On the Structure of the h-Vector of a Paving Matroid

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Abstract

We give two proofs that the h-vector of any paving matroid is a pure O-sequence, thus answering in the affirmative a conjecture made by R. Stanley, for this particular class of matroids. We also investigate the problem of obtaining good lower bounds for the number of bases of a paving matroid given its rank and number of elements.

1 Introduction

Matroids are important structures in combinatorics, particularly in relation to combinatorial optimization and graph theory, see [18, 25, 33]. With any matroid M there is an associated simplicial complex $\Delta(M)$ given by the independent sets of M. Such simplicial complexes are called *matroid complexes* and are known to be *shellable*, that is, the maximal faces are equicardinal and can be arranged in a certain order that helps inductive proofs. (We give a full definition of shellability in the next section). One key combinatorial invariant associated with a shellable complex is its h-vector which encodes information such as, for example, its face and Betti numbers. For these reasons shellable complexes have received much attention, see [3, 5, 6, 31, 34]. The concept of shellability is also important in theoretical computer science as the entries of the h-vector of a graphic matroid $M(G)$ are the coefficients of the H-form of the reliability polynomial of the underlying graph G, see [11].

A non-empty set of monomials $\mathcal{M}$ is a *multicomplex* if whenever $m \in \mathcal{M}$ and $m'|m$, then $m' \in \mathcal{M}$. A finite or infinite sequence h=(h_0, h_1, …, h_d) of integers is called an *O-sequence* if there exists a multicomplex containing exactly h_i monomials of degree i. An O-sequence is *pure* if there exists a multicomplex containing h_i monomials of degree i such that all the maximal elements in the multicomplex have the same degree. Properties of pure O-sequences are mentioned in Section 2.

In 1977, Richard Stanley made the following conjecture linking h-vectors of matroid complexes and O-sequences [28], (see also [29]).

Conjecture 1.1. *The h-vector of a matroid complex is a pure O-sequence.*

No progress was made on this conjecture for some considerable time. But in 1997, work of Norman Biggs [1, 2] together with [22] implicitly proved Conjecture 1.1 for cographic matroids. For an explicit exposition see [23]. More recently, the conjecture was proved for rank two matroids in [30], for lattice-path matroids in [26], for cotransversal matroids in [24] and most recently for rank three matroids in [13].

A *paving matroid* is one in which all circuits have size at least $r(M)$. Interest in paving matroids goes back to 1976 when Dominic Welsh [32] asked if most matroids are paving. This question was motivated by numerical results obtained in [7], where a catalogue of all matroids with up to eight elements was presented. The numerical data was updated in [21] to include matroids with nine elements, and the results made the problem even more intriguing. More recently, the authors of [20] conjecture that asymptotically almost every matroid is paving, that is, the proportion of n-element matroids which are paving tends to one as n tends to infinity.

In this work we give a proof that paving matroids satisfy Conjecture 1.1. Should paving matroids genuinely form a significant proportion of all matroids, then our result will be of a different kind from all the previous work on Conjecture 1.1, as all previous work only considers classes of matroids whose size is insignificant compared with the total number of matroids.

This article is organized as follows. In Section 2 we give definitions and basic properties of matroids, h-vectors and O-sequences. In the next section we prove Stanley's conjecture for paving matroids. The direct approach to Stanley's conjecture is to attempt to get good bounds on the number of bases of a paving matroid in terms of its number of elements and rank and on the minimum number of elements in a pure multicomplex of degree r in d indeterminates which contains every monomial of degree $r-1$. This was our original approach to the problem but we were unable to obtain good enough explicit bounds. However, there appear to be some intriguing open questions concerning these problems including potential links with various other well-studied combinatorial objects. A subclass of paving matroids, namely sparse paving matroids, was introduced by Jerrum in [16] and has recently received attention in [21]. In Section 5 we obtain a good lower bound for the number of bases of a sparse paving matroid in terms of the rank r and number n of elements. We have examples showing that this bound is tight for infinitely many values of r and n. We then move on to consider bounds on the sizes of pure multicomplexes of degree r in d indeterminates which contain every monomial of degree $r-1$ and conjecture a link with the number of aperiodic binary necklaces. The last section contains our conclusions.

2 Preliminaries

In this section we introduce some definitions and key properties of shellable complexes and matroids. We assume some familiarity with matroid theory. For an excellent exposition of shellability of matroid complexes see [4] and for matroids see [25].

2.1 h-vectors

Let Δ be a *simplicial complex* on the vertex set $V = \{x_1, \ldots, x_n\}$. Thus, Δ is a collection of subsets of V such that for all i, $\{x_i\} \in \Delta$, and if $F \in \Delta$ and $F' \subseteq F$, then $F' \in \Delta$. The subsets in Δ are called *faces* and the *dimension* of a face with $i+1$ elements is i. The *dimension* of Δ is the maximum dimension of a face in Δ. Associated with Δ we have its *face vector* or *f-vector* $(f_0, f_1, \ldots, f_d)$, where f_i is the number of faces of size i (or dimension $i-1$) in Δ. The *face enumerator* is the generating function of the entries of the f-vector, defined by

$$f_\Delta(x) = \sum_{i=0}^{d} f_i x^{d-i}.$$

The maximal faces of Δ are called *facets*. When all the facets have the same cardinality, Δ is said to be *pure*. From now on we will only consider pure $d-1$-dimensional simplicial complexes.

Given a linear ordering $F_1, F_2, \ldots, F_t$ of the facets of a simplicial complex Δ, let Δ_i denote the subcomplex generated by the facets $F_1, F_2, \ldots, F_i$, that is, $F \in \Delta_i$ if and only if $F \subseteq F_j$ for some j with $1 \leq j \leq i$.

For a pure simplicial complex Δ, a *shelling* is a linear order of the facets $F_1, F_2, \ldots, F_t$ such that, for $2 \leq l \leq t$,

$$\{F : F \subseteq F_l \text{ and } F \in \Delta_{l-1}\}$$

forms a pure $(\dim(\Delta) - 1)$-dimensional simplicial complex, where $\Delta_0 = \emptyset$. A complex is said to be *shellable* if it is pure and admits a shelling.

For $1 \leq l \leq t$, define $R(F_l) = \{x \in F_l \mid \; F_l \setminus x \in \Delta_{l-1}\}$. The number of facets such that $|R(F_l)| = i$ is denoted by h_i and, importantly, does not depend on the particular shelling, see [4]. The vector $(h_0, h_1, \ldots, h_d)$ is called the *h-vector* of Δ. The *shelling polynomial* is the generating function of the entries of the h-vector, given by

$$h_\Delta(x) = \sum_{i=0}^{d} h_i x^{d-i}.$$

It is well known, see for example [4], that the face enumerator and the shelling polynomial satisfy the relation

$$h_\Delta(x+1) = f_\Delta(x)$$

and so the coefficients satisfy

$$f_k = \sum_{i=0}^{k} h_i \binom{d-i}{k-i}$$

and

$$h_k = \sum_{i=0}^{k} (-1)^{i+k} f_i \binom{d-i}{k-i}, \tag{1}$$

for $0 \le k \le d$.

2.2 Matroids and their complexes

A *matroid* is an ordered pair $M{=}(E, \mathcal{I})$ such that E is a finite set and $\mathcal{I}$ is a collection of subsets of E satisfying the following three conditions:

1. $\emptyset \in \mathcal{I}$;
2. if $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$;
3. if I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Maximal independent sets are called *bases* and it follows easily from the conditions above that all bases have the same cardinality. This common cardinality is called the *rank* of the matroid and is usually denoted by $r(M)$ or just r.

One fundamental example is the class of *uniform matroids*. The uniform matroid with rank r and n elements is denoted by $U_{r,n}$. A set of its elements is independent if and only if it has size at most r.

We recall some basic definitions of matroid theory. A minimal subset C of E that is not independent is called a *circuit*. The *closure* $\overline{A}$ of a subset A of E is defined by

$$\overline{A} = A \cup \{a \,|\, M \text{ has a circuit } C \text{ such that } a \in C \subseteq A \cup \{a\}\}.$$

A subset S is *spanning* if $\overline{S} = E$. A subset H is a *hyperplane* if it is a maximal non-spanning set. For all the other concepts of matroid theory we refer the reader to Oxley's book [25].

If $M{=}(E, \mathcal{I})$ is a matroid, the family of all independent sets forms a simplicial complex of dimension $r(M) - 1$, which we denote by $\Delta(M)$. The facets of $\Delta(M)$ are the bases of M and therefore $\Delta(M)$ is pure. Complexes of this kind are called *matroid complexes.* Matroid complexes are known to be shellable, see [4].

Loops of a matroid are circuits of rank zero and therefore do not belong to any independent set. Consequently they do not play any role in $\Delta(M)$ and so to investigate Conjecture 1.1, we can safely just consider loopless matroids.

Furthermore, *coloops* of a matroid are elements contained in every basis. Equivalently, they belong to every facet of $\Delta(M)$. Suppose M is formed from M' by deleting a coloop. Then $r(M) = r(M')-1$, but more pertinently if the h-vector of M is $(h_0, h_1, \dots, h_r)$, then the h-vector of M' is $(h_0, h_1, \dots, h_r, 0)$. Thus, all the relevant information concerning the h-vector of a matroid can still be obtained after deleting all its coloops. Consequently, for our purposes we only need to consider coloopless matroids.

2.3 Pure O-sequences

An explicit characterization of O-sequences can be found in [28]. However, a complete characterization is not known for pure O-sequences, but Hibi [15] has shown that a pure O-sequence $(h_0, h_1, \dots, h_d)$ must satisfy the following conditions.

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor} \tag{2}$$

and

$$h_i \leq h_{d-i}, \text{ whenever } 0 \leq i \leq \lfloor d/2 \rfloor. \tag{3}$$

Hibi also conjectured that the h-vector of a matroid complex must satisfy inequalities (2) and (3).

The following result concerning the h-vector of a matroid complex is due to Brown and Colbourn [8].

Theorem 2.1. *The h-vector of a connected rank-d matroid satisfies the following inequalities.*

$$(-1)^j \sum_{i=0}^{j} (-b)^i h_i \geq 0, \qquad 0 \leq j \leq d, \tag{4}$$

for any real number $b \geq 1$ with equality possible only if b=1.

This theorem shows that the converse of Stanley's conjecture is not true because the sequence $(1, 4, 2)$ is a pure O-sequence but does not satisfy the conditions of the theorem.

Later, Chari [10] proved a stronger result which generalizes Theorem 2.1 and solves Hibi's conjecture. The fact that the h-vector of a coloop free matroid satisfies inequalities (2)–(4) can also be proved [9] using the Tutte polynomial.

3 Stanley's conjecture for paving matroids

A paving matroid $M = (E, \mathcal{I})$ is a matroid whose circuits all have size at least $r(M)$. If M is a rank-r paving matroid, the face vector of $\Delta(M)$ is easy to compute. Every subset of size $i < r$ is a face of $\Delta(M)$ and the facets are the bases of M. Then, we get the following result, which is implicit in [4].

Proposition 3.1. *The h-vector of a rank-r paving matroid with n elements and $b(M)$ bases is $(h_0, \ldots, h_r)$ where $h_k = \binom{n-r+k-1}{k}$ for $0 \leq k \leq r-1$ and $h_r = b(M) - \binom{n-1}{r-1}$.*

Proof. Using (1) and $f_i = \binom{n}{i}$ for $0 \leq i \leq r-1$ we see that

$$h_k = \sum_{i=0}^{k} (-1)^{i+k} \binom{r-i}{k-i} \binom{n}{i}.$$

for $0 \leq k \leq r-1$. Using the identity $(-1)^a \binom{b}{a} = \binom{a-b-1}{a}$ we get

$$h_k = \sum_{i=0}^{k} \binom{k-r-1}{k-i} \binom{n}{i}.$$

Now using the Vandermonde convolution formula $\binom{a+b}{k} = \sum_{i=0}^{k} \binom{a}{i}\binom{b}{k-i}$ we get

$$h_k = \binom{n-r+k-1}{k}.$$

Because $\sum_{i=0}^{r} h_i = b(M)$, we get

$$h_r = b(M) - \sum_{i=0}^{r-1} h_i = b(M) - \sum_{i=0}^{r-1} \binom{n-r+i-1}{i} = b(M) - \binom{n-1}{r-1}.$$

□

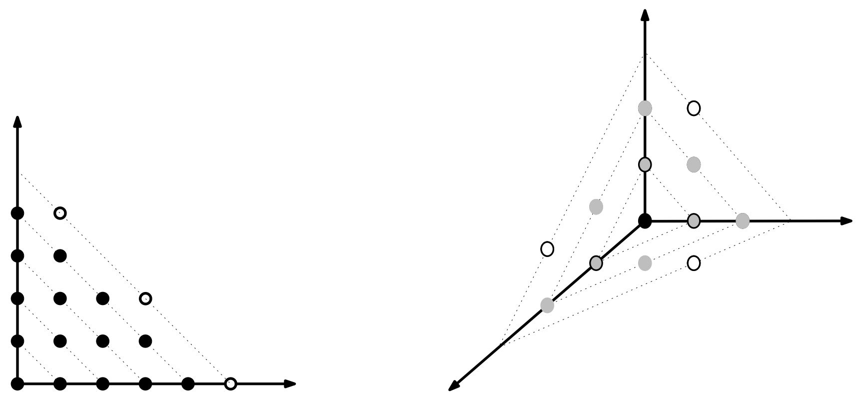

Figure 1: On the left-hand side the point (i,j) represents the monomial x^iy^j. Thus the 18 points represent a multicomplex over 2 variables with $f(5,2)$ monomials of degree 5 and all monomials of degree 4. On the right-hand side the point (i,j,k) represents the monomial $x^iy^jz^k$. Thus the 13 points represent a multicomplex over 3 variables with $f(3,3)$ monomials of degree 3 and all monomials of degree 2.

The idea for proving that the h-vector of a coloopless paving matroid is the O-sequence of a pure multicomplex is simple. We define the multicomplex $\mathcal{M}_{r,d}$ to be the pure multicomplex in which the maximal elements are all monomials of degree r in d indeterminates $z_1,\dots,z_d$. This multicomplex has O-sequence $(h_0,\dots,h_r)$, where $h_k=\binom{d+k-1}{k}$.

Now, for $r\geq 1$ and $d\geq 1$, define the function

$$f(r,d)=\min\{h_r\,|\,(h_0,\dots h_r)\text{ is the pure O-sequence of }\mathcal{M}\supset\mathcal{M}_{r-1,d}\}.$$

This means that $f(r,d)$ is the minimum number of monomials of degree r in a pure multicomplex of degree r which contains every monomial of degree $r-1$ in the d indeterminates $z_1,\dots,z_d$. In Figure 1 we present two examples. So for any positive integers d and r, if $h_k=\binom{d+k-1}{k}$ for $0\leq k\leq r-1$ and $f(r,d)\leq h_r\leq\binom{d+r-1}{r}$, the sequence $(h_0,h_1,\dots,h_r)$ is a pure O-sequence.

If M is a paving matroid with n elements and rank r, then by taking $d=n-r$, we see that the h-vector of M satisfies $h_k=\binom{d+k-1}{k}$ for $0\leq k\leq r-1$. To prove Stanley's conjecture for the class of loopless, coloopless paving matroids, it will be sufficient to prove that for all $r>0$ and $d>0$, $f(r,d)\leq h_r\leq\binom{d+r-1}{r}$ or equivalently that for all r and n with $0<r<n$

$$f(r,n-r)\leq b(M)-\binom{n-1}{r-1}\leq\binom{n-1}{r}.$$

By the remarks at the end of Section 2.2, this is enough to establish the conjecture for the class of all paving matroids. The second inequality is trivial since $b(M) \leq \binom{n}{r}$, so we focus on the first inequality.

Some initial values of f are easy to get.

Lemma 3.2. *For $r \geq 1$ and $d \geq 1$ we have $f(1,d) = 1$, $f(2,d) = \lceil d/2 \rceil$, $f(r,1) = 1$ and $f(r,2) = \lceil r/2 \rceil$.*

Lemma 3.3. *For $r \geq 2$ and $d \geq 2$, $f(r,d) \leq f(r,d-1) + f(r-1,d)$.*

Proof. Let $\mathcal{M}'$ be a multicomplex in indeterminates $z_1, \ldots, z_{d-1}$ containing $\mathcal{M}_{r-1,d-1}$, having h-vector $(h'_0, \ldots, h'_r)$ satisfying $h'_r = f(r, d-1)$. Let $\mathcal{M}''$ be a multicomplex in indeterminates $z_1, \ldots, z_d$ containing $\mathcal{M}_{r-2,d}$, having h-vector $(h''_0, \ldots, h''_{r-1})$ satisfying $h''_r = f(r-1, d)$.

Consider the multicomplex $\mathcal{M}$ that is the union of $\mathcal{M}'$ and

$$z_d\mathcal{M}'' = \{z_d\, m |\, m \in \mathcal{M}''\}.$$

Then, $\mathcal{M}$ contains all the monomials over $z_1, \ldots, z_{d-1}$ of degree at most $r-1$ and all the monomials over $z_1, \ldots, z_d$ of degree at most $r-1$ where z_d has degree at least 1. These are precisely all the monomials over $z_1, \ldots, z_d$ of degree at most $r-1$. Therefore $\mathcal{M}$ contains $\mathcal{M}_{r-1,d}$.

It remains to prove that $\mathcal{M}$ is a multicomplex. Let $m \in \mathcal{M}$ and $m'|m$. Then m' is a monomial in indeterminates $z_1, \ldots, z_d$ and either $m' = m$ or m' has degree at most $r-1$. By using the previous part of the proof, in either case we obtain that $m \in \mathcal{M}$.

Finally, the O-sequence of $\mathcal{M}$ is $(h'_0, h'_1 + h''_0, \ldots, h'_r + h''_{r-1})$.

□

Let $\mathcal{P}_{r,n}$ be the class of coloopless, loopless rank-r paving matroids on n elements. Note that by asking for loopless paving matroids, we are just eliminating rank-1 paving matroids with loops. We define

$$g(r,n) = \min\left\{b(M) - \binom{n-1}{r-1} \mid M \in \mathcal{P}_{r,n}\right\}.$$

Observe that $g(r,n)$ equals the minimum value of h_r among all h-vectors of matroids in $\mathcal{P}_{r,n}$. Thus, to prove Stanley's conjecture for paving matroids it is enough to show that $g(r,n) \geq f(r, n-r)$.

Lemma 3.4. *For all $n \geq 1$, $g(1,n) \geq f(1, n-1)$.*

Proof. Up to isomorphism, the only matroid in $\mathcal{P}_{1,n}$ is $U_{1,n}$, thus $g(1,n) = n-1$ and $f(1, n-1) = 1$. □

The *2-stretching* of a matroid M is the matroid obtained by replacing each element of M by 2 elements in series. We use the following result from [12].

Lemma 3.5. *Let M be a rank-r coloopless paving matroid. If for every element e of M, $M \setminus e$ has a coloop, then one of the following three cases happens.*

1. *M is isomorphic to $U_{r,r+1}$, $r \geq 1$.*
2. *M is the 2-stretching of a uniform matroid $U_{s,s+2}$, for some $s \geq 1$.*
3. *M is isomorphic to $U_{1,2} \oplus U_{1,2}$.*

Lemma 3.6. *Let M be a rank-r coloopless paving matroid with n elements. If for every element e of M, $M \setminus e$ has a coloop, then $b(M) - \binom{n-1}{r-1} = f(r, n-r)$.*

Proof. It follows from the previous lemma that we just have to check three cases. If $M \cong U_{r,r+1}$, $b(M) = r + 1 = \binom{r}{r-1} + f(r, 1)$.

If the matroid M is the 2-stretching of $U_{s,s+2}$, it has rank $2s + 2$, with $2s + 4$ elements and $2(s + 2)(s + 1)$ bases. Thus $b(M) - \binom{2s+3}{2s+1} = s + 1$. On the other hand, $f(2s + 2, 2) = s + 1$, and we have equality.

Finally, if $M \cong U_{1,2} \oplus U_{1,2}$, then $b(M) = 4$ and $f(2, 2) = 1$ which implies equality. $\square$

Theorem 3.7. *If $0 < r < n$ we have $g(r, n) \geq f(r, n - r)$.*

Proof. We prove the statement by induction on $r + n$. If $r = 1$, the result follows by Lemma 3.4. On the other hand suppose that $r = n - 1$. In any coloopless matroid, for each edge e, there must be a basis not containing e. Consequently any such matroid with n elements and rank $n - 1$ must have at least n bases. Hence

$$g(n-1, n) \geq n - \binom{n-1}{n-2} = 1 = f(n-1, 1).$$

Now suppose that $1 < r < n - 1$ and that the theorem is true for all r' and n' with $r' + n' < r + n$. Let M be a matroid in $\mathcal{P}_{r,n}$ such that $b(M) = \binom{n-1}{r-1} + g(r, n)$. Suppose first that $M \setminus e$ has no coloops for some $e \in E(M)$. Then

$$\begin{aligned} g(r, n) = b(M) - \binom{n-1}{r-1} &= b(M \setminus e) - \binom{n-2}{r-1} + b(M/e) - \binom{n-2}{r-2} \\ &\geq g(r, n-1) + g(r-1, n-1) \geq f(r, n-r-1) + f(r-1, n-r) \\ &\geq f(r, n-r). \end{aligned}$$

If M has no such element e then the result follows by Lemma 3.5. $\square$

Corollary 3.8. *The h-vector of the matroid complex of a paving matroid is a pure O-sequence.*

4 Bounds on the number of bases of a sparse paving matroid

One intriguing problem is to determine more about the functions f and g from the previous section. This appears to be a rather hard problem, in particular we have not been able to find tight bounds on the number of bases of a paving matroid in terms of its rank and number of elements. In this section we find a tight bound for the number of bases for a subclass of paving matroids, namely the sparse paving matroids.

We will require the following result on paving matroids which is an exercise in [25] (Page 132, Exercise 8).

Proposition 4.1. *Paving matroids are closed under minors. Moreover a matroid M is paving if and only if it does not contain the matroid $U_{2,2} \oplus U_{0,1}$ as a minor.*

Sparse paving matroids were introduced by Jerrum in [16, 21]. A rank-r matroid M is *sparse paving* if M is paving and for every pair of circuits C_1 and C_2 of size r we have $|C_1 \triangle C_2| > 2$. For example, all uniform matroids are sparse paving matroids.

There is a simple characterization of paving matroids which are sparse in terms of the sizes of their hyperplanes.

Theorem 4.2. *Let M be a paving matroid of rank $r \geq 1$. Then M is sparse paving if and only if all the hyperplanes of M have size r or $r-1$.*

Proof. For the forward implication let H be a hyperplane of M and I be a maximal independent set contained in H. If there are two elements $e \neq f$ in $H \setminus I$, then $C_1 = I \cup \{e\}$ and $C_2 = I \cup \{f\}$ are circuits of size r but $|C_1 \triangle C_2| = 2$, contrary to the assumption that M is sparse paving. Thus, any hyperplane has size either $r-1$ or r.

To prove the converse first note that the closure of any circuit of size r is a hyperplane. By assumption, hyperplanes have size at most r and so any circuit of size r is a hyperplane. Suppose that C_1 and C_2 are distinct circuits of size r in M. Then $I = C_1 \cap C_2$ is an independent set and because I is the intersection of two hyperplanes, its rank is at most $r-2$. So, $|C_1 \cap C_2| \leq r-2$ and $|C_1 \triangle C_2| > 2$. $\square$

Note that we can say a little more about the circuits and hyperplanes of size r in a sparse paving matroids of rank r. In the proof of the preceding theorem we show that any circuit of size r is a hyperplane. Conversely any proper subset of a hyperplane of size r is independent and so such a hyperplane must be a circuit. So the circuits of size r are precisely the hyperplanes of size r.

The fact that the class of sparse paving matroids is closed under duality appears to be (recent) folklore but we are unable to find a reference.

Theorem 4.3. *If M is an n-element sparse paving matroid, then M^* is also sparse paving.*

Proof. If M has rank zero or rank n, then M^* is isomorphic to $U_{n,n}$ or $U_{0,n}$ respectively. Both of these matroids are sparse paving.

Let us suppose that M has rank $1 \leq r \leq n-1$. By duality, C is a circuit of a matroid N over E if and only if $E \setminus C$ is a hyperplane of N^*. From Theorem 4.2 it follows that all the hyperplanes of M have size r or $r-1$. Consequently all the circuits of M^* have size $n-r$ or $n-r+1$ and so M^* is paving. Furthermore all the hyperplanes of M^* have size $n-r$ or $n-r-1$ and so by Theorem 4.2 M^* is sparse paving. □

The next result was first proved by Jerrum [16]. It follows immediately from Theorem 4.3 and the fact that the collection of circuits of $M \setminus e$ is the collection of circuits of M that do not contain e.

Theorem 4.4. *Sparse paving matroids are closed under minors*

Theorem 4.5. *A matroid M is sparse paving if and only if it does not have $U_{2,2} \oplus U_{0,1}$ nor $U_{0,2} \oplus U_{1,1}$ as minors.*

Proof. By Proposition 4.1, it is enough to prove that a paving matroid M is sparse if and only if M does not contain $U_{0,2} \oplus U_{1,1}$ as a minor.

If a rank-r paving matroid M contains $U_{0,2} \oplus U_{1,1}$ as a minor, then M^* contains $U_{2,2} \oplus U_{0,1}$ as a minor and by Proposition 4.1 it is not paving. Thus M cannot be sparse paving by Theorem 4.3.

Suppose M is a rank-r paving matroid with n elements that is not sparse. By Theorem 4.2 there must exist a hyperplane H of size at least $r+1$. Let I be a maximal independent subset of H and let $\{e, f\} \in H \setminus I$. Now let $g \notin H$. If we delete the elements in $E \setminus (H \cup \{g\})$ and contract the elements in $H \setminus \{e, f\}$ we get a $U_{0,2} \oplus U_{1,1}$ minor. □

In order to get more properties of sparse paving matroids, we need the following definition from [25]. Given integers $k > 1$ and $m > 0$, a collection

$\mathcal{T} = \{T_1, \ldots, T_k\}$ of subsets of a set E, such that each member of $\mathcal{T}$ has at least m elements and each m-element subset of E is contained in a unique member of $\mathcal{T}$, is called an m-partition of E. The following proposition is also from [25].

Proposition 4.6. *If $\mathcal{T}$ is an m-partition of E, then $\mathcal{T}$ is the set of hyperplanes of a paving matroid of rank $m+1$ on E. Moreover, for $r \geq 2$, the set of hyperplanes of every rank-r paving matroid on E is an $(r-1)$-partition of E.*

The collection of hyperplanes of a sparse paving matroid M of rank $r \geq 2$ are the circuits of size r together with the independent sets of size $r-1$ not contained in any circuit of size r. Because the hyperplanes of M form an $(r-1)$-partition, any subset A of size $r-1$, that is not a hyperplane, (so A is an independent set contained in some circuit of size r) is contained in a unique circuit of size r.

Any Steiner system $S(r-1, r, n)$ corresponds to a sparse paving matroid by taking the bases to be all sets of size r not appearing as blocks of the Steiner system. As the number of blocks in a $S(r-1, r, n)$ is $\frac{1}{r}\binom{n}{r-1}$, we see that the number of bases of the corresponding sparse paving matroid is $\binom{n}{r} - \frac{1}{r}\binom{n}{r-1} = \frac{n-r}{r}\binom{n}{r-1}$. The next results show that this is a lower bound for the number of bases of a sparse paving matroid. Because the Steiner systems $S(2, 3, 6p+1)$, $S(2, 3, 6p+3)$ (see [17]) and $S(3, 4, 6p+2)$, $S(3, 4, 6p+4)$ (see [14]) exist for all p, there is an infinite number of matroids that achieve our bound.

The following lemma is straightforward.

Lemma 4.7. *If M is a sparse paving matroid with rank 1 and n elements, then M is isomorphic to either $U_{1,n}$ or $U_{1,n-1} \oplus U_{0,1}$.*

Theorem 4.8. *Let M be a rank-r matroid with n elements and $r \geq 1$. If M is a sparse paving matroid then it has at least $\frac{n-r}{r}\binom{n}{r-1}$ bases.*

Proof. If $r = 1$, M is isomorphic to either $U_{1,n}$ or $U_{1,n-1} \oplus U_{0,1}$ by Lemma 4.7. Both of these matroids have at least $n-1$ bases.

Let us suppose that $r \geq 2$. Because M is paving, every subset of size $r-1$ is independent and, because it is sparse, the remarks preceding the theorem imply that any set of size $r-1$ is in at most one circuit of size r. Now, form the bipartite graph of bases and independent sets of size $r-1$. That is, the vertices are the independent sets of sizes r or $r-1$ and there is an edge (B, I) if and only if the basis B contains the independent set I. The degree of any independent set I of size $r-1$ is at least $n-r$. So the number of edges in the bipartite graph is at least $(n-r)\binom{n}{r-1}$. As the degree of any basis in this graph is r, the result follows. $\square$

Many invariants that are usually difficult to compute for a general matroid are easy for sparse paving matroids. For example, observe that if M is sparse paving, all subsets of size $k<r$ are independent, and all subsets of size $k>r$ are spanning. On the other hand the subsets of size r are either bases or circuit–hyperplanes. Thus, the Tutte polynomial of a rank-r sparse matroid M with n elements and λ circuit–hyperplanes is given by

$$T_M(x,y)=\sum_{i=0}^{r-1}\binom{n}{i}(x-1)^{r-i}+\binom{n}{r}+\lambda(xy-x-y)+\sum_{i=r+1}^{n}\binom{n}{i}(y-1)^{i-r}.$$

5 Bounds for number of bases of paving matroids and sizes of multicomplexes

In the previous section we gave a tight lower bound for the number of bases of a sparse paving matroid. Such a lower bound is more difficult to obtain in the case of paving matroids. One lower bound for the number of bases of a loopless, coloopless paving matroid of rank r with n elements is given by $f(r,n-r)+\binom{n-1}{r-1}$. In this section we investigate further the function f and give an alternative proof of Corollary 3.8.

5.1 The function $\mathbf{f(r,d)}$

We define two families of graphs. First, we define the graph $G_{r,d}$ to have one vertex corresponding to each monomial of degree r over d variables and to have an edge $\{m,m'\}$ if and only if there exist distinct variables x and y such that $m'=\frac{m}{x}y$. The second family is similar. We define $TG_{r,d}$ to have one vertex for each monomial of degree at most r over d variables and to have an edge $\{m,m'\}$ if and only if there exist different variables x and y such that $m'=\frac{m}{y}x$ or there exists a variable y such that $m'=\frac{m}{y}$.

Clearly, $G_{i,d}$ is an induced subgraph of $TG_{r,d}$ for all $0\le i\le r$. Recall that a set U of vertices *dominates* a set U' of vertices in a graph if every vertex in $U'\setminus U$ is adjacent to a vertex in U. The problem of finding $f(r,d)$ can be translated to the problem of finding the vertex subset of $G_{r,d}$ of minimum size that dominates the vertex set of $G_{r-1,d}$.

For this purpose we define the *standard colouring* ϱ_d of $G_{r,d}$. Let us suppose the d variables are $\{x_0,\dots,x_{d-1}\}$. To each variable x_i we associate the colour $\varrho_d(x_i)=i \bmod d$ and then we extend this colouring linearly to all monomials, that is, for a monomial $m = x_0^{t_0} \cdots x_{d-1}^{t_{d-1}}$, the value of $\varrho_d(m)$ is $0\,t_0 + \ldots + (d-1)t_{d-1} \bmod d$.

Lemma 5.1. *The standard colouring ϱ_d is a proper colouring and $\chi(G_{r,d}) \leq d$.*

Proof. If (m, m') is an edge of $G_{r,d}$, then there exist $i \neq j$ such that $m' = \frac{m}{x_i} x_j$. So, $\varrho_d(m) - \varrho_d(m') = i - j \not\equiv 0 \pmod{d}$. Thus, m and m' receive different colours and ϱ_d is a proper d-colouring of $G_{r,d}$. □

Proposition 5.2. *The chromatic number $\chi(G_{r,d})$ equals the clique number $\omega(G_{r,d})$ and both equal d.*

Proof. From the previous lemma we know that $\chi(G_{r,d}) \leq d$. Clearly the vertices in $\{x_0^r, x_0^{r-1}x_1, \ldots, x_0^{r-1}x_{d-1}\}$ form a clique. Thus $\omega(G_{r,d}) \geq d$. But for any graph G we have $\omega(G) \leq \chi(G)$ and the result follows. □

Observe that in the previous proof we show that x_0^r is in a d-clique. Actually, any monomial in $G_{r,d}$ is in as many cliques of size d as the number of different variables in the monomial. That is, if $x_i | m$, then the vertices $\{\frac{m}{x_i}x_0, \ldots, \frac{m}{x_i}x_{d-1}\}$ form a clique. Thus, any colour class of a d-colouring of $G_{r,d}$ dominates $V(G_{r,d})$. So any colour class of a d-colouring of $G_{r,d}$ is a dominating independent set and thus, it is a maximal independent set and a minimal dominating set.

Another important observation is that a colour class of a d-colouring of $G_{r,d}$ dominates the vertex subset $V(G_{r-1,d})$ in $TG_{r,d}$. This is because the neighbours of a monomial m of rank $r-1$ in $V(G_{r,d})$ are $\{m\,x_0, \ldots, m\,x_{d-1}\}$ and form a d-clique. So they must intersect each colour class of a d-colouring of $G_{r,d}$.

We now define the function $\overline{f}(r,d)$ to be the minimum size of a chromatic class in the standard coloring ϱ_d of $G_{r,d}$. The previous paragraph proves the following.

Proposition 5.3. *For all $r \leq d$, we have $f(r,d) \leq \overline{f}(r,d)$.*

Now, it is easy to give an upper bound for $f(r,d)$.

Proposition 5.4. *For all $r \leq d$, we have $f(r,d) \leq \binom{r+d-1}{d-1}/d$.*

Proof. $\binom{r+d-1}{d-1}/d$ is the average size of a colour class in a d colouring of $G_{r,d}$. □

While trying to find a formula for $\overline{f}(r,d)$, our computations appeared to point to the number of aperiodic necklaces with r black beads and d white beads, also known as the number of binary Lyndon words of length $r+d$ and density r. *Binary necklaces* or *necklaces of beads with colours black and white* are circular sequences of 0's and 1's, where two sequences obtained by a rotation are considered the same. That is, the necklaces of length n are the orbits of the action of the cyclic group C_n on circular sequences of 0's and 1's of length n. A necklace of length n is called *aperiodic* if the orbit has size n.

The number of aperiodic necklaces with n beads, r black and d white, is

$$L_2(r,d) = \frac{1}{r+d} \sum_{k|(r+d,r)} \mu(k) \binom{(r+d)/k}{r/k},$$

where (a,b) denotes the greatest common divisor of the integers a and b and μ is the classical Möbius function. This formula is well known and is a typical example of the Möbius inversion formula, see [19]. In particular, note that when d and r are coprimes, the formula simplifies to $\binom{n}{r}/n = \binom{n-1}{r-1}/r = \binom{n-1}{d-1}/d$.

Theorem 5.5. *If n and r are coprime, then $\overline{f}(r,n-r)$ equals the number of aperiodic necklaces of n beads, r black and $d=n-r$ white.*

Proof. Consider φ the action of the cyclic group C_d over $G_{r,d}$ given by

$$\varphi(x_0^{t_0} \cdots x_{d-1}^{t_{d-1}}) = x_1^{t_0} \cdots x_{d-1}^{t_{d-2}} x_0^{t_{d-1}}.$$

The orbits of this action correspond to necklaces with r black beads and d white beads. Variables correspond to white beads and to the right of the black bead corresponding to x_i we place as many black beads as the exponent of x_i, for $0 \le i \le d-1$. The orbits of size d correspond to aperiodic necklaces.

Let us see the effect of φ on the standard colouring ϱ, that is, we want to find $\varrho(\varphi(m))$ for a monomial $m = x_0^{t_0} \cdots x_{d-1}^{t_{d-1}}$. We have $\varrho(\varphi(m)) - \varrho(m) \equiv r \mod d$. Thus, every orbit has size d and all the monomials in the orbit have different colours. We conclude that in this case the number of aperiodic necklaces equals the common size of any colour class in the standard colouring ϱ of $G_{r,d}$. $\square$

Conjecture 5.6. $f(r,d) = \overline{f}(r,d) = L_2(r,d)$.

Notice that if I' is a set of monomials of size $f(r,d)$ which dominates the vertices in $G_{r-1,d}$, it is a dominating set in $G_{r,d}$. This is because, if $m \in V(G_{r,d})$, then for some x_i the monomial $m' = m/x_i$ is in $V(G_{r-1,d})$. But the set of neighbours of m' in $V(G_{p,d})$ is $S = \{m' x_0, \ldots, m' x_{d-1}\}$ and $m \in S$. As an element m'' of I' has to be in S and S induces a complete graph, we conclude that m'' and m are adjacent.

5.2 An alternative proof of Theorem 3.7

For a connected paving matroid we can use the Brown-Colbourn Theorem 2.1 mentioned earlier to bound h_r for $r \geq 1$ from below by

$$S(r,n) = (-1)^{r-1} \sum_{i=0}^{r-1} (-1)^i \binom{n-r+i-1}{i}.$$

A few values of S are given by the following, where we adopt the usual convention that $\binom{a}{0} = 1$ for all a and $\binom{a}{b} = 0$ for all integers a and b such that $b > a$ and $b > 0$.

Proposition 5.7.

- $S(1,n) = 1$ *for all* $n \geq 1$.
- $S(2,n) = n - 3$ *for all* $n \geq 2$.
- $S(n,n) = (-1)^{n-1}$ *for all* $n \geq 1$.
- $S(n-1,n) = n - 1 \bmod 2$ *for* $n \geq 2$.
- $S(n-2,n) = \lfloor \frac{n-1}{2} \rfloor$

Proof.

- $S(1,n) = (-1)^0 \binom{n-2}{0} = 1$ for all $n \geq 1$.
- $S(2,n) = (-1)(\binom{n-3}{0} - \binom{n-2}{1}) = n - 3$ for all $n \geq 2$.
- $S(n,n) = (-1)^{n-1} \sum_{i=0}^{n-1} (-1)^i \binom{i-1}{i} = (-1)^{n-1}$ for all $n \geq 1$.
- $S(n-1,n) = (-1)^{n-2} \sum_{i=0}^{n-2} (-1)^i \binom{i}{i} = (-1)^{n-2} \sum_{i=0}^{n-2} (-1)^i$, that is 1 if n is even and 0 otherwise.
- $S(n-2,n) = (-1)^{n-3} \sum_{i=0}^{n-3} (-1)^i \binom{i+1}{i} = (-1)^{n-3} \sum_{i=0}^{n-3} (-1)^i (i+1)$. This is $(-1)^{n-3}(-(n-1)/2) = (n-2)/2$ if n is even and $(n-1)/2$ if n is odd.

$\square$

The sequence $\{S(r,n)\}$ has a similar recursion to the binomial coefficients.

Theorem 5.8. *For $n > r > 0$*

$$S(r+1, n+1) = S(r+1, n) + S(r, n)$$

Proof. This follows directly from Pascal–Stifel's formula $\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r}$. $\square$

This result is enough to show that the integer sequence $\{S(r,n)\}$ is sequence A108561, in [27], where in the notation used there $T(n,r) = S(r+1, n+1)$. Both sequences satisfy the same recurrence and the same boundary conditions.

How does $S(r,n)$ compare with $f(r, n-r)$? We can prove the following.

Theorem 5.9. *If $1 \leq r \leq n-2$ we have $f(r, n-r) \leq S(r,n)$*

Proof. We prove the theorem by induction on n. For $r = 1$, $f(n, n-r) = 1 = S(r,n)$ for all n. For $r = n-2$, we have $f(r, n-r) = \lceil \frac{n-2}{2} \rceil = \lfloor \frac{n-1}{2} \rfloor = S(r,n)$. If $1 < r < n-2$, then using Lemma 3.3, we have $f(r, n-r) \leq f(r, n-r-1) + f(r-1, n-r)$. Using induction this is at most $S(r, n-1) + S(r-1, n-1)$ which equals $S(r,n)$. $\square$

A coloopless paving matroid that is not connected must have rank at most one, so the previous result implies that $f(r, n-r) \leq g(r,n)$ whenever $2 \leq r \leq n-2$. Thus, we have an alternative proof of Corollary 3.8, because it is easy to check the inequality for the remaining values of r and n.

6 Conclusion

We have proved Stanley's conjecture for paving matroids. This adds another case to the stream of results that prove the conjecture for a particular family of matroids [23, 24, 26, 30]. Hopefully, all this work will lead to a proof of this conjecture, or at least to a better understanding of the h-vector of a matroid.

The problem of giving good lower bounds on the number of bases of a loopless, coloopless paving matroid appears to be a challenging but interesting problem. The lower bound found by computing $f(r, n-r) + \binom{n-1}{r-1}$ is not tight in most cases. When $n - r = 2$ and n is even, we get a lower bound of $n(n-2)/2$ which is achieved by the dual matroid of the 2-thickening of $U_{2,m}$; when n is odd we get a lower bound of $(n-1)^2/2$ which is achieved by the dual matroid of the free extension of the 2-thickening of $U_{2,m}$. So, in this case, the lower bound is tight. But when $n-r=3$ the situation is quite different. When the rank is 2, the bound gives 6 bases, which is achieved by the paving matroid $U_{1,3} \oplus U_{1,2}$. When the rank is 3 the lower bound gives 13. There are 8 coloopless paving matroids with 6 elements and rank 3, yet the minimum number of bases is 15. Even if we use the alternative lower bound $S(r,n) + \binom{n-1}{r-1}$ we only get a lower bound of 14 in this case.

The function f is very intriguing and seems very difficult to compute from the definition. We can prove that $f(r,d) = \overline{f}(r,d)$ for $d = 1,2,3$ and all $r \geq 1$ and also for $d = 4$ for $1 \leq r \leq 6$. We have checked that $\overline{f}(r,d) = L_2(r,d)$ for many small values of r and d with $(r,d) > 1$ by using Maple. Conjecture 5.6 would imply, for example, that $f(r,d) = f(d,r)$ which geometrically is not so easy to see and we have been unable to prove it.

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