

Filtering and Fault Detection for Nonlinear Systems with Polynomial Approximation

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Abstract

This paper is concerned with polynomial filtering and fault detection problems for a class of nonlinear systems subject to additive noises and faults. The nonlinear functions are approximated with polynomials of a chosen degree. Different from the traditional methods, the approximation errors are not discarded but formulated as low-order polynomial terms with norm-bounded coefficients. The aim of the filtering problem is to design a least squares filter for the formulated nonlinear system with uncertain polynomials, and an upper bound of the filtering error covariance is found and subsequently minimized at each time step. The desired filter gain is obtained by recursively solving a set of Riccati-like matrix equations, and the filter design algorithm is therefore applicable for online computation. Based on the established filter design scheme, the fault detection problem is further investigated where the main focus is on the determination of the threshold on the residual. Due to the nonlinear and time-varying nature of the system under consideration, a novel threshold is determined that accounts for the noise intensity and the approximation errors, and sufficient conditions are established to guarantee the fault detectability for the proposed fault detection scheme. Comparative simulations are exploited to illustrate that the proposed filtering strategy achieves better estimation accuracy than the conventional polynomial extended Kalman filtering approach. The effectiveness of the associated fault detection scheme is also demonstrated.

Key words: Polynomial filtering; nonlinear systems; polynomial approximation; fault detection; adaptive threshold; recursive algorithm, Kronecker power.

1 Introduction

The past few decades have seen the nonlinear state estimation problem as a recurring research theme due to the pervasive existence of nonlinearities in almost all real-world industrial systems. If not adequately dealt with, the intrinsic nonlinearities may lead to undesirable dynamic behaviors such as oscillation or even instability. Indeed, the nonlinear analysis issue has been the main stream of research for systems control and estimation problems attracting researchers from a variety of disciplines. So far, much effort has been devoted to the estimation/filtering problems for nonlinear systems, see e.g. [2, 4, 7, 11–13, 19, 21, 23, 27, 35, 36] and the references therein. Among others, the renowned extended Kalman filter (EKF) algorithm has proved to be an effective method to solve the estimation problem for nonlinear systems in the least mean square sense. Recently, considerable attention has been paid to the performance

improvement of the traditional EKF with respect to the insensitivity to the parameter uncertainties as well as the capability of handling nonlinearities [22, 32, 38]. When the system states and observations are polynomial with additive Gaussian white noises, the mean-square filter has been designed where the statistical characteristics of Gaussian distribution have been made use of to recursively calculate the filtering error covariance [1, 3, 5].

Polynomial extended Kalman filter (PEKF) is an extension of EKF with aim to cater for inherent nonlinearities using polynomial approximations. Traditional EKF is only concerned with the *linear* term and simply ignores the linearization error, while PEKF considers the Carleman approximation of a nonlinear system of a given order μ [26]. The order could be determined according to the form of the nonlinearity and the estimation performance specification. In this sense, the PEKF is more applicable than EKF as far as the accuracy is concerned. When the order $\mu = 1$, PEKF reduces to conventional EKF. A PEKF is designed to cope with an augmented state which is made of Kronecker powers of the original

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state [15]. Due to its higher accuracy than that of EKF, the PEKF has stirred quite a lot research attention and many corresponding results have been reported in the literature [14, 16, 17, 28]. Nevertheless, the PEKF approach still ignores the Carleman approximation errors which would give rise to certain conservatism especially when the nonlinearities are severe.

In theory, well-behaved nonlinear functions could only be approximated *accurately* by polynomials whose orders approach *infinity*. In engineering systems, however, the polynomials with extremely high orders are difficult to be realized. A feasible way is to determine the finite order of the polynomials for satisfactory approximation of the nonlinear dynamics according to the degree of the nonlinearities and the nature of the research problem. As such, the unavoidable high-order approximation errors would be impacting on the estimation performances that should be taken into account. While EKF and PEKF work quite well for system with relatively low degree of nonlinearities, the classical EKF algorithm ignores the linearization errors and most available PEKF approaches discard the Carleman approximation errors. It is noted that the approximation errors differ greatly from each other due to various forms of nonlinearities. Instead of being simply dropped, the approximation errors do offer further room for improving the estimation accuracy if properly coped with. This seemingly natural idea, unfortunately, would inevitably bring us substantial challenges when calculating the covariances of the estimation errors in the least mean square sense since the approximation errors could not be exactly known. Moreover, coupled with both the low-order terms and external disturbances, the approximate errors would become very complicated to analyze. These identified difficulties motivate us to address the PEKF problem *by allowing for the Carleman approximation error* with aim to obtain higher approximation accuracy and better estimation performance.

The fault detection (FD) problem is another active research topic in control engineering due primarily to the increasingly higher and higher safety requirements. Yet, the FD problem provides an ideal platform to demonstrate the applicability of the polynomial filter technique to be developed. Since a properly designed filter could generate residual signal so as to efficiently detect abnormal changes in the system, filter/observer based FD has become a common technique. In filter/observer based FD methods, a fault would be detected and diagnosed effectively via comparing the actual system output with the filter output signal since the faults would normally bring in unexpected variations in the system measurement. To date, a great number of results have been published on this issue, see [18, 33, 41, 44] and the references therein. Residual evaluation, which consists of threshold and decision logic, is critical to the performance of FD. For systems varying with time or parameters, the *adaptive threshold* has been of particular research importance

for its better trackability and faster self-adjustment as compared with the constant threshold.

So far, many existing results have focused on adaptive threshold generation for linear systems [29–31, 43]. However, the corresponding results for nonlinear systems have been scattered in spite of their engineering significance [24, 34]. In [40, 42], the nonlinear dynamics have been assumed to be uniformly bounded and the bounds have been utilized to determine the adaptive threshold. In the context of polynomial fault detection for nonlinear systems, the adaptive threshold determination problem is essentially difficult mainly for two reasons: 1) the expressions of the disturbances and approximation errors are sophisticated and their influences on the threshold remain unclear; and 2) it is non-trivial to calculate the bounds of the disturbances and approximation errors in order to propose a reasonable fault detection threshold. Up to date, the polynomial fault detection scheme has not been fully investigated, not to mention the case where the Carleman approximation errors are taken into consideration. This constitutes another motivation of our present work.

Summarizing the discussions made above, in this paper, both the polynomial filtering and fault detection problems are thoroughly investigated for a class of nonlinear systems. The Carleman approximation of a given order is introduced to approximate the nonlinear functions. In contrast to existing literature, the high-order approximation errors are explicitly taken into account in terms of low-order polynomials with uncertain but bounded coefficients. A time-varying filter is first designed at each time step to guarantee that the filtering error covariance is bounded in the fault-free case. Such a bound is subsequently minimized with respect to a properly designed filter gain. To show the applicability of the proposed filter scheme, a fault detection strategy is then proposed consisting of the calculation of adaptive threshold and decision logic. The filter gain and adaptive threshold are determined using the information from the approximation errors and additive disturbances. With the fault detection strategy, the fault detectability is also investigated.

The main contribution of the paper can be highlighted as follows: 1) a polynomial estimation scheme for a class of nonlinear systems is presented by taking account of the Carleman approximation errors, which leads to higher estimation accuracy; 2) an upper bound of the estimation error covariance in the polynomial scheme is calculated and minimized by designing an appropriate filter gain; 3) adaptive fault detection threshold is developed for the desired polynomial filter to realize efficient detection and the fault detectability is analyzed; and 4) the proposed algorithms for both the filter gain design and the adaptive threshold computation are recursive and therefore suitable for online applications. The rest of paper is organized as follows. In Section 2, the Carleman ap-

proximation and the approximation error analysis are introduced for nonlinear systems. The polynomial filter design problem is solved in Section 3 and the adaptive fault detection scheme is established in Section 4. A simulation example is illustrated in Section 5 and the paper is concluded in Section 6.

Notations. The notation used in the paper is standard. \mathbb{R}^n and $\mathbb{R}^{n \times m}$ denote, respectively, the n -dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscripts A^T and A^{-1} denote the transpose and inverse of matrix A , respectively. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). I is the identity matrix with compatible dimension. $\mathbb{E}\{x\}$ stands for the expectation of the stochastic variable x . $\|A\|$ denotes the spectral norm of matrix A , and $\|x\|$ refers to the Euclidean norm of vector x . \otimes is the Kronecker product defined as

$$A \otimes B = \begin{bmatrix} a_{1,1}B & \cdots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \cdots & a_{m,n}B \end{bmatrix}. x^{[m]} \text{ represents the } m\text{th}$$

Kronecker power of vector x , where $x^{[m]} = x^{[m-1]} \otimes x$ and $x^{[0]} = 1$ [6, 10].

2 Problem Formulation

Consider the following class of stochastic discrete-time nonlinear systems:

$$\begin{cases} x_{k+1} = g(x_k, u_k) + v_k + f_k, \\ y_k = h(x_k) + w_k, \end{cases} \quad (1)$$

where $x_k \in \mathbb{R}^n$ is the system state; $y_k \in \mathbb{R}^b$ is the measurement output; $u_k \in \mathbb{R}^l$ is the control input; $f_k \in \mathbb{R}^n$ is the additive fault; $g: \mathbb{R}^n \times \mathbb{R}^l \mapsto \mathbb{R}^n$ and $h: \mathbb{R}^n \mapsto \mathbb{R}^b$ are $\mu + 1$ times continuously differentiable nonlinear maps. The state noise $v_k \in \mathbb{R}^n$ and the output noise $w_k \in \mathbb{R}^b$ are uncorrelated zero-mean sequences. The initial state x_0 is random and independent of the noises. It is assumed that

$$\mathbb{E}\{x_0^{[i]}\} = \zeta_{0,i}, \quad \mathbb{E}\{v_k^{[i]}\} = \xi_{v,k,i}, \quad \mathbb{E}\{w_k^{[i]}\} = \xi_{w,k,i}, \quad (2)$$

for all $i = 1, 2, \dots, 2\mu$, where $\zeta_{0,i}$, $\xi_{v,k,i}$, and $\xi_{w,k,i}$ are known vectors. Meanwhile, $x_0^{[i]}$, $v_k^{[i]}$, and $w_k^{[i]}$ are assumed to be distributed in bounded domains and the following inequalities hold definitely:

$$\|x_0^{[i]}\| \leq s_{0,i}, \quad \|v_k^{[i]}\| \leq s_{v,k,i}, \quad \|w_k^{[i]}\| \leq s_{w,k,i}, \quad (3)$$

where $s_{0,i}$, $s_{v,k,i}$ and $s_{w,k,i}$ are known scalars for all $i = 1, 2, \dots, \mu$. The fault f_k is assumed to be norm-bounded

so that the measurements and the estimated states will not go infinity in a finite time in the faulty case.

2.1 Polynomial approximation of nonlinear functions

Consider the sequences $x_{k+1}^{[m]}$ and $y_k^{[m]}$, the Kronecker powers of the state and measurement around the state estimate \tilde{x}_k for $m = 0, 1, \dots, \mu$. With Taylor polynomials around \tilde{x}_k , when the nonlinear functions are $\mu + 1$ times continuously differentiable, we have

$$\begin{cases} x_{k+1}^{[m]} = \sum_{i=0}^{\mu} G_{m,i}(\tilde{x}_k, u_k, v_k, f_k)(x_k - \tilde{x}_k)^{[i]} \\ \quad + G_{m,\mu+1}(x_k, \tilde{x}_k, u_k, v_k, f_k)(x_k - \tilde{x}_k)^{[\mu+1]}, \\ y_k^{[m]} = \sum_{i=0}^{\mu} H_{m,i}(\tilde{x}_k, w_k)(x_k - \tilde{x}_k)^{[i]} \\ \quad + H_{m,\mu+1}(x_k, \tilde{x}_k, w_k)(x_k - \tilde{x}_k)^{[\mu+1]}, \end{cases} \quad (4)$$

where $\tilde{x}_{\theta_k} = \Theta_k x_k + (I - \Theta_k) \tilde{x}_k$, $\Theta_k = \text{diag}\{\theta_{1k}, \dots, \theta_{nk}\}$, $\theta_{ik} \in [0, 1]$ for all $i = 1, 2, \dots, n$, and

$$\begin{aligned} & G_{m,i}(\tilde{x}_k, u_k, v_k, f_k) \\ &= \frac{1}{i!} \left(\nabla_x^{[i]} \otimes (g + v + f)^{[m]} \right) \Big|_{(x=\tilde{x}_k, u=u_k, v=v_k, f=f_k)}, \end{aligned} \quad (5)$$

$$\begin{aligned} & H_{m,i}(\tilde{x}_k, w_k) \\ &= \frac{1}{i!} \left(\nabla_x^{[i]} \otimes (h + w)^{[m]} \right) \Big|_{(x=\tilde{x}_k, w=w_k)}, \end{aligned} \quad (6)$$

where the operation $\nabla_x^{[i]} \otimes$ applied to a function $\chi(x)$ is defined as:

$$\nabla_x^{[0]} \otimes \chi = \chi, \quad (7)$$

$$\nabla_x^{[i+1]} \otimes \chi = \nabla_x \otimes (\nabla_x^{[i]} \otimes \chi), i > 0, \quad (8)$$

with $\nabla_x = [\partial/\partial x_1, \dots, \partial/\partial x_n]$.

In order to obtain a least squares filter and determine the adaptive threshold for fault detection, now we consider system (1) with $f_k = 0$. Firstly we focus on the expression of the remainder terms. Denote

$$\nabla_x^{[\mu]} \otimes (g + v)^{[\mu]} = [\varepsilon_{1,\mu,m}(x, u, v), \dots, \varepsilon_{n^m,\mu,m}(x, u, v)]^T, \quad (9)$$

where $\varepsilon_{i,\mu,m}(x, u, v) : \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^n \mapsto \mathbb{R}^{n^\mu}$ for all $i = 1, \dots, n^m$. It can be easily verified that

$$\begin{aligned} & G_{m,\mu+1}(x_k, \tilde{x}_k, u_k, v_k, 0)(x_k - \tilde{x}_k)^{[\mu+1]} \\ &= \mathfrak{G}(x_k, \tilde{x}_k, u_k, v_k)(x_k - \tilde{x}_k)^{[\mu]}, \end{aligned} \quad (10)$$

where

$$\mathfrak{G}(x_k, \tilde{x}_k, u_k, v_k) = \frac{1}{(\mu+1)!} \times \begin{bmatrix} (x_k - \tilde{x}_k)^T (\nabla_x^T \otimes \varepsilon_{1,\mu,m}^T(\tilde{x}_{\theta_k}, u_k, v_k)) \\ \vdots \\ (x_k - \tilde{x}_k)^T (\nabla_x^T \otimes \varepsilon_{n^m,\mu,m}^T(\tilde{x}_{\theta_k}, u_k, v_k)) \end{bmatrix}. \quad (11)$$

We assume that x_k is bounded in an ellipsoid of center \tilde{x}_k and shape matrix E_k , i.e. $x_k = \tilde{x}_k + E_k z_k$, for some $z_k \in \mathbb{R}^n$ and $\|z_k\| \leq 1$. Since the nonlinear functions $\nabla_x^T \otimes \varepsilon_{i,\mu,m}^T(\tilde{x}_{\theta_k}, u_k, v_k)$ are assumed to be continuous, their norm reaches an extremum on the bounded ellipsoid domain, thus there exist constants $\bar{m}_{1,k}, \dots, \bar{m}_{n^m,k}$, such that for $i = 1, 2, \dots, n^m$,

$$\|\nabla_x^T \otimes \varepsilon_{i,\mu,m}^T(\tilde{x}_{\theta_k}, u_k, v_k)\| \leq \bar{m}_{i,k}. \quad (12)$$

With $x_k = \tilde{x}_k + E_k z_k$ and $\|z_k\| \leq 1$, it follows directly that

$$\|(x_k - \tilde{x}_k)^T (\nabla_x^T \otimes \varepsilon_{i,\mu,m}^T(\tilde{x}_{\theta_k}, u_k, v_k))\| \leq \bar{m}_{i,k} \|E_k\|, \quad (13)$$

therefore there exist vectors $\varsigma_{i,k} \in \mathbb{R}^{n^\mu}$ such that $\|\varsigma_{i,k}\| \leq 1$ and

$$(x_k - \tilde{x}_k)^T (\nabla_x^T \otimes \varepsilon_{i,\mu,m}^T(\tilde{x}_{\theta_k}, u_k, v_k)) = \bar{m}_{i,k} \|E_k\| \varsigma_{i,k}^T. \quad (14)$$

Denoting $\Xi_k = [\varsigma_{1,k}, \dots, \varsigma_{n^m,k}]^T$, we have

$$\mathfrak{G}(\tilde{x}_{\theta_k}, u_k, v_k) = \frac{1}{(\mu+1)!} \|E_k\| \text{diag}\{\bar{m}_{1,k}, \dots, \bar{m}_{n^m,k}\} \Xi_k. \quad (15)$$

From the definition of Ξ_k and $\|\varsigma_{i,k}\| \leq 1$, it is obvious that $\|\Xi_k\| \leq \sqrt{n^m}$. Therefore, denote

$$L_{g,m,k} = \frac{\sqrt{n^m}}{(\mu+1)!} \|E_k\| \text{diag}\{\bar{m}_{1,k}, \dots, \bar{m}_{n^m,k}\}, \quad (16)$$

then $\mathfrak{G}(x_k, \tilde{x}_k, u_k, v_k)$ can be written as

$$\mathfrak{G}(x_k, \tilde{x}_k, u_k, v_k) = L_{g,m,k} \Delta_{g,m,k}, \quad (17)$$

where $L_{g,m,k}$ and $\Delta_{g,m,k}$ are problem-dependent matrices with $\|\Delta_{g,m,k}\| \leq 1$. Similarly, we can have that

$$\begin{aligned} & H_{m,\mu+1}(x_k, \tilde{x}_k, w_k)(x_k - \tilde{x}_k)^{[\mu+1]} \\ &= L_{h,m,k} \Delta_{h,m,k} (x_k - \tilde{x}_k)^{[\mu]}, \end{aligned} \quad (18)$$

where $L_{h,m,k}$ and $\Delta_{h,m,k}$ are problem-dependent matrices with $\|\Delta_{h,m,k}\| \leq 1$ as well.

Remark 1 The way of processing the high-order approximation errors is along the similar line of coping with the linearization errors in [8, 20, 39] where linear terms with norm-bounded matrix coefficients have been utilized to formulate the linearization errors. Different from [8, 20, 39], in this paper, polynomial terms of orders higher than μ , namely, the approximation errors, are formulated as terms of lower orders with norm-bounded matrix coefficients. The unknown bounded matrices $\Delta_{g,m,k}$ and $\Delta_{h,m,k}$ are functions of both x_k and \tilde{x}_k , and they are not expressed as $\Delta_{g,m,k}(x_k, \tilde{x}_k)$ and $\Delta_{h,m,k}(x_k, \tilde{x}_k)$ only for simplicity. Notice that the approximation errors have been ignored in previous PEKF studies, which may lead to unsatisfactory estimation performance especially when the encountered nonlinearities are severe. Obviously, the approximation errors have a great influence on the estimation accuracy, and the novel error formulation (18) helps to achieve a better approximation and estimation performance in the sequel, which will be demonstrated in the simulation.

2.2 The polynomial nonlinear systems

Before proceeding further, the following lemma is introduced.

Lemma 1 [9] For any integer $h \geq 0$ and for any $a, b \in \mathbb{R}^n$, we have

$$(a+b)^{[h]} = \sum_{k=0}^h M_h^k (a^{[k]} \otimes b^{[h-k]}), \quad (19)$$

with a set of suitably defined matrices $M_h^k \in \mathbb{R}^{n^h \times n^h}$ ($k = 0, 1, \dots, h$).

With Lemma 1, (17), (18) and the fact that

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD), \quad (20)$$

(4) in the fault-free case can be written as follows:

$$\begin{cases} x_{k+1}^{[m]} = \sum_{j=0}^{\mu} A_{m,j,k} x_k^{[j]} + L_{g,m,k} \Delta_{g,m,k} \sum_{j=0}^{\mu} \Upsilon_{j,k} x_k^{[j]} \\ \quad + v_{m,k}, \\ y_k^{[m]} = \sum_{j=0}^{\mu} C_{m,j,k} x_k^{[j]} + L_{h,m,k} \Delta_{h,m,k} \sum_{j=0}^{\mu} \Upsilon_{j,k} x_k^{[j]} \\ \quad + w_{m,k}, \end{cases} \quad (21)$$

where

$$\begin{aligned} A_{m,j,k} &= \sum_{i=j}^{\mu} \sum_{p=0}^m \frac{1}{i!} M_m^p \left(\left(\nabla_x^{[i]} \otimes g^{[p]} \right) \otimes \xi_{v,k,m-p} \right) \\ &\quad \times M_i^j \left(I_{n^j} \otimes (-\tilde{x}_k)^{[i-j]} \right), \end{aligned} \quad (22)$$

$$C_{m,j,k} = \sum_{i=j}^{\mu} \sum_{p=0}^m \frac{1}{i!} M_m^p \left(\left(\nabla_x^{[i]} \otimes h^{[p]} \right) \otimes \xi_{w,k,m-p} \right) \times M_i^j \left(I_{n^j} \otimes (-\tilde{x}_k)^{[i-j]} \right), \quad (23)$$

$$v_{m,k} = \sum_{i=0}^{\mu} \sum_{p=0}^m \frac{1}{i!} M_m^p \left(\left(\left(\nabla_x^{[i]} \otimes g^{[p]} \right) (x_k - \tilde{x}_k)^{[i]} \right) \otimes \left(v_k^{[m-p]} - \xi_{v,k,m-p} \right) \right), \quad (24)$$

$$w_{m,k} = \sum_{i=0}^{\mu} \sum_{p=0}^m \frac{1}{i!} M_m^p \left(\left(\left(\nabla_x^{[i]} \otimes h^{[p]} \right) (x_k - \tilde{x}_k)^{[i]} \right) \otimes \left(w_k^{[m-p]} - \xi_{w,k,m-p} \right) \right), \quad (25)$$

$$\Upsilon_{j,k} = M_{\mu}^j \left(I_{n^j} \otimes (-\tilde{x}_k)^{[\mu-j]} \right). \quad (26)$$

Denote the following augmented state and measurement:

$$x_k^{(\mu)} = \begin{bmatrix} 1 \\ x_k \\ x_k^{[2]} \\ \vdots \\ x_k^{[\mu]} \end{bmatrix}, y_k^{(\mu)} = \begin{bmatrix} 1 \\ y_k \\ y_k^{[2]} \\ \vdots \\ y_k^{[\mu]} \end{bmatrix}, \quad (27)$$

then (21) can be written as:

$$\begin{cases} x_{k+1}^{(\mu)} = A_k x_k^{(\mu)} + L_{g,k} \Delta_{g,k} \Upsilon_k x_k^{(\mu)} + v_k^{(\mu)}, \\ y_k^{(\mu)} = C_k x_k^{(\mu)} + L_{h,k} \Delta_{h,k} \Upsilon_k x_k^{(\mu)} + w_k^{(\mu)}, \end{cases} \quad (28)$$

where

$$A_k = \begin{bmatrix} A_{0,0,k} & \dots & A_{0,\mu,k} \\ A_{1,0,k} & \dots & A_{1,\mu,k} \\ \vdots & \ddots & \vdots \\ A_{\mu,0,k} & \dots & A_{\mu,\mu,k} \end{bmatrix}, \quad C_k = \begin{bmatrix} C_{0,0,k} & \dots & C_{0,\mu,k} \\ C_{1,0,k} & \dots & C_{1,\mu,k} \\ \vdots & \ddots & \vdots \\ C_{\mu,0,k} & \dots & C_{\mu,\mu,k} \end{bmatrix},$$

$$\begin{aligned} v_k^{(\mu)} &= (v_{0,k}^T, \dots, v_{\mu,k}^T)^T, \quad w_k^{(\mu)} = (w_{0,k}^T, \dots, w_{\mu,k}^T)^T, \\ \Upsilon_k &= [\Upsilon_{0,k}, \dots, \Upsilon_{\mu,k}], \quad L_{g,k} = \text{diag}\{L_{g,0,k}, \dots, L_{g,\mu,k}\}, \\ L_{h,k} &= \text{diag}\{L_{h,0,k}, \dots, L_{h,\mu,k}\}, \\ \Delta_{g,k} &= \text{diag}\{\Delta_{g,0,k}, \dots, \Delta_{g,\mu,k}\}, \\ \Delta_{h,k} &= \text{diag}\{\Delta_{h,0,k}, \dots, \Delta_{h,\mu,k}\}, \end{aligned} \quad (29)$$

and it is straightforward to see that $\|\Delta_{g,k}\| \leq 1$ and $\|\Delta_{h,k}\| \leq 1$. It is noted that when $g(x_k, u_k)$ (respec-

tively, $h(x_k)$) is linear, the parameter $L_{g,k}$ (respectively, $L_{h,k}$) is zero. From the facts that $\mathbb{E}\{v_k^{[i]}\} = \xi_{v,k,i}$ and $\mathbb{E}\{w_k^{[i]}\} = \xi_{w,k,i}$, it follows that $v_k^{(\mu)}$ and $w_k^{(\mu)}$ are both zero-mean. However, it is quite difficult to calculate $\mathbb{E}\{v_k^{(\mu)}(v_k^{(\mu)})^T\}$ and $\mathbb{E}\{w_k^{(\mu)}(w_k^{(\mu)})^T\}$ due to that $v_k^{(\mu)}$ and $w_k^{(\mu)}$ involve the system state, the state estimate, and the nonlinear dynamics.

2.3 The filter and the fault detection problems

For system (28), the filter to be designed is of the following form:

$$\tilde{x}_{k+1}^{(\mu)} = A_k \tilde{x}_k^{(\mu)} + K_k \left(y_k^{(\mu)} - C_k \tilde{x}_k^{(\mu)} \right), \quad (30)$$

where $\tilde{x}_k^{(\mu)}$ is the estimate of $x_k^{(\mu)}$ at time step k with $\tilde{x}_0^{(\mu)} = \mathbb{E}\{x_0^{(\mu)}\}$. K_k is the filter gain to be determined.

We are now in a position to state the addressed polynomial filter and fault detection problems as follows. 1) We are interested in designing the filter gain K_k in (30) for the system (28) such that the resulting estimation error covariance is bounded and such a bound is subsequently minimized. 2) Based on the proposed filter design scheme, the associated fault detection problem is to generate a residual signal whose threshold is adaptively computed by reflecting the approximation errors and external disturbances.

Remark 2 Based on the degrees of the nonlinearities and the engineering requirement, the integer μ could be chosen in advance. A bigger μ would lead to higher approximation accuracy at the cost of heavier computation burden. In this sense, the determination of μ should be made according to the trade-off between the estimation accuracy and the computation expense. Therefore, the polynomial filter to be developed is applicable in many nonlinear cases since the order could be selected adaptively. When $\mu = 1$ and the approximation errors are not considered, the filter to be designed reduces to the conventional EKF.

Remark 3 Though the polynomial approximation errors could be represented in a mathematical way, they remain indeterminate to the designers because of the unknown matrix coefficients. As a result, the accurate estimation error covariance could not be obtained. A natural alternative is to find an upper bound of the estimation error covariance and then minimize it by a properly designed filter gain K_k . This way, the feasibility of the algorithm is enhanced and the robustness with respect to the approximation errors is maintained. After the determination of K_k , an adaptive fault detection strategy would be established accordingly.

Remark 4 It is observed from (21)-(28) that, in the proposed filter (30), A_k and C_k are quite complicated since they are nonlinear functions of the estimated states, addressed nonlinear dynamics as well as the statistics of disturbances. Rather than $A(\tilde{x}_k^{(\mu)})$ and $C(\tilde{x}_k^{(\mu)})$, they are denoted by A_k and C_k for simplicity only. By no means are they similar to the known system matrices in the linear case. In the estimation process, these two parameters have to be updated at each time instant by fairly complicated computation.

3 Polynomial Filter Design

The following lemmas are essential in establishing the main results.

Lemma 2 [37] Given matrices A , H , E and F with appropriate dimensions such that $FF^T \leq I$. Let X be a symmetric positive definite matrix and γ be an arbitrary positive constant such that $\gamma^{-1}I > EXE^T$. Then the following inequality holds

$$(A + HFE)X(A + HFE)^T \leq A(X^{-1} - \gamma E^T E)^{-1}A^T + \gamma^{-1}HH^T. \quad (31)$$

Lemma 3 [20] For any two vectors $x, y \in \mathbb{R}^n$, the following inequality holds

$$xy^T + yx^T \leq \varepsilon xx^T + \varepsilon^{-1}yy^T, \quad (32)$$

where $\varepsilon > 0$ is a scalar.

Denote the estimation error as

$$e_k = x_k^{(\mu)} - \tilde{x}_k^{(\mu)}, \quad (33)$$

and the estimation error covariance conditional on the observations $y_j (j = 0, \dots, k)$ as

$$P_{k+1} = \mathbb{E} \{ e_{k+1} e_{k+1}^T | y_0, \dots, y_k \}. \quad (34)$$

For presentation convenience, $\tilde{x}_k^{(\mu)}$, e_k and P_k are partitioned as follows:

$$\tilde{x}_k^{(\mu)} = \begin{bmatrix} \tilde{x}_{0,k}^{(\mu)} \\ \tilde{x}_{1,k}^{(\mu)} \\ \vdots \\ \tilde{x}_{\mu,k}^{(\mu)} \end{bmatrix}, e_k = \begin{bmatrix} e_{0,k} \\ e_{1,k} \\ \vdots \\ e_{\mu,k} \end{bmatrix},$$

$$P_k = \begin{bmatrix} P_{0,0,k} & P_{0,1,k} & \cdots & P_{0,\mu,k} \\ P_{1,0,k} & P_{1,1,k} & \cdots & P_{1,\mu,k} \\ \vdots & \vdots & \ddots & \vdots \\ P_{\mu,0,k} & P_{\mu,1,k} & \cdots & P_{\mu,\mu,k} \end{bmatrix}, \quad (35)$$

where $e_{i,k}, \tilde{x}_{i,k}^{(\mu)} \in \mathbb{R}^{n^i}$ and $P_{i,j,k} \in \mathbb{R}^{n^i \times n^j}$ for any $i, j = 0, 1, \dots, \mu$. It can be seen that $P_{i,j,0} = \text{st}_{n^i, n^j}^{-1}(\zeta_{0,i+j} - \zeta_{0,i}\zeta_{0,j}^T)$, where $\text{st}_{a,b}^{-1}$ is the inverse of the stack operator defined as: for a vector $\rho = [\rho_1, \dots, \rho_{a \times b}]$, $\text{st}_{a,b}^{-1}(\rho) = [\rho_{ij}]_{a \times b}$, where $\rho_{ij} = \rho_{(j-1)a+i}$.

The first goal of this paper is to design a filter in the form of (30) for system (28) such that an upper bound of the covariance of the filtering error P_k can be provided and minimized. For this purpose, let us now deal with the covariance of the filtering error in the following lemma.

Lemma 4 Denoting

$$\begin{aligned} \check{A}_k &= (A_k - K_k C_k) + \sqrt{2} [L_{g,k}, -K_k L_{h,k}] \\ &\quad \times \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right]^T \Upsilon_k, \end{aligned} \quad (36)$$

$$\Psi_k^v = \mathbb{E} \left\{ v_k^{(\mu)} (v_k^{(\mu)})^T \right\}, \quad (37)$$

$$\Psi_k^w = \mathbb{E} \left\{ w_k^{(\mu)} (w_k^{(\mu)})^T \right\}, \quad (38)$$

the covariance of the filtering error in (34) obeys the following recursion:

$$\begin{aligned} P_{k+1} &= \mathbb{E} \{ \check{A}_k P_k \check{A}_k^T \} + 2 [L_{g,k}, -K_k L_{h,k}] \\ &\quad \times \mathbb{E} \left\{ \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right]^T \Upsilon_k \tilde{x}_k^{(\mu)} (\tilde{x}_k^{(\mu)})^T \Upsilon_k^T \right. \\ &\quad \times \left. \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right] \right\} [L_{g,k}, -K_k L_{h,k}]^T \\ &\quad + \sqrt{2} \mathbb{E} \left\{ \check{A}_k e_k (\tilde{x}_k^{(\mu)})^T \Upsilon_k^T \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right] \right\} \\ &\quad \times [L_{g,k}, -K_k L_{h,k}]^T + \sqrt{2} [L_{g,k}, -K_k L_{h,k}] \\ &\quad \times \mathbb{E} \left\{ \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right]^T \Upsilon_k \tilde{x}_k^{(\mu)} e_k^T \check{A}_k^T \right\} \\ &\quad + K_k \Psi_k^w K_k^T + \Psi_k^v. \end{aligned} \quad (39)$$

Proof: Substituting (28) and (30) into (33) yields

$$\begin{aligned} e_{k+1} &= \left((A_k - K_k C_k) + \sqrt{2} [L_{g,k}, -K_k L_{h,k}] \right. \\ &\quad \times \left. \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right]^T \Upsilon_k \right) e_k + v_k^{(\mu)} - K_k w_k^{(\mu)} \end{aligned}$$

$$+ \sqrt{2} [L_{g,k}, -K_k L_{h,k}] \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right]^T \Upsilon_k \tilde{x}_k^{(\mu)}. \quad (40)$$

Noticing that $v_k^{(\mu)}$ and $w_k^{(\mu)}$ are zero-mean disturbances and independent of e_k , (39) can be obtained directly with the definition of \bar{A}_k . The proof now is complete.

As discussed before, the accurate estimation error covariance in Lemma 4 could not be obtained due to the unknown matrix coefficients. To solve the estimation problem, an upper bound of the estimation error covariance will be derived in the next theorem, and the filter gain will then be designed to minimize the bound at each time step.

Theorem 1 Let $\gamma_{1,k}, \gamma_{2,k}, \gamma_{3,k}, \varepsilon$ be positive scalars. With initial condition $\bar{P}_0 = P_0$, assume that the following discrete Riccati-like equation

$$\begin{aligned} \bar{P}_{k+1} = & (1 + \varepsilon) A_k (\bar{P}_k^{-1} - \gamma_{1,k} \Upsilon_k^T \Upsilon_k)^{-1} A_k^T + 2(\gamma_{1,k}^{-1} \\ & \times (1 + \varepsilon) + \gamma_{2,k}^{-1} (1 + \varepsilon^{-1})) L_{g,k} L_{g,k}^T + \bar{\Psi}_k^v \\ & - Z_k^T Y_k^{-1} Z_k \end{aligned} \quad (41)$$

has positive definite solutions such that the following constraints

$$\gamma_{1,k}^{-1} I - \Upsilon_k \bar{P}_k \Upsilon_k^T > 0, \quad (42)$$

$$\gamma_{2,k}^{-1} I - \Upsilon_k \tilde{x}_k^{(\mu)} (\tilde{x}_k^{(\mu)})^T \Upsilon_k^T > 0, \quad (43)$$

$$\gamma_{3,k}^{-1} I - \Upsilon_k \bar{X}_k \Upsilon_k^T > 0, \quad (44)$$

are satisfied, where

$$\begin{aligned} Y_k = & (1 + \varepsilon) C_k (\bar{P}_k^{-1} - \gamma_{1,k} \Upsilon_k^T \Upsilon_k)^{-1} C_k^T + 2(\gamma_{1,k}^{-1} \\ & \times (1 + \varepsilon) + \gamma_{2,k}^{-1} (1 + \varepsilon^{-1})) L_{h,k} L_{h,k}^T + \bar{\Psi}_k^w, \end{aligned} \quad (45)$$

$$Z_k = (1 + \varepsilon) C_k (\bar{P}_k^{-1} - \gamma_{1,k} \Upsilon_k^T \Upsilon_k)^{-1} A_k^T, \quad (46)$$

$$\begin{aligned} \bar{\Psi}_{a,b,k}^v = & \sum_{i=0}^{\mu} \sum_{j=0}^{\mu} \sum_{p=0}^a \sum_{q=0}^b \frac{1}{i!j!} M_a^p \left(\left(\left(\nabla_x^{[i]} \otimes g^{[p]} \right) \bar{\Omega}_{i,j,k} \right. \right. \\ & \times \left. \left(\nabla_x^{[j]} \otimes g^{[q]} \right)^T \right) \otimes \left(\text{st}_{n^{a-p}, n^{b-q}}^{-1} (\xi_{v,k, a-p+b-q}) \right. \\ & \left. \left. - \xi_{v,k, a-p} \xi_{v,k, b-q}^T \right) \right) (M_b^q)^T, \end{aligned} \quad (47)$$

$$\begin{aligned} \bar{\Psi}_{a,b,k}^w = & \sum_{i=0}^{\mu} \sum_{j=0}^{\mu} \sum_{p=0}^a \sum_{q=0}^b \frac{1}{i!j!} M_a^p \left(\left(\left(\nabla_x^{[i]} \otimes h^{[p]} \right) \bar{\Omega}_{i,j,k} \right. \right. \\ & \times \left. \left(\nabla_x^{[j]} \otimes h^{[q]} \right)^T \right) \otimes \left(\text{st}_{n^{a-p}, n^{b-q}}^{-1} (\xi_{w,k, a-p+b-q}) \right. \end{aligned}$$

$$\left. - \xi_{w,k, a-p} \xi_{w,k, b-q}^T \right) (M_b^q)^T, \quad (48)$$

$$\begin{aligned} \bar{\Omega}_{i,j,k} = & \sum_{p=0}^i \sum_{q=0}^j M_i^p \left(\left(I_{n^p} \otimes (-\tilde{x}_k)^{[i-p]} \right) \bar{X}_{p,q,k} \right. \\ & \left. \left(I_{n^q} \otimes (-\tilde{x}_k)^{[j-q]} \right)^T \right) (M_j^q)^T, \end{aligned} \quad (49)$$

$$\begin{aligned} \bar{X}_{k+1} = & A_k (\bar{X}_k^{-1} - \gamma_{3,k} \Upsilon_k^T \Upsilon_k)^{-1} A_k^T + \gamma_{3,k}^{-1} L_{g,k} L_{g,k}^T \\ & + \bar{\Psi}_k^v, \end{aligned} \quad (50)$$

$$\bar{X}_0 = \mathbb{E} \left\{ x_0^{(\mu)} \left(x_0^{(\mu)} \right)^T \right\}.$$

Similar to P_k , matrices \bar{X}_k , \bar{P}_k , $\bar{\Psi}_k^v$ and $\bar{\Psi}_k^w$ are partitioned as follows:

$$\begin{aligned} \bar{X}_k = & \begin{bmatrix} \bar{X}_{0,0,k} & \bar{X}_{0,1,k} & \cdots & \bar{X}_{0,\mu,k} \\ \bar{X}_{1,0,k} & \bar{X}_{1,1,k} & \cdots & \bar{X}_{1,\mu,k} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{X}_{\mu,0,k} & \bar{X}_{\mu,1,k} & \cdots & \bar{X}_{\mu,\mu,k} \end{bmatrix}, \\ \bar{P}_k = & \begin{bmatrix} \bar{P}_{0,0,k} & \bar{P}_{0,1,k} & \cdots & \bar{P}_{0,\mu,k} \\ \bar{P}_{1,0,k} & \bar{P}_{1,1,k} & \cdots & \bar{P}_{1,\mu,k} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{\mu,0,k} & \bar{P}_{\mu,1,k} & \cdots & \bar{P}_{\mu,\mu,k} \end{bmatrix}, \\ \bar{\Psi}_k^v = & \begin{bmatrix} \bar{\Psi}_{0,0,k}^v & \bar{\Psi}_{0,1,k}^v & \cdots & \bar{\Psi}_{0,\mu,k}^v \\ \bar{\Psi}_{1,0,k}^v & \bar{\Psi}_{1,1,k}^v & \cdots & \bar{\Psi}_{1,\mu,k}^v \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\Psi}_{\mu,0,k}^v & \bar{\Psi}_{\mu,1,k}^v & \cdots & \bar{\Psi}_{\mu,\mu,k}^v \end{bmatrix}, \\ \bar{\Psi}_k^w = & \begin{bmatrix} \bar{\Psi}_{0,0,k}^w & \bar{\Psi}_{0,1,k}^w & \cdots & \bar{\Psi}_{0,\mu,k}^w \\ \bar{\Psi}_{1,0,k}^w & \bar{\Psi}_{1,1,k}^w & \cdots & \bar{\Psi}_{1,\mu,k}^w \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\Psi}_{\mu,0,k}^w & \bar{\Psi}_{\mu,1,k}^w & \cdots & \bar{\Psi}_{\mu,\mu,k}^w \end{bmatrix}, \end{aligned} \quad (51)$$

where $\bar{X}_{i,j,k}, \bar{P}_{i,j,k}, \bar{\Psi}_{i,j,k}^v \in \mathbb{R}^{n^i \times n^j}$, $\bar{\Psi}_{i,j,k}^w \in \mathbb{R}^{m^i \times m^j}$ for any $i, j = 0, 1, \dots, \mu$. Then, with the filter gain given by

$$K_k = Z_k^T Y_k^{-1}, \quad (52)$$

the matrix \bar{P}_k is an upper bound of P_k in the fault-free case. Moreover, the filter gain given by (52) minimizes the upper bound at each time step.

Proof: For notational simplicity, we denote

$$\Psi_{i,j,k}^v = \mathbb{E} \{ v_{i,k} v_{j,k}^T \}, \quad \Psi_{i,j,k}^w = \mathbb{E} \{ w_{i,k} w_{j,k}^T \},$$

$$X_k = \mathbb{E} \left\{ x_k^{(\mu)} \left(x_k^{(\mu)} \right)^T \right\},$$

$$\Omega_{i,j,k} = \mathbb{E} \left\{ (x_k - \tilde{x}_k)^{[i]} \left((x_k - \tilde{x}_k)^{[j]} \right)^T \right\}. \quad (53)$$

Ψ_k^v and Ψ_k^w have been defined in Lemma 4.

Before proving that \bar{P}_k is an upper bound of P_k , we are going to show that $X_k \leq \bar{X}_k$, $\Psi_k^v \leq \bar{\Psi}_k^v$, and $\Psi_k^w \leq \bar{\Psi}_k^w$. These results could be obtained by induction. With $\bar{X}_0 = \mathbb{E} \left\{ x_0^{(\mu)} \left(x_0^{(\mu)} \right)^T \right\}$, we assume that $i = 1, 2, \dots, k$, $X_i \leq \bar{X}_i$. Now it remains to show that $\Psi_k^v \leq \bar{\Psi}_k^v$, $\Psi_k^w \leq \bar{\Psi}_k^w$, and $X_{k+1} \leq \bar{X}_{k+1}$.

Noting the fact that v_k is independent of x_k , it follows from Lemma 1, (20) and (24) that

$$\begin{aligned} \Psi_{a,b,k}^v &= \sum_{i=0}^{\mu} \sum_{j=0}^{\mu} \sum_{p=0}^a \sum_{q=0}^b \frac{1}{i!j!} M_a^p \left(\left(\left(\nabla_x^{[i]} \otimes g^{[p]} \right) \Omega_{i,j,k} \right. \right. \\ &\quad \times \left. \left(\nabla_x^{[j]} \otimes g^{[q]} \right)^T \right) \otimes \mathbb{E} \left\{ \left(v^{[a-p]} - \xi_{v,k,a-p} \right) \right. \\ &\quad \times \left. \left(v^{[b-q]} - \xi_{v,k,b-q} \right)^T \right\} \left. \right) (M_b^q)^T, \end{aligned} \quad (54)$$

and

$$\begin{aligned} \Omega_{i,j,k} &= \sum_{p=0}^i \sum_{q=0}^j M_i^p \left(\left(I_{n^p} \otimes (-\tilde{x}_k)^{[i-p]} \right) \mathbb{E} \left\{ x_k^{[p]} \right. \right. \\ &\quad \times \left. \left. \left(x_k^{[q]} \right)^T \right\} \left(I_{n^q} \otimes (-\tilde{x}_k)^{[j-q]} \right)^T \right) (M_j^q)^T. \end{aligned} \quad (55)$$

In fact, $v_{m,k}$ can be written as:

$$\begin{aligned} v_{m,k} &= \sum_{i=0}^{\mu} \sum_{p=0}^m \frac{1}{i!} M_m^p \left(\left(\nabla_x^{[i]} \otimes g^{[p]} \right) \right. \\ &\quad \times \left. \left(v_k^{[m-p]} - \xi_{v,k,m-p} \right) \right) (x_k - \tilde{x}_k)^{[i]}. \end{aligned} \quad (56)$$

Recalling Lemma 1 and the definition of $x_k^{(\mu)}$, one has

$$v_{m,k} = \mathfrak{X}_m(\tilde{x}_k, u_k, v_k) x_k^{(\mu)}, \quad (57)$$

where $\mathfrak{X}_m(\cdot, \cdot, \cdot)$ is a proper nonlinear mapping. Then, $v_k^{(\mu)}$ can be expressed as

$$v_k^{(\mu)} = \mathfrak{X}(\tilde{x}_k, u_k, v_k) x_k^{(\mu)}, \quad (58)$$

where

$$\mathfrak{X}(\tilde{x}_k, u_k, v_k) = [\mathfrak{X}_0^T(\tilde{x}_k, u_k, v_k), \dots, \mathfrak{X}_\mu^T(\tilde{x}_k, u_k, v_k)]^T. \quad (59)$$

Now, it follows from the assumption $X_k \leq \bar{X}_k$ and the independence between v_k and x_k that

$$\begin{aligned} \Psi_k^v &= \mathbb{E} \left\{ \mathfrak{X}(\tilde{x}_k, u_k, v_k) \mathbb{E} \left\{ x_k^{(\mu)} \left(x_k^{(\mu)} \right)^T \right\} \mathfrak{X}^T(\tilde{x}_k, u_k, v_k) \right\} \\ &\leq \mathbb{E} \left\{ \mathfrak{X}(\tilde{x}_k, u_k, v_k) \bar{X}_k \mathfrak{X}^T(\tilde{x}_k, u_k, v_k) \right\}. \end{aligned} \quad (60)$$

The above inequality shows that, by replacing the term $\mathbb{E} \left\{ x_k^{[p]} \left(x_k^{[q]} \right)^T \right\}$ with $\bar{X}_{p,q,k}$ in (55), we would get an upper bound of Ψ_k^v . Therefore, from (47) and (49), it follows directly that $\Psi_k^v \leq \bar{\Psi}_k^v$. Similarly, we can get $\Psi_k^w \leq \bar{\Psi}_k^w$.

It remains to show that $X_{k+1} \leq \bar{X}_{k+1}$. With (28), we have

$$\begin{aligned} X_{k+1} &= \mathbb{E} \left\{ (A_k + L_{g,k} \Delta_{g,k} \Upsilon_k) X_k (A_k + L_{g,k} \Delta_{g,k} \Upsilon_k)^T \right. \\ &\quad \left. + \Psi_k^v \right\}. \end{aligned} \quad (61)$$

Since it has been proved that $\Psi_k^v \leq \bar{\Psi}_k^v$, we have

$$\begin{aligned} X_{k+1} &\leq \mathbb{E} \left\{ (A_k + L_{g,k} \Delta_{g,k} \Upsilon_k) X_k (A_k + L_{g,k} \Delta_{g,k} \Upsilon_k)^T \right. \\ &\quad \left. + \bar{\Psi}_k^v \right\}. \end{aligned} \quad (62)$$

With the assumption that $X_k \leq \bar{X}_k$ and the condition (44), it follows from Lemma 2 that

$$\begin{aligned} X_{k+1} &\leq A_k (\bar{X}_k^{-1} - \gamma_{3,k} \Upsilon_k^T \Upsilon_k)^{-1} A_k^T + \gamma_{3,k}^{-1} L_{g,k} L_{g,k}^T \\ &\quad + \bar{\Psi}_k^v = \bar{X}_{k+1}. \end{aligned} \quad (63)$$

So far, we have proved that $X_k \leq \bar{X}_k$, $\Psi_k^v \leq \bar{\Psi}_k^v$, and $\Psi_k^w \leq \bar{\Psi}_k^w$, and we are going to deal with P_k next. The corresponding results can also be obtained by induction. It is already known that $\bar{P}_0 = P_0$. Then, assuming that, for $i = 1, 2, \dots, k$, $P_i \leq \bar{P}_i$ and it remains to prove that $P_{k+1} \leq \bar{P}_{k+1}$.

With Lemma 3, we can have the following inequality:

$$\begin{aligned} &\sqrt{2} \check{A}_k e_k \left(\tilde{x}_k^{(\mu)} \right)^T \Upsilon_k^T \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right] \\ &\quad \times [L_{g,k}, -K_k L_{h,k}]^T + \sqrt{2} [L_{g,k}, -K_k L_{h,k}] \\ &\quad \times \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right]^T \Upsilon_k \tilde{x}_k^{(\mu)} e_k^T \check{A}_k^T \\ &\leq \varepsilon \check{A}_k e_k e_k^T \check{A}_k^T + 2\varepsilon^{-1} [L_{g,k}, -K_k L_{h,k}] \end{aligned}$$

$$\begin{aligned} & \times \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right]^T \Upsilon_k \tilde{x}_k^{(\mu)} \left(\tilde{x}_k^{(\mu)} \right)^T \Upsilon_k^T \\ & \times \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right] [L_{g,k}, -K_k L_{h,k}]^T. \end{aligned} \quad (64)$$

Substituting (64) into (39) yields

$$\begin{aligned} P_{k+1} & \leq (1 + \varepsilon) \mathbb{E} \{ \check{A}_k P_k \check{A}_k^T \} + 2(1 + \varepsilon^{-1}) [L_{g,k}, -K_k L_{h,k}] \\ & \times \mathbb{E} \left\{ \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right]^T \Upsilon_k \tilde{x}_k^{(\mu)} \left(\tilde{x}_k^{(\mu)} \right)^T \Upsilon_k^T \right. \\ & \times \left. \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right] \right\} [L_{g,k}, -K_k L_{h,k}]^T \\ & + K_k \Psi_k^w K_k^T + \bar{\Psi}_k^v. \end{aligned} \quad (65)$$

From the upper bounds of the covariances of $v_k^{(\mu)}$ and $w_k^{(\mu)}$ and the assumption that $P_k \leq \bar{P}_k$, we have

$$\begin{aligned} P_{k+1} & \leq (1 + \varepsilon) \mathbb{E} \{ \check{A}_k \bar{P}_k \check{A}_k^T \} + 2(1 + \varepsilon^{-1}) [L_{g,k}, -K_k L_{h,k}] \\ & \times \mathbb{E} \left\{ \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right]^T \Upsilon_k \tilde{x}_k^{(\mu)} \left(\tilde{x}_k^{(\mu)} \right)^T \Upsilon_k^T \right. \\ & \times \left. \left[\frac{1}{\sqrt{2}} \Delta_{g,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right] \right\} [L_{g,k}, -K_k L_{h,k}]^T \\ & + K_k \bar{\Psi}_k^w K_k^T + \bar{\Psi}_k^v. \end{aligned} \quad (66)$$

Noticing that $\left\| \left[\frac{1}{\sqrt{2}} \Delta_{f,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right] \right\| \leq 1$, it follows from (42), (43) and Lemma 2 that

$$\begin{aligned} P_{k+1} & \leq (1 + \varepsilon) \left((A_k - K_k C_k) (\bar{P}_k^{-1} - \gamma_{1,k} \Upsilon_k^T \Upsilon_k)^{-1} \right. \\ & \times (A_k - K_k C_k)^T + 2\gamma_{1,k}^{-1} (L_{g,k} L_{g,k}^T + K_k L_{h,k} \\ & \times L_{h,k}^T K_k^T) \left. \right) + 2\gamma_{2,k}^{-1} (1 + \varepsilon^{-1}) (L_{g,k} L_{g,k}^T \\ & + K_k L_{h,k} L_{h,k}^T K_k^T) + \bar{\Psi}_k^v + K_k \bar{\Psi}_k^w K_k^T. \end{aligned} \quad (67)$$

Furthermore, it can be seen from (45) and (46) that

$$\begin{aligned} P_{k+1} & \leq K_k Y_k K_k^T - K_k Z_k - Z_k^T K_k^T + (1 + \varepsilon) A_k (\bar{P}_k^{-1} \\ & - \gamma_{1,k} \Upsilon_k^T \Upsilon_k)^{-1} A_k^T + 2(\gamma_{1,k}^{-1} (1 + \varepsilon) + \gamma_{2,k}^{-1} (1 \\ & + \varepsilon^{-1})) L_{g,k} L_{g,k}^T + \bar{\Psi}_k^v. \end{aligned} \quad (68)$$

Considering $Y_k = Y_k^T > 0$ and completing the square with respect to K_k , we have

$$\begin{aligned} P_{k+1} & \leq (K_k - Z_k^T Y_k^{-1}) Y_k (K_k - Z_k^T Y_k^{-1})^T - Z_k^T Y_k^{-1} Z_k \\ & + 2 \left(\gamma_{1,k}^{-1} (1 + \varepsilon) + \gamma_{2,k}^{-1} (1 + \varepsilon^{-1}) \right) L_{g,k} L_{g,k}^T \\ & + (1 + \varepsilon) A_k (\bar{P}_k^{-1} - \gamma_{1,k} \Upsilon_k^T \Upsilon_k)^{-1} A_k^T + \bar{\Psi}_k^v. \end{aligned} \quad (69)$$

It is now obvious that, when $K_k = Z_k^T Y_k^{-1}$, the upper bound of P_{k+1} is minimized and

$$\begin{aligned} P_{k+1} & \leq -Z_k^T Y_k^{-1} Z_k + (1 + \varepsilon) A_k (\bar{P}_k^{-1} - \gamma_{1,k} \Upsilon_k^T \Upsilon_k)^{-1} \\ & \times A_k^T + 2 \left(\gamma_{1,k}^{-1} (1 + \varepsilon) + \gamma_{2,k}^{-1} (1 + \varepsilon^{-1}) \right) L_{g,k} L_{g,k}^T \\ & + \bar{\Psi}_k^v = \bar{P}_{k+1}, \end{aligned} \quad (70)$$

which concludes the proof.

Remark 5 The filter gain K_k is calculated at each time instant to minimize an upper bound of the filtering error covariance. The system (28) under consideration includes the polynomial approximation errors, thereby better reflecting the reality. Specific efforts have been made to handle the approximation errors. Though the unknown matrices $\Delta_{g,k}$ and $\Delta_{h,k}$ can not be introduced to design the filter, the matrices $L_{g,k}$ and $L_{h,k}$ reflect the effects of the approximation errors on the filter design in a quantitative way. The value of the state estimate $\tilde{x}_k^{(\mu)}$ can be directly used to design K_k since $\tilde{x}_k^{(\mu)}$ is already obtained at time step k . Due to the consideration of the polynomial approximation errors, the accurate covariances Ψ_k^v and Ψ_k^w cannot be obtained as done in [14, 15]. Instead, the upper bounds of the covariances have been calculated and employed to design the suboptimal filter. It is worth mentioning that, $\bar{\Phi}_k^w$, an upper bound of $\mathbb{E} \{ w_k^{(\mu)} (w_k^{(\mu)})^T \}$, could guarantee the invertibility of Y_k in (52). The parameter ε can be determined to balance the intrinsic characteristic of the proposed polynomial filter and the impact brought from the state estimates in the upper bound of the filtering error covariance. The proposed algorithm can be applied to nonlinear time-varying systems that are $\mu + 1$ times continuously differentiable. Furthermore, the gain K_k can be designed recursively by a Riccati-like equation, which is applicable for online computations.

Theorem 1 provides a recursive way to obtain the filter gain K_k . The algorithm is summarized in Algorithm 1 to show the calculation of K_k at each time step with a given order μ .

4 Fault Detection

To efficiently detect the fault in the system (1), we need to define a residual signal based on the proposed polynomial filter and then evaluate it properly. Here, we utilize the following signal as the residual:

$$r_k = y_k^{(\mu)} - C_k \tilde{x}_k^{(\mu)}. \quad (71)$$

In the next theorem, a time-varying fault detection threshold will be calculated for the residual defined in (71). The state estimate, approximation error and bounds of the noises are all taken into consideration.

Algorithm 1: the calculation of K_k

- Step 1.* Select the initial values \bar{P}_0 and $\tilde{x}_0^{(\mu)}$ based on the distribution of x_0 .
- Step 2.* Determine E_k and $\bar{m}_{i,k}(i = 1, \dots, n^\mu)$ according to the system dynamics, the disturbances, and the initial estimation error, and compute the Carleman approximation matrices $A_k, C_k, L_{g,k}, L_{h,k}$, and Υ_k based on (16)-(18) and (21)-(29).
- Step 3.* Determine $\gamma_{1,k}, \gamma_{2,k}$, and $\gamma_{3,k}$ with (42)-(44).
- Step 4.* Compute $\bar{\Omega}_k, \bar{\Phi}_k^w$, and $\bar{\Phi}_k^v$ according to (47)-(51).
- Step 5.* Choose a positive scalar ε , and calculate Y_k and Z_k based on (45) and (46).
- Step 6.* Obtain the desired filter gain K_k with (52).
- Step 7.* Update the estimate $\tilde{x}_{k+1}^{(\mu)}$ by (30).
- Step 8.* Calculate \bar{P}_{k+1} and \bar{X}_{k+1} using (41) and (50).
- Step 9.* Set $k = k + 1$ and go to *Step 2*.
-

Theorem 2 For system (1), let us consider the filter (30) designed in Theorem 1. Then, the residual in (71) satisfies the following inequality in the fault free case:

$$\|r_k\| \leq (\|C_k\| + \|L_{h,k}\| \|\Upsilon_k\|) \delta_{e,k} + \|L_{h,k}\| \|\Upsilon_k \tilde{x}_k^{(\mu)}\| + \delta_{w,k} := \delta_{r,k}, \quad (72)$$

where

$$\begin{aligned} \delta_{e,k+1} = & \left(\|A_k - K_k C_k\| + \sqrt{2} \| [L_{g,k}, -K_k L_{h,k}] \| \|\Upsilon_k\| \right) \\ & \times \delta_{e,k} + \sqrt{2} \| [L_{g,k}, -K_k L_{h,k}] \| \|\Upsilon_k \tilde{x}_k^{(\mu)}\| \\ & + \|K_k\| \delta_{w,k} + \delta_{v,k}, \end{aligned} \quad (73)$$

$$\begin{aligned} \delta_{v,m,k} = & \sum_{i=0}^{\mu} \sum_{p=0}^m \frac{1}{i!} \|M_m^p\| \left\| \nabla_x^{[i]} \otimes g^{[p]} \right\| \bar{\rho}_{i,k}(s_{v,k,m-p} \\ & + \|\xi_{v,k,m-p}\|), \end{aligned} \quad (74)$$

$$\begin{aligned} \delta_{w,m,k} = & \sum_{i=0}^{\mu} \sum_{p=0}^m \frac{1}{i!} \|M_m^p\| \left\| \nabla_x^{[i]} \otimes h^{[p]} \right\| \bar{\rho}_{i,k}(s_{w,k,m-p} \\ & + \|\xi_{w,k,m-p}\|), \end{aligned} \quad (75)$$

$$\bar{\rho}_{i,k} = \sum_{q=0}^i \left\| M_i^q \left(I_{n^q} \otimes (-\tilde{x}_k)^{[i-q]} \right) \right\| \left(\|\tilde{x}_{q,k}^{(\mu)}\| + \delta_{e,k} \right), \quad (76)$$

$$\delta_{v,k} = \sqrt{\sum_{m=0}^{\mu} (\delta_{v,m,k})^2}, \quad (77)$$

$$\delta_{w,k} = \sqrt{\sum_{m=0}^{\mu} (\delta_{w,m,k})^2}, \quad (78)$$

and the initial values of $\delta_{e,k}$ are given by

$$\delta_{e,i,0} = s_{0,i} + \|\zeta_{0,i}\|, \delta_{e,0} = \sqrt{\sum_{i=0}^{\mu} \delta_{e,i,0}^2}. \quad (79)$$

Then $\delta_{r,k}$ is the threshold that we can utilize to detect the fault.

Proof: Substituting (28) and (30) into (71), we have

$$r_k = C_k e_k + L_{h,k} \Delta_{h,k} \Upsilon_k (e_k + \tilde{x}_k^{(\mu)}) + w_k^{(\mu)}. \quad (80)$$

With $\|\Delta_{h,k}\| \leq 1$ and triangle inequality, we can get

$$\begin{aligned} \|r_k\| \leq & (\|C_k\| + \|L_{h,k}\| \|\Upsilon_k\|) \|e_k\| + \|L_{h,k}\| \|\Upsilon_k \tilde{x}_k^{(\mu)}\| \\ & + \|w_k^{(\mu)}\|. \end{aligned} \quad (81)$$

If we can prove that $\|e_k\| \leq \delta_{e,k}$ and $\|w_k^{(\mu)}\| \leq \delta_{w,k}$, then (72) follows directly from (81).

It follows from (25) that,

$$\begin{aligned} \|w_{m,k}^{(\mu)}\| \leq & \sum_{i=0}^{\mu} \sum_{p=0}^m \frac{1}{i!} \|M_m^p\| \left\| \nabla_x^{[i]} \otimes h^{[p]} \right\| \left\| (x_k - \tilde{x}_k)^{[i]} \right\| \\ & \times \left\| w_k^{[m-p]} - \xi_{w,k,m-p} \right\|, \end{aligned} \quad (82)$$

and the upper bound of $\left\| (x_k - \tilde{x}_k)^{[i]} \right\|$ can be determined as follows:

$$\begin{aligned} \left\| (x_k - \tilde{x}_k)^{[i]} \right\| &= \left\| \sum_{q=0}^i M_i^q \left(I_{n^q} \otimes (-\tilde{x}_k)^{[i-q]} \right) x_k^{[q]} \right\| \\ &\leq \sum_{q=0}^i \left\| M_i^q \left(I_{n^q} \otimes (-\tilde{x}_k)^{[i-q]} \right) \right\| \\ &\quad \times \left\| \tilde{x}_{q,k}^{(\mu)} + e_{q,k} \right\|. \end{aligned} \quad (83)$$

Noticing the fact that $\|e_{q,k}\| \leq \|e_k\|$, we have

$$\begin{aligned} \left\| (x_k - \tilde{x}_k)^{[i]} \right\| &\leq \sum_{q=0}^i \left\| M_i^q \left(I_{n^q} \otimes (-\tilde{x}_k)^{[i-q]} \right) \right\| \\ &\leq \sum_{q=0}^i \left\| M_i^q \left(I_{n^q} \otimes (-\tilde{x}_k)^{[i-q]} \right) \right\| \\ &\quad \times \left(\|e_k\| + \|\tilde{x}_k^{(\mu)}\| \right). \end{aligned} \quad (84)$$

With (84), we can see that if $\|e_k\| \leq \delta_{e,k}$, then $\|w_{m,k}^{(\mu)}\| \leq \delta_{w,m,k}$ and $\|w_k^{(\mu)}\| \leq \delta_{w,k}$. Similarly, we have that if $\|e_k\| \leq \delta_{e,k}$, then $\|v_k^{(\mu)}\| \leq \delta_{v,k}$. Thus, to obtain (72),

all we need to verify is that $\|e_k\| \leq \delta_{e,k}$. This can be proved by induction. It is obvious that $\|e_0\| \leq \delta_{e,0}$. Assuming that $\|e_i\| \leq \delta_{e,i}$ ($i = 1, 2, \dots, k$), we are going to demonstrate that $\|e_{k+1}\| \leq \delta_{e,k+1}$.

With (40) and $\left\| \left[\frac{1}{\sqrt{2}} \Delta_{f,k}^T, \frac{1}{\sqrt{2}} \Delta_{h,k}^T \right] \right\| \leq 1$, it follows that

$$\begin{aligned} \|e_{k+1}\| &\leq \left(\|A_k - K_k C_k\| + \sqrt{2} \| [L_{g,k}, -K_k L_{h,k}] \| \|\Upsilon_k\| \right) \\ &\quad \times \|e_k\| + \sqrt{2} \| [L_{g,k}, -K_k L_{h,k}] \| \left\| \Upsilon_k \tilde{x}_k^{(\mu)} \right\| \\ &\quad + \|v_k^{(\mu)}\| + \|K_k\| \|w_k^{(\mu)}\|. \end{aligned} \quad (85)$$

Based on our assumption that $\|e_k\| \leq \delta_{e,k}$, we have $\|w_k^{(\mu)}\| \leq \delta_{w,k}$ and $\|v_k^{(\mu)}\| \leq \delta_{v,k}$. Then it is straightforward to see that $\|e_{k+1}\| \leq \delta_{e,k+1}$. Now we can draw the conclusion that $\|e_k\| \leq \delta_{e,k}$. Substituting $\|e_k\| \leq \delta_{e,k}$ and $\|w_k^{(\mu)}\| \leq \delta_{w,k}$ into (81) yields (72). The proof now is complete.

The algorithm is summarized in Algorithm 2 to show the determination of the threshold $\delta_{r,k}$ at each time step with a given order μ . Based on Theorem 2, a natural

Algorithm 2: the determination of $\delta_{r,k}$

- Step 1.* Determine the initial values $\delta_{r,0}$ with (72) and (79).
 - Step 2.* Calculate the Euclidean norm of the matrices $\|A_k\|$, $\|C_k\|$, $\|L_{g,k}\|$, $\|L_{h,k}\|$, $\|\Upsilon_k\|$, and $\|K_k\|$ based on (21)-(28) and (52).
 - Step 3.* Compute $\bar{\rho}_{i,k}$ with (76).
 - Step 4.* Choose $\delta_{v,k}$ and $\delta_{w,k}$ according to (74),(75), (77) and (78).
 - Step 5.* Obtain $\delta_{r,k}$ based on (72).
 - Step 6.* Update $\delta_{e,k+1}$ based on (73).
 - Step 7.* Set $k = k + 1$ and go to *Step 2*.
-

fault detection strategy follows directly as follows:

$$\begin{cases} \|r_k\| \leq \delta_{r,k} & \implies \text{no fault} \\ \|r_k\| > \delta_{r,k} & \implies \text{fault detected} \end{cases} \quad (86)$$

With the proposed filter and fault detection threshold, the fault detectability is analyzed in the sequel.

Theorem 3 *For system (1), consider the filter (30) designed in Theorem 1 and the fault detection strategy given in (86). The fault will be detected at time step $k + 1$ if the following inequality holds:*

$$\begin{aligned} \alpha_{v,f,k} &> \frac{1}{\|C_{k+1}\| - \|L_{f,h,k+1}\| \|\Upsilon_{k+1}\|} \left(\delta_{r,k+1} + \delta_{w,k+1} \right. \\ &\quad \left. + \|L_{f,h,k+1}\| \|\Upsilon_{k+1} \tilde{x}_{k+1}^{(\mu)}\| \right) + \|K_k\| \|w_k^{(\mu)}\| \end{aligned}$$

$$\begin{aligned} &+ \left(\|A_k - K_k C_k\| + \sqrt{2} \| [L_{f,g,k}, -K_k L_{f,h,k}] \| \right. \\ &\quad \times \|\Upsilon_k\| \left. \right) \delta_{e,k} + \sqrt{2} \| [L_{f,g,k}, -K_k L_{f,h,k}] \| \\ &\quad \times \left\| \Upsilon_k \tilde{x}_k^{(\mu)} \right\|, \end{aligned} \quad (87)$$

where

$$\begin{aligned} \alpha_{v,f,m,k} &= (\|f_k\| - s_{v,k,1})^m - \|\xi_{v,k,m}\| \\ &\quad - \sum_{i=0}^{\mu} \sum_{p=1}^m \frac{1}{i!} \left\| M_m^p \left(\nabla_x^{[i]} \otimes g^{[p]} \right) \right\| \bar{\rho}_{i,k} (\|f_k\| \\ &\quad + s_{v,k,1})^{m-p} + \|\xi_{v,k,m-p}\|, \end{aligned} \quad (88)$$

$$\alpha_{v,f,k} = \sqrt{\sum_{m=0}^{\mu} (\alpha_{v,f,m,k})^2}, \quad (89)$$

$L_{f,g,k}$ and $L_{f,h,k}$ are the problem-dependent matrices in the faulty case, which are corresponding to $L_{g,k}$ and $L_{h,k}$ in the fault-free case, respectively.

Proof: In the presence of f_k , $v_{m,k}$ can be written as

$$\begin{aligned} v_{m,k} &= \sum_{i=0}^{\mu} \sum_{p=0}^m \frac{1}{i!} M_m^p \left(\left(\nabla_x^{[i]} \otimes g^{[p]} \right) (x_k - \tilde{x}_k)^{[i]} \right) \\ &\quad \otimes \left((v_k + f_k)^{[m-p]} - \xi_{v,k,m-p} \right). \end{aligned} \quad (90)$$

Then it follows directly that

$$\begin{aligned} \|v_{m,k}\| &\geq (\|f_k\| - s_{v,k,1})^m - \|\xi_{v,k,m}\| \\ &\quad - \sum_{i=0}^{\mu} \sum_{p=1}^m \frac{1}{i!} \left\| M_m^p \left(\nabla_x^{[i]} \otimes g^{[p]} \right) \right\| \bar{\rho}_{i,k} (\|f_k\| \\ &\quad + s_{v,k,1})^{m-p} + \|\xi_{v,k,m-p}\| = \alpha_{v,f,m,k}. \end{aligned} \quad (91)$$

Subsequently, we have

$$\|v_k^{(\mu)}\| \geq \alpha_{v,f,k}. \quad (92)$$

With (40), we have

$$\begin{aligned} \|e_{k+1}\| &\geq \|v_k^{(\mu)}\| - \sqrt{2} \| [L_{f,g,k}, -K_k L_{f,h,k}] \| \left\| \Upsilon_k \tilde{x}_k^{(\mu)} \right\| \\ &\quad - \left(\|A_k - K_k C_k\| + \sqrt{2} \| [L_{f,g,k}, -K_k L_{f,h,k}] \| \right. \\ &\quad \times \|\Upsilon_k\| \left. \right) \|e_k\| - \|K_k\| \|w_k^{(\mu)}\|. \end{aligned} \quad (93)$$

Substituting (87) into (93) yields

$$\begin{aligned} \|e_{k+1}\| &> \frac{1}{\|C_{k+1}\| - \|L_{f,h,k+1}\| \|\Upsilon_{k+1}\|} \left(\delta_{r,k+1} \right. \\ &\quad \left. + \delta_{w,k+1} + \|L_{f,h,k+1}\| \|\Upsilon_{k+1} \tilde{x}_{k+1}^{(\mu)}\| \right). \end{aligned} \quad (94)$$

Considering (80), we have

$$\begin{aligned} \|r_{k+1}\| &\geq \|C_{k+1}\| - \|L_{f,h,k+1}\| \|\Upsilon_{k+1}\| \|e_{k+1}\| \\ &\quad - \|L_{f,h,k+1}\| \|\Upsilon_{k+1}\| \tilde{x}_{k+1}^{(\mu)} - \|w_{k+1}^{(\mu)}\|. \end{aligned} \quad (95)$$

From (94) and (95), it follows that

$$\|r_{k+1}\| \geq \delta_{r,k+1}. \quad (96)$$

Based on the proposed fault detection strategy (86), the fault can be detected at time step $k+1$ and that concludes the proof.

Remark 6 *Theorem 2 puts forward a recursive method to determine the fault detection threshold for system (28). With the adaptive threshold, abnormal changes in systems can be detected accordingly. The fault may be some state-dependent abrupt changes (e.g. actuator or sensor failures in a control system). The approximation errors, additive disturbances, and initial estimation error have all been considered in the threshold. The spectral norms of matrices and the upper bounds of Euclidean norms of vector disturbances have been introduced to calculate the threshold. In this way, the false alarms can be avoided in the proposed fault detection scheme. Moreover, the threshold is related to the measurement and estimated states, and hence needs to be updated online at each time step, which makes the method adaptive. With the proposed filter and fault detection strategy, the fault detectability is analyzed in Theorem 3 as well. The fault detectability condition established in Theorem 3 takes the Carleman approximation and the approximation error into account and therefore looks complicated. Nevertheless, such a condition can be easily verified and also provides inequality constraints on the fault size. The explicit consideration of the approximation error in the filter and fault detector designs constitutes the main difference between our work and those in [14, 15], and the derivation of the threshold and fault detectability analysis for nonlinear systems with polynomial approximation are new. If the stability analysis of the proposed filter approach becomes a concern, some additional assumptions can be made on the system parameters so as to ensure the boundedness of the estimation errors [25, 32]. In the next section, a numerical simulation would be carried out to show the effectiveness of the presented fault detection strategy.*

5 Illustrative Example

Consider the following nonlinear discrete system:

$$\begin{cases} x_{1,k+1} = 0.85x_{1,k} + 0.5x_{2,k} \sin(x_{1,k}) + v_{1,k} + f_k, \\ x_{2,k+1} = 1.15u_k - 0.5x_{1,k} \sin(x_{2,k}) + v_{2,k}, \\ y_k = x_{2,k} + w_k, \\ u_k = y_k. \end{cases}$$

The state noises and measurement noise are independent and obey the following distributions (for $i = 1, 2$):

$$\begin{cases} P(v_{i,k} = -1 \times 10^{-3}) = 0.6, \\ P(v_{i,k} = 0) = 0.2, \\ P(v_{i,k} = 3 \times 10^{-3}) = 0.2, \end{cases}$$

and

$$\begin{cases} P(w_k = -7 \times 10^{-3}) = 0.3, \\ P(w_k = 3 \times 10^{-3}) = 0.7. \end{cases}$$

When the order $\mu = 2$, how to construct the polynomial filter is explained step by step as follows. For the addressed system, we have

$$\begin{cases} g(x_k, u_k) = \begin{bmatrix} 0.85x_{1,k} + 0.5x_{2,k} \sin(x_{1,k}) \\ 1.15u_k - 0.5x_{1,k} \sin(x_{2,k}) \end{bmatrix} \\ h(x_k) = x_{2,k}, \end{cases}$$

With the linear measurement, we can easily obtain that $L_{h,k} = 0$ and

$$C_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{E}\{w_k^{[2]}\} & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

With (22) and (29), A_k can be determined. Based on (26), we have

$$\Upsilon_k = \begin{bmatrix} \tilde{x}_{1,k}^2 & -2\tilde{x}_{1,k} & 0 & 1 & 0 & 0 & 0 \\ \tilde{x}_{1,k}\tilde{x}_{2,k} & -\tilde{x}_{2,k} & -\tilde{x}_{1,k} & 0 & 1 & 0 & 0 \\ \tilde{x}_{1,k}\tilde{x}_{2,k} & -\tilde{x}_{2,k} & -\tilde{x}_{1,k} & 0 & 0 & 1 & 0 \\ \tilde{x}_{2,k}^2 & 0 & -2\tilde{x}_{2,k} & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix E_k is set to be $0.075I$ based on the system dynamics, the disturbances, and the initial estimation error, and it can be seen that the assumption on the estimation error (i.e. $x_k = \tilde{x}_k + E_k z_k$ for some $z_k \in \mathbb{R}^n$ and $\|z_k\| \leq 1$) holds with such a choice of E_k . Based on E_k and the values of $\bar{m}_{i,k}$ for $i = 1, 2, \dots, n^\mu$, the matrix $L_{g,k}$ can be chosen as $0.1I$. After getting all the aforementioned parameters, Theorem 1 can be applied to recursively calculate the desired filter gain.

In the fault-free case, the average estimation error covariances with the proposed method and the ones with

the conventional PEKF algorithm [14, 15] are both listed in Table 1. Monte-Carlo simulations with 50 runs are carried out in the cases where $\mu = 1$ and $\mu = 2$. From the table, two conclusions can be drawn: 1) approximating the system with a polynomial of a higher order implies a more accurate approximation of the original nonlinear system and gives better estimation results, no matter whether the approximation errors are considered or not; and 2) the proposed filter outperforms classical PEKF, thanks to the specific efforts we have made to deal with the high-order approximation errors.

Table 1
Average Trace of the Estimation Error Covariance

	$\mu = 1$	$\mu = 2$
Traditional PEKF	6.1763×10^{-4}	4.9531×10^{-4}
Proposed Filter	5.5649×10^{-4}	3.9534×10^{-4}

To demonstrate the fault detection strategy, we add the following fault to the system:

$$f_k = \begin{cases} 0.003, & \text{if } k > 20, \\ 0, & \text{otherwise.} \end{cases}$$

Figs. 1-2 show the actual states and the estimates when $\mu = 2$. It can be concluded that, the proposed filter performs well when there is no fault in the system (when $k \leq 20$), and the obvious difference between the actual state and estimated state in the faulty case (when $k > 20$) could help us to detect the fault. The Euclidean norm of residual and the adaptive threshold are illustrated in Fig. 3. It can be seen that using the adaptive threshold, the additive fault could be detected immediately after it occurs. This result presents the effectiveness of our adaptive fault detection strategy.

Remark 7 The computational complexity of the proposed algorithm is $O(n^{3\mu})$. The numerical experiments are performed on a 2.83 GHz quad-core desktop computer with 3.50 GB RAM using 32 bit MATLAB. When $\mu = 1$, the execution time is 15ms; when $\mu = 2$, the execution time is 94ms. So it is obvious that the computational burden of the proposed algorithm increases exponentially with a bigger degree of polynomials. When $\mu = 2$, the average Euclidean norm of the residuals obtained with the proposed filter after the fault occurs is 7.250×10^{-3} in 50 Monte-Carlo simulations. With traditional PEKF, the average Euclidean norm of the residuals is 5.280×10^{-3} . So the proposed filter are more sensitive to the fault and can better solve the fault detection problem.

6 Conclusion

The filtering and fault detection problems for a class of nonlinear systems have been addressed with a polynomial approach in the paper. The nonlinear functions

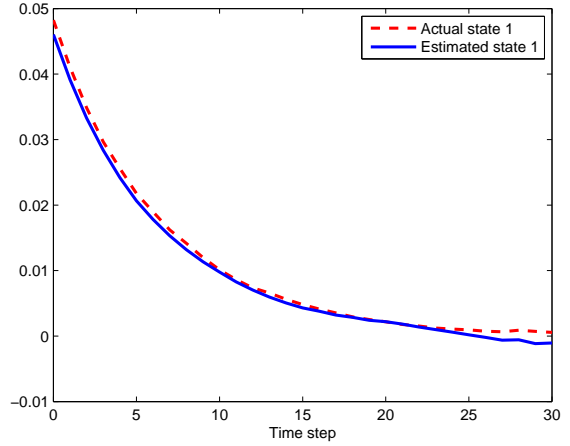


Fig. 1. The actual state x_1 and its estimate

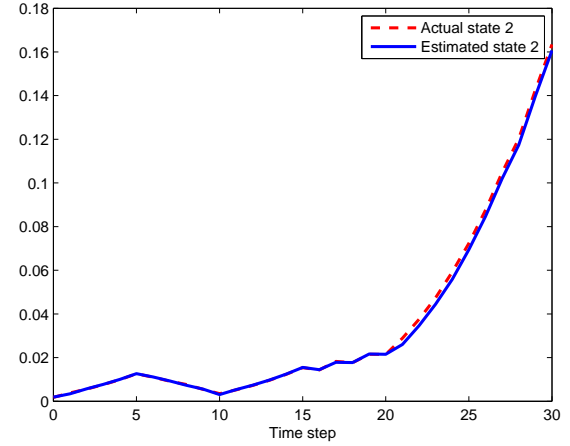


Fig. 2. The actual state x_2 and its estimate

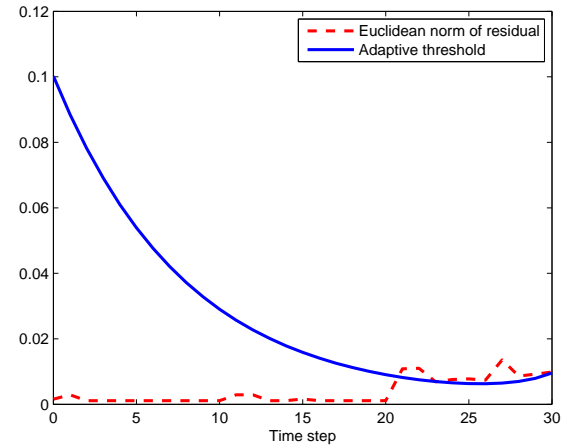


Fig. 3. The Euclidean norm of residual and the threshold

have been approximated with polynomials of a chosen degree, and the approximation errors, which result from high-order remainder terms of series expansions, have been written as low-order terms with norm-bounded coefficients. An upper bound of the filtering error covariance has been derived in the presence of the approximation errors. Then the filter has been designed to minimize the bound at each time step in the fault-free case. The desired filter gain could be obtained with a set of Riccati-like recursive matrix equations, thus the algorithm is applicable for online computation. A fault detection strategy with adaptive threshold has been proposed to efficiently detect the possible fault, taking account of the noises and the approximation errors. Sufficient conditions have been established to guarantee the fault detectability for the proposed fault detection scheme. A numerical simulation has been given to illustrate that the proposed method could achieve accurate state estimation and effective fault detection. It should be noted that two of the future research topics would be to investigate the fault diagnosis and fault tolerant control problems after the fault detection.

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