Elementary solution to the time-independent quantum navigation problem

Dorje C. Brody[1,2] **and David M. Meier**[1]
[1]Department of Mathematics, Brunel University, Uxbridge UB8 3PH, UK
[2]St Petersburg National University of Information Technologies, Mechanics and Optics, Kronwerkskii ave 49, St Petersburg 197101, Russia

Abstract. A quantum navigation problem concerns the identification of a time-optimal Hamiltonian that realises a required quantum process or task, under the influence of a prevailing 'background' Hamiltonian that cannot be manipulated. When the task is to transform one quantum state into another, finding the solution in closed form to the problem is nontrivial even in the case of time-independent Hamiltonians. An elementary solution, based on trigonometric analysis, is found here when the Hilbert space dimension is two. Difficulties arising from generalisations to higher-dimensional systems are discussed.

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Motivated in part by the advances in quantum technologies, significant progress has been made in finding the time-optimal scheme to implement a unitary operation that achieves the transformation of one quantum state into another, subject to a given set of constraints [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. In practical situations such as in nuclear magnetic resonance or more generally in atomic physics, which are relevant for example to applications in quantum computation, it is typically assumed that external influences such as a background field or potential are absent. The consideration of time-optimal control in the presence of external fields, however, is important since for many experimental applications it can be disproportionately expensive to completely eliminate external influences. In such a context, Russell & Stepney [15] considered a time-minimisation problem of transporting one unitary operator $\hat{U}_I$ into another operator $\hat{U}_F$, subject to the existence of a background Hamiltonian $\hat{H}_0$ that cannot be manipulated. The task here therefore is to find the (time-dependent) control Hamiltonian $\hat{H}_1(t)$ such that $\hat{H} = \hat{H}_0 + \hat{H}_1$ transforms $\hat{U}_I$ into $\hat{U}_F$ in the shortest possible time. Evidently, there has to be a bound on the energy resource, which in their problem is given by the trace norm of the control—the 'full throttle' condition: $\mathrm{tr}(\hat{H}_1^2) = 1$ at all time. In addition, to ensure the existence of viable controls it is assumed that the background Hamiltonian is not dominant, i.e.

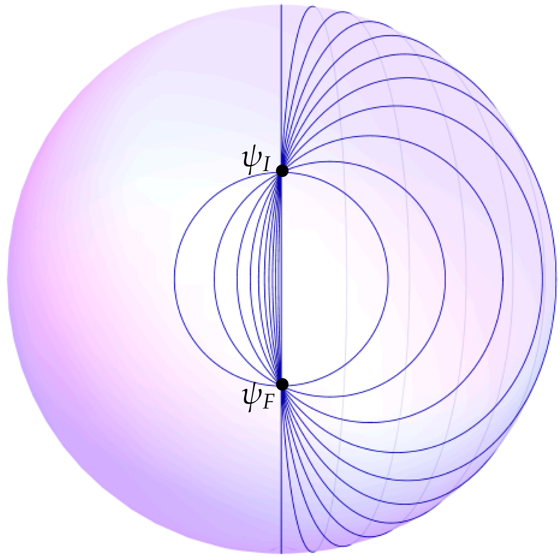

Figure 1. *Circles on the sphere passing through a pair of points.* It is evident that the totality of latitudinal circles passing through the two points $|\psi_I\rangle$ and $|\psi_F\rangle$ share the property that the axes perpendicular to the circles lie on the plane that bisects all the circles.

$\mathrm{tr}(\hat{H}_0^2) < 1$. Inspired by the classical problem of navigation in the ocean in the presence of wind or currents [16, 17], this is referred to as the quantum Zermelo navigation problem [15]. The solution to this problem of constructing a unitary gate under an external field was obtained recently [18, 19]. The more-challenging problem of finding the time-optimal transformation $|\psi_I\rangle \to |\psi_F\rangle$ of quantum states under a similar setup has subsequently been solved [20].

In the present paper we investigate an analogous problem of finding the time-optimal control Hamiltonian $\hat{H}_1$ that achieves the transformation $|\psi_I\rangle \to |\psi_F\rangle$, subject to the existence of an ambient Hamiltonian $\hat{H}_0$, but in the time-independent context. It turns out that when $\hat{H}_1$ cannot vary in time, then the problem of finding the 'time-optimal' control that generates a given unitary gate $\hat{U}_I \to \hat{U}_F$ becomes trivial (shown below), while that of finding the optimal control to generate the transformation $|\psi_I\rangle \to |\psi_F\rangle$ remains nontrivial. Nevertheless, in the case of two-level systems, on account of the fact that the configuration of the states can be visualised on a Bloch sphere, we are able to derive an elementary solution that requires nothing more than trigonometric manipulations. Our solution in fact extends to higher dimensions if the background Hamiltonian $\hat{H}_0$ ('wind') happens to leave invariant the Hilbert subspace spanned by the given two states $|\psi_I\rangle$ and $|\psi_F\rangle$; whereas the solution to the more general cases in higher dimensions remains open.

We begin our analysis by remarking that if a time-independent Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$ were to transform $|\psi_I\rangle$ into $|\psi_F\rangle$ in a two-dimensional Hilbert space, then since the action of $\hat{H}$ amounts to a rigid rotation of the associated Bloch sphere about some axis, the two states $|\psi_I\rangle$ and $|\psi_F\rangle$ must lie on the same latitudinal circle with respect to the axis of rotation determined by $\hat{H}$. A set of such circles is sketched in figure 1. Therefore, the totality of rotation axes permitting such transformations lie on the great circle passing the point $\frac{1}{\sqrt{2}}(|\psi_I\rangle + |\psi_F\rangle)$ that is orthogonal to the great circle joining the two points on the Bloch sphere corresponding to the states $|\psi_I\rangle$ and $|\psi_F\rangle$. Without loss of generality, let us work in the frame such that the two states can be expressed in the form

$$|\psi_I\rangle = \begin{pmatrix} \cos\frac{1}{4}(\pi-\theta) \\ \\ \sin\frac{1}{4}(\pi-\theta) \end{pmatrix}, \qquad |\psi_F\rangle = \begin{pmatrix} \cos\frac{1}{4}(\pi+\theta) \\ \\ \sin\frac{1}{4}(\pi+\theta) \end{pmatrix}, \tag{1}$$

where θ is the angular separation of the two states $|\psi_I\rangle$ and $|\psi_F\rangle$. In other words, we work with the coordinates such that both the initial and the target states lie on a longitudinal great circle, and such that the equator bisects the join of $|\psi_I\rangle$ and $|\psi_F\rangle$. By embedding the Bloch sphere in $\mathbb{R}^3$ we then find that the two points on the sphere corresponding to the vectors $|\psi_I\rangle$ and $|\psi_F\rangle$ lie on the xz-plane, located symmetrically about the xy-plane. Writing $\boldsymbol{\psi}_I$ and $\boldsymbol{\psi}_F$ for the two vectors in $\mathbb{R}^3$ corresponding to the two states, we thus have

$$\boldsymbol{\psi}_I = \frac{1}{2}\begin{pmatrix} \cos\frac{\theta}{2} \\ 0 \\ \sin\frac{\theta}{2} \end{pmatrix}, \qquad \boldsymbol{\psi}_F = \frac{1}{2}\begin{pmatrix} \cos\frac{\theta}{2} \\ 0 \\ -\sin\frac{\theta}{2} \end{pmatrix}, \tag{2}$$

since the radius of the Bloch sphere is $\frac{1}{2}$. This configuration is schematically illustrated in figure 2.

With the above choice of coordinates it should be evident that any rotation of the sphere about an axis that lies on the xy-plane will in time transport $|\psi_I\rangle$ into $|\psi_F\rangle$. Conversely, no rotation about an axis that does not lie on the xy-plane will ever transport $|\psi_I\rangle$ into $|\psi_F\rangle$. In the absence of the background 'wind' $\hat{H}_0$, therefore, if the objective is to minimise the time subject to finite energy resource, then since the voyage time is the distance divided by speed, *a priori* one has to deal with a complicated optimisation problem of minimising this ratio. (In the present context the distance is measured with respect to the Fubini-Study metric, whereas the speed is determined in accordance with the Anandan-Aharonov relation [21].) Fortunately, in the case of a unitary evolution, the path that minimises the distance is precisely the path that maximises the evolution speed [4], so there is no need to evoke a simultaneous optimisation; all one needs is to find the shortest path. But geodesic curves on a sphere are given by the great circles, so without any calculation it is evident that the optimal Hamiltonian is given by the one corresponding to a rotation about the y-axis [7].

In the presence of a background wind $\hat{H}_0$, however, the situation is different: In this case, depending on the choice of $\hat{H}$, that is, the choice of the rotation axis on the xy-plane, the energy resource available to the Hamiltonian $\hat{H}$ is different. As a

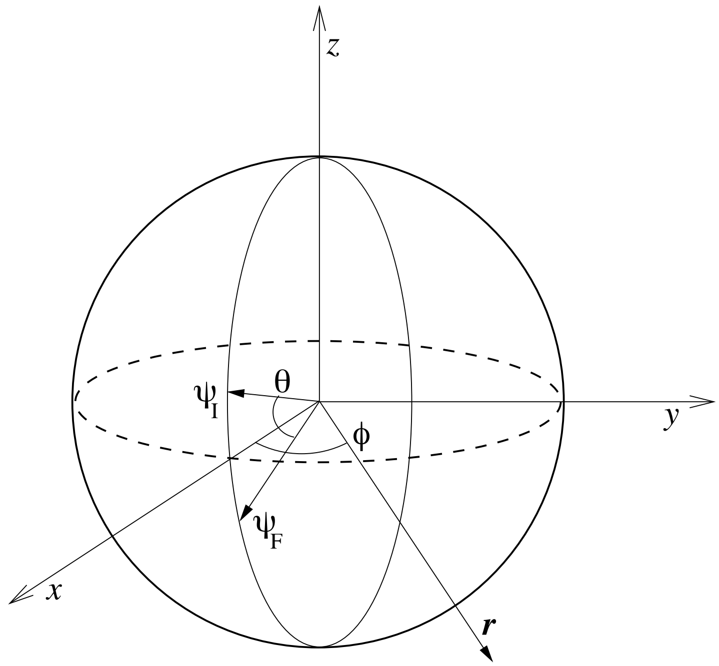

Figure 2. *Configuration of the initial and target states.* Since the action of a time-independent Hamiltonian $\hat{H} = \boldsymbol{r} \cdot \hat{\boldsymbol{\sigma}}$ amounts to a rigid rotation of the Bloch sphere, if the unitary operator $\mathrm{e}^{-\mathrm{i}\hat{H}t}$ were to transform a given initial state $|\psi_I\rangle$ into a target final state $|\psi_F\rangle$ at some time, the axis $\boldsymbol{r}$ of rotation has to lie on the equator that bisects the great circle joining $|\psi_I\rangle$ and $|\psi_F\rangle$. When $\boldsymbol{r}$ points in the direction of the y-axis, the orbit $|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\hat{H}t}|\psi_I\rangle$ is a geodesic curve, but owing to the prevailing 'wind' $\hat{H}_0$, the journey along the shortest path does not result in the shortest time.

consequence, one can find a Hamiltonian $\hat{H}$ such that although the path $|\psi(t)\rangle$ joining $|\psi_I\rangle$ and $|\psi_F\rangle$ is not the shortest, there is sufficient energy resource to overcome the extra mileage such that the voyage time will be shorter than that corresponding to the rotation about the y-axis. The objective, therefore, is to find the axis for which the voyage time is minimised.

With these observations at hand, let us write the background Hamiltonian $\hat{H}_0$ in the form

$$\hat{H}_0 = \sqrt{\frac{\epsilon}{2}}(x\hat{\sigma}_x + y\hat{\sigma}_y + z\hat{\sigma}_z), \tag{3}$$

where $x^2 + y^2 + z^2 = 1$ and where $0 < \epsilon < 1$. It follows that $\mathrm{tr}(\hat{H}_0^2) = \epsilon < 1$. Whatever the control Hamiltonian $\hat{H}_1$ might be, the total Hamiltonian has to take the form

$$\hat{H} = \frac{\omega}{2}(\cos\phi\,\hat{\sigma}_x + \sin\phi\,\hat{\sigma}_y) \tag{4}$$

for some ω satisfying the constraint. In other words, the axis of rotation generated by $\hat{H}$ is at some angle ϕ from the x-axis on the xy-plane. Since $\hat{H}_1 = \hat{H} - \hat{H}_0$, the constraint $\mathrm{tr}(\hat{H}_1^2) = 1$ on the control Hamiltonian implies that

$$\omega^2 - 2\sqrt{2\epsilon}(x\cos\phi + y\sin\phi)\omega - 2(1-\epsilon) = 0. \tag{5}$$

We shall find that the voyage time τ such that the condition $\mathrm{e}^{-\mathrm{i}\hat{H}\tau}|\psi_I\rangle = |\psi_F\rangle$ is met will also depend on the variables ω and ϕ. Thus, our objective is to minimise τ subject to the constraint (5).

It should be remarked parenthetically that we have chosen both $\hat{H}_0$ and $\hat{H}_1$ to be trace free. This is because a physically meaningful constraint on the energy resource, in the case of a quantum system modelled on a finite-dimensional Hilbert space, is linked to the gap between the highest and the lowest attainable energy eigenvalues, not to the value of the ground-state energy [4]. We shall therefore be working, without loss of generality, with trace-free Hamiltonians. Observe also that the no-dominance condition $\epsilon < 1$ for the wind can in fact be relaxed to a weaker condition

$$\epsilon \leq \max_{\phi} \frac{1}{1 - (x\cos\phi + y\sin\phi)^2} \tag{6}$$

so that (5) admits a real root for ω.

To proceed, it should be evident from the foregoing formulation that the voyage time is proportional to the angle, call it α, of rotation about the $\hat{H}$-axis that turns the vector $\boldsymbol{\psi}_I$ into $\boldsymbol{\psi}_F$ in $\mathbb{R}^3$. Specifically, since the angular frequency generated by the Hamiltonian $\hat{H}$ of (4) is ω, this in turn determines the voyage time according to the relation $\alpha = \omega t$. It follows that the problem reduces to working out elementary trigonometric relations. Let us define the vector $\boldsymbol{r}$ by

$$\boldsymbol{r} = \frac{\omega}{2}\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \tag{7}$$

so that $\hat{H} = \boldsymbol{r}\cdot\hat{\boldsymbol{\sigma}}$. Thus $\boldsymbol{r}$ determines the axis of rotation in $\mathbb{R}^3$ generated by $\hat{H}$.

To determine α, let us first identify the angular separation ρ between $\boldsymbol{r}$ and $\boldsymbol{\psi}_I$ (which, of course, is the same as that between $\boldsymbol{r}$ and $\boldsymbol{\psi}_F$ on account of the symmetry). To assist the analysis, in figure 3 we give the perspective of the configuration around the $\boldsymbol{r}$-axis. Since $\boldsymbol{r}\cdot\boldsymbol{\psi}_I = |\boldsymbol{r}|\,|\boldsymbol{\psi}_I|\cos\rho$, we find

$$\cos\rho = \cos\phi\cos\tfrac{1}{2}\theta. \tag{8}$$

The final step required is to identify the vector $\boldsymbol{c}$ depicted in figure 3 that points in the direction of $\boldsymbol{r}$ such that the two points $\boldsymbol{\psi}_I$ and $\boldsymbol{\psi}_F$ lie on the plane perpendicular to $\boldsymbol{r}$ at $\boldsymbol{c}$. But clearly this is given by

$$\boldsymbol{c} = \tfrac{1}{2}\cos\rho\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}, \tag{9}$$

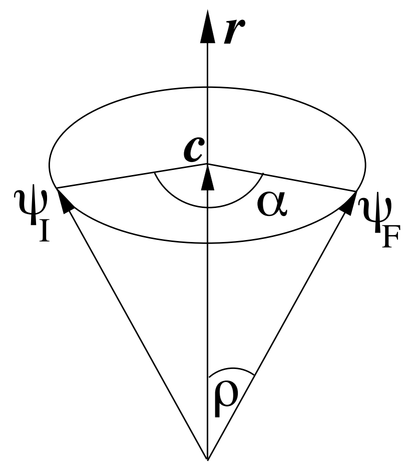

Figure 3. *Identification of the rotation angle.* The angle α of rotation about the $\boldsymbol{r}$-axis required to turn the vector $\boldsymbol{\psi}_I$ into $\boldsymbol{\psi}_F$ is determined by first identifying the vector $\boldsymbol{c}$ such that $\boldsymbol{\psi}_I - \boldsymbol{c} \perp \boldsymbol{r}$. Then we have $(\boldsymbol{\psi}_I - \boldsymbol{c}) \cdot (\boldsymbol{\psi}_F - \boldsymbol{c}) \propto \cos\alpha$, from which α can be obtained.

from which it follows, after some algebra, that

$$\cos\alpha = \frac{\boldsymbol{\psi}_I - \boldsymbol{c}}{|\boldsymbol{\psi}_I - \boldsymbol{c}|} \cdot \frac{\boldsymbol{\psi}_F - \boldsymbol{c}}{|\boldsymbol{\psi}_F - \boldsymbol{c}|} = \frac{\sin^2\phi - \tan^2\frac{\theta}{2}}{\sin^2\phi + \tan^2\frac{\theta}{2}}. \tag{10}$$

As indicated above, since the angular frequency is ω, the first time τ at which the state $|\psi_I\rangle$ is turned into $|\psi_F\rangle$ is given by $\omega\tau = \alpha$, that is,

$$\tau = \frac{1}{\omega}\cos^{-1}\left(\frac{\sin^2\phi - \tan^2\frac{\theta}{2}}{\sin^2\phi + \tan^2\frac{\theta}{2}}\right). \tag{11}$$

On the other hand, the constraint (5) allows us to express ω in terms of ϕ. Putting these together, the first voyage time $\tau = \tau(\phi)$ can be expressed explicitly as a function of the angle ϕ that determines the axis of rotation generated by $\hat{H}$, which in turn determines $\hat{H}_1$. Specifically, we have

$$\omega = \sqrt{2\epsilon(x\cos\phi + y\sin\phi)^2 + 2(1-\epsilon)} + \sqrt{2\epsilon}\,(x\cos\phi + y\sin\phi), \tag{12}$$

which together with (11) gives $\tau(\phi)$, and this in turn must be minimised for fixed x, y, ϵ and θ.

In figure 4 we plot $\tau(\phi)$ as a function of ϕ for a choice of parameters x, y, ϵ and θ. Since the problem is reduced to a one-dimensional minimisation task, the optimal value ϕ^* for the axis of rotation can easily be determined numerically, which, when substituted in (12) and in (4), identifies the optimal overall Hamiltonian $\hat{H}(\phi^*)$, from

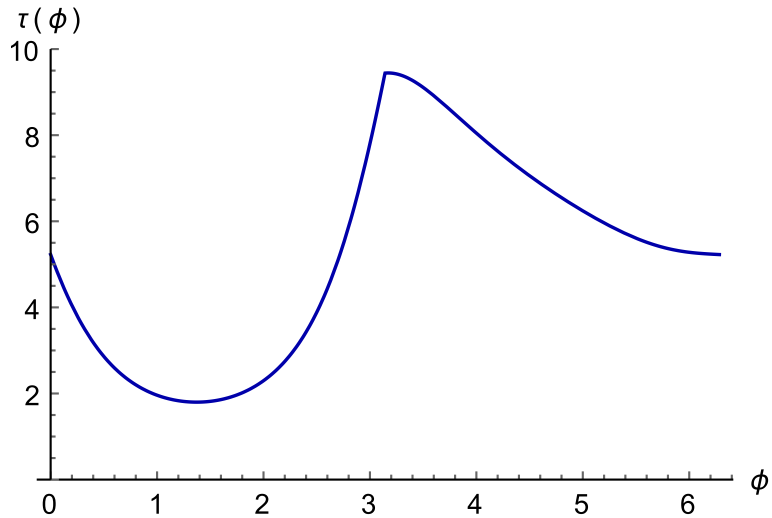

Figure 4. *Voyage time $\tau(\phi)$ as a function of the angle ϕ.* As the axis of rotation generated by $\hat{H}$, parameterised by ϕ, is varied, the voyage time changes accordingly. Here, as an example, we plot $\tau(\phi)$ for the parameter choice: $x = 0.1$, $y = 0.23$, $z = (1 - x^2 - y^2)^{\frac{1}{2}} \approx 0.97$, $\epsilon = 0.9$, and $\theta = \pi/2$. In this example, the optimal angle is $\phi^* \approx 0.44\,\pi$. At $\phi = \pi$ there is a cusp (irrespective of the parameter values), since the orientation of the path $|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\hat{H}t}|\psi_I\rangle$ changes as ϕ passes through π.

which the optimal control can be determined by the relation $\hat{H}_1(\phi^*) = \hat{H}(\phi^*) - \hat{H}_0$. As for the time required to achieve the transformation, this is given by $\tau(\phi^*)$. In the special case for which the states $|\psi_I\rangle$ and $|\psi_F\rangle$ are orthogonal, the calculation simplifies and we obtain the expression

$$\tau(\phi^*) = \frac{\pi}{\sqrt{2(1-\epsilon z^2)} + \sqrt{2\epsilon(1-z^2)}} \tag{13}$$

for the shortest time to achieve the required transformation. The expression (13) can be viewed as the 'windy' generalisation of the passage time to turn a state into another state that is orthogonal to it (cf. [4]). This completes our analysis of finding the time-optimal Hamiltonian that generates the transformation $|\psi_I\rangle \rightarrow |\psi_F\rangle$ of quantum states, subject to the existence of a prevailing 'wind' $\hat{H}_0$.

We conclude by remarking on the generalisation to higher dimensions, as well as on the problem of optimally generating a given unitary gate. In either case, in the time-independent context the evolution operator is given by $\mathrm{e}^{-\mathrm{i}\hat{H}T}$ for some Hamiltonian $\hat{H}$ and voyage time T. Thus the task is to find the best choice of $\hat{H}$ that minimises T such that either

$$\mathrm{e}^{-\mathrm{i}\hat{H}T}|\psi_I\rangle = |\psi_F\rangle \tag{14}$$

or

$$\mathrm{e}^{-\mathrm{i}\hat{H}T}\hat{U}_I = \hat{U}_F, \tag{15}$$

is realised, depending on which problem one is considering. Now for a system modelled on an n-dimensional Hilbert space, the space of pure states is the associated projective Hilbert space of $n-1$ complex dimensions. Thus, the specification of a state requires the specification of $2n-2$ degrees of freedom. On the other hand, the specification of a Hamiltonian, up to trace, requires n^2-1 degrees of freedom. Together with the fact that T is also unknown, we have, in (14), n^2 unknowns; while, noting that there is also the trace-norm condition, there are $(2n-2)+1 = 2n-1$ conditions. It follows that the solution to the problem of the type (14) involves an optimisation over $n^2-(2n-1) = (n-1)^2$ parameters, which in general is nontrivial. For $n = 2$, this reduces to a single-parameter optimisation, and an explicit representation of τ in terms of trigonometric functions can be found, as shown above. For $n > 2$, our solution remains valid if $\hat{H}_0$ leaves invariant the two-dimensional Hilbert subspace spanned by $|\psi_I\rangle$ and $|\psi_F\rangle$, on account of the observation made in [7]; whereas in the general case, the voyage time τ will depend on $(n-1)^2$ parameters, hence a numerical search in a higher-dimensional parameter space is required to identify the optimal τ^*. It remains open whether a similarly simple analytical form of τ can be found in higher dimensions. In any event, the problem of the kind represented in (14) is in general nontrivial, even in the time-independent context.

As for the construction of a unitary gate of the kind sketched in (15), on the other hand, the situation is markedly different. Here the number of unknowns remains the same, but the number of constraints in (15), together with the trace condition, completely counterbalances the number of unknowns (recall that while (14) is a vector relation, (15) is a matrix relation), and there is no degree of freedom left to optimise. That is, the optimal Hamiltonian is given exactly by $\hat{H}^* = \mathrm{i}T^{-1}\ln(\hat{U}_F\hat{U}_I^{-1})$, where T is fixed by the trace-norm condition. Specifically, writing $\hat{X} = \mathrm{i}\ln(\hat{U}_F\hat{U}_I^{-1})$ for simplicity, we have, on account of $\mathrm{tr}(\hat{H}_1^2) = \mathrm{tr}((T^{-1}\hat{X}-\hat{H}_0)^2) = 1$,

$$\frac{1}{T} = \frac{\sqrt{[\mathrm{tr}(\hat{H}_0\hat{X})]^2 + [1-\mathrm{tr}(\hat{H}_0^2)]\mathrm{tr}(\hat{X}^2)} + \mathrm{tr}(\hat{H}_0\hat{X})}{\mathrm{tr}(\hat{X}^2)} \tag{16}$$

for the voyage time required to realise the transformation $\hat{U}_I \to \hat{U}_F$. Thus, in the case of time-independent Hamiltonians, the problem of finding a time-optimal Hamiltonian to generate a unitary gate, under the influence of a background Hamiltonian $\hat{H}_0$, is empty—only with a time-dependent control $\hat{H}_1(t)$ the 'bound' in (16) can be overcome [18, 19].

In summary, we have obtained an explicit representation for the solution to the time-independent navigation problem for quantum states in the case of two-level systems. In higher dimensions the problem does not appear to offer such explicit representations for the solution, although we have illustrated how the problem can be analysed numerically. In relation to the time-optimal construction of a unitary gate, we have shown that the problem can be solved instantly in arbitrary dimensions when the Hamiltonian is restricted to be time independent. It would be of interest to extend the analysis to the case for which there is additional external noise. Such considerations amount to the Zermelo-extension of the optimal control in noisy environments (cf. [22, 23]).

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