are made for completed construction or delivery of physical assets, that an Independent Qualification Body (IQB) assess that construction or delivery for quality and fulfilment of requirements. It is common practice to have the IQBs appointed as part of the overall initial contract signing process. Thus for each candidate IQB there is a decision akin to the above example as to why they were or were not appointed (clearly appoint is akin to buy). The proposed tool would not only facilitate the conduct and conclusion-drawing of each decision or design debate, but could also provide documentation for an audit trail.

## 8.7 Contributions: Strengths and Limitations

This section summarises the contributions of this thesis, effectively drawing out the strengths of the thesis as a whole. In addition this section outlines the limitations of the contributions to put them into a broader perspective. I present this summary of contributions in a chapter by chapter fashion:

**Basic Framework** Chapter 2's contributions are not significant in themselves, but rather are helpful for expressing the ideas of later sections.

- Confirmation is the main contribution of this chapter. It is helpful in that it introduces and formalises the positive relationship between arguments with the same claim. A limitation is that the other positive relationship of claim to support is not addressed in the basic framework and is only touched upon in the final chapter.
- A formal framework for describing argument aggregation is provided. This is a fully concrete framework and as such is akin to only one instance of the many possible frameworks that can be derived from the ABA or Dung abstractions. An advantage of such a concrete approach is that allows a very specific understanding about properties that will be encountered in an implementation.
- Contradiction is a minor contribution and may be viewed as simply one step in the common set of steps of aggregating arguments.

**Undercut Framework** Chapter 3's contribution is preclusive undercut.

- Preclusive undercut is the main contribution of this chapter and one of the relatively useful contributions of this thesis.
- Of particular interest with preclusion is that given that it is well motivated in professional debate, is the observation that it does not fit easily with the abstract approach of Dung.

Thus preclusion provides a different and complementary research direction from the abstract aspect of the argumentation literature.

- The chapter also contributes a literature survey showing that there is a lack of consensus in the literature as to what constitutes an undercut. As the field matures such a consensus should be expected to emerge.

**Reflection Between Arguments** Chapter 4 introduces and provides the central contribution of the thesis, that given a chain of two arguments with one attacking the other, then provided certain simple assumptions there will always exist a third argument. Given notions of defeat and reinstatement which are prevalent in the literature, e.g. (Prakken & Vreeswijk, 2002), there is logic to show that every defeated argument will inevitably be reinstated.

- This analysis demonstrates that if reflection is not taken into account then practical implementations of various theoretical semantics are likely to give counter-intuitive results.
- The key contribution of the thesis is to identify predictable repetitions of arguments that occur when arguments are chained together in simple lines of two or three attacking arguments.
- This basic reflection between two arguments takes on four forms which I call Base Propositions One, Two, Three and Four. Undercut engenders rebuttal. Rebuttal engenders undercut. Rebuttal also engenders rebuttal and Undercut engenders undercut.
- While the existence of reflected arguments and their various properties are quite remarkable it is difficult to say exactly what their importance or relevance must be to the larger field of argumentation and aggregation. Abstract argumentation approaches, because they ignore any inner structure of arguments, are able to avoid questions of reflection.
- If a debate tracking tool relies on humans to compose the arguments then it appears that humans use their intelligence to filter out the reflected arguments. If an automated tool is employed to generate all of the arguments that logically follow from a knowledgebase then reflection becomes crucial.
- What is needed is to filter out or remove all reflected arguments and reflected attacks, leaving only direct arguments and direct attacks. This chapter establishes this requirement and Chapter 6 provides a data structure to meet that requirement.

In this context I mention that a management consulting exercise I did for a London investment bank found for a practical, reasonably small but non-trivial professional debate that filtering out all reflections while leaving in all direct arguments becomes difficult to do manually.

A limitation of this analysis of reflection is that I have not been able to prove that I have found all of the forms of reflection. It is usually safe to say in scientific endeavours such as this that the chances are that there are further forms yet to be discovered – which could then possibly alter the understandings established herein.

**Reflection Between Confirmations** Chapter 5's contribution is to deepen the appreciation of reflection, showing in particular that when the notions of reflection and preclusion are analysed together then more sophisticated patterns and behaviours are seen.

- Reflections between confirmations, independent of the notion of preclusion, exhibit a form of expanded reflection I call multi-pair enlargement. As shown in the worked example of Section 8.6 even a simple debate can lead to a large amount of multi-pair enlargement.
- There are thus a variety of mechanisms that can cause expanded or reduced reflection, all of which can interact in even simple examples to give quite complex reflection patterns.
- While all of the reflections of the previous chapter fitted a particular template I called Type I reflection, the inclusion of preclusion adds a subtler form of reflection mandating the need for Type II reflections.
- In addition to Type I and II reflections which can be said to flow from head of a chain toward the tail, (i.e. from nearer the root a some debate tracking tree to nearer a leaf) there is another form of reflection that flows up chains of arguments from tail to head. I call these Type III reflections and show that they have distinct and noteworthy properties.
- When the notion of preclusion is analysed in combination with these Type III reflections it is seen that a fourth distinct form of reflection, Type IV, is found. Given that there is much to know about Type I reflection (e.g. Base Propositions One, Two, Three and Four) then there is likely to be more to know about these more advanced forms of reflection.
- It is also apparent that direct and reflected arguments can be intermingled in a single confirmation so mechanisms are needed to separate them.
- In the work of the previous chapter, reflected arguments were analysed specific to a given reflector and thus had a relative flavour to their behaviour. In contrast, in this chapter analysis is provided to cover all possible reflectors, thereby moving the understanding of reflections from a relative footing to what could be called an absolute form.
- Even if a confirmation contains only direct arguments and they are direct relative to all possible reflectors, then it still may not contain all such direct arguments. Consequently it is necessary to introduce maximal forms that ensure that all appropriate direct arguments are included in a confirmation.

As with the last chapter it is unfortunate that it has not been possible to prove that all forms of reflection have been unearthed and thus some of these characteristics could change in the light of further analysis. Another non-ideal aspect of this chapter is that while distorted reflections are established as existing, it is not immediately clear a) if that is really a problem or b) if it is a problem then what to do about it. I follow the theme of arguing that it is a problem and that the solution must be to prune out or remove all reflected arguments, but my theme is not a proven result.

**Tracking a Debate** Chapter 6 describes how to create and populate a data structure to record the passage of a debate. The focus here is on tracking all relevant or pertinent arguments while at the same time excluding all irrelevant or duplicated arguments. Clearly value judgements are entailed in defining what constitutes the debate, the whole debate and nothing but the debate. Thus a merit of this thesis is that a proposal is put forward for tracking the debate, the whole debate and nothing but the debate and a limitation is there must exist alternatives which have not been analysed here.

- The chapter starts by analysing debate tracking structures from the literature, particularly trees, to show the indication that they all may well, or appear to, contain reflection.
- The chapter then goes on to pick a particular tree for detailed analysis and shows a number of ways in which this tree is problematic. Reasoning is presented to persuade that the problems described are not unique to this particular tree, but rather there is every likelihood that they occur across the literature – nowhere in the literature is there description of a mechanism to stop them from occurring.
- One of the things I like most about this thesis is that is practically grounded in providing a way to track debates that is free from the reflection problems identified.
- While my debate tree structure has the plus of excluding reflection it has the drawback of being not as simple as those in the literature – its definition takes more lines of logic.
- On the other hand an advantage of my debate tree is that it tracks a whole debate as a single tree rather than as the forest of trees provided by various other authors.

An interesting puzzle that this debate tree raises is how to best portray it to users. It would appear that a three layer approach is warranted where the bottom layer is the mathematical formalism presented herein, the top layer is some easier-to-understand visual language and the middle layer is a mapping between the two.

**Judging a Debate** Chapter 7 describes how to draw a conclusion from a debate, recorded as a debate tree, as to whether the motion carries or is defeated. My judge functions differ from abstract semantics in that they a) deliver a single item rather than a set of items or set of sets and b) that single item is a proposition rather than an argument. In this way I argue that judges map more closely to the needs of professional debate than do semantics.

- Judges in the literature can be categorised into two main groups: existential and quantitative. Existential judges only attend to the existence of arguments, while quantitative judges are sensitive also to the number of arguments. This thesis dwells on and advocates the quantitative approach as it resolves more types of inconsistency than does the existential approach.
- While this section does provide an illustrative judge that works with the debate tree of the previous chapter, this judge has various limitations. Considerably more research is warranted, I would suggest, to provide intelligent judge functions that are not subject to these limitations.

- A particularly challenging, but desirable, property of judge functions is that the outcomes of a series of debates are consistent when taken together. This property is called cumulativity in the literature and merits further research. I do not provide it.

- A key question asked by this chapter, which lies at the core of this thesis, is does reflection change the outcome of debates? Rather frustratingly even though there is an indication that reflection changes the outcomes of both existential and quantitative judges I have not been able to rigourously prove that this is a fact.

This chapter has the limitation of being not as formal as those that precede it and much of its descriptions of desirable judge properties is only textual narrative. From that perspective the chapter is rather unsatisfactory as there is no way of seeing if my proposed judge is definitively any better or worse than alternatives. It appears to be better as it takes into account reflection, preclusion and confirmation, but there are clearly so many possible judges that how to pick the best or most suitable is a puzzle.

I would like to say in conclusion, my opinion is that the field of argumentation within artificial intelligence should keep itself tied to solid motivations drawn from professional debate and thus to forms of theory that can be mapped to applications. It should answer the questions of a) where do arguments come from and b) given an arbitrary item $x$, is $x$ an argument or not? Certainly I support and welcome theoretical and abstract work, but do question when this takes paths that inhibit its inter-linking with professional debate. In contrast, what I like about my thesis is that it starts with theoretical research and then maps that into something very concrete and practical. I like that it leads to and proposes an implementation that can be used in facilitating real-world professional debates.

## 8.8 Closing Remarks and Conclusion for Discussion

This chapter has covered the topics for further research, assisting arguments, the design for an implementation, an example of a debate and a summarisation of the contributions of this thesis as a whole.

An interesting aspect of this thesis is that it raises questions about some of the assumptions underpinning the abstract school of argumentation that is playing such a dominant role in the current argumentation literature. This questioning has the key points of:

- Should an argument attack be allowed to attack more than one argument? I provide practical motivation and a formalism for this approach, showing how it occurs in professional debates. In contrast the abstract argumentation literature confines itself to an attack being on one and only one argument. Certainly one abstract argument can attack several arguments, but there is no linking to show the dependency or disjunction between the attacks.

- Should graded or partial defeat be allowed? The preclusive undercut would suggest that graded defeat has its applications in professional debate.

- Should judge functions be allowed to count arguments? Again there is practical motivation here, for example from the writings of Benjamin Franklin and John Fox, however the current abstract literature does not embrace the notion of quantitative judgement.

A second and central notion in this thesis is of course that of reflection. I would suggest that once reflection is understood and appreciated then taking it into account is not particularly problematic. The challenges and issues arise if reflection is not taken into account.

The relationship of reflection to the abstract argumentation literature is somewhat paradoxical, because without a concrete form for arguments there is no way of saying whether reflection occurs or not. If every attacked, and therefore defeated, argument is automatically subject to a counter-attack, then some of the semantics in the literature would behave differently. Without a consideration of reflection, however, there appears to be an impact on various semantics and this, I would suggest, warrants further research.

Clearly my line of investigation in this thesis parts company from the mainstream of the argumentation literature, i.e. the abstract and semi-abstract approaches with their various semantics. My direction could be described as focussing on the counting of arguments and on thus on approaches that part company even from the mainstream of logic. I would argue back that I employ classical logic for individual arguments and that to aggregate arguments in a way that fits with professional debate then quantitative approaches are clearly motivated and warranted. The crucial benefit of the approach I give here is that it appears to be considerably better at resolving inconsistency than approaches which rely solely on the existence of arguments and not on their numbers.

I feel the threshold I introduce also has its charm in that it adds a new dimension to argument aggregation, that of filtering out noise or weakly argued claims. Certainly in law, in preparation for trial, there is a crucial need to sort through the evidence and arguments to separate the strong from the weak. Graded defeat would thus appear to be an unavoidable aspect of argumentation if it is to mirror practical situations.

Weights of arguments, stemming from weights of assumptions, while not addressed in this thesis is the requirement of professional debate that must not be forgotten going forward. All arguing, based on equal strength or flat knowledgebases must be accepted as limited. From this perspective my work is perhaps weaker than those who embrace priorities or preferences in their argumentation. I would argue however, than the ability of my judge function to order a set of motions from strongest to weakest is a useful step in the direction of graded defeat. I would also hold that in lieu of an improved theoretical basis for deciding where weights originate from and what they mean that my classical logic based approach has the level of integrity that is necessary for professional debate. The core issue for weights is that one must decide where the weights come from and how to formulate or construct them. Likewise, the abstract literature does not address where argument come from or how to formulate them. The minimal consistent subset approach to argumentation, with classical logic proof rules, does show where arguments come from and does allow a robust treatment of reflection. To be clear about reflection you have to be clear about the proof rules, so I would suggest also the semi-abstract ABA approach in its pure form is also not well suited to professional debate.

As my final point of conclusion I would argue that practical implementations of argumentation tools need to be used in real-world professional debates to clarify the requirements of the field of argumentation as a whole. All such tools must be based, of course, on solid theory, but without the tools it is hard to evaluate the appropriateness of the underpinnings of these various theories. Let the next step be professional debate tools based on the theory herein or similar, to clarify and refine the requirements needed for artificial intelligence based argumentation to be deeply accepted within the realm of professional debate.

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