THE DYNAMICS OF THE STOCHASTIC SHADOW GIERER-MEINHARDT SYSTEM

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ABSTRACT. We consider the dynamics of the stochastic shadow Gierer-Meinhardt system with one-dimensional standard Brownian motion. We establish the global existence and uniqueness of solutions. We also prove a large deviation result.

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1. INTRODUCTION

In his pioneering work ([31]) in 1952, Turing explained the onset of pattern formation in reaction-diffusion systems by a spatial instability of an unpatterned state leading to a pattern. This approach is now commonly called *Turing diffusion-driven instability*. Since then many reaction-diffusion models have been studied to explore pattern formation. One of the most widely used class of such models are those of activator-inhibitor type. Among them the Gierer-Meinhardt system is one of the most popular models. After a suitable re-scaling it can be stated as follows:

$$\begin{cases} \partial_t A = d_1 \Delta A - A + \frac{A^p}{H^q} & \text{in } \mathcal{O}, \\ \tau \partial_t H = d_2 \Delta H - H + \frac{A^\alpha}{H^\beta} & \text{in } \mathcal{O}, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 & \text{on } \partial\mathcal{O}, \end{cases} \tag{1.1}$$

where $A(t,x) : \mathbb{R}^+ \times \mathcal{O} \to \mathbb{R}^+$ and $H(t,x) : \mathbb{R}^+ \times \mathcal{O} \to \mathbb{R}^+ \setminus \{0\}$. Further, $\mathcal{O} \subset \mathbb{R}^d$ is a smooth and bounded domain and p, q, α, β are all positive constants with the condition $\frac{p-1}{\alpha} < \frac{q}{\beta+1}$. We assume that the diffusivities $d_1 > 0$, $d_2 > 0$ and time-relaxation parameter $\tau > 0$ are all constants.

Gierer and Meinhardt originally suggested this system in 1972 to model (re)generation phenomena in *hydra* [10]. In the Gierer-Meinhardt model, A is the activator and H is the inhibitor. The parameters d_1, d_2 and τ can be tuned to study some interesting phenomena such as Turing instability and peak steady states. It is assumed that the two components A and H, representing the concentrations of certain biochemicals, are first produced by an external source. Then they interact as represented by the coupled nonlinear terms in the system. Further, they both decay, and they diffuse with different diffusion constants. It can be shown that Turing instability can only occur if the ratio of the activator and inhibitor diffusivities satisfies

$$\frac{d_1}{d_2} < C \quad \text{for a certain constant } C < 1.$$

The Gierer-Meinhardt system has been studied extensively by many authors, both in the biological and physical communities, and more recently also in mathematics, to elucidate its role in pattern formation. We refer to [32] for more background on the model and its investigation.

Since this paper is concerned with the dynamical behavior of the solutions of (1.1), we first review what is known about their properties.

The dynamics of (1.1) remains far from being completely understood. Let us mention a few results in this direction. Global existence has been shown by Rothe for the three-dimensional case with the powers $p = 2,\ q = 1,\ \alpha = 2,\ \beta = 0$ ([29]), and by Jiang for $\frac{p-1}{\alpha} < 1$ ([12]). Blow-up in (1.1) can occur for $\frac{p-1}{\alpha} > 1$ since this even happens for the corresponding kinetic system ([22]).

If the inhibitor diffuses over the whole domain very quickly, it is possible to assume that it is constant throughout the domain. Formally taking the limit $d_2 \to \infty$ results in the shadow system, for which the inhibitor component H is replaced by its spatial average $\xi = \overline{H} = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} H\,dx$, where $|\mathcal{O}|$ is the measure of $\mathcal{O}$. Note that ξ is constant in space, but it can still change in time. The behavior of the system (1.1) stands in marked contrast to its shadow system. Taking the limit $D \to \infty$ in (1.1), we get

$$
\begin{cases}
\partial_t A = d_1 \Delta A - A + \frac{A^p}{\xi^q} & \text{in } \mathcal{O},\\
\tau \dot{\xi} = -\xi + \frac{\overline{A^\alpha}}{\xi^\beta}, & \\
\frac{\partial A}{\partial \nu} = 0 & \text{on } \partial\mathcal{O},
\end{cases}
\tag{1.2}
$$

where $\overline{A^\alpha} = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} A^\alpha\,dx$. It was suggested by Keener ([13]) to study the system (1.2) and the name "shadow system" was proposed by Nishiura ([23]).

The dynamics for (1.2) has been less well studied than for (1.1). Global existence and finite-time blow-up have been explored by Li and Ni ([18]). In particular, they show that for $\frac{p-1}{\alpha} < \frac{2}{d+2}$ there is a unique global solution, whereas for $\frac{p-1}{\alpha} > \frac{2}{d}$ blow-up can occur. The range $\frac{2}{d} \geq \frac{p-1}{\alpha} \geq \frac{2}{d+2}$ remains open.

Since the systems (1.1) and (1.2) are both deterministic, their evolutions are completely determined by the initial data. This is obviously not consistent with phenomena in nature, where random influences from the environment often play an important role. These random effects are conceived as stochastic fluctuations in stochastic modeling.

Motivated by these issues, Kelkel and Surulescu ([14]) proposed a stochastic Gierer-Meinhardt type system with saturation effects and source terms to study stochastic influences. They replaced the highly singular nonlinear reaction terms $\frac{A^\alpha}{H^\beta}$ and $\frac{A^p}{H^q}$ in (1.1) by certain Lipschitz nonlinearities. For the system (1.1) we can model external random effects as follows:

$$
\begin{cases}
\partial_t A = d_1 \Delta A - A + \frac{A^p}{H^q} + A\partial_t W_1 & \text{in } \mathcal{O}, \\
\tau \partial_t H = d_2 \Delta H - H + \frac{A^\alpha}{H^\beta} + H\partial_t W_2 & \text{in } \mathcal{O}, \\
\frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 & \text{on } \partial\mathcal{O},
\end{cases}
\tag{1.3}
$$

where W_1 and W_2 are independent space-time white noises ([4]). The particular form of $A\partial_t W_1$ and $H\partial_t W_2$ is chosen to keep the stochastic solutions positive (see also [14]).

It is very challenging to prove the existence and uniqueness for stochastic partial differential equations with linear type multiplicative noises, see e.g. [9, 21] for more details. To the best of our knowledge, the only other paper for stochastic Gierer-Meinhardt type systems is [14], which includes two coupled stochastic PDEs with bounded and Lipschitz nonlinearity. In [14] the authors proved the *local* existence of the *positive* stochastic solution by Da Prato-Zabczyk's approach ([4]).

As a first attempt, we remove the random influences on the activator A to simplify (1.3). Letting $d_2 \to \infty$, the model (1.3) is further simplified as

$$
\begin{cases}
\partial_t A = d_1 \Delta A - A + \frac{A^p}{\xi^q} & \text{in } \mathcal{O}, \\
\tau \dot{\xi} = -\xi + \frac{\overline{A^\alpha}}{\xi^\beta} + \xi \dot{B}_t, & \\
\frac{\partial A}{\partial \nu} = 0 & \text{on } \partial\mathcal{O},
\end{cases}
\tag{1.4}
$$

Note that the noise term models some large-scale fluctuations which spread instantly over the whole domain. This could represent a random process which happens on a length-scale larger than the domain, e.g. the influence which the change of the concentration of a biochemical on a whole organ has on a single cell.

We adopt the notation in [18] and write the stochastic shadow Gierer-Meinhardt system rigorously as follows:

$$
\begin{cases}
\partial_t u = \Delta u - u + \frac{u^p}{\xi^q}, \\
\mathrm{d}\xi = -\xi \mathrm{d}t + \frac{\overline{u^\alpha}}{\xi^\beta}\mathrm{d}t + \sqrt{\varepsilon}\xi \mathrm{d}B_t, \\
\frac{\partial u}{\partial \nu} = 0, \\
u(0) = v, \\
\xi(0) = \zeta,
\end{cases}
\tag{1.5}
$$

where $u(t,x,\omega) : \mathbb{R}^+ \times \mathcal{O} \times \Omega \to \mathbb{R}^+$, $\xi(t,\omega) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+ \setminus \{0\}$. Here $\varepsilon > 0$ is some constant and B_t is one-dimensional standard Brownian motion. Note that in (1.5) we have set the diffusion constant of the activator $d_1 = 1$ which can always be achieved by a rescaling of the domain.

The system in our papers differs from that in [14] in several respects: the nonlinearity in Eq. (1.5) is not bounded and far from being Lipschitz; we do not include positive production terms for activator and inhibitor; we consider the shadow system; we shall prove the *global* existence of the strong positive solution.

Eq. (1.5) is a stochastic system which includes one deterministic PDE and one SDE with long-range interactions. To our knowledge, this seems to be the first paper to study a stochastic shadow system.

On the other hand, Eq. (1.5) can be taken as a highly degenerate stochastic PDE (see [19] for more details). Its ergodicity is a very challenging problem which will be studied in future papers (see [19, 17] for some work in this direction).

Our first main result on global existence can be stated as follows:

Theorem 1.1. *Let p, q, α, β satisfy the following condition*

$$\frac{p-1}{\alpha} < \frac{q}{\beta+1}, \quad \frac{p-1}{\alpha} < \frac{2}{d+2}.$$

Eq. (1.5) *has a unique global solution $(u, \xi) \in C([0,T]; C(\mathcal{O}, \mathbb{R}) \times \mathbb{R})$ for all $T > 0$ such that for all $t > 0$*

$$u(t,x) \geq 0 \quad \forall x \in \mathcal{O}, \qquad \xi(t) \geq \mathrm{e}^{-\frac{3}{2}t-\varepsilon|B_t|}\zeta.$$

Our next main result is the small noise large deviation principle given in Theorem 4.4 below. Our result implies that as $\varepsilon \to 0$ in Eq. (1.5), the stochastic system converges to its deterministic part with exponential speed $O(e^{-c/\varepsilon})$. It suggests that as the external random noise is small, the random influences can be ignored. This is also consistent with our intuition.

As for the references of large deviation results on stochastic systems, we give the following list of articles which is far from complete: [1]-[8], [20], [24]-[30], [34]-[36]. We shall follow the approach in [18] to prove Theorem 1, some ideas along the same lines have also appeared in [12, 23]. The random force in Eq. (1.5) produces some additional stochastic terms, which can be very large or even become infinite. To control these terms, we shall use a martingale inequality and modify the energy estimate in [18] by adding suitable stochastic terms and figuring out an explicit inequality. For the large deviation result, we shall follow the variational approach in [1] by checking the two assumptions of Theorem 4.4 therein (see Propositions 4.5 and 4.6 below). To prove these two propositions, we also need to use a martingale inequality and some special energy estimates.

The structure of this paper is as follows. In Section 2 we show local existence and uniqueness of solutions. In Section 3 we prove global existence and uniqueness. In Section 4 we prove the large deviation result. Finally, in Section 5 we discuss our results and give an outlook to open problems and further research.

2. Local existence and uniqueness of the shadow stochastic Gierer-Meinhardt system

Without loss of generality, we assume that $\varepsilon = 1$ in this and the next section. Write

$$B_t^* = \sup_{0 \leq s \leq t} |B_s| \quad \forall t > 0.$$

Let $N > 0$ be a constant and define the following stopping time

$$\tau_N(\omega) = \inf\{t > 0 : |B_t(\omega)| \geq N\}.$$

It is clear that

$$\{\omega \in \Omega : \tau_N(\omega) \leq t\} = \{\omega \in \Omega : B_t^*(\omega) \geq N\}. \tag{2.1}$$

It is well known that $\sup_{0\leq s\leq t} B_s$ satisfies

$$\mathbb{P}\left(\sup_{0\leq s\leq t} B_s \in (x, x + \mathrm{d}x)\right) = \frac{2}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{x^2}{2t}}\mathrm{d}x, \quad x > 0.$$

Since

$$\begin{aligned}\mathbb{P}\left(B_t^* > x\right) &\leq \mathbb{P}\left(\sup_{0\leq s\leq t} B_s > \frac{x}{2}\right) + \mathbb{P}\left(\sup_{0\leq s\leq t} (-B_s) > \frac{x}{2}\right)\\ &= 2\mathbb{P}\left(\sup_{0\leq s\leq t} B_s > \frac{x}{2}\right) = \frac{4}{\sqrt{2\pi t}}\int_{\frac{x}{2\sqrt{t}}}^{\infty} \mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}x,\end{aligned}$$

the distribution of B_t^* has a density function f_t satisfying

$$f_t(x) \leq \frac{4}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{x^2}{8t}}. \tag{2.2}$$

For notational simplicity, we shall drop the variable ω in the random variables or random sets below if no confusion arises. Further define

$$S(t) = \mathrm{e}^{(\Delta-1)t}, \quad R(t, B_t) = \mathrm{e}^{-\frac{3}{2}t+B_t}, \tag{2.3}$$

where Δ is the Laplace operator with Neumann boundary condition and $C(\bar{\mathcal{O}}, \mathbb{R}^d)$ is the space of all bounded continuous functions $f : \mathcal{O} \to \mathbb{R}^d$ with the uniform norm. It is easy to check that $C(\mathcal{O}, \mathbb{R}^d)$ is closed under the uniform norm. For notational simplicity, we shall write

$$\|f\|_C = \|f\|_{C(\mathcal{O},\mathbb{R}^d)} \quad \forall\ f \in C(\mathcal{O}, \mathbb{R}^d).$$

It is clear that the following relations hold:

$$\begin{aligned}&\|S(t)f\|_C \leq \|f\|_C \quad \forall t > 0\ \ \forall f \in C(\mathcal{O}, \mathbb{R}^d),\\ &\|f^p\|_C \leq \|f\|_C^p \quad \forall p \geq 1\ \ \forall f \in C(\mathcal{O}, \mathbb{R}^d).\end{aligned} \tag{2.4}$$

For any (u, ξ), recall

$$\|(u, \xi)\|_{C([0,T];C\times\mathbb{R})} = \|u\|_{C([0,T];C)} + \|\xi\|_{C([0,T];\mathbb{R})} \quad \forall\ T > 0.$$

Let X, Y both be some quantities, we shall simply denote $Y \lesssim X$ if there exists some (not important) constant C such that $Y \leq CX$.

Lemma 2.1. *For every $N>0$, there exists some T depending on $N, \|v\|_C$ and ζ such that for all $\omega \in \Omega$ up to a negligible set, Eq. (1.5) has a unique solution $(u,\xi) \in C([0,T\wedge\tau_N]; C(\mathcal{O},\mathbb{R})\times\mathbb{R})$ such that for all $t\in[0,T\wedge\tau_N]$*

$$\begin{aligned} u(t) &= S(t)v + \int_0^t S(t-s)\left(\frac{u^p(s)}{\xi^q(s)}\right)\mathrm{d}s, \\ \xi(t) &= R(t,B_t)\zeta + \int_0^t R(t-s,B_t-B_s)\left(\frac{\overline{u^\alpha}(s)}{\xi^\beta(s)}\right)\mathrm{d}s, \end{aligned} \tag{2.5}$$

with the property

$$\begin{aligned} &u(t,x)\geq 0 \qquad \forall\, t\in[0,T\wedge\tau_N]\ \forall\, x\in\mathcal{O}; \\ &\xi(t)\geq \mathrm{e}^{-\frac{3}{2}t-N}\zeta \qquad \forall\, t\in[0,T\wedge\tau_N]. \end{aligned} \tag{2.6}$$

Moreover, $(u(t),\xi(t))$ satisfies the first two equations in Eq. (1.5) for each $t\in(0,T\wedge\tau_N]$. In particular,

$$\xi(t) = \zeta - \int_0^t \xi(s)\mathrm{d}s + \int_0^t \frac{\overline{u^\alpha}(s)}{\xi^\beta(s)}\mathrm{d}s + \int_0^t \xi(s)\mathrm{d}B_s \quad \forall t\in[0,T\wedge\tau_N].$$

Proof. For all $\omega\in\Omega$ up to a negligible set, define the following space

$$\begin{aligned} \mathcal{A}_{T,M,N,\omega} = \Big\{ &(u(\omega),\xi(\omega)) \in C([0,T\wedge\tau_N(\omega)]; C(\mathcal{O},\mathbb{R})\times\mathbb{R}^+) : \\ &u(\omega,t)\geq 0,\ \xi(\omega,t)\geq \mathrm{e}^{-\frac{3}{2}t-N}\zeta,\ \ \forall\, 0\leq t\leq T\wedge\tau_N(\omega); \\ &u(0)=v,\ \xi(0)=\zeta;\ \|(u,\xi)(\omega)\|_{C([0,T\wedge\tau_N(\omega)];C\times\mathbb{R})}\leq M. \Big\}, \end{aligned}$$

where $T\in(0,1]$ is some number depending on M,N,v,ζ to be determined later and

$$M > 2 + \|v\|_C + \mathrm{e}^N\zeta.$$

We shall drop all the ω in the definition of $\mathcal{A}_{T,M,N,\omega}$ in the argument below for notational simplicity.

For all $(u_1,\xi_1),(u_2,\xi_2)\in\mathcal{A}_{T,M,N}$, define

$$\mathrm{d}_T\left((u_1,\xi_1),(u_2,\xi_2)\right) = \|(u_1,\xi_1)-(u_2,\xi_2)\|_{C([0,T\wedge\tau_N];C\times\mathbb{R})}.$$

It is easy to check that under the distance d_T the set $\mathcal{A}_{T,M,N}$ is a closed metric space.

For each $(u,\xi)\in\mathcal{A}_{T,M,N}$, define

$$\begin{aligned} \left[\mathcal{F}_1(u,\xi)\right](t) &= S(t)v + \int_0^t S(t-s)\left(\frac{u^p(s)}{\xi^q(s)}\right)\mathrm{d}s, \\ \left[\mathcal{F}_2(u,\xi)\right](t) &= R(t,B_t)\zeta + \int_0^t R(t-s,B_t-B_s)\left(\frac{\overline{u^\alpha}(s)}{\xi^\beta(s)}\right)\mathrm{d}s, \end{aligned} \tag{2.7}$$

where S and R are defined in (2.3). For further use, we simply denote

$$\mathcal{F}(u,\xi) = \left(\mathcal{F}_1(u,\xi),\mathcal{F}_2(u,\xi)\right).$$

We shall prove below that

(i) There exists some $\hat{T}$ depending on $N, M, \|v\|_C$ and ζ such that

$$\mathcal{F}(u,\xi) \in \mathcal{A}_{T,M,N} \tag{2.8}$$

for any $(u,\xi) \in \mathcal{A}_{T,M,N}$ with $T = \hat{T}$.

(ii) There exists some $\tilde{T}$ depending on $N, M, \|v\|_C$ and ζ such that

$$\mathrm{d}_T(\mathcal{F}(u_1,\xi_1), \mathcal{F}(u_2,\xi_2)) \le \frac{1}{2}\mathrm{d}_T((u_1,\xi_1),(u_2,\xi_2)) \tag{2.9}$$

for any $(u_1,\xi_1),(u_2,\xi_2) \in \mathcal{A}_{T,M,N}$ with $T = \tilde{T}$.

By the definition of $\mathcal{A}_{T,M,N}$, taking $T = \min\{\tilde{T}, \hat{T}\}$, it is clear that (2.8) holds for any $(u,\xi) \in \mathcal{A}_{T,M,N}$ and that (2.9) holds for any $(u_1,\xi_1),(u_2,\xi_2) \in \mathcal{A}_{T,M,N}$. Thus, we apply the Banach fixed point theorem to obtain a local unique solution in the sense of (2.5). Differentiating both sides of (2.5) ([11]), we immediately get that (u,ξ) satisfies the first two equations of Eq. (1.5) and that the desired stochastic integral equation holds.

Now we only need to show the statements (i) and (ii) from above. Let C be some positive constant depending only on α, β, p, q, whose exact values may vary from case to case.

Let us first show (i). For any $(u,\xi) \in \mathcal{A}_{\hat{T},M,N}$ with $\hat{T}$ to be determined below, it is clear $\mathcal{F}(u,\xi)(0) = (v,\zeta)$. Since $S(t)$ maps a positive function to a positive one, it is easy to see

$$[\mathcal{F}_1(u,\xi)]\,(t) \ge 0 \qquad \forall t \in [0, \hat{T}\wedge\tau_N].$$

By (2.4), for all $t \in [0, \hat{T}\wedge\tau_N]$ we have

$$\begin{aligned}
\|\,[\mathcal{F}_1(u,\xi)]\,(t)\|_C &\le \|v\|_C + \mathrm{e}^{\frac{3}{2}q+Nq}\zeta^{-q}\int_0^t \|u(s)\|_C^p \mathrm{d}s \\
&\le \|v\|_C + \mathrm{e}^{\frac{3}{2}q+Nq}\zeta^{-q}M^p t,
\end{aligned}$$

and

$$\begin{aligned}
|\,[\mathcal{F}_2(u,\xi)]\,(t)| &\le \mathrm{e}^{-\frac{3}{2}t+B_t}\zeta + \mathrm{e}^{\frac{3}{2}\beta t+N\beta}\int_0^t \mathrm{e}^{-\frac{3}{2}(t-s)+B_t-B_s}\|u(s)\|_C^\alpha \mathrm{d}s \\
&\le \mathrm{e}^N\zeta + \mathrm{e}^{\frac{3}{2}\beta+N\beta+2N}M^\alpha t
\end{aligned}$$

Taking $\hat{T} = \min\{T_1, T_2\}$ with $T_1 = \mathrm{e}^{-\frac{3}{2}q-Nq}\zeta^q M^{-p}$ and $T_2 = \mathrm{e}^{-\frac{3}{2}\beta-N\beta-2N}M^{-\alpha}$, from the above two inequalities we get

$$\|\mathcal{F}(u,\xi)\|_{C([0,\hat{T}\wedge\tau_N];C\times\mathbb{R})} \le 2 + \|v\|_C + \mathrm{e}^N\zeta \le M.$$

Hence, $\mathcal{F}(u,\xi) \in \mathcal{A}_{\hat{T},M,N}$.

Next we show (ii). For any $(u_1,\xi_1),(u_2,\xi_2)\in\mathcal{A}_{\tilde T,M,N}$ with $\tilde T$ to be determined below, observe that for all $t\in[0,\tilde T\wedge\tau_N]$

$$\|\left[\mathcal{F}_1(u_1,\xi_1)\right](t)-\left[\mathcal{F}_1(u_2,\xi_2)\right](t)\|_C\le\int_0^t\left\|\frac{u_1^p(s)}{\xi_1^q(s)}-\frac{u_2^p(s)}{\xi_2^q(s)}\right\|_C\mathrm{d}s\le I_1(t)+I_2(t),$$

where

$$I_1(t)=\int_0^t\frac{\|u_1^p(s)-u_2^p(s)\|_C}{\xi_1^q(s)}\mathrm{d}s,$$
$$I_2(t)=\int_0^t\|u_2^p(s)\|_C\left|\frac{1}{\xi_1^q(s)}-\frac{1}{\xi_2^q(s)}\right|\mathrm{d}s.$$

Writing $u_{1,2,\lambda}(s)=\lambda u_1(s)+(1-\lambda)u_2(s)$ for $\lambda\in[0,1]$, by (2.4) we have

$$\begin{aligned}\|u_1^p(s)-u_2^p(s)\|_C&\le p\int_0^1\left\|\left(u_{1,2,\lambda}(s)\right)^{p-1}\left(u_1(s)-u_2(s)\right)\right\|_C\mathrm{d}\lambda\\&\le p\int_0^1\|u_{1,2,\lambda}(s)\|_C^{p-1}\|u_1(s)-u_2(s)\|_C\mathrm{d}\lambda\\&\le pM^{p-1}\|u_1(s)-u_2(s)\|_C.\end{aligned}\tag{2.10}$$

Thus

$$I_1(t)\le p\mathrm{e}^{\frac32 q+Nq}\zeta^{-q}M^{p-1}t\|u_1-u_2\|_{C([0,t];C)}\qquad\forall t\in[0,\tilde T\wedge\tau_N].$$

Writing $\xi_{1,2,\lambda}(s)=\lambda\xi_1(s)+(1-\lambda)\xi_2(s)$ for $\lambda\in[0,1]$, we have

$$\begin{aligned}I_2(t)&\le q\int_0^t M^p\int_0^1\frac{|\xi_1(s)-\xi_2(s)|}{\left(\xi_{1,2,\lambda}(s)\right)^{q+1}}\mathrm{d}\lambda\mathrm{d}s\\&\le q\mathrm{e}^{(\frac32+N)(q+1)}\zeta^{-(q+1)}M^pt\|\xi_1-\xi_2\|_{C([0,t];\mathbb{R})}\qquad\forall t\in[0,\tilde T\wedge\tau_N],\end{aligned}$$

which, together with the estimate of I_1, implies that for all $t\in[0,\tilde T\wedge\tau_N]$

$$\begin{aligned}&\|\mathcal{F}_1(u_1,\xi_1)-\mathcal{F}_1(u_2,\xi_2)\|_{C([0,t];C)}\\&\le C\mathrm{e}^{(\frac32+N)q}\zeta^{-q}M^{p-1}\left(1+\mathrm{e}^{\frac32+N}M\zeta^{-1}\right)t\|(u_1,\xi_1)-(u_2,\xi_2)\|_{C([0,t];C\times\mathbb{R})}.\end{aligned}\tag{2.11}$$

A similar argument as above gives that for all $t\in[0,\tilde T\wedge\tau_N]$,

$$\begin{aligned}&\|\mathcal{F}_2(u_1,\xi_1)-\mathcal{F}_2(u_2,\xi_2)\|_{C([0,t];\mathbb{R})}\\&\le C\mathrm{e}^{2N+(\frac32+N)\beta}\zeta^{-\beta}M^{\alpha-1}(1+\mathrm{e}^{\frac32+N}M\zeta^{-1})t\|(u_1,\xi_1)-(u_2,\xi_2)\|_{C([0,t];C\times\mathbb{R})}.\end{aligned}\tag{2.12}$$

From the above two inequalities, there exists some $\tilde T$ depending on M,N,ζ such that

$$\|\mathcal{F}(u_1,\xi_1)-\mathcal{F}(u_2,\xi_2)\|_{C([0,\tilde T\wedge\tau_N];C\times\mathbb{R})}\le\frac12\|(u_1,\xi_1)-(u_2,\xi_2)\|_{C([0,\tilde T\wedge\tau_N];C\times\mathbb{R})}$$

i.e.,

$$\mathrm{d}_{\tilde T}(\mathcal{F}(u_1,\xi_1),\mathcal{F}(u_2,\xi_2))\le\frac12\mathrm{d}_{\tilde T}((u_1,\xi_1),(u_2,\xi_2)).$$

□

3. Global existence and uniqueness of the shadow stochastic Gierer-Meinhardt system

3.1. Some a'priori estimates.
To prove the global existence and uniqueness theorem, we assume that $(u(t), \xi(t))_{0\le t\le 1}$ is a solution of Eq. (1.5) such that

$$u \in C([0,1]; C(\mathcal{O}, \mathbb{R})), \quad \xi \in C([0,1], \mathbb{R}) \quad a.s.,$$

and prove the following a'priori estimates of (u, ξ).

Lemma 3.1. *We have*

$$\xi(t) \ge \mathrm{e}^{-\frac{3}{2}t + B_t}\zeta \quad \forall t > 0, \tag{3.1}$$

$$\inf_{0\le s\le t} \xi(s) \ge \mathrm{e}^{-\frac{3}{2}t - B_t^*}\zeta \quad \forall t > 0, \tag{3.2}$$

$$\sup_{0\le t\le 1} \xi(t) \lesssim \mathrm{e}^{B_1^*}\zeta + \mathrm{e}^{2B_1^*}\left(\sup_{0\le t\le 1} \overline{u^\alpha}(t)\right)^{\frac{1}{1+\beta}}. \tag{3.3}$$

Proof. Applying Itô formula to $\xi^{1+\beta}(t)$ we have

(3.4)
$$\mathrm{d}\xi^{1+\beta}(t) = \frac{1}{2}(1+\beta)(\beta-2)\xi^{1+\beta}(t)\mathrm{d}t + (1+\beta)\xi^{1+\beta}(t)\mathrm{d}B_t + (1+\beta)\overline{u^\alpha}(t)\mathrm{d}t,$$

which implies

$$\begin{aligned}\xi^{1+\beta}(t) &= \mathrm{e}^{-\frac{3}{2}(1+\beta)t + (1+\beta)B_t}\zeta^{1+\beta} \\ &\quad + (1+\beta)\int_0^t \mathrm{e}^{-\frac{3}{2}(1+\beta)(t-s) + (1+\beta)(B_t - B_s)}\overline{u^\alpha}(s)\mathrm{d}s,\end{aligned} \tag{3.5}$$

which clearly implies the desired three inequalities. □

Let $\delta > 0$ be some fixed number and define

$$\mathcal{M}_\delta(t) = \int_0^t \xi^{-\delta}(s)\mathrm{d}B_s, \quad \mathcal{M}_\delta^* = \sup_{0\le t\le 1} \mathcal{M}_\delta(t).$$

Lemma 3.2. *For all $M > 0$ we have*

$$\mathbb{E}\mathcal{M}_\delta^* \le C, \tag{3.6}$$

where C depends only on δ, ζ. Moreover, we have

$$\mathcal{M}_\delta^* < \infty \quad a.s..$$

Proof. It follows from the martingale inequality and Itô isometry that

$$\begin{aligned}\mathbb{E}\mathcal{M}_\delta^* &\leq \left[\mathbb{E}\sup_{0\leq t\leq 1}\left|\int_0^t \xi^{-\delta}(s)\mathrm{d}B_s\right|^2\right]^{\frac{1}{2}} \\ &\leq \sqrt{2}\left[\mathbb{E}\left|\int_0^1 \xi^{-\delta}(s)\mathrm{d}B_s\right|^2\right]^{\frac{1}{2}} = \sqrt{2}\left[\int_0^1 \mathbb{E}\xi^{-2\delta}(s)\mathrm{d}s\right]^{\frac{1}{2}}.\end{aligned}$$

This and (3.1) further give

$$\mathbb{E}\mathcal{M}_\delta^* \leq \sqrt{2}\zeta^{-\delta}\int_0^1 \mathbb{E}e^{3\delta t-2\delta B_t}\mathrm{d}s,$$

which immediately implies the desired inequality. □

Lemma 3.3. *Let* $\delta > 0$. *We have*

$$\int_0^1 \frac{\overline{u^\alpha}(s)}{\xi^{1+\beta+\delta}(s)}\mathrm{d}s \leq \Lambda(\delta,\zeta,B,\mathcal{M}_\delta^*), \tag{3.7}$$

where

$$\Lambda(\delta,\zeta,B,\mathcal{M}_\delta^*) = \delta^{-1}\zeta^{-\delta} + \frac{3+\delta}{2}e^{\frac{3}{2}\delta+\delta B_1^*}\zeta^{-\delta} + \mathcal{M}_\delta^*.$$

Proof. Applying Itô formula to $\xi^{-\delta}(t)$, we get

$$\xi^{-\delta}(t) - \zeta^{-\delta} = \frac{\delta(3+\delta)}{2}\int_0^t \xi^{-\delta}(s)\mathrm{d}s - \delta\int_0^t \frac{\overline{u^\alpha}(s)}{\xi^{1+\delta+\beta}(s)}\mathrm{d}s - \delta\int_0^t \xi^{-\delta}(s)\mathrm{d}B_s,$$

which gives

$$\begin{aligned}\int_0^t \frac{\overline{u^\alpha}(s)}{\xi^{1+\delta+\beta}(s)}\mathrm{d}s &\leq \delta^{-1}\zeta^{-\delta} + \frac{3+\delta}{2}\int_0^t \xi^{-\delta}(s)\mathrm{d}s + \sup_{0\leq t\leq 1}\left|\int_0^t \xi^{-\delta}(s)\mathrm{d}B_s\right| \\ &\leq \delta^{-1}\zeta^{-\delta} + \frac{3+\delta}{2}\int_0^t e^{\frac{3}{2}\delta s-\delta B_s}\zeta^{-\delta}\mathrm{d}s + \sup_{0\leq t\leq 1}\left|\int_0^t \xi^{-\delta}(s)\mathrm{d}B_s\right|,\end{aligned}$$

where the last inequality is by (3.1). This immediately yields the desired inequality. □

Next we shall follow the spirit in [18] to prove the following energy estimates, which is the key point for establishing the global solution.

Lemma 3.4. *Let* $\rho > 0$ *be some number such that*

$$\rho < \frac{q}{1+\beta}, \qquad \frac{p-1}{\alpha} < \rho < \frac{2}{d+2}. \tag{3.8}$$

Let $\ell > 0$ *and let*

$$\theta = \frac{1}{\ell}(p-1-\alpha\rho+\ell), \quad \gamma = \frac{d(\rho+\theta-1)}{2\theta}.$$

Let $\delta \in (0, \frac{q-\rho-\rho\beta}{\rho})$*. As* ℓ *is sufficiently large so that* $\theta \in (0,1)$*,* $\gamma \in (0,1)$ *and* $\frac{\rho}{1-\gamma\theta} \in (0,1)$*, we have*

$$\sup_{0\le t\le 1} \|u(t)\|_{L^\ell}^\ell \le C\left(\|v\|_{L^\ell}^{\frac{(1-\theta\gamma)\ell}{1-\theta}} + \Theta^{\frac{1-\theta\gamma}{1-\theta}}\Lambda^{\frac{\rho}{1-\theta}}(\delta,\zeta,B,\mathcal{M}_\delta^*)\right)\vee 1, \tag{3.9}$$

where C *depends on* p,q,α,β*. Further,* $\Lambda(\delta,\zeta,B,\mathcal{M}_\delta^*)$ *is defined in Lemma 3.3 and*

$$\Theta = \mathrm{e}^{\frac{3}{2}\frac{q-\rho(1+\beta+\delta)}{1-\theta\gamma}}\zeta^{\frac{\rho(1+\beta+\delta)-q}{1-\theta\gamma}}\mathrm{e}^{\frac{q-\rho(1+\beta+\delta)}{1-\theta\gamma}B_1^*}.$$

Proof. Without loss of generality, we assume $|\mathcal{O}| = 1$ in this proof. Let ℓ be a large number to be chosen later and write

$$w(t) = u^{\ell/2}(t).$$

Then a straightforward calculation gives

$$\partial_t\|w\|_{L_2}^2 = -\frac{4d(\ell-1)}{\ell}\|\nabla w\|_{L_2}^2 - \ell\|w\|_{L_2}^2 + \frac{\ell}{\xi^q}\int_{\mathcal{O}} u^{p-1+\ell}\mathrm{d}x. \tag{3.10}$$

Note that $\theta \in (0,1)$ as ℓ is large and $\lim_{\ell\to\infty}\theta = 1$. By the second inequality of (3.8), we have

$$0<\gamma<1 \quad \text{as } \ell \text{ is sufficiently large.} \tag{3.11}$$

By Hölder inequality and the following Gagliardo-Nirenberg inequality

$$\|w\|_{L^{\frac{2\theta}{1-\rho}}} \le C\left(\|\nabla w\|_{L^2} + \|w\|_{L^2}\right)^\gamma \|w\|_{L^2}^{1-\gamma}, \tag{3.12}$$

we have

(3.13)
$$\begin{aligned}
\frac{1}{\xi^q}\int_{\mathcal{O}} u^{p-1+\ell}\mathrm{d}x &= \frac{1}{\xi^q}\int_{\mathcal{O}} u^{\alpha\rho}u^{p-1-\alpha\rho+\ell}\mathrm{d}x \\
&\le \xi^{\rho(1+\beta+\delta)-q}\left(\int_{\mathcal{O}} w^{\frac{2\theta}{1-\rho}}\mathrm{d}x\right)^{1-\rho}\left(\frac{\overline{u^\alpha}}{\xi^{1+\beta+\delta}}\right)^\rho \\
&\le C\xi^{\rho(1+\beta+\delta)-q}\left(\|\nabla w\|_{L^2} + \|w\|_{L^2}\right)^{2\theta\gamma}\|w\|_{L^2}^{2\theta(1-\gamma)}\left(\frac{\overline{u^\alpha}}{\xi^{1+\beta+\delta}}\right)^\rho.
\end{aligned}$$

Note that $\gamma \in (0,1)$, the above and Young inequalities give

$$\begin{aligned}
\frac{1}{\xi^q}\int_{\mathcal{O}} u^{p-1+\ell}\mathrm{d}x &\le \theta\gamma c^{\frac{1}{\gamma\theta}}\left(\|\nabla w\|_{L^2} + \|w\|_{L^2}\right)^2 \\
&\quad + C\xi^{\frac{\rho(1+\beta+\delta)-q}{1-\theta\gamma}}\left(\frac{\overline{u^\alpha}}{\xi^{1+\beta+\delta}}\right)^{\frac{\rho}{1-\gamma\theta}}\|w\|_{L^2}^{\frac{2\theta(1-\gamma)}{1-\theta\gamma}}.
\end{aligned}$$

This inequality, together with (3.10), yields that as c is sufficiently small

$$\begin{aligned} \partial_t \|w\|_{L^2}^2 &\leq C\xi^{\frac{\rho(1+\beta+\delta)-q}{1-\theta\gamma}} \left(\frac{\overline{u^\alpha}}{\xi^{1+\beta+\delta}}\right)^{\frac{\rho}{1-\gamma\theta}} \|w\|_{L^2}^{\frac{2\theta(1-\gamma)}{1-\theta\gamma}} \\ &\leq C\left(\inf_{0\leq s\leq 1} \xi(s)\right)^{\frac{\rho(1+\beta+\delta)-q}{1-\theta\gamma}} \left(\frac{\overline{u^\alpha}}{\xi^{1+\beta+\delta}}\right)^{\frac{\rho}{1-\gamma\theta}} \|w\|_{L^2}^{\frac{2\theta(1-\gamma)}{1-\theta\gamma}} \quad \forall t\in[0,1], \end{aligned} \tag{3.14}$$

where the last inequality is by the fact that $\frac{\rho(1+\beta+\delta)-q}{1-\theta\gamma} < 0$ (due to the assumption of δ). Thanks to (3.2), we have

$$\left(\inf_{0\leq t\leq 1} \xi(t)\right)^{\frac{\rho(1+\beta+\delta)-q}{1-\theta\gamma}} \leq \Theta. \tag{3.15}$$

Writing $\eta(t) = \|w(t)\|_{L^2}^2$, it follows from (3.14) and (3.15) that

$$\partial_t \eta(t) \leq C\Theta \left(\sup_{0\leq t\leq 1} \eta(t)\right)^{\frac{\theta(1-\gamma)}{1-\theta\gamma}} \left(\frac{\overline{u^\alpha}(t)}{\xi^{1+\beta+\delta}(t)}\right)^{\frac{\rho}{1-\gamma\theta}} \quad \forall t\in[0,1]. \tag{3.16}$$

Thanks to the second inequality in (3.8), we have $\frac{\rho}{1-\gamma\theta} < 1$ as ℓ is sufficiently large, thus the above and Hölder inequalities give

$$\sup_{0\leq t\leq 1} \eta(t) \leq \eta(0) + C\Theta \left(\int_0^1 \frac{\overline{u^\alpha}(s)}{\xi^{1+\beta+\delta}(s)} \mathrm{d}s\right)^{\frac{\rho}{1-\gamma\theta}} \left(\sup_{0\leq t\leq 1} \eta(t)\right)^{\frac{\theta(1-\gamma)}{1-\theta\gamma}} \tag{3.17}$$

If $\sup_{0\leq t\leq 1} \eta(t) > 1$, (3.17) implies

$$\left(\sup_{0\leq t\leq 1} \eta(t)\right)^{\frac{1-\theta}{1-\theta\gamma}} \leq \eta(0) + C\Theta \left(\int_0^1 \frac{\overline{u^\alpha}(s)}{\xi^{1+\beta+\delta}(s)} \mathrm{d}s\right)^{\frac{\rho}{1-\gamma\theta}}$$

and thus

$$\sup_{0\leq t\leq 1} \eta(t) \leq \eta^{\frac{1-\gamma\theta}{1-\theta}}(0) + C\Theta^{\frac{1-\theta\gamma}{1-\theta}} \left(\int_0^1 \frac{\overline{u^\alpha}(s)}{\xi^{1+\beta+\delta}(s)} \mathrm{d}s\right)^{\frac{\rho}{1-\theta}}.$$

This and Lemma 3.3 give

$$\sup_{0\leq t\leq 1} \eta(t) \leq C\left(\|v\|_{L^\ell}^{\frac{(1-\theta\gamma)\ell}{1-\theta}} + \Theta^{\frac{1-\theta\gamma}{1-\theta}} \Lambda^{\frac{\rho}{1-\theta}}(\delta,\zeta,B,\mathcal{M}_\delta)\right) \quad \text{if} \quad \sup_{0\leq t\leq 1} \eta(t) > 1.$$

Combining this with the case $\sup_{0\leq t\leq 1} \eta(t) \leq 1$ immediately yields the desired inequality. $\square$

3.2. **Existence and uniqueness of the global solution.** Before proving the global existence and uniqueness of the solution, we recall some facts from ([16, pp. 15-16]). Consider Δ with Neumman boundary as an operator on $L^\theta(\mathcal{O})$ with $\theta \geq 1$. Then the associated Helmholtz operator is given by

$$\mathcal{H} = I - \Delta.$$

We can define $\mathcal{H}^\alpha$ for all α since $S(t)$ is an analytic operator. Let $D(\mathcal{H}^\alpha_\theta)$ be the domain of $\mathcal{H}^\alpha$ equipped with the norm $\|.\|_{D(\mathcal{H}^\alpha_\theta)} = \|.\|_{L^\theta} + \|\mathcal{H}^\alpha.\|_{L^\theta}$. There exists some $t_0 > 0$ such that

$$\|\mathcal{H}^\alpha S(t).\|_{D(\mathcal{H}^\alpha_\theta)} \lesssim t^{-\alpha}\|.\|_{L^\theta} \quad \forall\, t \in (0, t_0]. \tag{3.18}$$

As $\alpha > \frac{d}{2\theta}$, $D(\mathcal{H}^\alpha)$ is continuously embedded in $C(\mathcal{O})$.

Proof of Theorem 1.1. We first concentrate on proving the global existence and uniqueness of the solution and follow the spirit in [16].

By the a'priori estimates of (3.3) and (3.2), to show the global existence of Eq. (1.5), it suffices to show that u can be globally extended. Suppose that there exists some measurable set $A \subset \Omega$ with $\mathbb{P}(A) > 0$ such that for each $\omega \in A$ there exists some T^*_ω such that

$$\lim_{t \uparrow T^*_\omega} \|u(t)\|_C = \infty.$$

Without loss of generality, we may assume $t_0 < T^*_\omega < 1$, where t_0 is the constant in (3.18). Let $t^* = T^*_\omega - \frac{t_0}{2}$. Then, choosing p such that $\frac{d}{2p} < 1$ and some $\alpha \in (\frac{d}{2p}, 1)$, by (3.2) and (3.18), for all $t \in (t^*, T^*_\omega - \varepsilon]$ with any $\varepsilon \in (0, t_0/4)$ we have

(3.19)

$$\begin{aligned}
\|u(t)\|_{D(\mathcal{H}^\alpha_\theta)} &\leq \|S(t-t^*)u(t^*)\|_{D(\mathcal{H}^\alpha_\theta)} + \int_{t^*}^t \left\| S(t-s)\frac{u(s)^p}{\xi(s)^q} \right\|_{D(\mathcal{H}^\alpha_\theta)} \mathrm{d}s \\
&\lesssim (t-t^*)^{-\alpha}\|u(t^*)\|_{L^\theta} + \int_{t^*}^t (t-s)^{-\alpha}\frac{\|u(s)\|^p_{L^{p\theta}}}{\xi(s)^q}\mathrm{d}s \\
&\lesssim (t-t^*)^{-\alpha}\|u(t^*)\|_{L^\theta} + \mathrm{e}^{\frac{3}{2}q + qB^*_1}\zeta^{-q}(t-t^*)^{1-\alpha} \sup_{0 \leq s \leq T^*_\omega - \varepsilon} \|u(s)\|^p_{L^{p\theta}},
\end{aligned}$$

where $\sup_{0 \leq s \leq T^*_\omega - \varepsilon} \|u(s)\|^p_{L^{p\theta}} \leq \tilde{C}$ and $\tilde{C}$ only depends on v, ζ, $p, q, \theta, \alpha, \beta, \omega$ by Lemma 3.4. Since $\varepsilon \in (0, t_0/4)$ and $t^* = T^*_\omega - \frac{t_0}{2}$, from the above inequality we get

$$\|u(T^*_\omega - \varepsilon)\|_{D(\mathcal{H}^\alpha_\theta)} \lesssim t_0^{-\alpha}\|v\|_{L^\theta} + \mathrm{e}^{\frac{3}{2}qt + qB^*_1}\zeta^{-q}t_0^{1-\alpha}\tilde{C}.$$

By Sobolev embedding, we further get

$$\|u(T^*_\omega - \varepsilon)\|_C \lesssim t_0^{-\alpha}\|u(t^*)\|_{L^\theta} + \mathrm{e}^{\frac{3}{2}qt + qB^*_1}\zeta^{-q}t_0^{1-\alpha}\tilde{C}.$$

Since $\varepsilon > 0$ can be arbitrarily small, we have

$$\|u(T^*_\omega -)\|_C \lesssim {t_0}^{-\alpha}\|u(t^*)\|_{L^\theta} + \mathrm{e}^{\frac{3}{2}qt + qB^*_1}\zeta^{-q}{t_0}^{1-\alpha}\tilde{C}.$$

This gives a contradiction. Hence, Eq. (1.5) admits a global unique solution for all $\omega \in \Omega$ a.s..

Now we prove the estimates in the theorem. From (3.1), it is easy to see that

$$\xi(t) \geq \mathrm{e}^{-\frac{3}{2}t-|B_t|}\zeta.$$

On the other hand, by (2.6) and a bootstrap argument, we obtain for all $\omega \in \Omega$ up to a negligible set,

$$u(t,x) \geq 0 \quad \forall\, t \geq 0 \;\; x \in \mathcal{O}.$$

□

4. Large deviation result

In this section, we prove the large deviation results. We begin by recalling the definition of the large deviation principle. Let $\{X^\varepsilon, \varepsilon > 0\}$ be a family of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a Polish space $\mathcal{E}$. Denote expectation with respect to $\mathbb{P}$ by $\mathbb{E}$. The large deviation principle is concerned with exponential decay of $\mathbb{P}(X^\varepsilon \in O)$ as $\varepsilon \to 0$.

Definition 4.1. ***(Rate function)*** A function $I : \mathcal{E} \to [0, \infty]$ is called a rate function on $\mathcal{E}$, if for each $M < \infty$ the level set $\{x \in \mathcal{E} : I(x) \leq M\}$ is a compact subset of $\mathcal{E}$. For $O \in \mathcal{B}(\mathcal{E})$, we define $I(O) \doteq \inf_{x\in O} I(x)$.

Definition 4.2. ***(Large deviation principle)*** Let I be a rate function on $\mathcal{E}$. The sequence $\{X^\varepsilon\}$ is said to satisfy the large deviation principle on $\mathcal{E}$ with rate function I if the following two conditions hold.

a. Large deviation upper bound. For each closed subset F of $\mathcal{E}$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \leq -I(F).$$

b. Large deviation lower bound. For each open subset G of $\mathcal{E}$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in G) \geq -I(G).$$

Remark 4.3. Note that the I above is a function from sets to real numbers. To define the rate function I, it suffices to define its value at each point.

4.1. **Large deviation result and the method.** Without loss of generality, we shall prove the LDP result for the dynamics in the time interval $[0, 1]$. Before stating our large deviation result, let us first recall the following preliminaries.

The Cameron-Martin space associated to the Brownian motion B_t is as follows:

$$H = \{h \in H^1([0,1];\mathbb{R}) : h(t) = \int_0^t \dot{h}(s)\mathrm{d}s,\; \|\dot{h}\|_{L^2([0,1],\mathbb{R})} < \infty\}.$$

Then H is a Hilbert space with the norm

$$\|h\|_H = \|\dot{h}\|_{L^2([0,1],\mathbb{R})} \quad \forall h \in H.$$

It is clear to see

$$|h(t)-h(s)| \le \|h\|_H \quad \forall 0 \le s < t \le 1. \tag{4.1}$$

Fix $N > 0$, and denote

$$\mathcal{A}_N^d = \{h \in H,\ \|h\|_H \le N\}.$$

Then $\mathcal{A}_N^d$ is a compact Polish space endowed with the weak topology of H. We denote the weak convergence in $\mathcal{A}_N^d$ by $\cdot \rightharpoonup \cdot$. Then for $\{h_n\}_n \subset H$ and $h \in H$, $h_n \rightharpoonup h$ if

$$\lim_{n\to\infty} \int_0^1 \phi(s)\dot{h}_n(s)\mathrm{d}s = \int_0^1 \phi(s)\dot{h}(s)\mathrm{d}s \quad \forall\, \phi \in L^2([0,1];\mathbb{R}).$$

Define

$$\begin{aligned}\mathcal{A}^s = \{h;\ h : \Omega\times[0,1] \to \mathbb{R} \text{ satisfies } h(\omega,.) \in H\ \ \forall\omega\in\Omega \\ \text{and } h(.,t) \text{ is } \mathcal{F}_t \text{ measurable } \forall t \in [0,1]\}\end{aligned}$$

and for all $N > 0$

$$\mathcal{A}_N^s = \{h \in \mathcal{A}^s : \|h(\omega)\|_H \le N \quad \forall \omega\in\Omega\}.$$

Let $h \in H$, consider the following differential equation

$$\begin{aligned}\partial_t u_h &= \Delta u_h - u_h + \frac{u_h^p}{\xi_h^q}, \\ \mathrm{d}\xi_h &= -\xi_h \mathrm{d}t + \frac{\overline{u_h^\alpha}}{\xi_h^\beta}\mathrm{d}t + \xi_h \mathrm{d}h(t),\end{aligned} \tag{4.2}$$

with the same boundary and initial conditions as in Eq. (1.5).

Let $\varepsilon \in [0,1]$ and let $(h_\varepsilon)_{0\le\varepsilon\le1} \subset \mathcal{A}^s$, to study the large deviation of Eq. (1.5), we also need to consider the following stochastic PDEs:

$$\begin{aligned}\partial_t u_{\varepsilon,h_\varepsilon} &= \Delta u_{\varepsilon,h_\varepsilon} - u_{\varepsilon,h_\varepsilon} + \frac{u_{\varepsilon,h_\varepsilon}^p}{\xi_{\varepsilon,h_\varepsilon}^q}, \\ \mathrm{d}\xi_{\varepsilon,h_\varepsilon} &= -\xi_{\varepsilon,h_\varepsilon}\mathrm{d}t + \frac{\overline{u_{\varepsilon,h_\varepsilon}^\alpha}}{\xi_{\varepsilon,h_\varepsilon}^\beta}\mathrm{d}t + \sqrt{\varepsilon}\xi_{\varepsilon,h_\varepsilon}\mathrm{d}B_t + \xi_{\varepsilon,h_\varepsilon}\mathrm{d}h_\varepsilon(t),\end{aligned} \tag{4.3}$$

with the same boundary and initial conditions as in Eq. (1.5). By the same argument as in the previous section, we can prove the global existence and uniqueness of the solutions to Eqs. (4.2) and (4.3).

Now we are at the position to state our large deviation result.

Theorem 4.4 (Large deviation principle)**.** *Let $\{(u_\varepsilon, \xi_\varepsilon)\}$ be the solution of the equation*

$$\begin{cases} \partial_t u_\varepsilon = \Delta u_\varepsilon - u_\varepsilon + \frac{u_\varepsilon^p}{\xi_\varepsilon^q}, \\ \mathrm{d}\xi_\varepsilon = -\xi_\varepsilon \mathrm{d}t + \frac{\overline{u_\varepsilon^\alpha}}{\xi_\varepsilon^\beta}\mathrm{d}t + \sqrt{\varepsilon}\xi_\varepsilon \mathrm{d}B_t, \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0, \\ u_\varepsilon(0) = v, \\ \xi_\varepsilon(0) = \zeta. \end{cases} \tag{4.4}$$

Then $\{(u_\varepsilon, \xi_\varepsilon)\}$ satisfies a large deviation principle in $C([0,1]; C \times \mathbb{R})$ with the rate function I given as follows: for any $(u, \xi) \in C([0,1]; C \times \mathbb{R})$ we have

$$I((u,\xi)) := \inf_{\{h\in H:(u_h,\xi_h)=(u,\xi)\}} \left(\frac{1}{2}\|h\|_H^2\right),$$

with the convention $\inf\{\emptyset\} = \infty$, where (u_h, ξ_h) is the solution to Eq. (4.2).

We shall follow the method in [1, Theorem 4.4] to prove the above LDP. According to this method, we only need to show the following two propositions.

Proposition 4.5. *Let $g_n,\ h \in \mathcal{A}_N^d$ and (u_{g_n}, ξ_{g_n}) be the solution of Eq. (4.2) with h replaced by g_n. Up to taking a subsequence, we have*

$$\lim_{g_n \rightharpoonup h} \|(u_{g_n}, \xi_{g_n}) - (u_h, \xi_h)\|_{C([0,1];C\times\mathbb{R})} = 0.$$

Proposition 4.6. *For a family $\{h_\varepsilon\} \subset \mathcal{A}_N^s$ for which h_ε converges in distribution to h under the weak topology of H, up to taking a subsequence, the solution $(u_{\varepsilon,h_\varepsilon}, \xi_{\varepsilon,h_\varepsilon})$ of (4.3) converges in distribution to (u_h, ξ_h); more precisely, for all bounded continuous functions $f : C([0,1]; C \times \mathbb{R}) \to \mathbb{R}$, up to taking a subsequence, the following relation holds:*

$$\lim_{\varepsilon\to 0} \mathbb{E} f(u_{\varepsilon,h_\varepsilon}, \xi_{\varepsilon,h_\varepsilon}) = \mathbb{E} f(u_h, \xi_h). \tag{4.5}$$

4.2. **Proof of Proposition 4.5.** Before showing Proposition 4.5, we prove the following lemmas which provide the preliminaries for using Sobolev embedding and Arzelà-Ascoli Theorem.

Lemma 4.7. *For all $t \in [0,1]$, we have the following estimates*

$$\xi_h(t) \geq \mathrm{e}^{-t-\|h\|_H}\zeta, \tag{4.6}$$

$$\xi_h(t) \lesssim \mathrm{e}^{\|h\|_H}\zeta + \mathrm{e}^{\|h\|_H}\left(\sup_{0\le t\le 1} \overline{u_h^\alpha}(t)\right)^{\frac{1}{1+\beta}}. \tag{4.7}$$

Proof. From Eq. (4.2), we have

$$\mathrm{d}\xi_h^{1+\beta}(t) = -(1+\beta)\xi_h^{1+\beta}(t)\mathrm{d}t + (1+\beta)\xi_h^{1+\beta}(t)\mathrm{d}h(t) + (1+\beta)\overline{u_h^\alpha}(t)\mathrm{d}t, \tag{4.8}$$

which clearly implies

$$\xi_h^{1+\beta}(t) = \mathrm{e}^{-(1+\beta)t+(1+\beta)h(t)}\zeta^{1+\beta} + (1+\beta)\int_0^t \mathrm{e}^{-(1+\beta)(t-s)+(1+\beta)(h(t)-h(s))}\overline{u_h^{\alpha}}(s)\mathrm{d}s.$$

This equality and (4.1) clearly imply the desired two inequalities. □

Lemma 4.8. *We have*

$$\int_0^t \frac{\overline{u_h^{\alpha}}(s)}{\xi_h^{1+\delta+\beta}(s)}\mathrm{d}s \le \Lambda(\delta,\zeta,h) \quad \forall t\in[0,1],$$

where

$$\Lambda(\delta,\zeta,h) = \delta^{-1}\zeta^{-\delta} + \mathrm{e}^{\delta(1+\|h\|_H)}\zeta^{-\delta} + \mathrm{e}^{\delta(1+\|h\|_H)}\|h\|_H.$$

Proof. Differentiating $\xi_h^{-\delta}(t)$ we get

$$\xi_h^{-\delta}(t) - \zeta^{-\delta} = \delta\int_0^t \xi_h^{-\delta}(s)\mathrm{d}s - \delta\int_0^t \frac{\overline{u_h^{\alpha}}(s)}{\xi_h^{1+\delta+\beta}(s)}\mathrm{d}s - \delta\int_0^t \xi_h^{-\delta}(s)\mathrm{d}h_s,$$

which, together with (4.6) and Hölder inequality, gives

$$\begin{aligned}\int_0^t \frac{\overline{u_h^{\alpha}}(s)}{\xi_h^{1+\delta+\beta}(s)}\mathrm{d}s &\le \delta^{-1}\zeta^{-\delta} + \int_0^t \xi_h^{-\delta}(s)\mathrm{d}s + \left|\int_0^t \xi_h^{-\delta}(s)\dot{h}_s\mathrm{d}s\right| \\ &\le \delta^{-1}\zeta^{-\delta} + \mathrm{e}^{\delta(1+\|h\|_H)}\zeta^{-\delta} + \left(\int_0^t \xi_h^{-2\delta}(s)\mathrm{d}s\right)^{\frac12}\|h\|_H \\ &\le \delta^{-1}\zeta^{-\delta} + \mathrm{e}^{\delta(1+\|h\|_H)}\zeta^{-\delta} + \mathrm{e}^{\delta(1+\|h\|_H)}\|h\|_H\end{aligned}$$

for all $t\in[0,1]$. This completes the proof. □

Lemma 4.9. *Let ρ,ℓ,θ,γ be the same as in Lemma 3.4. Let $\delta\in(0,\frac{q-\rho-\rho\beta}{\rho})$. As ℓ is sufficiently large so that $\theta\in(0,1)$, $\gamma\in(0,1)$ and $\frac{\rho}{1-\gamma\theta}\in(0,1)$, we have*

$$\sup_{0\le t\le 1}\|u_h(t)\|_{L^\ell}^{\ell} \le C\left(\|v\|_{L^\ell}^{\frac{(1-\theta\gamma)\ell}{1-\theta}} + \tilde{\Theta}^{\frac{1-\theta\gamma}{1-\theta}}\Lambda^{\frac{\rho}{1-\theta}}(\delta,\zeta,h)\right)\vee 1,$$

where C depends on α,β,p,q, $\Lambda(\delta,\zeta,h)$ is defined in Lemma 4.8 and

$$\tilde{\Theta} = \mathrm{e}^{\frac{q-\rho(1+\beta+\delta)}{1-\theta\gamma}}\zeta^{\frac{\rho(1+\beta+\delta)-q}{1-\theta\gamma}}\mathrm{e}^{\frac{q-\rho(1+\beta+\delta)}{1-\theta\gamma}\|h\|_H}.$$

Proof. Repeating the argument for deriving (3.17) and using (4.1), we get

$$\sup_{0\le t\le 1}\eta(t) \le \eta(0) + C\tilde{\Theta}\left(\int_0^1 \frac{\overline{u_h^{\alpha}}(s)}{\xi_h^{1+\beta+\delta}(s)}\mathrm{d}s\right)^{\frac{\rho}{1-\gamma\theta}}\left(\sup_{0\le t\le 1}\eta(t)\right)^{\frac{\theta(1-\gamma)}{1-\theta\gamma}},$$

where $\eta(t) = \|u_h(t)\|_{L^\ell}^{\ell}$. By the same argument as that after (3.17), we get the desired inequality. □

Lemma 4.10. *Let* (u_h, ξ_h) *be the solution of Eq.* (4.2)*. We have*

$$\sup_{h\in\mathcal{A}_N^d} \|(u_h,\xi_h)\|_{C([0,1];C\times\mathbb{R})} \le C, \tag{4.9}$$

where C *depends on* $N, \zeta, \|v\|_C, \alpha, \beta, p, q$.

Proof. Similar as in the proof of Lemma 2.1, set

$$\mathcal{A}_{T,M,N} = \Big\{(u,\xi) \in C([0,T]; C(\mathcal{O},\mathbb{R})\times\mathbb{R}) : u(t)\ge 0,\ \xi(t)\ge \mathrm{e}^{-t-N}\zeta, \forall 0\le t\le T;$$
$$u(0)=v,\ \xi(0)=\zeta;\ \|(u,\xi)\|_{C([0,T];C\times\mathbb{R})}\le M\Big\}$$

with $M > 2+\|v\|_C+\mathrm{e}^N\zeta$ and $T>0$ being some number depending on $N, M, \alpha, \beta, p, q$. By a similar argument as in the proof of Lemma 2.1, we have

$$\sup_{h\in\mathcal{A}_N^d} \|(u_h,\xi_h)\|_{C([0,T];C\times\mathbb{R})} \le M. \tag{4.10}$$

To complete the proof, we only need to bound the solution on the time interval $[T,1]$. On the one hand, by (4.6), (4.7) and Lemma 4.9, there exists some $\bar{C}$ depending only on v, ζ, N such that

$$\sup_{h\in\mathcal{A}_N^d} \|\xi_h\|_{C([0,1];\mathbb{R})} \le \overline{C}. \tag{4.11}$$

Repeating the argument as in the proof of Theorem 1.1 and choosing $\alpha > \frac{d}{2\theta}$, we have some $\hat{C}$ depending only on $v, \zeta, \alpha, \beta, N$ such that

$$\sup_{h\in\mathcal{A}_N^d}\ \sup_{T\le t\le 1} \|u_h\|_{D(\mathcal{H}_p^\alpha)} \le \hat{C}.$$

This inequality and Sobolev embedding theorem further give

$$\sup_{h\in\mathcal{A}_N^d} \|u_h\|_{C([T/2,1];C)} \le \widetilde{C}, \tag{4.12}$$

where $\tilde{C}$ depends only on $v, \zeta, \alpha, \beta, N$. Hence,

$$\sup_{h\in\mathcal{A}_N^d} \|(u_h,\xi_h)\|_{C([0,1];C\times\mathbb{R})} \le \tilde{C}+\bar{C}. \tag{4.13}$$

The proof is complete. □

Proof of Proposition 4.5. Let all constants C below be some numbers depending on $N, \zeta, \|v\|_C, \alpha, \beta, p, q$, whose exact values may vary from line to line. Recall $S(t) = e^{(\Delta-1)t}$ and denote $\Lambda_{n,m}(t) = u_{g_n}(t) - u_{g_m}(t)$. Observe

$$\Lambda_{n,m}(t) = \int_0^t S(t-s)\left(\frac{u_{g_n}^p(s)}{\xi_{g_n}^q(s)} - \frac{u_{g_m}^p(s)}{\xi_{g_m}^q(s)}\right)\mathrm{d}s.$$

Thanks to Lemma 4.7 and Lemma 4.10, we have

(4.14)
$$
\begin{aligned}
\|\Lambda_{n,m}(t)\|_C &\leq \int_0^t \left\| \frac{u^p_{g_n}(s)}{\xi^q_{g_n}(s)} - \frac{u^p_{g_m}(s)}{\xi^q_{g_m}(s)} \right\|_C \mathrm{d}s \\
&\leq \int_0^t \frac{\left\|u^p_{g_n}(s) - u^p_{g_m}(s)\right\|_C}{\xi^q_{g_n}(s)} \mathrm{d}s + \int_0^t \|u_{g_m}(s)\|_C^p \left| \frac{1}{\xi^q_{g_n}(s)} - \frac{1}{\xi^q_{g_m}(s)} \right| \mathrm{d}s \\
&\leq C\int_0^t \Lambda_{m,n}(s)\mathrm{d}s + C\int_0^t |\xi_{g_n}(s) - \xi_{g_m}(s)|\mathrm{d}s.
\end{aligned}
$$

For all $s, t \in [0,1]$ and $g_n \in \mathcal{A}^d_N$, by Lemma 4.10 and the second equation of (4.2), we have

$$
\begin{aligned}
|\xi_{g_n}(t) - \xi_{g_n}(s)| &\leq \int_s^t \xi_{g_n}(r)\mathrm{d}r + \int_s^t \frac{\overline{u^\alpha_{g_n}}(r)}{\xi^\beta_{g_n}(r)}\mathrm{d}r + \int_s^t \xi_{g_n}(r)\,|\dot{g}_n(r)|\,\mathrm{d}r \\
&\leq C(t-s) + C(t-s) + \left(\int_s^t |\xi_{g_n}(r)|^2\mathrm{d}r\right)^{\frac{1}{2}} \left(\int_0^1 |\dot{g}_n(r)|^2\mathrm{d}r\right)^{\frac{1}{2}} \\
&\leq C(t-s) + C(t-s)^{\frac{1}{2}}.
\end{aligned}
$$

The above inequality clearly implies that $\{\xi_{g_n},\ n \geq 1\}$ is equi-continuous. By Arzelà-Ascoli Theorem, there exist some $\xi \in C([0,1];\mathbb{R})$ and a subsequence of $\{\xi_{g_n},\ n \geq 1\}$ (say $\{\xi_{g_n},\ n \geq 1\}$ without loss of generality) such that

(4.15)
$$
\lim_{n\to\infty} \|\xi_{g_n} - \xi\|_{C([0,1];\mathbb{R})} = 0.
$$

It follows from (4.6) and (4.9) that for all $t \in [0,1]$

$$
\xi(t) \geq \mathrm{e}^{-t-\|h\|_H}\zeta.
$$

Moreover, (4.15) and (4.14) clearly imply that $\{u_{g_n},\ n \geq 1\}$ is a Cauchy sequence in $C([0,1];C)$. Hence, there exists some $u \in C([0,1];C)$ so that

(4.16)
$$
\lim_{n\to\infty} \|u_{g_n} - u\|_{C([0,1];C)} = 0.
$$

Since

$$
u_{g_n}(t) = S(t)v + \int_0^t S(t-s)\frac{u^p_{g_n}(s)}{\xi^q_{g_n}(s)}\mathrm{d}s,
$$

letting $n \to \infty$ we get

(4.17)
$$
u(t) = S(t)v + \int_0^t S(t-s)\frac{u^p(s)}{\xi^q(s)}\mathrm{d}s.
$$

On the other hand, by (4.15) and $g_n \rightharpoonup h$ in H,

$$\begin{aligned}&\int_0^t \xi_{g_n}(s)\dot{g}_n(s)\mathrm{d}s - \int_0^t \xi(s)\dot{h}(s)\mathrm{d}s \\ =&\int_0^t [\xi_{g_n}(s)-\xi(s)]\dot{g}_n(s)\mathrm{d}s + \int_0^t \xi(s)[\dot{g}_n(s)-\dot{h}(s)]\mathrm{d}s \to 0\end{aligned}$$

as $n \to \infty$. Letting $n \to \infty$, the above limit and the following relation

$$\xi_{g_n}(t) = \zeta - \int_0^t \xi_{g_n}(s)\mathrm{d}s + \int_0^t \frac{\overline{u^\alpha_{g_n}}(s)}{\xi^\beta_{g_n}(s)}\mathrm{d}s + \int_0^t \xi_{g_n}(s)\dot{g}_n(s)\mathrm{d}s$$

give

$$\xi(t) = \zeta - \int_0^t \xi(s)\mathrm{d}s + \int_0^t \frac{\overline{u^\alpha}(s)}{\xi^\beta(s)}\mathrm{d}s + \int_0^t \xi(s)\dot{h}(s)\mathrm{d}s.$$

This relation, together with (4.17), implies that (u,ξ) solves Eq. (4.2). Thanks to the uniqueness, we have $(u,\xi) = (u_h,\xi_h)$ and thus

$$\lim_{g_n \rightharpoonup h} \|(u_{g_n},\xi_{g_n}) - (u_h,\xi_h)\|_{C([0,1];C\times\mathbb{R})} = 0.$$

□

4.3. **Proof of Proposition 4.6.** Before showing Proposition 4.6, let us first prove the following lemmas which give the preliminaries for using Skorohod embedding and an asymptotic tightness criterion in probability theory.

Lemma 4.11. *Let $\varepsilon > 0$ be such that $2-\beta\varepsilon > 0$ and $h_\varepsilon \in \mathcal{A}^s_N$. We have the following estimates*

$$\xi_{\varepsilon,h_\varepsilon}(t) \geq \mathrm{e}^{-\frac{(2-\varepsilon\beta)}{2}t - N + \sqrt{\varepsilon}B_t}\zeta \quad \forall t\in[0,1], \tag{4.18}$$

$$\inf_{0\le t\le 1} \xi_{\varepsilon,h_\varepsilon}(t) \geq \mathrm{e}^{-1-N-\sqrt{\varepsilon}B^*_1}\zeta. \tag{4.19}$$

Moreover, we have

$$\sup_{0\le t\le 1} \xi_{\varepsilon,h_\varepsilon}(t) \lesssim \mathrm{e}^{N+\sqrt{\varepsilon}B^*_1}\zeta^{1+\beta} + \mathrm{e}^{N+2\sqrt{\varepsilon}B^*_1}\left(\sup_{0\le t\le 1}\overline{u^\alpha_{\varepsilon,h_\varepsilon}}(t)\right)^{\frac{1}{1+\beta}}. \tag{4.20}$$

Proof. We simply write $u = u_{\varepsilon,h_\varepsilon}$, $\xi = \xi_{\varepsilon,h_\varepsilon}$ and $h = h_\varepsilon$. By Itô formula, we have

$$\begin{aligned}\mathrm{d}\xi^{1+\beta}(t) =& -\frac{1}{2}(1+\beta)(2-\varepsilon\beta)\xi^{1+\beta}(t)\mathrm{d}t + (1+\beta)\xi^{1+\beta}(t)\mathrm{d}h(t)\\ &+\sqrt{\varepsilon}(1+\beta)\xi^{1+\beta}(t)\mathrm{d}B_t + (1+\beta)\overline{u^\alpha}(t)\mathrm{d}t,\end{aligned} \tag{4.21}$$

which clearly implies

$$\begin{aligned}\xi^{1+\beta}(t) &= \mathrm{e}^{-\frac{(1+\beta)(2-\varepsilon\beta)}{2}t+(1+\beta)h(t)+\sqrt{\varepsilon}(1+\beta)B_t}\zeta^{1+\beta} \\ &\quad + (1+\beta)\int_0^t \mathrm{e}^{-\frac{(1+\beta)(2-\varepsilon\beta)}{2}(t-s)+(1+\beta)(h(t)-h(s))+\sqrt{\varepsilon}(1+\beta)(B_t-B_s)}\overline{u^\alpha}(s)\mathrm{d}s \\ &\le \mathrm{e}^{(1+\beta)\|h\|_H+\sqrt{\varepsilon}(1+\beta)B_1^*}\zeta^{1+\beta} + (1+\beta)\int_0^t \mathrm{e}^{(1+\beta)\|h\|_H+2\sqrt{\varepsilon}(1+\beta)B_1^*}\overline{u^\alpha}(s)\mathrm{d}s,\end{aligned}$$

where the last inequality is by (4.1). The above inequality clearly implies the three desired inequalities. □

Let $\delta > 0$, define

$$\mathcal{M}_{\varepsilon,\delta}(t) = \int_0^t \xi_{\varepsilon,h_\varepsilon}^{-\delta}(s)\mathrm{d}B_s, \quad \mathcal{M}_{\varepsilon,\delta}^* = \sup_{0\le t\le 1} |\mathcal{M}_{\varepsilon,\delta}(t)|.$$

Lemma 4.12. *Let $\mu > 0$ and $\delta > 0$, for all $\varepsilon \in [0,1]$ we have*

$$\mathbb{E}\left(\mathcal{M}_{\varepsilon,\delta}^*\right)^\mu \le C, \tag{4.22}$$

where C depends only on μ, N, δ and ζ. Moreover, we have

$$\mathcal{M}_{\varepsilon,\delta}^* < \infty \quad a.s.. \tag{4.23}$$

Proof. We only have to show the desired inequality for the case $\mu > 2$ since the case of $0 < \mu \le 2$ is an immediate corollary from the former. We simply write $\xi_\varepsilon = \xi_{\varepsilon,h_\varepsilon}$.

By Burkholder-Davis-Gundy inequality and Hölder inequality, we have

$$\mathbb{E}\left(\mathcal{M}_{\varepsilon,\delta}^*\right)^\mu \le C\mathbb{E}\left[\int_0^1 \xi_\varepsilon^{-2\delta}(s)\mathrm{d}s\right]^{\frac{\mu}{2}} \le C\left[\int_0^1 \mathbb{E}\xi_\varepsilon^{-\mu\delta}(s)\mathrm{d}s\right],$$

which, together with (4.19), gives

$$\mathbb{E}\left(\mathcal{M}_{\varepsilon,\delta}^*\right)^\mu \le C\mathbb{E}\mathrm{e}^{\mu\delta+\mu\delta N+\mu\delta\sqrt{\varepsilon}B_1^*}\zeta^{-\mu\delta}.$$

The desired inequality immediately follows from the above inequality and (2.2). The second inequality is a direct corollary from the first one. □

Lemma 4.13. *Let $\varepsilon > 0$ be such that $2 - \beta\varepsilon > 0$ and let $h_\varepsilon \in \mathcal{A}_N^s$. For all $\delta > 0$, we have*

$$\int_0^1 \frac{\overline{u_{\varepsilon,h_\varepsilon}^\alpha}(s)}{\xi_{\varepsilon,h_\varepsilon}^{1+\beta+\delta}(s)}\mathrm{d}s \le \Lambda(\zeta,\varepsilon,B,N,\delta,\mathcal{M}_{\varepsilon,\delta}^*),$$

where

$$\Lambda(\zeta,\varepsilon,B,N,\delta,\mathcal{M}_{\varepsilon,\delta}^*) = \delta^{-1}\zeta^{-\delta} + \frac{(2+\varepsilon+\delta\varepsilon+2N)\mathrm{e}^{\delta+\delta N+\delta\sqrt{\varepsilon}B_1^*}}{2}\zeta^{-\delta} + \sqrt{\varepsilon}\mathcal{M}_{\varepsilon,\delta}^*$$

Proof. For notational simplicity, we shall write $\xi(t) = \xi_{\varepsilon,h_\varepsilon}(t)$ and $u(t) = u_{\varepsilon,h_\varepsilon}(t)$. Applying Itô formula to $\xi^{-\delta}(t)$, we get

$$\begin{aligned}\xi^{-\delta}(t)-\zeta^{-\delta}=&\frac{\delta\,(2+\varepsilon+\delta\varepsilon)}{2}\int_0^t\xi^{-\delta}(s)\mathrm{d}s-\delta\int_0^t\frac{\overline{u^\alpha}(s)}{\xi^{1+\delta+\beta}(s)}\mathrm{d}s\\&-\delta\int_0^t\xi^{-\delta}(s)\dot{h}_s\mathrm{d}s-\delta\sqrt{\varepsilon}\int_0^t\xi^{-\delta}(s)\mathrm{d}B_s,\end{aligned}$$

which gives

$$\begin{aligned}\int_0^t\frac{\overline{u^\alpha}(s)}{\xi^{1+\delta+\beta}(s)}\mathrm{d}s\le{}&\delta^{-1}\zeta^{-\delta}+\frac{2+\varepsilon+\delta\varepsilon}{2}\int_0^t\xi^{-\delta}(s)\mathrm{d}s\\&+\sup_{0\le t\le1}\left|\int_0^t\xi^{-2\delta}(s)\mathrm{d}s\right|^{\frac12}\|h\|_H+\sqrt{\varepsilon}\sup_{0\le t\le1}\left|\int_0^t\xi^{-\delta}(s)\mathrm{d}B_s\right|\\&\le\delta^{-1}\zeta^{-\delta}+\frac{(2+\varepsilon+\delta\varepsilon+2\|h_\varepsilon\|_H)\mathrm{e}^{\delta+\delta\|h_\varepsilon\|_H+\delta\sqrt{\varepsilon}B_1^*}}{2}\zeta^{-\delta}\\&+\sqrt{\varepsilon}\sup_{0\le t\le1}\left|\int_0^t\xi^{-\delta}(s)\mathrm{d}B_s\right|,\end{aligned}$$

where the last inequality is by (4.19). This clearly implies the desired inequality. □

Lemma 4.14. *Let ρ,ℓ,θ,γ be the same as those in Lemma 3.4. Let $h_\varepsilon\in\mathcal{A}_N^s$ and $\delta\in(0,\frac{q-\rho-\rho\beta}{\rho})$. As ℓ is sufficiently large so that $\theta\in(0,1)$, $\gamma\in(0,1)$ and $\frac{\rho}{1-\gamma\theta}\in(0,1)$, we have*

$$\sup_{0\le t\le1}\|u_{\varepsilon,h_\varepsilon}(t)\|_{L^\ell}^\ell\le C\left(\|v\|_{L^\ell}^{\frac{(1-\theta\gamma)\ell}{1-\theta}}+\hat{\Theta}^{\frac{1-\theta\gamma}{1-\theta}}\Lambda^{\frac{\rho}{1-\theta}}(\zeta,\varepsilon,B,N,\delta,\mathcal{M}^*_{\varepsilon,\delta})\right)\vee1,$$

where C depends on α,β,p,q, $\Lambda(\zeta,\varepsilon,B,N,\delta,\mathcal{M}^*_{\varepsilon,\delta})$ is defined in Lemma 4.13 and

$$\hat{\Theta}=\mathrm{e}^{(1+N)\frac{q-\rho(1+\beta+\delta)}{1-\theta\gamma}}\zeta^{\frac{\rho(1+\beta+\delta)-q}{1-\theta\gamma}}\mathrm{e}^{\frac{q-\rho(1+\beta+\delta)}{1-\theta\gamma}\sqrt{\varepsilon}B_1^*}.$$

Proof. Repeating the argument for getting (3.17) and using (4.1), we get

$$\sup_{0\le t\le1}\eta(t)\le\eta(0)+C\hat{\Theta}\left(\int_0^1\frac{\overline{u^\alpha_{\varepsilon,h_\varepsilon}}(s)}{\xi^{1+\beta+\delta}_{\varepsilon,h_\varepsilon}(s)}\mathrm{d}s\right)^{\frac{\rho}{1-\gamma\theta}}\left(\sup_{0\le t\le1}\eta(t)\right)^{\frac{\theta(1-\gamma)}{1-\theta\gamma}},\tag{4.24}$$

where $\eta(t)=\|u_{\varepsilon,h_\varepsilon}(t)\|_{L^\ell}^\ell$. Repeating the argument after (3.17), we immediately get the desired inequality. □

Proof of Proposition 4.6. For notational simplicity, we shall write $u_\varepsilon=u_{\varepsilon,h_\varepsilon}$ and $\xi_\varepsilon=\xi_{\varepsilon,h_\varepsilon}$. We choose $\ell>0$ in Lemma 4.14 to be sufficiently large so that $\ell>2\alpha$ and fix it. We also fix the number $\rho,\theta,\gamma,\delta$ in Lemma 4.14. By their definitions, $\ell,\rho,\theta,\gamma,\delta$ are all some fixed numbers depending on α,β,p,q. Let all C below be some numbers depending on $\zeta, v, \alpha, \beta, p, q$ and N, whose exact values may vary from one another. We shall prove the proposition by the following two steps.

(Step 1) We shall prove in Step 2 below that there exist some $\xi \in C([0,1],\mathbb{R})$ and a subsequence $\{\xi_{\varepsilon_n}\}$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that

$$\lim_{n\to\infty} \xi_{\varepsilon_n} = \xi \quad \text{in distribution under the topology } C([0,1],\mathbb{R}). \tag{4.25}$$

By Skorohod embedding theorem, there exist a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and random variables $\{\hat{\xi}_{\varepsilon_n}\}$ and $\hat{\xi}$ which have the same distributions as $\{\xi_{\varepsilon_n}\}$ and ξ, respectively, such that

$$\lim_{n\to\infty} \|\hat{\xi}_{\varepsilon_n} - \hat{\xi}\|_{C([0,1];\mathbb{R})} = 0 \quad a.s..$$

Consider the equations

$$\partial_t \hat{u}_{\varepsilon_n} = \Delta \hat{u}_{\varepsilon_n} - \hat{u}_{\varepsilon_n} + \frac{\hat{u}^p_{\varepsilon_n}}{\hat{\xi}^q_{\varepsilon_n}}, \quad \hat{u}_{\varepsilon_n}(0) = v,$$

$$\partial_t \hat{u} = \Delta \hat{u} - \hat{u} + \frac{\hat{u}^p}{\hat{\xi}^q}, \quad \hat{u}(0) = v, \tag{4.26}$$

both with the same boundary condition, by the same argument as in the proof of Proposition 4.5, we get

$$\lim_{n\to\infty} \|\hat{u}_{\varepsilon_n} - \hat{u}\|_{C([0,1];C)} = 0 \quad a.s.. \tag{4.27}$$

It is clear that the distribution of $(\hat{u}_{\varepsilon_n}, \hat{\xi}_{\varepsilon_n})$ is the same as those of $(u_{\varepsilon_n}, \xi_{\varepsilon_n})$. By (4.36) below, we have

$$\lim_{\varepsilon\to 0+} \mathbb{E}\sqrt{\varepsilon} \sup_{0\le t\le 1} \left| \int_0^t \xi_\varepsilon \mathrm{d}B_s \right| = 0.$$

Hence, up to taking a subsequence, we have

$$\lim_{n\to\infty} \sqrt{\varepsilon_n} \sup_{0\le t\le 1} \left| \int_0^t \xi_{\varepsilon_n} \mathrm{d}B_s \right| = 0.$$

By the same argument as in the proof of Proposition 4.5, we get

$$\hat{\xi}(t) = \zeta - \int_0^t \hat{\xi}(s)\mathrm{d}s + \int_0^t \frac{\overline{\hat{u}^\alpha}(s)}{\hat{\xi}^\beta(s)}\mathrm{d}s + \int_0^t \hat{\xi}(s)\dot{h}(s)\mathrm{d}s. \tag{4.28}$$

Now (4.26) and (4.28) yield that $(\hat{u}, \hat{\xi})$ satisfies Eq. (4.2). By uniqueness of the solution, $(\hat{u}, \hat{\xi})$ and (u_h, ξ_h) have the same distribution. Hence, we have completed the proof up to showing (4.25).

(Step 2) Now we show (4.25). To this end, it suffices to prove the following asymptotic tightness criterion ([15, Theorem 2.1]):

(i) For any $0 \le t_1 < t_2 < \dots < t_n \le 1$ with $n \in \mathbb{N}$, the distribution of $(\xi_\varepsilon(t_1), \dots, \xi_\varepsilon(t_n))_{0\le\varepsilon\le 1}$ is tight.

(ii) For all $\lambda > 0$

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbb{P}\left\{\sup_{\substack{0\le s<t\le 1\\ |t-s|\le\delta}}|\xi_\varepsilon(t)-\xi_\varepsilon(s)|>\lambda\right\}=0. \tag{4.29}$$

First of all, for all $\nu > 0$, by Hölder inequality and Lemma 4.14 we have

$$\left[\sup_{0\le t\le 1}\overline{u_\varepsilon^\alpha}(t)\right]^\nu \le \left(\sup_{0\le t\le 1}\|u_\varepsilon(t)\|_{L^\ell}^\ell\right)^{\frac{\nu\alpha}{\ell}} \le C\left[\mathrm{e}^{c_1 B_1^*}+\mathrm{e}^{c_2 B_1^*}\left(\mathcal{M}_{\varepsilon,\delta}^*\right)^{c_3}\right],$$

where c_1, c_2, c_3 all depend on α, β, p, q, ν. Thanks to Lemma 4.12 and (2.2), using Hölder and the above inequalities, we have

$$\mathbb{E}\left[\sup_{0\le t\le 1}\overline{u_\varepsilon^\alpha}(t)\right]^\nu \le C. \tag{4.30}$$

Thanks to (4.19) and (2.2), by a similar but easier argument, we get

$$\mathbb{E}\left[\inf_{0\le t\le 1}\xi_\varepsilon(t)\right]^{-\nu} \le C. \tag{4.31}$$

By Hölder inequality and (4.20), we have

$$\begin{aligned}\sup_{0\le t\le 1}\xi_\varepsilon^2(t) &\lesssim \mathrm{e}^{2N+2\sqrt{\varepsilon}B_1^*}+\mathrm{e}^{2N+4\sqrt{\varepsilon}B_1^*}\left(\sup_{0\le t\le 1}\overline{u_\varepsilon^\alpha}(t)\right)^{\frac{2}{1+\beta}}\\ &\lesssim \mathrm{e}^{2N+2B_1^*}+\mathrm{e}^{2N+4B_1^*}\left(\sup_{0\le t\le 1}\overline{u_\varepsilon^\alpha}(t)\right)^{\frac{2}{1+\beta}}.\end{aligned}$$

Thanks to (4.30) and (2.2), using Hölder and the above inequalities, we have

$$\mathbb{E}\sup_{0\le t\le 1}\xi_\varepsilon^2(t)\le C. \tag{4.32}$$

For all small $c > 0$, choosing $K = \sqrt{\frac{C}{c}}$ and by Chebyshev inequality, there exists some $K > 0$ such that

$$\mathbb{P}\left(\sup_{0\le t\le 1}\xi_\varepsilon(t)>K\right)\le\frac{\mathbb{E}\sup_{0\le t\le 1}\xi_\varepsilon^2(t)}{K^2}=c$$

and thus

$$\mathbb{P}\left(\sup_{0\le t\le 1}\xi_\varepsilon(t)\le K\right)\ge 1-c.$$

For any $0 \le t_1 < t_2 < ... < t_n \le 1$ with $n \in \mathbb{N}$, we have

$$\mathbb{P}\left(\xi_\varepsilon(t_1)\le K, ..., \xi_\varepsilon(t_n)\le K\right)\ge 1-c.$$

Since $c > 0$ is arbitrary, the distribution of $(\xi_\varepsilon(t_1), ..., \xi_\varepsilon(t_n))$ is tight. Hence, (i) above holds.

Next we check that (ii) also holds. Observe

$$\begin{aligned}\sup_{|s-t|\le\delta}|\xi_\varepsilon(t)-\xi_\varepsilon(s)| &\le \delta\left[\sup_{0\le t\le1}\xi_\varepsilon(t)+\sup_{0\le t\le1}\frac{\overline{u_\varepsilon^\alpha}(t)}{\xi_\varepsilon^\beta(t)}\right]\\ &\quad+\sup_{|s-t|\le\delta}\left|\int_s^t \xi_\varepsilon(r)\dot h_\varepsilon(s)\mathrm{d}s\right|+2\sqrt{\varepsilon}\sup_{0\le t\le1}\left|\int_0^t \xi_\varepsilon \mathrm{d}B_s\right|.\end{aligned}\tag{4.33}$$

By Hölder inequality, we get

$$\sup_{|s-t|\le\delta}\left|\int_s^t \xi_\varepsilon(r)\dot h_\varepsilon(s)\mathrm{d}s\right|\le\sup_{|s-t|\le\delta}\left[\int_s^t \xi_\varepsilon^2(r)\mathrm{d}s\right]^{\frac12}\|h_\varepsilon\|_H\le N\sqrt{\delta}\sup_{0\le t\le1}\xi_\varepsilon(t),$$

which, together with (4.32), yields

$$\mathbb{E}\sup_{|s-t|\le\delta}\left|\int_s^t \xi_\varepsilon(r)\dot h_\varepsilon(s)\mathrm{d}s\right|\le C\delta^{\frac12}.\tag{4.34}$$

Observing

$$\sup_{0\le t\le1}\frac{\overline{u_\varepsilon^\alpha}(t)}{\xi_\varepsilon^\beta(t)}\le\left(\sup_{0\le t\le1}\overline{u_\varepsilon^\alpha}(t)\right)\left(\inf_{0\le t\le1}\xi_\varepsilon^{-\beta}(t)\right),$$

by (4.30), (4.31) and Hölder inequality, this further gives

$$\mathbb{E}\sup_{0\le t\le1}\frac{\overline{u_\varepsilon^\alpha}(t)}{\xi_\varepsilon^\beta(t)}\le C.\tag{4.35}$$

Moreover, by Hölder and martingale inequalities and Itô identity, we get
(4.36)

$$\begin{aligned}\mathbb{E}\sqrt{\varepsilon}\sup_{0\le t\le1}\left|\int_0^t \xi_\varepsilon \mathrm{d}B_s\right| &\le\sqrt{\varepsilon}\left[\mathbb{E}\sup_{0\le t\le1}\left|\int_0^t \xi_\varepsilon \mathrm{d}B_s\right|^2\right]^{\frac12}\\ &\le\sqrt{2\varepsilon}\left[\mathbb{E}\left|\int_0^1 \xi_\varepsilon \mathrm{d}B_s\right|^2\right]^{\frac12}=\sqrt{2\varepsilon}\left[\int_0^1 \mathbb{E}|\xi_\varepsilon|^2\mathrm{d}s\right]^{\frac12}\le C\sqrt{\varepsilon},\end{aligned}$$

where the last inequality is by (4.32). Combining (4.32), (4.35), (4.34), (4.36) with (4.33), we immediately obtain

$$\mathbb{E}\sup_{|s-t|\le\delta}|\xi_\varepsilon(t)-\xi_\varepsilon(s)|\le C(\delta+\sqrt{\delta}+\sqrt{\varepsilon}).$$

By Chebyshev inequality,

$$\mathbb{P}\left\{\sup_{\substack{0\le s<t\le1\\|t-s|\le\delta}}|\xi_\varepsilon(t)-\xi_\varepsilon(s)|>\lambda\right\}\le C\lambda^{-1}(\delta+\sqrt{\delta}+\sqrt{\varepsilon}),$$

which immediately implies (ii).

□

4.4. **Proof of the large deviation theorem.**

Proof. By Theorem 4.4 in [1], and Proposition 4.5 and Proposition 4.6, we can obtain Theorem 4.4. The I in the theorem is an immediate consequence of [1, (4.3)]. □

5. Discussion and Outlook

Finally, let us mention some directions of our future research on the stochastic Gierer-Meinhardt system. Some important questions including the following have been left open in this study and we plan to explore them next. When does blow-up of solutions occur? Can related results be derived for stochastic processes other than one-dimensional standard Brownian motion? Can our results be extended from the stochastic shadow Gierer-Meinhardt system to the full Gierer-Meinhardt system? Do similar results hold for other pattern-forming systems such as the Gray-Scott or Schnakenberg models?

For pattern formation in the deterministic Gierer-Meinhardt model many interesting phenomena have been established such as Turing instability, peaked steady states with single or multiple spikes, and various kinds of bifurcations. We are interested in the question what will happen if some random forces are added to these models. Due to the randomness in the system, the peaked patterns and their bifurcations will be random rather than deterministic and we expect that the nature of their interactions will change. Depending on the exact conditions, they can likely be destabilized by the stochastic effects and new patterns might emerge. Our next goal is to investigate the trajectories of random patterns and their bifurcations and gain further insight into the mechanisms controlling these interactions ([33]).

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References

[1] A. Budhiraja and P. Dupuis, A variational representation for positive functionals of infinite dimensional brownian motion, Probab. Math. Statist. 20 (2000), 39-61.
[2] A. Budhiraja, P. Dupuis and V. Maroulas.: Large deviations for infinite dimensional stochastic dynamical systems, Ann. Probab. 36 (2008), 1390-1420.
[3] S. Cerrai and M. Rockner: Large deviations for stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term, Ann. Probab. 32 (2004), 1100-1139.
[4] G. Da Prato and J. Zabczyk, Stochastic equations in infinite dimensiions, Encyclpedia of Mathematics and its Applications 45, Cambridge Press (1992).
[5] A. Dembo and O. Zeitouni, Large deviations techniques and applications, Jones and Bartlett Publishers, Boston, London (1992).

[6] J. Duan and A. Millet: Large deviations for the Boussinesq equations under random influences, Stochastic Process. Appl. 119 (2009) 2052-2081.
[7] J. Feng and T. G. Kurtz: Large Deviations of Stochastic Processes. Mathematical Surveys and Monographs, vol. 131. American Mathematical Society, Providence (2006).
[8] M. I. Freidlin: Random perturbations of reaction-diffusion equations: the quasi-deterministic approximations, Trans. Am. Math. Soc. 305 (1988), 665-697.
[9] B. Gess: Random attractors for stochastic porous media equations perturbed by space-time linear multiplicative noise, Ann. Probab., 42 (2014), 818-864.
[10] A. Gierer and H. Meinhardt, A theory of biological pattern formation, Kybernetik (Berlin) 12 (1972), 30-39.
[11] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland Publishing Co., Amsterdam, (1981).
[12] H. Jiang, Global existence of solutions of an activator-inhibitor system, Discrete Contin. Dyn. Syst. 14 (2006), 737-751.
[13] J. P. Keener, Activators and inhibitors in pattern formation, Stud. Appl. Math. 59 (1978), 1-23.
[14] J.Kelkel and C.Surulescu, On a stochastic reaction-diffusion system modeling pattern formation on seashells, J. Math. Biology 60 (2010), 765-796.
[15] M. R. Kosorok, Introduction to Empirical Processes and Semiparametric Inference, Springer Series in Statistics (2008).
[16] K. Kristiansen, Reaction-diffusion models in mathematical biology, Master thesis, Technology University of Denmark.
[17] S. Kuksin and A. Shirikyan, Coupling approach to white-forced nonlinear PDEs, J. Math. Pures Appl. 81 (2002), 567-602.
[18] F. Li and W. M. Ni, On the global existence and finite time blow-up of shadow systems, J. Differential Equations 247 (2009), 1762-1776.
[19] M. Hairer and J. C. Mattingly: Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing, Ann. Math. 164 (2006), 993-1032.
[20] W. Liu, M. Röckner and X.C. Zhu: Large deviation principles for the stochastic quasi-geostrophic equations, Stochastic Process. Appl. 123 (2013), 3299-3327.
[21] S. A. Mohammed and T. Zhang Stochastic Burgers equation with random initial velocities: a Malliavin calculus approach. SIAM J. Math. Anal. 45 (2013), no. 4, 2396-2420.
[22] W. M. Ni, K. Suzuki and I. Takagi: The dynamics of a kinetic activator-inhibitor system, J. Differential Equations 229 (2006), 426-465.
[23] Y. Nishiura, Global structure of bifurcating solutions of some reaction-diffusion systems, SIAM J. Math. Anal. 13 (1982), 555-593.
[24] S. Peszat: Large deviation principle for stochastic evolution equations, Probab. Theory Relat. Fields 98 (1994), 113-136.
[25] A. A. Pukhalskii: On the theory of large deviations, Theory Probab. Appl. 38 (1993), 490-497.
[26] J. Ren and X. Zhang: Freidlin-Wentzell's large deviations for stochastic evolution equations, J. Funct. Anal. 254 (2008), 3148-3172.
[27] M. Röckner and T. Zhang, Stochastic evolution equations of jump type: existence, uniqueness and large deviation principles, Potential Anal. 26 (2007), 255-279.
[28] M. Röckner, F. Y. Wang and L. Wu, Large Deviations for Stochastic Generalized Porous Media Equations, Stoch. Proc. Appl. 116 (2006), 1677-1689.
[29] F. Rothe, Global Solutions of Reaction-Diffusion Systems, Lecture Notes in Mathematics, Vol. 1072, Springer-Verlag Berlin Heidelberg, 1984.
[30] A. Swiech and J. Zabczyk, Large deviations for stochastic PDE with Lévy noise, J. Funct. Anal. 260 (2011), 674-723.

[31] A. M. Turing, The chemical basis of morphogenesis, Phil. Trans. Roy. Soc. Lond. B 237 (1952), 37-72.
[32] J. Wei and M. Winter, Mathematical Aspects of Pattern Formation in Reaction-Diffusion Systems, Vol. 189, Springer-Verlag London, 2014.
[33] M. Winter and L. Xu, Some properties of the stochastic shadow Gierer-Meinhardt system, in progress.
[34] T. Xu and T. Zhang, White noise driven SPDEs with reflection: existence, uniqueness and large deviation principles. Stochastic Process. Appl. 119 (2009), 3453-3470.
[35] X. Yang, J. Zhai and T. Zhang: Large deviations for SPDEs of jump type, arXiv:1211.0466.
[36] T. Zhang, On small time asymptotics of diffusions on Hilbert spaces, Ann. Probab. 28 (2002), 537-557.

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