A Behavioural Approach to Financial Portfolio Selection Problem: an Empirical Study Using Heuristics

A thesis submitted for the degree of
Doctor of Philosophy

by

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Declaration

I certify that this thesis submitted for the degree of Doctor of Philosophy in Mathematical Science is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed: ________________________________

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"Nothing in life is quite as important as you think it is while you're thinking about it."

Daniel Kahneman

Abstract

The behaviourally based portfolio selection problem with investor's loss aversion and risk aversion biases in portfolio choice under uncertainty are studied. The main results of this work are developed heuristic approaches for the prospect theory and cumulative prospect theory models proposed by Kahneman and Tversky in 1979 and 1992 as well as an empirical comparative analysis of these models and the traditional mean variance and index tracking models. The crucial assumption is that behavioural features of the (cumulative) prospect theory model provide better downside protection than traditional approaches to the portfolio selection problem.

In this research the large scale computational results for the (cumulative) prospect theory model have been obtained. Previously, as far as we aware, only small laboratory (2-3 artificial assets) tests has been presented in the literature. In order to investigate empirically the performance of the behaviourally based models, a differential evolution algorithm and a genetic algorithm which are capable to deal with large universe of assets have been developed. The specific breeding and mutation as well as normalisation have been implemented in the algorithms. A tabulated comparative analysis of the algorithms' parameter choice is presented.

The performance of the studied models have been tested out-of-sample in different conditions using the bootstrap method as well as simulation of the distribution of a growing market and simulation of the t-distribution with fat tails which characterises the dynamics of a decreasing or crisis market. A cardinality and CVaR constraints have been implemented to the basic mean variance and prospect theory models. The comparative analysis of the empirical results has been made using several criteria such as CPU time, ratio between mean portfolio return and standart deviation, mean portfolio return, standard deviation σ, VaR and CVaR as alternative measures of risk. The strong influence of the reference point, loss aversion and risk aversion on the prospect theory model's results have been found.

The prospect theory model with the reference point being the index is compared to the index tracking model. The portfolio diversification benefit has been found. However, the aggressive behaviour in terms of returns of the prospect theory model with the reference point being the index leads to worse performance of this model in a bearish market compared to the index tracking model. The tabulated comparative analysis of the performance of all studied models is provided in this research for in-sample and out-of-sample tests.

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Chapter 1

Introduction

1.1 Portfolio selection: history and development

The portfolio optimisation problem is a two objective problem. On the one hand, it is a question of how to determine an amount (proportion, weight) of money to invest in each type of asset within the portfolio in order to receive the highest possible return (or utility) by the end of the investment period. While on the other hand, an appropriate level of risk should be achieved together for an acceptable level of return.

Modern Portfolio Theory (MPT) began with a paper [53] and a book [54] written by the Nobel laureate Harry Markowitz. Many researchers consider the emergence of this theory as the birth of modern financial economics (see, for example [70]). The cornerstones of Markowitz's theory are the concepts of return, risk and diversification. It is widely accepted [70] that an investment portfolio is a collection of income-producing assets that have been acquired to meet a financial goal. However, an investment portfolio as a concept did not exist before the late 1950s.

In 1938 John Burr Williams in his book "The Theory of Investment Value" [86] introduced the dividend discount model. The author suggested to solve the investment problem by finding a good stock and buying it at the best price. Many investors followed this advice and investing was perceived as a form of gambling for the rich people.

In 1949 Benjamin Graham wrote the book "The Intelligent Investor" [35], in which he advised that the investors in their decisions should take into account a company's fundamentals, i.e. company shares' real (supported by the value of the company assets) price. The investor's goal then according to Graham's investing philosophy is to find fundamentally good companies' shares at a cheap price. This concept is known as "margin of safety".

Markowitz in 1952 used mean return, variance (as a risk measure of the distribution of returns) and covariance (as a measure of the degree to which returns on two risky assets move in tandem [53]) to derive an efficient frontier where for each optimal portfolio its variance is minimised for a given portfolio expected return (or, inversely, portfolio expected return is maximised for a given variance). Hence, the optimal portfolio can be chosen in accordance with the investor's preferences and their attitude to risk and return.

One of Markowitz biggest contributions to the financial theory is the concept of diversification as a way to reduce risk. Scientific thoughts from previous years encouraged Markowitz and his followers to conceptualise the framework of portfolio selection, and, eventually, led to the solution of the portfolio optimisation problem.

Remarkably, there is a long history behind the Expected Utility Theory (EUT) that started in 1738 when Daniel Bernoulli investigated the St. Petersburg paradox. He was the first scientist who separated the definitions of "price" and "utility" in terms of determining an item's value. Price is an assessment of an item and depends only on the item itself and its characteristics, i.e. price is the objective value. In contrast, utility is subjective and "is dependent on the particular circumstances of the person making the estimate" [15]. EUT follows the assumptions of the neoclassical theory of individual choice in cases when risk appears. It was formally developed by John von Neumann and Oscar Morgenstern in their book "Theory of Games and Economic Behavior"(1944) [57].

The theory's main concern is the representation of individual attitudes towards risk [46]. Since the 1950s, several papers appeared showing that the empirical evidence on individuals' patterns of choice under risk are inconsistent with the expected utility theory, see e.g. [62]. It is also shown [65] that the players' behaviour systematically violates the independence axiom. At the same time the EUT is unable to explain many paradoxes that take place in economic practice (for example, Allais Paradox [6]).

The number of EUT's drawbacks led to the appearance of the Behavioural Portfolio Theory (BPT) – a new fundamental framework which was designed to compensate for the misguidings of the EUT. To date it is the best theory explaining the behaviour of the players and investors in the experiment in decision making under risk. In contrast to EUT, BPT fills in some gaps in explaining controversial economic phenomena, such as Ellsberg Paradox [26].

The recent financial crisis has shown the shortcomings of the individual market instruments and the low level of validity in investment decisions. This can be explained by the dismissive investors' attitude in assessing the real risks, they usually just follow their own intuition. In the investment practice, the situation of unaccounted risks is fairly common, hence, the investors need to have a reliable mathematical tool for justification of investment decisions. In this thesis we consider BPT as a tool which takes into account the behavioural errors.

BPT was developed by Shefrin and Statman in 2000 [74]. The main idea of the theory is the maximisation of the value of the investor's portfolio in which several goals are met and these goals are considered with different levels of risk aversion. BPT is based on two main theories: Security-Potential/Aspiration Theory (SP/A) and Prospect Theory (PT). SP/A theory, established by Lola Lopez in 1987 (see [51]), is a general choice (not only financial) risk framework and not specified for the portfolio selection problem. In our research we focus on the PT [45] devoted to human behaviour in financial decision making under uncertainty.

PT adopts the main idea from the expected utility theory and adds up the vital psychological components, which take into account human behaviour in the decision making process. It also fixes different types of inaccuracies that took place in previously developed behaviour based theories, e.g the independence axiom and the inconsistence assumption of a uniform attitude towards risk, see [74].

As far as we aware, despite many papers devoted to PT, only a few of them have considered its practical application in economics, in particular, in financial markets. It can be explained, according to Barberis [10], by the fact, that PT is not ready to be used as a real economic model.

In this thesis, we apply the PT model to several empirical and experimental data sets in order to find an optimal solution to the portfolio selection problem. We also test the results out-of-sample and compare the PT model's performance with the results obtained in the framework of the Markowitz mean variance model and the index tracking problem.

1.2 Main objectives of the thesis

The goal of this thesis is to identify potential benefits of behaviourally based prospect theory model depending on different market situations in comparison with traditionally accepted portfolio optimisation models. The main objectives according to the main goal are as follows:

- Development of the appropriate solution approaches to prospect theory and its extended version, cumulative prospect theory;
- To identify the optimal solution approach by means of comparative analysis and selection of optimal parameters;
- To investigate the performance of the studied models (prospect theory and cumulative prospect theory models) in comparison with mean variance and index tracking models in different settings:

  - to consider them with a cardinality constraint;
  - to consider them with a CVaR constraint;

- To analyse the performance of the models in out-of-sample data for different market conditions (using simulated data of bullish and bearish market dynamics).

1.3 Thesis structure

The thesis consists of five chapters, a bibliography and appendices. Chapter 1 is an auxiliary part of the work that provides introductory knowledge, the main objectives and motivation. This chapter contains a brief insight into the history of the portfolio selection problem.

Chapter 2 provides a literature survey for the main theories separately and mathematical formulations of the considered portfolio optimisation models as well as the definitions of some risk measures.

In Chapter 3 we develop several solution approaches for nonlinear portfolio optimisation problems. In Section 3.1 we describe two basic heuristic algorithms, namely the differential evolution algorithm and the genetic algorithm. These are able to deal with prospect theory and cumulative prospect theory problems which are non-convex problems as well as with cardinality and CVaR constrained prospect theory problems. We use an extended version of the differential evolution algorithm namely the differential evolution algorithm with smoothing of the utility function using splines in order to verify the solution and find the optimal solution approach. In Section 3.2 solution approach to the mean-variance-CVaR model is considered.

In Chapter 4 we present our empirical study and comparative analysis. Section 4.1 is devoted to basic settings for our empirical studies such as data used in the research, parameters of the models and parameters of the heuristic solution approaches. In Section 4.2 the performance of the mean variance and behaviourally based models is analysed. We split the models into 3 groups: basic models, models with cardinality constraints and models with a CVaR constraint. The empirical results obtained for each model are analysed in-sample and out-of-sample. We simulate the dynamics of a bullish and bearish market for the out-of-sample tests as well as apply the bootstrap method. In Section 4.3 the empirical results and comparative analysis of the index tracking model and prospect theory model with index tracking are presented.

We describe the most important findings and conclusion in Chapter 5. The main contribution of this thesis as well as the ideas for future work are presented in this chapter.

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1. "Prospect Theory Based Portfolio Optimisation: an Empirical Study and Analysis", at Research Student Symposium, School of Information Systems, Computing and Mathematics, Brunel University, London, UK, 2014

2. "Behavioural finance and mathematical modelling", at Student Mobility Programme, Brunel Business School, Brunel University, London, UK, 2014

3. "Portfolio Optimisation Model with Prospect Theory Investor Preferences: Benefits and Difficulties", at the School of Management, Keele University, Keele, UK, 2013

4. "Portfolio Optimisation with Prospect Theory Investor Preferences" at Research Student Symposium, School of Information Systems, Computing and Mathematics, Brunel University, London, UK, 2012

Chapter 2

Literature review

First we set out some general notation that we use for all of our models. In this chapter and in the rest of the thesis, we will use the following notation:

N - number of assets,

S - number of scenarios (time periods),

K - cardinality limit (desirable number of assets in the portfolio),

p_s - probability of scenario s, $\sum_s p_s = 1$,

$\bar{r}_i$ - mean return of asset i,

r_{is} - return of asset i in scenario s, $i = 1, \dots, N, \quad s = 1, \dots, S$,

r_0 - reference point,

$\omega_i \geq 0$ - weight of asset i in the portfolio,

$x = (\omega_1, \dots, \omega_N)$ - a portfolio and $\sum_{i=1}^{N} \omega_i = 1$,

$X = \{x = (\omega_1, \dots, \omega_N) \in \mathbb{R}_+^N\}$ - set of all portfolios,

$r_s(x)$ - return of portfolio x in scenario s,

d - desirable level of return,

z - constraint on CVaR,

$\succeq$ denotes a preference relation over the set of prospects, wherein $\succ$ is a strict preference relation and $\sim$ is an indifference relation.

2.1 Modern portfolio theory

In this section we consider several basic assumptions of modern portfolio theory that are important for our research. MPT says that investors act rationally and that they are risk-averse. This assumption comes from the efficient market hypothesis and means that people choose alternatives that are economically more beneficial for them. Originally, Markowitz explained rationality of an investor in terms of certainty and return. He devoted his technique to people who prefer certainty to uncertainty and who prefer higher return to less return. At the same time, he accepted that there is a type of investor who acts more as a speculator. However, he did not assume that even a rational investor becomes risk seeking in specific circumstances. According to Markowitz the MPT does not work for such cases [54].

Among the latest attempts to incorporate human mentality into a logical and practical mean variance scheme we would like to mention the results published in "Portfolio Optimization with Mental Accounts" (2010). In this study the authors considered a portfolio as a set of subportfolios with different financial goals and risk/return investor preferences. Mathematically the idea is to use the mean variance quadratic utility function with a risk aversion coefficient. Then they implied different levels of risk-aversion depending on the specific goals of the subportfolio into the Markowitz model [24]. "These generalizations of MVT (Mean-Variance Portfolio Theory) and BPT (Behavioural Portfolio Theory) via a unified MA (Mental Accounting) framework result in a fruitful connection between investor consumption goals and portfolio production" [24]. However, some questions are still left unanswered. For example, the diversification problem was completely ignored in the paper.

In general, the diversification problem is a question of how many assets in the portfolio will be necessary and sufficient to provide an efficient portfolio in respect of the transaction costs. Moreover, the optimal level of diversification should provide convenience for the portfolio management. We will consider this problem in further discussions.

Modern portfolio theory works under the assumption that asset returns are jointly normally distributed random variables [42]. However, in the early 1960s several scientists demonstrated that the Gaussian distribution is not suitable for the description of the return distribution. For example, Mandelbrot [52] and Fama [28] presented the models of the empirical heavy tailed character of the financial asset returns. These empirical returns demonstrated significant kurtosis, asymmetric skewness and heavy tails.

Nowadays, it is a widely accepted fact that return distributions have fat tails (leptokurtic returns). These fat tails are defined as rare but significant market events which can cause extreme gains or losses in a portfolio. In the normal distribution framework the probability of such an event is equal to 0.1%, in reality, these fat-tail events occur more frequently.

It is known that the Markowitz mean variance model provides an optimisation procedure which is based on historical average returns in order to estimate future portfolio returns. It means that the mean variance portfolio is calculated using mean and covariance matrices on data which reflects market trends in the past. However, historical estimates often provide poor prediction of future behavior of the assets in the real market conditions [16]. That is why many empirical studies of the portfolio selection problem include not only in-sample results but out-of-sample testings.

Another weakness of the MPT are unaccounted transaction costs, which could affect significantly the financial performance of the portfolio in the investment process [60]. On the one hand, in fast changing market conditions the rebalancing stage plays a very important role in keeping the portfolio optimal. This activity leads to an increase in the transaction costs and, hence, decrease in the current profit. On the other hand, ignoring the transaction cost in a portfolio selection model often leads to an inefficient portfolio in practice [59].

In this thesis we show how to solve this problem in terms of the investor's preferences of diversification level. Using a cardinality constraint in the problem formulation which is a limit on the number of assets in the portfolio, we restrict the transaction costs.

It is empirically confirmed that diversification beyond the level of 8-10 assets in a portfolio may not be rewarding [32], [27], [43]. From a mathematical point of view the optimal portfolio in MPT is always well-diversified because risk minimisation depends on the covariance matrix of return. The larger the number of assets held in the portfolio the greater the combined value of the risk becomes for the stocks with different parameters of the return distribution. It is found [77], [73], that the variance-covariance matrix of returns of a large size portfolio tends to conceal significant singularities or near-singularities, hence, the number of securities in the portfolio should be limited.

Taking into account the assumptions considered above we can conclude that MPT is both sufficiently general and static for a significant range of practical situations and simple enough for theoretical analysis and numerical solution. At the same time, the portfolio selection problem becomes even more complicated in modern economic conditions which demand more flexible and multi-factor models and tools to satisfy investor's preferences while MPT's assumptions lead to some serious limitations. MPT "is very useful, but it is descriptive, not prescriptive, and relies on assumptions that may not always be valid", according to Curtis [22].

Below we mathematically formulate the mean variance model with a cardinality constraint.

2.1.1 Formulation of the mean variance model

The variance of $r(x)$ is defined as $\sigma^2(r(x)) = E[(r(x) - E(r(x)))^2]$. The variance of the portfolio return $r(x) = \omega_1 r_1 + \ldots + \omega_N r_N$ is derived from the vector

$x = (\omega_1, \ldots, \omega_N)$ and can be written as:

$$\sigma^2(r(x)) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij}\ \omega_i\ \omega_j,$$

where $\sigma_{ij} = \sigma_i\ \sigma_j\ \varrho_{ij}$ (here ϱ_{ij} is the correlation coefficient between r_i and r_j) is the covariance of r_i and r_j and x is the vector of variable weights (unknown quantities) $\omega_i,\ i = 1, \ldots, N$, of assets in the portfolio.

In our research we consider the mean variance model [53] where variance is minimised with a fixed (prescribed) level of portfolio expected return. This model allows the investor to include all the available assets in the market. In the case when the number of assets in the portfolio is restricted by the investor preferences, the cardinality constraint should be introduced.

Let K be the desirable number of assets in the portfolio, let us define the indicator $\varphi_i,\ i = 1, \ldots, N$:

$$\varphi_i = \begin{cases} 1, & \text{if asset } i \text{ is included in the portfolio,} \\ 0, & \text{otherwise,} \end{cases}$$

with $l_i\ \varphi_i \leq \omega_i \leq u_i\ \varphi_i, \quad i = 1, \ldots, N,$

where l_i and u_i are a numerical boundaries which reflect the lower and upper level of investment in the asset if the asset is to be invested in.

It should be noted that one can transfer this model with a cardinality constraint into the basic MV model if we put $K = N$. For the sake of simplicity we can use a unified formulation for both, basic and cardinality constrained MV model. Then the mean variance portfolio optimisation problem with a cardinality constraint can be written as:

$$\text{minimise}\ \ \mathrm{MV_{cc}}(x) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij}\ \omega_i\ \omega_j, \tag{2.1}$$

subject to constraints:

$$\bar{r}(x) = \sum_{i=1}^{N} \bar{r}_i \, \omega_i \geq d, \tag{2.2}$$

$$\sum_{i=1}^{N} \omega_i = 1, \tag{2.3}$$

$$l_i \, \varphi_i \leq \omega_i \leq u_i \, \varphi_i, \quad i = 1, \ldots, N, \tag{2.4}$$

$$\sum_{i=1}^{N} \varphi_i \leq K, \tag{2.5}$$

$$\varphi_i \in \{0, 1\}, \; i = 1, \ldots, N. \tag{2.6}$$

Here constraint (2.2) ensures that the optimal portfolio has an expected return d, constraint (2.3) imposes that the investment weights sum to one (budget constraint). Inequality (2.4) describes a buy-in threshold and restricts asset investment. It is easy to see that if an asset i is not held, i.e. $\varphi_i = 0$, then the corresponding weight $\omega_i = 0$. If an asset i is held, i.e. $\varphi_i = 1$, then (2.4) ensures that the value of ω_i lies between the appropriate lower and upper limits, l_i and u_i respectively [87]. Inequality (2.5) ensures that the number of assets in the optimal portfolio is at most K. The binary definition (2.6) reflects the inclusion (or exclusion) of an asset in the portfolio.

According to the problem formulation and theoretical basis the mean variance model manages the risk of the portfolio taking into account the covariance matrix and standard deviation of assets. Modern portfolio theory and the work of Harry Markowitz on diversification and risk of a portfolio established the Capital Asset Pricing Model (CAPM) which distinguishes two types of portfolio risk: systematic and unsystematic. Systematic risk is considered as a market risk, i.e. is undiversifiable and common for all assets in the market while unsystematic risk is associated with each security. In terms of CAPM the optimal portfolio which aims to achieve the lowest risk together with any possible return is the market portfolio which, in fact, could be a market index. Following the assumption of CAPM the index tracking problem for portfolio selection is a replication of the "ideal" market portfolio in order to reduce unsystematic risk. In the next section we consider the index tracking portfolio selection problem.

2.2 Formulation of the index tracking model

Index tracking, known as a form of passive fund management, aims to produce optimal portfolios which replicate the index dynamics providing a balance between risk and return. However, the index tracking model normally includes almost all available assets in the market that leads to large transaction costs and a portfolio which is very difficult to manage because of its diversity [13]. Thus, the cardinality constrained index tracking model is also considered in this thesis. We explore this model in comparison with behaviourally based models in terms of diversification and tracking error issues.

In our research we use a simple index tracking model in the form of full replication as we are minimising the tracking error in order to reduce the difference between the index return and the portfolio return.

Let at time s
rm_s - index return,
$o_s = max(r_s(x) - rm_s,\ 0)$ - portfolio return amount over the index return,
$u_s = max(rm_s - r_s(x),\ 0)$ - portfolio return amount under the index return.
Tracking error (TE) for a given time period is equal to $|r_s(x) - rm_s|$. Clearly, at time s at least one of o_s or u_s is equal to 0, i.e. we can define a new quantity

$$\mathrm{TE}_s = o_s + u_s = \begin{cases} o_s, & \text{if } o_s \geq 0, \\ u_s, & \text{otherwise.} \end{cases} \tag{2.7}$$

Let us define the tracking error in the simplest possible way: as the difference between the index and portfolio returns over all time periods $s = 1, \ \dots, S$:

$$\mathrm{TE} = \sum_{s=1}^{S} \mathrm{TE}_s. \tag{2.8}$$

Here we would like to mention that tracking error can be defined in different ways, for example, in [68] the tracking error is defined as the root mean square of the difference between index and portfolio returns.

As was mentioned previously we can use the formulation of the cardinality constrained model for the basic model as well when we put $K = N$. Then the index tracking problem with cardinality constraint can be formulated as [64]:

$$\text{minimise } \mathrm{IT}_{\mathrm{cc}}(x) = \text{minimise } \mathrm{TE}(x) = \sum_{s=1}^{S}(o_s + u_s), \tag{2.9}$$

subject to the constraints

$$\sum_{i=1}^{N} \omega_i r_{is} = rm_s + o_s - u_s,\ \ s = 1, \ldots, S \tag{2.10}$$

$$\sum_{i=1}^{N} \omega_i = 1, \tag{2.11}$$

$$l_i\ \varphi_i \leq \omega_i \leq u_i\ \varphi_i,\ \ \ i = 1, \ldots, N, \tag{2.12}$$

$$\sum_{i=1}^{N} \varphi_i \leq K, \tag{2.13}$$

$$\varphi_i \in \{0, 1\},\ \ i = 1, \ldots, N, \tag{2.14}$$

$$\omega_i \geq 0,\ \ i = 1, \ldots, N, \tag{2.15}$$

$$o_s, u_s \geq 0,\ \ s = 1, \ldots, S. \tag{2.16}$$

Equations (2.10) check the difference between returns of the optimal portfolio and the index for each time period. Constraint (2.11) imposes that the investment weights sum to one (budget constraint) similar to the MV model. The constraints (2.12), (2.13) and (2.14) are formulated similar to the MV model and are used for restricting the number of assets in the portfolio.

2.3 Prospect theory

Prospect theory is a behavioural economic theory that describes decisions between alternatives that involve risk, where the probabilities of outcomes are known. It was developed as a descriptive model of decision making under uncertainty by two psychologists, Daniel Kahneman and Amos Tversky, and published in the Econometrica in 1979 [45]. The authors relied on a series of small experiments to identify the manner in which people make choice in the face of risk. The theory says that people make decisions based on the potential value of losses and gains rather than the final outcome, and that people evaluate these losses and gains using heuristics. Although the original formulation of prospect theory was only defined for lotteries with two non-zero outcomes, it can be generalised to n outcomes. Generalisations have been used by various authors (see, for example [72], [18], [29], [82]).

The original PT choice process consists of two phases. During the first phase, which is called editing, an agent defines their own (subjective) meaning of a gain and a loss by setting a reference point r_0 for the portfolio return, which represents zero gain (or zero loss) for this particular person. During the second stage, which is called the evaluating phase, our investor calculates the values of the prospect theory utility based on the potential outcomes and their respective probabilities, and chooses the maximal one.

Together with the original version of prospect theory in this section we also consider its extended version called Cumulative Prospect Theory (CPT), proposed in 1992 by Tversky and Kahneman [79]. According to the authors, CPT can be applied not only for the discrete, but also for the continuous distributions, and it allows incorporation of different decision weights for gains and losses. However, some researchers believe that CPT may be descriptively not as strong as PT (see, e.g. [65]). In this research we investigate the performance of both versions (PT and CPT) in order to identify the best models for different types of experimental data according to several criteria.

We would like to note, that only a few (C)PT studies contain numerical results. It can be explained by the computational difficulties connected to the complexity of the (C)PT objective function. Due to this fact only simple cases (2-3 artificially created assets) of the portfolio selection problem are available in the literature. Among them [45], [34], [39], [49], [50] for the prospect theory and [9], [14], [38], [89], [62] for cumulative prospect theory. Moreover, as far as we aware, all of them were based on normally distributed testing data. However, it is well known, that many asset allocation problems involve non-normally distributed returns since commodities typically have fat tails and are skewed. Our research aims to fill in this gap.

Clearly, the lack of numerical data for (cumulative) prospect theory leads to the lack of comparison analysis of traditional (mean-variance) approaches with behaviourally based approaches (PT and CPT). The first effort to compare these two models was made in 2004 [50]. The idea was to select the portfolio with the highest prospect theory utility amongst the other portfolios in the mean variance efficient frontier. Following this route, Pirvu and Schulze in 2012 presented the results confirming that an analytical solution is mostly equivalent to maximising the CPT objective function along the mean variance efficient frontier [62]. In this thesis we compare performances of both models separately using different types of data and simulation tests.

Below we mathematically formulate both the PT and CPT models with and without cardinality constraint.

2.3.1 Formulation of the prospect theory model

Consider the game:

$$(r_{-m}, p_{-m}), (r_{-m+1}, p_{-m+1}), \dots, (r_0, p_0), \dots, (r_{n-1}, p_{n-1}), (r_n, p_n), \tag{2.17}$$

where $(r_s, p_s),\ s = -m, -m+1, \dots, -1,\ 0,\ 1, \dots, n-1,\ n,$ means that the gambler wins r_s with probability p_s, of course, the sum of all probabilities is equal to 1, i.e. $\sum_{s=-m}^{n} p_s = 1$; r_0 denotes some numerical boundary called the reference point (constant) which depends on the investor's preference. Let r_s define the outcomes of the game (2.17) such that:

- if $s = 0$, i.e. $r_s = r_0$, then the investor's gain is 0,
- if $s > 0$, then $r_s > r_0$, hence the investor won from this investment,
- if $s < 0$, then $r_s < r_0$, hence the investor lost.

According to the prospect theory one needs to make additional mental adjustments in the original probability and outcome value functions p and r, which is equivalent to replacing a standard utility function by the prospect theory utility function. In order to do so we transform the original p and r into the prospect theory probability weight function $\pi(p)$ and value function $v(r)$. Figure 2.1 contains the graphs for the value function $v(r)$.

The prospect theory probability weighting function $\pi(p)$ measures, according to [45], "the impact of events on the desirability of prospects, and not merely the perceived likelihood of these events", i.e. expresses the weights of the decisions to the probabilities. Let us mention that $\pi(p)$ is an increasing function, $\pi(0) = 0$, $\pi(1) = 1$, and for very small values of probability p we have $\pi(p) \geq p$. The probability weighting function based on the observation that most people tend to overweigh small probabilities and underweigh large probabilities.

The prospect theory value function $v(r)$ describes the (behavioural) value of the gain/loss outcome. Kahneman and Tversky experimentally obtained the value function which was dependent on the initial value deviation. This function is usually asymmetric with respect to a given reference point r_0 (which reflects different investor's attitude to gains and losses), it is concave upward for gains and convex downward for losses. Moreover, generally the value function $v(r)$ grows steeper for losses than for gains, i.e. for $s > 0$ we have $v(r_s) \leq -v(r_{-s})$.

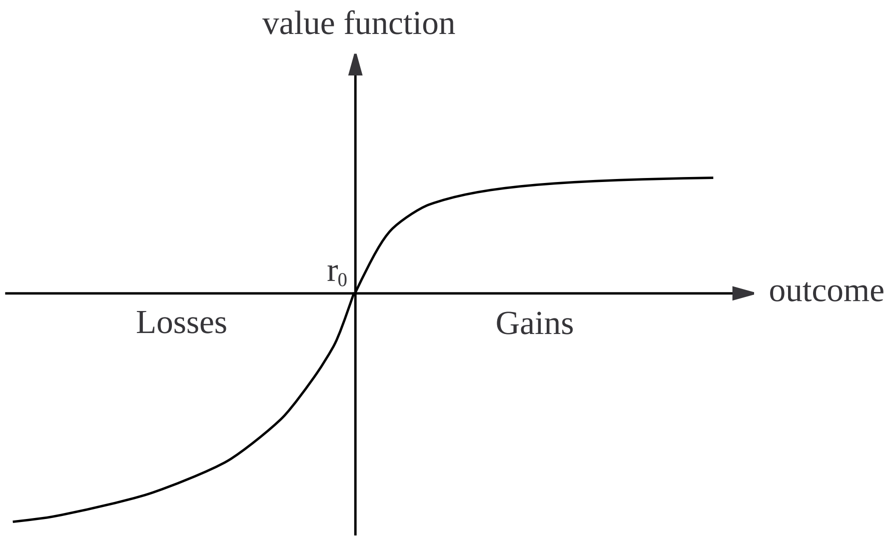

FIGURE 2.1: Prospect theory value function $v(r)$ with $\alpha = \beta = 0.88$ and $\lambda = 2.25$

The explicit formula for the prospect theory value function $v(r)$, given in [79], is:

$$v\,(r) = \begin{cases} (r - r_0)^{\alpha}, \text{ if } r \geq r_0, \\ -\lambda\,(r_0 - r)^{\beta}, \text{ if } r < r_0, \end{cases} \tag{2.18}$$

where $\alpha = \beta = 0.88$ are risk aversion coefficients with respect to gains and losses accordingly, $\lambda = 2.25$ is the loss aversion coefficient which underlines differences in the investor's perception of gains and losses. We note that the value function (2.18) is nonlinear with respect to return r and, hence, the portfolio variable x.

The prospect theory utility function can be written in terms of π and v as:

$$\mathrm{PT_U} = \sum_{s=-m}^{n} \pi(p_s)\; v(r_s) = \sum_{s=-m}^{n} p_s\; v\left(\sum_{i=1}^{N} r_{si}\;\omega_i\right). \tag{2.19}$$

Clearly, the formula (2.19) consists of two parts. The part in the gain domain (i.e. when $r \geq r_0$) is concave and the part in the loss domain (i.e. when $r \leq r_0$) is convex, capturing the risk-averse tendency for gains and risk-seeking tendency for losses as seen by many decision makers [65]. Let as mention, that for the sake of simplicity in our study we use $\pi(p) = p$. Clearly, the prospect theory utility function (2.19) is a nonlinear function.

The prospect theory model aims to find the best (optimal) portfolio which maximises the prospect theory utility function where decision variables are weights of available assets ω subject to constraints on a desirable level of return, budget and short sales. This is a nonlinear and non convex optimisation model as the objective function is nonlinear and non convex. In order to solve this problem we use heuristics which are an inexact solution approach.

According to the prospect theory portfolio selection problem looks as follows:

$$\text{maximise PT}(x) = \sum_{s=1}^{S} p_s \; v\left(\sum_{i=1}^{N} r_{si} \; \omega_i\right), \tag{2.20}$$

subject to the constraints

$$\bar{r}(x) = \sum_{i=1}^{N} \bar{r}_i \; \omega_i \geq d, \tag{2.21}$$

$$\sum_{i=1}^{N} \omega_i = 1, \tag{2.22}$$

$$\omega_i \geq 0, \; i = 1, \ldots, N. \tag{2.23}$$

We now detail some mathematical properties of the prospect theory model.

1. Completeness:

$$\text{For all portfolios } \; x_1, x_2 \;\; x_1 \succeq x_2 \;\text{ or }\; x_2 \succeq x_1.$$

2. Transitiveness:

$$x_1 \succeq x_2, \; x_2 \succeq x_3 \Rightarrow x_1 \succeq x_3.$$

3. Independence:

$$\forall x_1, x_2, x_3 \;\; \forall \varsigma \in (0,1) : x_1 \succ x_2 \Leftrightarrow \varsigma x_1 + (1-\varsigma) x_3 \succ \varsigma x_2 + (1-\varsigma) x_3.$$

For more details and proof of the properties given above see [79] and [84].

2.3.2 Formulation of the cumulative prospect theory model

Consider the game (2.17) under the following condition:

$$r_1 \leq \ldots \leq r_k \leq r_0 \leq r_{k+1} \leq \ldots \leq r_S,$$

i.e. all outcomes of the game $(r_1, \ldots, r_S)$ are arranged in ascending order. Therefore, for $j = 1, \ldots, k$ the loss is $r_0 - r_j$ and for $j = k+1, \ldots, S$ the gain is $r_j - r_0$.

Let us introduce the probability weighting function π, which is strictly increasing on $[0,1]$, $\pi(0) = 0$, $\pi(1) = 1$. For any prospect j, we define a positive prospect weights π^+ or a negative prospect weights π^- depending on the corresponding outcome. We now define the probability weighting functions π^- and π^+, which describe decision weights for gains and losses.

$$\pi^-(p_j) = \frac{p_j^\delta}{(p_j^\delta + (1-p_j)^\delta)^{1/\delta}}, \text{where } j = 1, \ldots, k, \tag{2.24}$$

and

$$\pi^+(p_j) = \frac{p_j^\gamma}{(p_j^\gamma + (1-p_j)^\gamma)^{1/\gamma}}, \text{where } j = k+1, \ldots, S, \tag{2.25}$$

where $\delta,\ \gamma \in (0,1)$ reflect quantitative values of risk seeking for losses and risk aversion for gains. These probability weighting functions $\pi^-(\cdot)$ and $\pi^+(\cdot)$ capture the overweighing of low probabilities if we put $\delta = 0.61$, $\gamma = 0.69$ in accordance with [79]. Figure 2.2 illustrates the cumulative prospect theory probability weighting function.

We would like to note that in the prospect theory model $\pi^+ = \pi^-$, hence, prospect theory assumes that decision weights for gains and losses are equal. In this connection we can consider CPT as a particular case of the prospect theory.

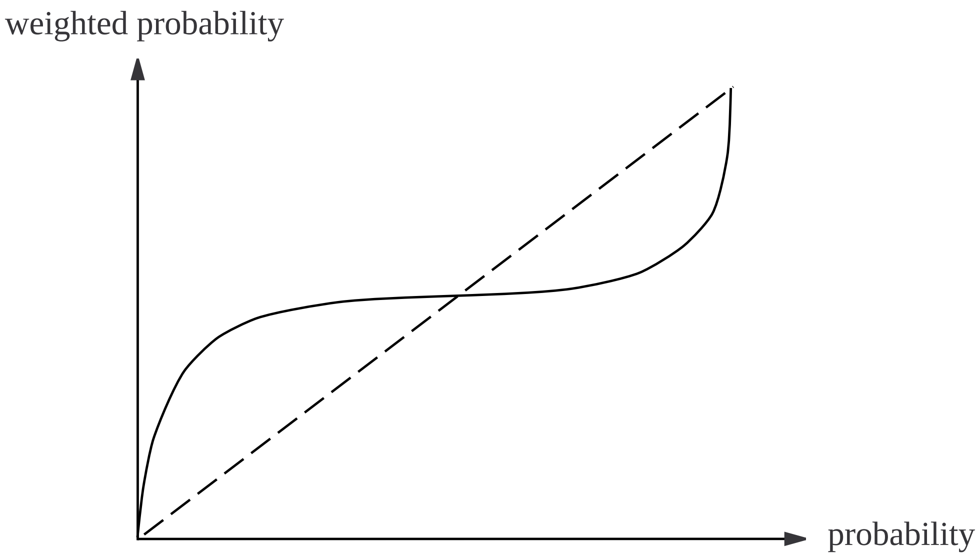

FIGURE 2.2: Cumulative prospect theory weighting function w

The portfolio optimisation problem for the cumulative prospect theory (CPT) model can be formulated as:

$$\text{maximise CPT}(x) = \sum_{s=1}^{S} \pi(p_s)\ v\left(\sum_{i=1}^{N} r_{si}\ \omega_i\right), \tag{2.26}$$

subject to the constraints

$$\bar{r}(x) = \sum_{i=1}^{N} \bar{r}_i\ \omega_i \geq d, \tag{2.27}$$

$$\sum_{i=1}^{N} \omega_i = 1, \tag{2.28}$$

$$\omega_i \geq 0,\ \ i = 1, \ldots, N, \tag{2.29}$$

where

$$\pi(p_s) = \begin{cases} \pi^{-}\left(\sum_{j=1}^{s} p_j\right) - \pi^{-}\left(\sum_{j=1}^{s-1} p_j\right), s = 1, \ldots, k, \\ \pi^{+}\left(\sum_{j=s}^{S} p_j\right) - \pi^{+}\left(\sum_{j=s+1}^{S} p_j\right), s = k+1, \ldots, S, \end{cases} \tag{2.30}$$

where p_j is the probability.

Here k is such that

$$r_1 \preceq \ldots \preceq r_k \preceq r_0 \preceq r_{k+1} \preceq \ldots \preceq r_S, \tag{2.31}$$

and $v(r(x))$ is the function of outcomes assessment defined by the following formula

$$v\left(r(x)\right) = \begin{cases} (r(x) - r_0)^\alpha, \text{ if } r(x) \geq r_0, \\ -\lambda \ (r_0 - r(x))^\beta, \text{ if } r(x) < r_0. \end{cases} \tag{2.32}$$

Let us mention that the CPT model possesses all the mathematical properties of the prospect theory model described in Section 2.3.1, namely, completeness, transitivity and independence. It also has a very important property, which is stochastic dominance of the preference relation [21], i.e. if for portfolios x_1 and x_2 we have $r_j(x_1) \succeq r_j(x_2)\ \forall j$ and $r_j(x_1) \succ r_j(x_2)$ for at least one j with $p_j > 0$ then $(r_1(x_1), p_1(x_1); \ldots; r_n(x_1), p_n(x_1)) \succeq (r_1(x_2), p_1(x_2); \ldots; r_n(x_2), p_n(x_2))$.

2.3.3 Prospect theory model with a cardinality constraint

Following the logic and notation of Section 2.1.1 we formulate the prospect theory model with a cardinality constraint as:

$$\text{maximise } \mathrm{PT}_{\mathrm{cc}}(x) = \sum_{s=1}^{S} p_s \ v\left(\sum_{i=1}^{N} r_{si}\ \omega_i\right), \tag{2.33}$$

subject to the constraints

$$\bar{r}(x) = \sum_{i=1}^{N} \bar{r}_i\ \omega_i \geq d, \tag{2.34}$$

$$\sum_{i=1}^{N} \omega_i = 1, \tag{2.35}$$

$$l_i\ \varphi_i \leq \omega_i \leq u_i\ \varphi_i, \quad i = 1, \ldots, N, \tag{2.36}$$

$$\sum_{i=1}^{N} \varphi_i \leq K, \tag{2.37}$$

$$\varphi_i \in \{0, 1\},\ i = 1, \ldots, N. \tag{2.38}$$

2.3.4 Prospect theory model for index tracking

Studying the prospect theory problem we found that the principle of the model is very similar to that of the index tracking portfolio optimisation problem. The main common feature is that behaviourally based models use a reference point as the limit for desired level of returns in each time period similar to an index tracking model which uses the index as a reference point. Thus it is easy to implement the idea of the index tracking problem into prospect theory by changing the value of the reference point. In this case we let r_0 be a vector of the index value for each time period of the data set not a scalar as it is in the original version of (cumulative) prospect theory. We also remove the limit on the desirable level of returns similar to the index tracking problem which focuses on the index value as a level of return for each time period. We call this model prospect theory with index tracking (PT with IT).

We also implemented a cardinality constraint in these models to address the issue of too diversified a portfolio. It is very interesting to compare not only the IT and PT with index tracking problems but these models with the limit on the number of the assets in the portfolio. We formulate the prospect theory model with index tracking and with a cardinality constraint as:

$$\text{maximise PT+IT}_{\text{cc}}(x) = \sum_{s=1}^{S} p_s\ v\left(\sum_{i=1}^{N} r_{si}\ \omega_i,\ rm_s\right), \tag{2.39}$$

subject to the constraints

$$\sum_{i=1}^{N} \omega_i = 1, \tag{2.40}$$

$$l_i\ \varphi_i \leq \omega_i \leq u_i\ \varphi_i,\quad i = 1, \ldots, N, \tag{2.41}$$

$$\sum_{i=1}^{N} \varphi_i \leq K, \tag{2.42}$$

$$\varphi_i \in \{0, 1\},\ i = 1, \ldots, N, \tag{2.43}$$

where

$$v\left(r(x), rm_s\right) = \begin{cases} (r(x) - rm_s)^{\alpha}, \text{ if } r(x) \geq rm_s, \\ -\lambda \left(rm_s - r(x)\right)^{\beta}, \text{ if } r(x) < rm_s. \end{cases} \tag{2.44}$$

As one can see in equation (2.44) the value function for the prosect theory model with index tracking is defined as a dynamic not constant due to the fact that instead of a constant reference point r_0 here we use a dynamic index rm which takes different values in each scenario (time period).

2.4 Measures of risk

The concept of risk plays one of the major roles in the portfolio selection problem. Markowitz was the first scientist who postulated the dependence between risk level and returns. He suggested to minimise risk subject to a desirable level of expected return and its dispersion. Some of the researchers supposed that it is possible to reach zero level of risk (see, for example, [86] and others). However, the truth is that risk can be reduced with the help of diversification, but not fully eliminated without changing the return [70].

In this study we consider one symmetric measure of risk, which is variance and two asymmetric ones, called Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR).

Definition 2.1. Variance
Consider a continuous random variable $\tilde{x}$ with density $f = f_{\tilde{x}}$, distribution $F = F_{\tilde{x}}$ and expected return $\mu := E(\tilde{x}) := \int_{-\infty}^{+\infty} xf(x)dx$. Then we define the *variance* of the variable $\tilde{x}$ as:

$$\sigma^2 := \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)dx \tag{2.45}$$

and its standard deviation is:

$$\sigma := \left[\int_{-\infty}^{+\infty} (x-\mu)^2 f(x)dx\right]^{1/2}. \tag{2.46}$$

Irving Fisher was probably the first scientist who suggested to use variance as a measure of risk in his paper [31] (see also [70]).

The variance as a measure of risk has an advantage of being simple. As already discussed, computational simplicity is very important for the portfolio selection problem. However, there are limitations for this case.

The first limitation is that it is not sensitive to higher moments (for example skewness, kurtosis) of the probability distribution and there are many distributions that have the same mean and same variance. In other words, the mean variance framework does not capture the complexity of risk.

Another issue is that in the mean variance framework gains and losses are considered symmetrical. Many statistical measures of risk do so but they do not seem to be adequate for finance: investors do not treat gains and losses symmetrically [45]. They care about "downside risk" (investor are loss averse).

The asymmetric nature of risk is not the only reason why a focus on the downside is important. If we assume that financial returns follow a multivariate normal (or elliptical) distribution, then any downside risk measures can be expressed as a function of mean and variance (or some other measure of scale when variance is undefined). As a consequence, under these assumptions, measuring variance would be sufficient. Empirical research (see, for example, [52] and [28]), however, has demonstrated that financial returns, and hedge fund returns in particular, are skewed and fat-tailed which means that the focus on the downside cannot be understated.

Definition 2.2. VaR

Let r be the specific level with which the value x of a given portfolio will be compared to, at the end of a given time period. If $x < r$, then there is a loss, whose value is $r - x$. The portfolio's loss is thus given by the random variable

$$\tilde{l} := r - \tilde{x}. \tag{2.47}$$

The probability that $\tilde{l} \leq l$ is given by the distribution function

$$F_{\tilde{l}}(l) := P(\tilde{l} \leq l) = \int_{-\infty}^{l} f_{\tilde{l}}(t)dt. \tag{2.48}$$

Using the loss distribution (2.48) for a given time period and a given confidence level $1-\alpha$, $0 \leq \alpha \leq 1$, the VaR of $\tilde{x}$ is defined as:

$$\text{VaR}(\tilde{x}) := F_{\tilde{l}}^{-1}(1-\alpha), \tag{2.49}$$

where $(1-\alpha) \cdot 100\%$ is a quantile of the portfolio's loss distribution.

The reasons and grounds for the development of VaR include the regulators' pressure for better control of financial risks, financial markets globalisation, which exposed institutions to more sources of risk, and technological developments that contributed to enterprise-wide risk management [44].

Basel III issued by the Basel Committee on Banking Supervision accumulates two years of regulatory reform including Basel 2.5. It introduced a new regulatory regime for capital, liquidity and banking supervision, where VaR is described as a compulsory measure of risk.

In the academic literature the most used two confidence levels are 95% and 99%. Researchers do not have preferences which level to apply in their models and calculations. However, in real economic application most financial institutes choose only 99% in order to protect their investments with higher level of reliability.

VaR is a single, summary, statistical measure of possible portfolio losses. For a given time horizon and a confidence level $1-\alpha$ the VaR of a portfolio is the loss of market value over the time horizon that is exceeded by the portfolio only with probability α.

In comparison to traditional measures of risk, VaR represents an aggregate view of a portfolio risk considering leverage, correlations, and positions. VaR can be applied to a variety of financial instruments, including derivatives [44].

Despite the fact that VaR is a very popular risk measure, it has some mathematical characteristics which are unfavorable for application of this measure of risk to real world financial problems. For instance, it has no subadditivity or convexity properties. This drawback is highly criticised due to the fact that according to the diversification principle of modern portfolio theory, a subadditive measure should generate lower measured risk for those portfolios which are diversified than for a nondiversified one [23]. As for the economy, under specific circumstances it can be more useful to divide a large company into two smaller ones and the VaR risk measure is not suitable for this case [78].

Also VaR is most often defined in terms of net outcomes or profit/loss. However, the money value is not constant through time in a financial market. This creates ignorance of the difference between the monetary value at one date and the monetary value at another date. However, for small time periods and a single currency it performs well. As VaR uses quantiles, it is necessary to pay attention to discontinuities and intervals of quantile numbers. However, VaR fails to account for concentration of risks [8].

Another point is that VaR is adequate only based on standard deviation of normal distributions. In this case it is proportional to the standard deviation. The VaR for a combination of two portfolios can be greater than the sum of the risks of the portfolios separately. Also, VaR is difficult to optimise in the situation when it is calculated using scenarios. In contrast to VaR, CVaR is known to have more beneficial properties than VaR in these cases [66].

Definition 2.3. CVaR

Let $\tilde{x}$ be a random variable responsible for the return of a portfolio x over a specified holding period and $A\% = \alpha \in (0, 1)$ is a percentage representing a sample of the worst case scenarios for the outcomes of $\tilde{x}$ (so called confidence interval, usually chosen as $\alpha = 0.01$ or $\alpha = 0.05$). Thus, figuratively CVaR at a specified level α is the "average losses in the worst $A\%$ of cases" [3], where "loss" means negative outcome of $\tilde{x}$. The CVaR at a level α of $\tilde{x}$ is a negative value of the mean of the α-tail distribution of $\tilde{x}$ (with respect to the extreme adverse outcomes) and its distribution function is rescaled to span $[0, 1]$:

$$\text{CVaR}_\alpha(\tilde{x}) := \int_{-\infty}^{\infty} z dF_{\tilde{x}}^{\alpha}(z), \tag{2.50}$$

where

$$F_{\tilde{x}}^{\alpha}(z) = \begin{cases} 0, & \text{if } \; z > \text{VaR}_\alpha(\tilde{x}), \\ \frac{F_{\tilde{x}}(z)-\alpha}{1-\alpha}, & \text{otherwise}, \end{cases}$$

for more details see, for example, [66] or [69].

There is an alternative definition of CVaR, called "upper CVaR" which reflects the conditional expectation of $\tilde{x}$ subject to $\tilde{x} > \text{VaR}_\alpha(\tilde{x})$:

$$\text{CVaR}_\alpha^+(\tilde{x}) = E[\tilde{x}|\tilde{x} > \text{VaR}_\alpha(\tilde{x})].$$

Mathematical properties of CVaR

Let $c \in \mathbb{R}$ and $\tilde{y}$, $\tilde{y}_1$, $\tilde{y}_2$ be random variables representing the returns of the portfolios y, y_1 and y_2 respectively, then CVaR has the following properties [3]:

1. Monotonicity: If $\tilde{y}_1 \leq \tilde{y}_2$, then $\text{CVaR}_\alpha(\tilde{y}_1) \leq \text{CVaR}_\alpha(\tilde{y}_2)$.
2. Sub-additivity: $\text{CVaR}_\alpha(\tilde{y}_1 + \tilde{y}_2) \leq \text{CVaR}_\alpha(\tilde{y}_1) + \text{CVaR}_\alpha(\tilde{y}_2)$.
3. Translation invariance: $\text{CVaR}_\alpha(\tilde{y} + c) = \text{CVaR}_\alpha(\tilde{y}) + c$.
4. Positive homogeneity: $\text{CVaR}_\alpha(c\ \tilde{y}) = c\ \text{CVaR}_\alpha(\tilde{y}),\ \text{for } c > 0$.
5. Convexity: $\text{CVaR}_\alpha(\lambda\tilde{y}_1 + (1-\lambda)\tilde{y}_2) \leq \lambda\ \text{CVaR}_\alpha(\tilde{y}_1) + (1-\lambda)\ \text{CVaR}_\alpha(\tilde{y}_2)$, for $0 < \lambda < 1$.

It is known [61] that properties 2 and 4 are equivalent to convexity. It is important to note that VaR does not satisfy these properties. This issue leads to limitation to its application.

The Basel Committee proposed in 2012 the use of expected shortfall (also known as CVaR) instead of VaR in market risk management. They suggest moving from VaR to expected shortfall, a risk measure that better captures "tail risk" [58].

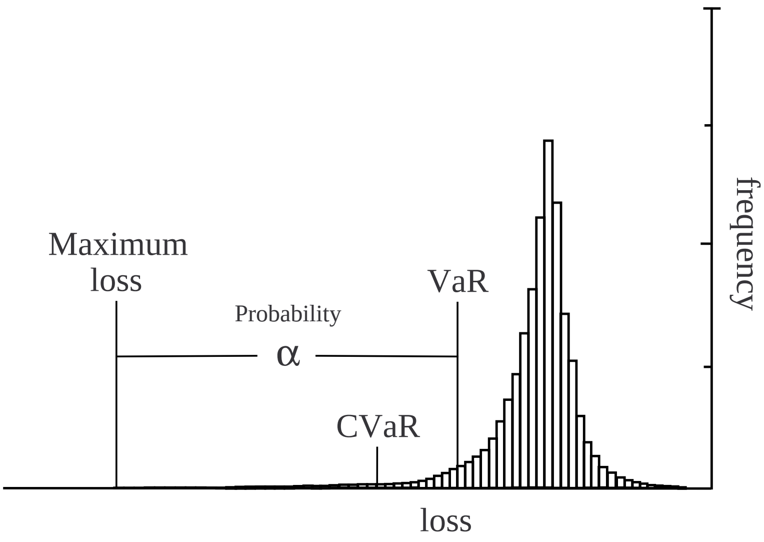

FIGURE 2.3: Value-at-Risk and Conditional Value-at-Risk on probability density function of asset returns

CVaR is an alternative measure of risk which quantifies the losses in the tail of the distribution. CVaR is often used together with VaR and this combination of instruments can be applied to the risk estimation for non-symmetric loss distributions (with high or low skewness). Figure 2.3 reflects the meaning of VaR and CVaR in terms of the distribution of returns.

CVaR and the formula for its minimisation were first delivered in the paper of Rockafellar and Uryasev in 2000. They showed numerical effectiveness using case studies, involving portfolio optimisation and option hedging [66].

Additionally, it was shown that imposing a CVaR constraint in the portfolio selection problem can deliver better results than imposing a VaR constraint. VaR does not show the extent of the losses that might occur beyond the threshold amount suggested by VaR. Unlike VaR, CVaR does quantify those losses that might occur in the tail of the distribution. CVaR is the expected loss given the loss is greater that or equal to VaR (see definition of CVaR) [7].

In order to be fair, it is necessary to provide examples of disadvantages of CVaR. It has implementation problems because CVaR is very sensitive to estimation error of the market observations (more sensitive than VaR, for example) and approximation error (this issue is unique to every scenario optimisation problem and does not exist in the MV approach). Also CVaR accuracy depends on accuracy of tail modeling. For more discussion about disadvantages of CVaR see [88] and [81].

The more significant issue is conceptional problems of CVaR, for instance, the fact that CVaR cannot integrate into the way investors consider risks. The reason is that CVaR averages both small and extremely large losses, hence it gives them the same weight in terms of the risk calculation, therefore it does not account for increasing risk aversion against extreme losses [71].

CVaR is used in return-risk analyses similar to Markowitz's (1952) mean-variance approach. For instance, it is easy to calculate a portfolio with a specified level of return and minimal CVaR or to impose a constraint on CVaR and find a portfolio with maximal expected return. In addition, we can impose several constraints on CVaR simultaneously with specifying different confidence levels (shaping the loss distribution by this). Hence, it represents a flexible and useful risk management tool [80].

One of the most important properties of CVaR in terms of applications is that CVaR can be expressed by a convenient minimisation (or maximisation) formula. This formula can be incorporated into optimisation problems with respect to $x \in X$ which are minimising risk or shaping it within bounds. Convexity is preserved in this case. If the random variables, under consideration, are discrete, the number of outcomes is finite, which can be represented as various outcomes under various scenarios, then CVaR optimisation is represented as a linear programming model of finite dimension [67].

2.4.1 Mean-CVaR model

Let $r(x)$ be a random variable that depends on a decision vector $x = (\omega_1, \omega_2, \cdots, \omega_N) \in A$, where A is a feasible set of portfolios, $r(x) = \omega_1 r_1 +$

$\ldots + \omega_N r_N$. Consider a function [67]:

$$F_\alpha(x, v) = \frac{1}{\alpha} E\{[-r(x) + v]^+\} - v, \tag{2.51}$$

where $\alpha \in (0, 1)$, $[u]^+ = \begin{cases} u, & \text{if } u \geq 0, \\ 0, & \text{if } u < 0. \end{cases}$

Let us note that:

1. Function F_α defined in (2.51) is finite, continuous and convex with respect to v and $\mathrm{CVaR}_\alpha(r(x)) = \min_{v \in \mathbb{R}} F_\alpha(x, v)$. We also would like to mention that the set $A_\alpha(x)$ (set of all the values of v such that the minimum is achieved) is a non-empty, compact (closed and bounded) and could possibly consist of one point.

2. Minimising CVaR_α with respect to $x \in A$ is equivalent to minimising F_α with respect to $(x, v) \in A \times \mathbb{R}$, i.e.:

$$\min_{x \in A} \mathrm{CVaR}_\alpha(r(x)) = \min_{(x,v) \in A \times \mathbb{R}} F_\alpha(x, v). \tag{2.52}$$

   It is important to note here, that a pair (x^*, v^*) minimises the right hand side of (2.52) if and only if x^* minimises its left hand side, $v^* \in A_\alpha(x^*)$.

3. $\mathrm{CVaR}_\alpha(r(x))$ is convex with respect to x as well as $F_\alpha(x, v)$ is convex with respect to (x, v).

Let $r(x)$ be a discrete random variable with S possible outcomes (scenarios) $r_1(x), \ldots, r_S(x)$ with probabilities $p_1, \ldots, p_S$ respectively. In this the case we let the reference point $r_0 = 0$ and rewrite formula (2.51) as:

$$F_\alpha(x, v) = \frac{1}{\alpha} \sum_{s=1}^{S} p_s[v - r_s(x)]^+ - v = \frac{1}{\alpha} \sum_{s=1}^{S} p_s \left[v - \sum_{i=1}^{N} \omega_i r_{is} \right]^+ - v. \tag{2.53}$$

Hence, we can formulate the mean-CVaR model for the portfolio selection problem [69]:

$$\text{minimise} \;\; \text{CVaR}(x) = \frac{1}{\alpha}\sum_{s=1}^{S} \; p_s \; y_s \; - \; v, \tag{2.54}$$

subject to constraints:

$$\bar{r}(x) = \sum_{i=1}^{N} \bar{r}_i \; \omega_i \geq d, \tag{2.55}$$

$$\sum_{i=1}^{N} \omega_i = 1, \tag{2.56}$$

$$\omega_i \geq 0, \;\; i = 1, \dots, N, \tag{2.57}$$

$$v - \sum_{i=1}^{N} \omega_i r_{is} \leq y_s, \;\; s = 1, \dots, S, \tag{2.58}$$

$$y_s \geq 0, \;\; s = 1, \dots, S, \tag{2.59}$$

where v is the VaR and y_s is the amount beyond VaR for scenario s.

2.4.2 Mean-variance-CVaR model

Modern approaches to the portfolio selection problem often lead to the creation of new mathematical models which take into account several risk measures simultaneously (see, for example [47], [48], [85], [37]). In this section we give a formulation for the portfolio selection problem in which random variables are described by three statistics [69]: expected value $E(r(x))$, variance $\sigma^2(r(x))$ and the CVaR at a specified confidence level $\alpha \in (0, 1)$. The mean-variance-CVaR model gives an optimal solution as a tradeoff between the mean variance efficient frontier and the mean-CVaR efficient frontier.

Let us define a preference relation for random variables $r(x)$ in terms of mean-variance-CVaR model as follows. Consider the portfolio selection problem with random variables $r(x_1)$ and $r(x_2)$ which are returns of portfolios x_1 and x_2 respectively, $x_1, x_2 \in A$. We say that $r(x_1) \succeq r(x_2)$ (i.e. portfolio x_1 is preferred to portfolio x_2) if and only if $E(r(x_1)) \geq E(r(x_2))$, $\sigma^2(r(x_1)) \leq \sigma^2(r(x_2))$, $\text{CVaR}_\alpha(r(x_1)) \leq \text{CVaR}_\alpha(r(x_2))$, where at least one inequality must be strict [69].

Hence, the non-dominated (efficient) solutions of the mean-variance-CVaR model are the Pareto efficient solutions of a multi-objective problem, where the expected value is maximised while the variance and the CVaR are minimised. Generally, the problem can be written as follows:

$$\text{maximise} \quad [E(r(x)), \; -\sigma^2(r(x)), \; -\text{CVaR}_\alpha(r(x))] \tag{2.60}$$

for $x \in A$ [67].

Let us consider a portfolio selection problem with S scenarios and N assets. Using formula (2.53) we formulate the mean-variance-CVaR model as:

$$\text{minimise} \;\; \text{MV}_{\text{CVaR}}(x) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} \; \omega_i \; \omega_j, \tag{2.61}$$

subject to constraints:

$$\bar{r}(x) = \sum_{i=1}^{N} \bar{r}_i \; \omega_i \geq d, \tag{2.62}$$

$$\sum_{i=1}^{N} \omega_i = 1, \tag{2.63}$$

$$\omega_i \geq 0, \; i = 1, \ldots, N, \tag{2.64}$$

$$v - \sum_{i=1}^{N} \omega_i r_{is} \leq y_s, \; s = 1, \ldots, S, \tag{2.65}$$

$$\frac{1}{\alpha} \sum_{s=1}^{S} \rho_s \; y_s - v \leq z, \tag{2.66}$$

$$y_s \geq 0, \; s = 1, \ldots, S, \tag{2.67}$$

where v, $(\omega_1, \ldots, \omega_N)$, $(y_1, \ldots, y_S)$ are decision variables, z is a real number constraint on CVaR level which lies between the $z_{\min}$ (the minimum possible level of

CVaR) and $z_{\max}$ (the maximum possible level of CVaR) [69].

2.4.3 Prospect theory model with CVaR constraint

Solution of the PT-CVaR model is a single-objective problem, where the expected prospect theory utility function is maximised with desirable level of return and a given level of CVaR on the return distribution.

Following the logic of Sections 2.3.1 the prospect theory model with limited CVaR is formulated as follows:

$$\text{maximise } \mathrm{PT}_{\mathrm{CVaR}}(x) = \sum_{s=1}^{S} \pi_s \; v_s(r(x)), \tag{2.68}$$

subject to the constraints

$$\bar{r}(x) = \sum_{i=1}^{N} \bar{r}_i \; \omega_i \geq d, \tag{2.69}$$

$$\sum_{i=1}^{N} \omega_i = 1, \tag{2.70}$$

$$\omega_i \geq 0, \;\; i = 1, \ldots, N, \tag{2.71}$$

$$v - \sum_{i=1}^{N} \omega_i r_{is} \leq y_s, \;\; s = 1, \ldots, S, \tag{2.72}$$

$$\frac{1}{\alpha} \sum_{s=1}^{S} \rho_s \; y_s - v \leq z, \tag{2.73}$$

$$y_s \geq 0, \;\; s = 1, \ldots, S. \tag{2.74}$$

Summary

In this chapter we consider 9 models which will be used for further testing and analysis. These models take into account the investor's preferences in different forms. Generally, the following human behavioural preferences which is implemented in the models will be studied:

1. tradeoff between risk and return;

2. loss aversion;

3. risk aversion;

4. level of diversification.

On the one hand, it is interesting to analyse the performance of rationally based and behaviourally based optimal portfolios from the return and risk point of view, while on the other hand, the solution approach to mathematically complicated portfolio optimisation problems with nonlinear objective functions and constraints is significantly valuable.

Chapter 3

Solution approach

In the previous chapter we considered four basic models: mean variance, index tracking, prospect theory and cumulative prospect theory models. The mean variance problem is convex and can be solved easily with a built-in solver using different software as well as the index tracking problem which is simple to deal with using a standard solver such as *FortMP* in AMPL. In contrast, the prospect theory and the cumulative prospect theory models are non-convex. Hence, the solution approach becomes more challenging.

We also consider a cardinality constraint as a limit on the number of assets in the optimal portfolio. We suppose that the investor may prefer a certain amount of stocks in their optimal portfolio instead of the entire set of assets available in the market. While, the portfolio optimisation problem with a cardinality constraint takes into account the investor's behavioural preferences, it leads to a very challenging mathematical problem from the solution approach point of view.

The mean variance and prospect theory portfolio optimisation problem with a limit on the number of assets is a non-linear mixed-integer program [19], [87]. Generally there are two approaches of formulation for solving the cardinality constrained mean variance problem. The basic approach is to formulate it as classic Markowitz problem subject to standard linking constraints on thresholds plus the cardinality constraint. In this case different heuristic methods or standard simplex method which are suitable for non-linear problems are applied [87].

An alternative approach is to reformulate it directly as a bi-objective problem. This technique allows the investor to analyse the tradeoff between cardinality and mean-variance. Such an approach determines the set of nondominated points of the bi-objective problems in which an objective is smooth and combines mean and variance in the form of a quadratic function and the other is non-smooth. For the solution of the bi-objective optimisation problem a derivative free optimisation algorithm was chosen [17].

In our research we use the first (basic) approach because of two reasons. The first issue is the problem formulation. We need a unified form of problem formulations because our comparative analysis involves more then one model. It is easier to implement a new constraint to the standard problem instead of changing the objective function each time.

The second reason is the solution approach. Some of our models are very complex and require specific algorithms. Heuristic approaches can deal with these types of problems even when it is extended with new constraints.

For the MV and IT cardinality constrained models we use the standard solver *CPLEX* (AMPL) which is developed to deal with integer, mixed-integer, linear programming and quadratic problems, including problems with quadratic constraints possibly involving integer variables. For the behaviourally based models we have developed an approach specified for non-convex objective function with complex behavioural component. We found that a heuristic is an appropriate solution approach for our task.

3.1 Behaviourally based models

It is important to note that problems (2.20)–(2.23), (2.26)–(2.29) and (2.33)–(2.43) are non convex and functions (2.20), (2.26) and (2.33) are non differentiable. In addition we consider the cardinality constrained PT model which potentially makes the problem more complex for solving. As long as it is very difficult to find an optimal solution for this type of problem many researchers and traders use heuristics that are inexact methods to solve this sort of portfolio optimisation problems.

In our research we have used two heuristic solution approaches for the basic and cardinality constrained portfolio optimisation problems with behavioural component. The first is based on the differential evolution algorithm and the second is a genetic algorithm. We consider the traditional differential evolution algorithm and the differential evolution with the smoothing non-convex objective function using spline interpolation. Also in the development of paper [20], we suggest the genetic algorithm which is based on meta-heuristic approaches [40], in order to find the "optimal" solution for the cardinality constrained portfolio optimisation problem.

For the sake of simplicity in our calculations we define the prospect theory weighting function as $\pi(p) = p$ and use the original value function $v(r)$ as proposed in [79]:

$$v\left(r\right)=\begin{cases}(r-r_0)^{\alpha}, \text{ if } r\geq r_0,\\ -\lambda\left(r_0-r\right)^{\beta}, \text{ if } r<r_0.\end{cases} \tag{3.1}$$

3.1.1 Differential evolution

A recent addition to the class of evolutionary heuristics is a method of differential evolution proposed by R. Storn and K. Price [76], [63]. In our research to solve the problem (2.33)–(2.43) we use this algorithm which is based on the evolutionary principle. In this section, we consider a differential evolution approach which aims to obtain an "optimal" solution for the (cumulative) prospect theory problem.

Let N be the number of all available assets. We need to find an optimal value of a uniformly distributed variable $x = (\omega_1,\ \omega_2, \ldots, \omega_N) \in D_K \subseteq \mathbb{R}^N$, where D_K is a set of feasible objective function values, i.e. we are looking for the value of $x \in D_K$, which provides a solution for the problem (2.20) and (2.26). In order to find this optimal value of x we need to maximise the expected value of $\mathrm{PT}_{\mathrm{cc}}(r(x))$ (which is equivalent to (C)PT$(r(x))$ if $K = N$) using the following steps.

1. Initialisation. We define the set

$$D_K = \{v \in D, \text{ such that exactly } K \text{ components of vector } v \text{ are positive}\}.$$

Let $P \in \mathbb{N}$. We generate an initial population $v_i = (\omega_{i1}, \ldots, \omega_{iN}),\ \forall\ i = 1, \ldots, P^2\ v_i \in D_K$.

2. Mutation and Crossover. Choose vectors $v_a,\ v_b,\ v_c$ randomly from the vectors $v_l,\ l = 1, \ldots, P^2$, such that they do not coincide with v_i and each other. Also pick a random number $R \in \{1, \ldots, N\}$. We construct the components of a new vector $\tilde{v}_i \in D$ as follows. With probability CR and if $R = j,\ j = 1, \ldots, N$ for the jth component, vector $\tilde{v}_{ij} = v_{aj} + (F + z_1)(v_{bj} - v_{cj} + z_2)$ and $\tilde{v}_{ij} = v_{ij}$ otherwise. Here parameters $F \in [0, 2]$ and $\mathrm{CR} \in [0, 1]$ are called the differential weight and the crossover probability respectively and should be chosen by the user; quantities z_1 and z_2 are either zero with a low probability (e.g. 0.0001 and 0.0002, respectively), or are normally distributed random variables with a mean of zero and a small standard deviation (for example 0.02). The parameters z_1 and z_2 are optional for the differential evolution algorithm. They are used to add up some "noise" to the calculation of the resulting vector and avoid getting into local extrema.

3. Selection. Using equation (2.33) we calculate the values $\mathrm{PT}_{\mathrm{cc}}(v_i)$ and $\mathrm{PT}_{\mathrm{cc}}(\tilde{v}_i)$ and choose the maximum called $\max(v_i)$ to proceed to the new population which is used in the next generation until the stopping criteria (e.g. number of generations, precision, etc.) is met.

4. Final Assessment. In the last generation $g = G$ find the vector which $v_i^* = \{v_i | \max\{\mathrm{PT_{cc}}(v_1), \ldots, \mathrm{PT_{cc}}(v_{P^2})\},\ E(\max\ \mathrm{PT_{cc}}(v_i)) \geq d\}$ (d constraint check). The vector v_i^* then is our best solution [41].

3.1.2 Differential evolution with smoothing of the utility function using splines

In this thesis we implement spline interpolation for the prospect theory utility function into our differential evolution approach in order to solve the prospect theory problem. We simply smooth the original utility function and apply the differential evolution algorithm to solve the problem.

Smoothing splines often apply for discrete or noisy data to provide smooth curves. We obtain a practical, effective method for estimating the optimum amount of smoothing from the data. Derivatives can be estimated from the data by differentiating the resulting (nearly) optimal smoothing spline [83].

Note that the function (2.20) is not differentiable at the point $r = r_0$. This alternative approach to the calculation of an efficient portfolio according to prospect theory, based on the smoothing of the objective function was proposed in [25]. The idea is to use a cubic spline instead of the value function (2.18) in a δ-neighbourhood of the point $r = r_0$, $\delta > 0$. In other words, one can replace the value function (2.18) by its smoothed version:

$$v_\delta\left(r\right) = \begin{cases} v(r), \text{ if } r \notin (r_0 - \delta, r_0 + \delta), \\ \mathfrak{v}(r), \text{ if } r \in (r_0 - \delta, r_0 + \delta), \end{cases} \tag{3.2}$$

where $\mathfrak{v}(r) = ar^3 + br^2 + cr + d$. Since the values of functions $v(r)$ and $\mathfrak{v}(r)$ and their derivatives should coincide at the endpoints of the δ-neighbourhood, i.e. at points $r_0 - \delta$ and $r_0 + \delta$, we can calculate the coefficients $a,\ b,\ c,\ d$, of the cubic polynomial $\mathfrak{v}(r)$ from the system of linear equations:

$$\begin{cases} \mathfrak{v}(-\delta + r_0) = v(-\delta + r_0), \\ \mathfrak{v}'(-\delta + r_0) = v'(-\delta + r_0), \\ \mathfrak{v}(\delta + r_0) = v(\delta + r_0), \\ \mathfrak{v}'(\delta + r_0) = v'(\delta + r_0). \end{cases} \tag{3.3}$$

So, we can rewrite the formula (2.20) as:

$$\mathrm{PT}(v_\delta(r(x))) \to \max_{x \in D} . \tag{3.4}$$

The function $v_\delta(r)$ is smooth and differentiable at a point $r = r_0$.

3.1.3 Genetic algorithm

A genetic algorithm is a searching mechanism which is based on evolutionary principles of natural selection and genetics. The theoretical background of genetic algorithms was developed by Holland [40]. It works with populations of solutions and uses the principles of survival of the fittest. In genetic algorithms the variables of the solution are coded into chromosomes. To make a natural selection and get good solutions, chromosomes are evaluated by a fitness-criterion. In the considered optimisation problems the measure of fitness is usually connected with the objective function. For more information see [55], [12], [2].

To maximise the objective function or utility function $\mathrm{PT}_{\mathrm{cc}}(x)$ given in formula (2.33) using a genetic algorithm we need to make the following steps.

1. Initialisation. We define the set

$$D_K = \{x \in D, \text{ such that exactly } K \text{ components of vector } x \text{ are positive}\}.$$

Let $P \in \mathbb{N}$. We generate an initial population $x_i = (\omega_{i1}, \ldots, \omega_{iN})$, $\forall\ i = 1, \ldots, P^2$ $x_i \in D_K$.

2. Selection. At each generation $g = 1, \ldots, G$ we calculate values $\mathrm{PT_{cc}}(x_1), \ldots, \mathrm{PT_{cc}}(x_{P^2})$ and put them in decreasing order, i.e. we obtain a decreasing sequence

$$\left(\mathrm{PT_{cc}}(x_{m_1}) \geq \ldots \geq \mathrm{PT_{cc}}(x_{m_{P^2}})\right),$$

where set $x_{m_1}, \ldots, x_{m_{P^2}}$ is a permutation of the initial set $x_1, \ldots, x_{P^2}$. We fix the maximum value of the objective function $\max \mathrm{PT_{cc}}(x_i)$. Only the first $2P$ elements move to the new population without changes, i.e. $x_{m_1}, \ldots, x_{m_{2P}}$. Denote this elements of a new population $y_1, \ldots, y_{2P}$.

3. Crossover and mutation. We randomly choose two vectors $\tilde{x}_j$ and $\hat{x}_k$ in the set $\{x_{m_{2P+1}}, \ldots, x_{m_{P^2}}\}$ and breed them to produce a "child". In order to do this we construct the $l-$th element ($l = 1, \ldots, N$) of the new vectors $a_i = (a_{i1}, \ldots, a_{iN})$, $i = 2P+1, \ldots, P^2$, $a_i \in D_K$, from vectors $\tilde{x}_j$ and $\hat{x}_k$, $\forall j, k = 2P+1, \ldots, P^2$, by choosing between $\tilde{x}_{jl}$ and $\hat{x}_{kl}$ following the rules:

- if $\tilde{x}_{jl} = \omega_j$ and $\hat{x}_{kl} = \omega_k$ (i.e. the asset is in both parents portfolios), than the asset in the child is as follows $a_{il} = \chi \cdot \omega_j + (1-\chi) \cdot \omega_k$, where χ is randomly generated number in [0,1];

- if $\tilde{x}_{jl} = 0$ and $\hat{x}_{kl} = 0$ (i.e. the asset is not in either parent portfolios), than $a_{il} = 0$ (this asset is not in the child);

- if $\tilde{x}_{jl} = \omega_j$ and $\hat{x}_{kl} = 0$ (i.e. the asset is in only one of the parent portfolios), than with probability $\pi \quad a_{il} = \omega_j$ (i.e. this asset is included in the portfolio with probability π).

To introduce mutation we change each element of the constructed vector a_i with a given small probability $\zeta > 0$ for the randomly generated number from [0,1]. Then we ensure that the number of non-zero elements of the new vector is less than or equal to K and normalise the elements of this vector. We also find the maximum of the vectors $a_i, \tilde{x}_j, \hat{x}_k$ and denote this as y_i. This is the most fit vector and now move this to the new population. Continue while the last y_{P^2} element of the new population matrix have been processed.

4. Assessment. We calculate the values $\mathrm{PT_{cc}}(y_1), \ldots, \mathrm{PT_{cc}}(y_{P^2})$ and compare the maximum values of the obtained objective function max $\mathrm{PT_{cc}}(y_i)$ to max $\mathrm{PT_{cc}}(x_i)$. The new population proceeds to the new generation (if $g < G$) if and only if max $\mathrm{PT_{cc}}(y_i) \geq$ max $\mathrm{PT_{cc}}(x_i)$.

5. Final Assessment. In the last generation $g = G$ find the vector $y_i^* = \{y_i | \max\{\mathrm{PT_{cc}}(y_1), \ldots, \mathrm{PT_{cc}}(y_{P^2})\},\ E(\max\ \mathrm{PT_{cc}}(y_i)) \geq d\}$ (d constraint check). The vector y_i^* then is the best solution.

The implementation of both algorithms: the differential algorithm and the genetic algorithm, basic and with the extension constraints and modification, are presented in Appendix A.

3.2 Models with the CVaR constraint

It is well known that CVaR is an efficient measure of risk in modern finance [80], [7], [67]. We discussed its advantages in Chapter 2. In this section the solution approach to the basic models such as mean variance and prospect theory with a CVaR constraint are presented.

Unlike the single-objective mean variance and (cumulative) prospect theory models considered in section 2.4.2 the mean-variance-CVaR model is multi-objective, because one needs to minimise two objectives, namely variance and CVaR, subject to a desirable mean return. In order to simplify the calculations we transform it into a single-objective problem following the logic of the Pareto efficiency [36], [75]. As a result, we formulate problem (2.61)–(2.67), where for a desired level of portfolio return (like in the mean variance model) we minimise only variance, but with additional constraints on CVaR (see section 2.4.2 for details).

It is known that (see, e.g. [69]) the value x^* is a Pareto optimal solution of the problem (2.60) if and only if x^* is an optimal solution of the problem (2.61)–(2.67) with $z = \mathrm{CVaR}_\alpha(x^*)$ and $d = E(x^*)$ if the covariance matrix is positive definite. Note, that the positive definiteness of the covariance matrix ensures strict convexity of the objective function (variance) and, hence, guarantees the uniqueness of the optimal solution.

We deal with the mean-variance-CVaR model by doing the following steps.

1. Calculate minCVaR.

   We find the minimum value of CVaR for the specified data sample without constraining the mean portfolio return. The output is an optimal objective value denoted as minCVaR.

2. Calculate d_{minCVaR}.

   The maximum expected return (mean) acceptable for the CVaR-minimised portfolio (i. e. portfolio with CVaR = minCVaR calculated in the previous step) can be derived by solving the problem of maximising the mean portfolio return subject to CVaR lower limit equal to minCVaR. Obtained value of maximum mean is denoted by d_{minCVaR}.

3. Calculate d_{minvar}.

   We calculate the maximum value of expected return that is obtained by solving the classical Markowitz optimisation problem (with no constraint on expected return), i. e. minimising variance, and denote it as d_{minvar}.

4. Calculate $[d_{\min}, d_{\max}]$ and choose d^*.

   Choose $d_{\min}$ as the maximum of d_{minvar} and d_{minCVaR}.

   $d_{\max} = \max(\bar{r}(x))$ is the maximum possible expected return that can be found as the optimal objective value in the problem of maximising portfolio's expected return without additional constraints, except compulsory constraint on asset weights sum in the portfolio, $\sum_{i=1}^{N} \omega_i = 1$.

   Choose $d^* \in [d_{\min}, d_{\max}]$.

5. Calculate $[z_{\min}, z_{\max}]$ and choose z^*.

   We solve the optimisation problem $\min \text{CVaR}$ subject to portfolio return level d^* and denote the result as $z_{\min}$.

   We minimise variance subject to portfolio return level d^* and define the optimal solution as x^*. Then we calculate the CVaR for the found portfolio x^* and denote the result as $z_{\max}$.

Choose $z^* \in [z_{\min}, z_{\max}]$.

6. Solution of the problem.

   Solve the problem (2.61)–(2.67) subject to the obtained portfolio mean return value d^* and chosen value of CVaR z^*.

We implement a CVaR constraint into the prospect theory model in order to analyse the performance and to compare the results with the mean-variance-CVaR model. The problem (2.68)–(2.74) can be solved using heuristic approaches developed in Section 3.1.

Chapter 4

Computational results

4.1 Empirical study

4.1.1 Data

We have solved the portfolio optimisation problems using publicly available data relating to five major market indices, available from the OR-Library [11]. The five market indices are the Hang Seng (Hong Kong), DAX 100 (Germany), FTSE 100 (UK), S&P 100 (USA) and the Nikkei 225 (Japan) for 290 time periods each (weekly data), taken from: http://people.brunel.ac.uk/m̃astjjb/jeb/orlib/portinfo.html. All of these problems were considered previously by Chang et al. (2000) (see [19]) and Woodside-Oriakhi at al. (see [87]). The size of these five test problems ranged from $N = 31$ (Hang Seng) to $N = 225$ (Nikkei 225) and are presented in Table 4.1.

	Data set	Number of stocks N	Number of time periods S
1	Hang Seng	31	290
2	DAX 100	85	290
3	FTSE 100	89	290
4	S&P 100	98	290
5	Nikkei 225	225	290

Table 4.1: Test problem dimension

The data used in this thesis is given in the form of matrices of asset prices. We transformed the original data sets into matrices of asset returns. It is widely accepted to use logarithm of the price ratio in order to derive the rate of returns, instead of using absolute asset price relations [56]. In our research the rate of return r is calculated using the prices p for each time period s as follows:

$$r_i = \ln\left(\frac{p_{is}}{p_{is-1}}\right), i = 1, \dots, N, \quad s = 1, \dots, S,$$

where N is the number of assets and S is the total number of time periods.

The in-sample computational results reported in this research are obtained using the first 100 time periods of the data sets described above. The remaining time periods (190) are used in bootstrap out-of-sample tests.

In this research we apply simulation of the data with a particular type of distribution as an out-of-sample test data for our models. We are interested in so called "bullish" market dynamics which indicates the investor's confidence that the positive trend of the prices will continue. It also characterises increasing investments and high activity of exchange trades which follows from a stable economic situation. In contrast a "bearish" market demonstrates pessimistic expectations which leads to stagnation and long-term decreasing of the prices. In order to investigate the performance of the models in different conditions we simulate these two trends in the matrix of the asset returns.

The out-of-sample data set which simulates bullish and bearish markets were obtained using the built in functions available in the Statistics Toolbox in Matlab. For bullish market simulations we apply the function *datasample*. This function $y = datasample(data, k)$ returns k observations sampled uniformly at random, with replacement, from the specific data set in *data*. In order to obtain the data set which possesses properties of a bullish market we simulate the returns based on historical data of market growth (data form 4.01.2005 to 30.12.2005; 252 time periods in total).

Bearish market simulations are made using the command *mvtrnd*. The statement $r = mvtrnd(kR, df, cases)$ returns a matrix of random numbers chosen from the multivariate t-distribution, where kR is matrix of historical returns from the crisis period, df is the degrees of freedom (in our computational study $df = 5$) and it is either a scalar (like we use in this research) or could be a vector with cases elements (case is the number of lines, equal to 100 for these tests). We chose a t-distribution because the tails of a Student t-distribution tend to zero slower than the tails of the normal distribution which reflects more the real market situation. For the simulations of bearish market we used historical data related to the FTSE 100 index of the global crisis period in 2008 available in Bloomberg Database (data from 1.01.2008 to 31.12.2008; 261 time periods in total) as an initial matrix for simulation. So we apply both, crisis historical data as a sample of data and a t-distribution simulation in order to underline the contrast in two different types of return distributions, bullish and bearish.

The mean variance and the index tracking models (basic formulation and with additional constraints) were solved using AMPL software with CPLEX (version 12.5.1.0) as a software package for solving large-scale optimisation problems. The prospect theory and cumulative prospect theory portfolio selection problems (basic formulation and with additional constraints) were implemented using Matlab software, as well as built-in and specially developed functions. All simulations and bootstraping were run in Matlab. The system runs under MS Windows 7 64-bit SP 1 and in our computational work we used an Intel Core i3-2310M pc with a 2.10 GHz processor and 8.0 GB RAM.

4.1.2 Parameters of the models

Hereinafter, we consider the prospect theory and the cumulative prospect theory models as a class of behaviourally based models for the sake of convenience because their properties investigated in the analysis are similar. For these models (also with additional constraints and index tracking) we use constant values of the parameters $\lambda = 2.25$, $\alpha = \beta = 0.88$ as proposed by Tversky and Kahneman in their paper [79]. For equations (**??**) and (**??**) we put $\delta = 0.61$, $\gamma = 0.69$ in accordance with [79].

Tversky and Kahneman consider cumulative prospect theory as a complex choice model. Estimation of such types of problems is very difficult because of the large number of parameters. In order to reduce this number they "focused on the qualitative properties of the data rather than on parameter estimates and measures of fit" [79] by using a nonlinear regression procedure for estimation of the parameters of equation (2.18), they found that "the median exponent of the value function was 0.88 for both gains and losses, in accordance with diminishing sensitivity" and "the median λ was 2.25 ... and the median values of δ and γ, respectively, were 0.61 and 0.69"[79].

In order to compare the performance of different models we used the same level of desired portfolio return d for basic and cardinality constrained models only. For each data set the parameter $d = \max\bar{r} - (\max\bar{r} - \min\bar{r})\cdot 0.25$, where $\max\bar{r}$ and $\min\bar{r}$ are maximum and minimum mean of assets returns for the specific data set. It should be mentioned that as one can see this level was chosen to be high enough to consider this condition as extreme for the proposed models. Taking into account the character of the prospect theory model which chooses more aggressive portfolios with high level of returns the choice of a high d is justified (see the discussion of the results in Section 4.2). For some sets of data (especially for big data sets) we have to adjust the parameter d (reduce the value of d) in order to provide the feasibility of the optimal portfolio for behaviourally based models.

The values of the parameter d can be seen in Table 4.2 which also includes the reference point and the bank interest rate. It should be noted, that the reference point in Table 4.2 is used for the behaviourally based models, basic and with other additional constraint except index tracking. These values for the reference point reflect average interest rate (IR) for different market economies (depends on corresponding market index of used data set). Following the definition of the prospect theory value function parameter r_0 is set for each time period and in the case of basic models (with and without cardinality and CVaR constraints) is constant. In contrast, for the index tracking problem it is dynamic and changes for each time period depending on the index value.

We used K as a parameter for cardinality constrained models (basic models MV and PT). In order to distinguish the index tracking problem from the other models let K^* be a limit on the number of assets for these index tracking problems. As diversification levels of the basic models and the index tracking based models are different we use different values for the parameters K and K^*. Cardinality constraint models according to its formulation have lower and upper limits on the asset weight. We use $l_i = 0.01$ and $u_i = 1$ for these limits.

Data set	d	IR	r_0	K	K^*
Hang Seng	0.0118	0.005	0.00005	7	15
DAX 100	0.006	0.0025	0.000025	10	20
FTSE 100	0.0077	0.0025	0.000025	10	25
S&P 100	0.0109	0.005	0.00005	5	25
Nikkei 225	0.0005	0.0001	0.000001	3	25

Table 4.2: Tabulated values of model parameters

For models with a CVaR constraint it is necessary to define the feasible set of solutions for parameter d^* (expected return) and z^* (CVaR constraint) for each data set as was described in Section 3.2. Testing MV and PT models with a CVaR constraint we found that the feasible set for the target return which was defined for MV is also suitable for the PT model. However, parameter z has different feasible sets for these two problems, so, it is impossible to define the same z^* for them. Complexity and particular properties as well as behaviour of the prospect theory objective function compared to the mean variance problem lead to different feasible sets for the solution for these two models.

Thus, we define z_{min} and z_{max} for the prospect theory separately as the real minimum and maximum value of CVaR for the prospect theory model without a constraint on CVaR based on G observations (in each generation). In Table 4.3 boundaries of feasible sets for parameter z and chosen z^* for the MV model as well as boundaries for the parameter d (which are d_{min} and d_{max}) and d^* for both, MV and PT with CVaR constraint models (due to the similarity) are presented. The boundaries of feasible sets of solutions with parameter z and chosen value for z^* for PT model with CVaR constraint are shown in Table 4.4.

We calculate CVaR for considered models with confidence level $\alpha = 95\%$ in this research.

Data set	d_{min}	d_{max}	d^*	z_{min}	z_{max}	z^*
Hang Seng	0.0081	0.0153	0.011	0.0472	0.0570	0.056
DAX 100	0.0034	0.01008	0.007	0.0124	0.0191	0.019
FTSE 100	0.0054	0.0119	0.007	0.0102	0.0149	0.014
S&P 100	0.0030	0.0169	0.006	0.0082	0.0145	0.014
Nikkei 225	0.0018	0.0056	0.002	0.0287	0.0364	0.036

TABLE 4.3: Tabulated values of parameters d^* and z^* for MV model with CVaR constraint

Data set	z_{min}	z_{max}	z^*
Hang Seng	0.0631	0.0754	0.0727
DAX 100	0.0233	0.0273	0.0268
FTSE 100	0.0242	0.0378	0.025
S&P 100	0.0218	0.041	0.038
Nikkei 225	0.0392	0.0542	0.0487

TABLE 4.4: Tabulated values of parameter z^* for PT model with CVaR constraint

4.1.3 Parameters of the heuristic approaches

Previously we note that both, prospect theory and cumulative prospect theory models, are mathematically complex problems and therefore they are difficult to deal with. In Section 3.1 we proposed different solution approaches to these models. In order to obtain an "optimal" solution for the behaviourally based models we use differential evolution, differential evolution with spline interpolation and a genetic algorithm.

It is known that the parameters of heuristics and metaheuristic algorithms have a great influence on the effectiveness and efficiency of these algorithms (see for example [4]). It is important to find correct parameter settings for each problem and data set. To obtain the best solution for the problems we illustrate this here with the algorithms using the first data set (Hang Seng) trying to choose the most appropriate value for each parameter and analyse the effectiveness of each algorithm in order to define the best for our research. The analysis and selection of the parameters for the chosen algorithm for the other sets of data are presented in Appendix B.

Our choice of parameter is based on three comparison criteria: computational time, utility as the value of the objective function PT(x) and range of PT(x) as a difference $\xi = \max \mathrm{PT(x)} - \min \mathrm{PT(x)}$. In order to study the stability of the algorithm we test each combination of parameters 10 times and compare mean CPU time, mean utility and ξ in the form of the difference $\max \mathrm{PT(x)} - \min \mathrm{PT(x)}$.

The optimal solution of the prospect theory problem is typically unknown and we have no benchmark for comparative analysis. So we define the optimal solution to be the best in the set of solutions we have obtained in our tests. In this section we also consider the performance of different approaches to the prospect theory problem in order to define the best in terms of several indicators described above.

Much research has been devoted to using heuristic approaches as an effective tool for dealing with non-convex problems. Maringer in 2008 presented a comparative analysis of quadratic, power and the prospect theory utility function performance with different levels of loss aversion [49]. He used a differential evolution approach in order to get a solution for the prospect theory model. The paper focused more on performance of the models and parameters of the optimal portfolio return distribution but not on the solution approach itself.

To the best of our knowledge there are no studies where the differential evolution with spline interpolation and a genetic algorithm have been applied to the prospect theory problem. From the mathematical point of view it is interesting to investigate the performance of different solution approaches applied to problem (2.33)–(2.43) which is non convex and function (2.33) which is non differentiable.

Differential evolution algorithm

The differential evolution algorithm efficiency depends on parameters such as the differential weight F, the crossover probability CR, the population size P and the number of generations G. It is necessary to start with the F parameter because the differential weight is the key parameter for the differential evolution algorithm. As we noticed this value significantly influences the mean value of the objective function and its dispersion. It is known that $F \in [0, 2]$ (see Section 3.1.1), however, in our case a value larger than 1 gives us a very unstable solution. Thus, we define the following values to test: 0.05, 0.15, 0.5 and 0.95. In the calculations in Table 4.5 for our specific function, the smaller the value of the differential weight the higher the value of objective function (utility) and the smaller the range of the solution ($\xi = 0$ leads to the best quality of the solution). The value 0.05 gives us the best results according to all three criteria.

It should be mentioned that in choosing parameter $F = 0.05$ we set CR $=$ 0.5, $P = 20$ and $G = 100$. This choice is based on preliminary analysis and recommendations available in the literature [63], [30]. Hereinafter while testing each parameter one by one we fix the values of other parameters ($F = 0.05$, CR $= 0.5$, $P = 20$ and $G = 100$) in order to show the difference in the results.

The next step is to choose the optimal value for the crossover probability. It is known that the CR $\in [0, 1]$ (see Section 3.1.1). We analyse three values for CR $= 0.3, 0.5, 0.8$. The results in Table 4.5 confirms that CR $= 0.5$ provides an acceptable CPU time (better than CR $= 0.8$) and a stable utility (better than CR $= 0.3$) which leads to a stable solution.

The parameters F and CR should be chosen for the specific objective function and features of the problem. In contrast, the values of G and P primarily depend on the size of the problem. For example, for a data set with 32 assets (including the index as an asset) we define values for G and P, so, for larger scale problems we use values in proportion to the best we find here. We consider the values of these parameters as a function of problem size. We now explain the choice of these parameters only for the smallest data set Hang Seng.

We test values $P = 15,\ 20,\ 25$ in order to define suitable parameters in terms of CPU time and optimality of the solution. As one can see in Table 4.5 the population size of 20 provides the best utility (quantitatively and in terms of stability) with reasonable computational time. The value $P = 25$ requires more time (+35.6 seconds) compared to $P = 20$, providing the same utility while a smaller population size leads to an unstable solution.

Within the DE algorithm we need to decide which number of generations is the best for this problem size. We define three points to test which are $G = 70, 100, 130$ in order to find a balance between solution quality and computational time. We choose 100 because it provides maximum utility with range 0 in an acceptable CPU time as shown in Table 4.5.

Parameter	Parameter value	CPU time	$PT(x)$	ξ
F	**0.05**	**61.8**	**0.6237**	**0**
	0.15	64.4	0.6235	0.0003
	0.5	66	0.62084	0.0013
	0.95	69.2	0.56534	0.0269
CR	0.3	61.6	0.62356	0.0002
	0.5	**61.8**	**0.6237**	**0**
	0.8	65.4	0.6237	0
P	15	35.2	0.62302	0.0031
	20	**61.8**	**0.6237**	**0**
	25	97.4	0.6237	0
G	70	43.2	0.62342	0.0005
	100	**61.8**	**0.6237**	**0**
	130	80.6	0.6237	0

Table 4.5: Differential evolution parameter comparison (Hang Seng data set)

Differential evolution algorithm with spline interpolation

Due to the fact that the principles of the DE with spline interpolation algorithm is identical to that of the DE, the results of testing provide the same trend. We only changed the value of the differential weight $F = 0.1$ because it gives better CPU time. One can find the results of testing in Table 4.6. Finally, the chosen parameters for the DE with spline interpolation algorithm applied to the PT model are $F = 0.1$, $CR = 0.5$, $P = 20$ and $G = 100$.

Parameter	Parameter value	CPU time	$PT(x)$	ξ
F	**0.1**	**63.5**	**0.6237**	**0**
	0.15	65.4	0.6235	0.0005
	0.5	67	0.62084	0.0017
	0.95	69.2	0.56534	0.036
CR	0.3	62.7	0.6134	0.0024
	0.5	**63.5**	**0.6237**	**0**
	0.8	67.3	0.6237	0
P	15	41.6	0.62302	0.0043
	20	**63.5**	**0.6237**	**0**
	25	103.8	0.6237	0
G	70	43.2	0.62342	0.0005
	100	**63.5**	**0.6237**	**0**
	130	85.7	0.6237	0

Table 4.6: Differential evolution with spline interpolation parameter comparison (Hang Seng data set)

Genetic algorithm

There are three main parameters in the genetic algorithm: the mutation probability z, the population size P and the number of generations G. These parameters are the most influencing on the outcome of the algorithm.

As shown in Table 4.7 we tested different values for each of these parameters in order to find the optimal settings. In the analysis we used constant parameters $z = 0.5$, $P = 15$ and $G = 70$ for the Hang Seng (Hong Kong) data set while testing each parameter in order to show the difference in the results. This choice is based on preliminary analysis and recommendations available in the literature.

First of all the mutation probability should be chosen. We took several different values for the parameter z. As one can see in Table 4.7 the CPU time does not change much and does not depend on the value of this parameter. It is obvious that $z = 0.5$ gives us a necessary and sufficient mutation component to obtain the best stability of the solution. The values larger ($z = 0.7$) or smaller ($z = 0.3$) provide the solution with lower level of stability. In addition, the value of the objective function in this case is not the best as well.

Population size is a very important parameter for any heuristic algorithm. One should find the right value of P for the specific problem. There are many recommendations in the literature which can help to choose suitable parameters for the genetic algorithm (see for example [33]) according to the specific objective function. Most of the guides suggest to use the number of variables and multiply it by 10 for such complex objective functions such as prospect theory. At the same time for the portfolio optimisation problem the recommended population size is around 100-200 [5]. In our case there are 32 assets (including the index as an asset) in a data set and we found testing the model that reasonable interval for the search is [10, 20] for such a small matrix. Taking into account that in our algorithm we use population size P^2 we obtained an interval [100, 400] which covers the first recommendation ($32 \cdot 10 = 320$) and the second one ([100,200]).

The population size greatly affects the CPU time. Again we are searching for a balance between computational time and stability because the quality is not improving much with an increasing value of P. However, the solution becomes more volatile once you decrease the population size (see results for $P = 10$ in the Table 4.7). We define $P = 15$ as the best for our experiments because it gives optimal utility and saves computational time compared to $P = 20$. Also $P = 15$ provides a good search space for exploration.

We study the interval [40,100] in order to define the optimal parameter value for the number of generations. Previously, we tested extremely high values such as 300 and 400 and the quality of the solution did not change much versus the value of 100 but the CPU time increases dramatically. One can see in Table 4.7 that the difference between the results obtained using $G = 70$ and $G = 100$ is not much too, so, we can save time for approximately the same range of the solution and the value of objective function while decreasing the value of G results in a deteriation the solution.

As was mentioned previously, we consider values of P and G parameters as a function of the problem size for the heuristic approaches and one should choose it proportionally to the problem size. The values of G and P parameters for the genetic algorithm for different sized problems can be found in Appendix B.

Parameter	Parameter value	CPU time	$PT(x)$	ξ
z	0.3	36.6	0.6219	0.0084
	0.5	**36.2**	**0.62354**	**0.0002**
	0.7	36.8	0.62352	0.0004
P	10	15.6	0.60916	0.0713
	15	**36.2**	**0.62354**	**0.0002**
	20	67.4	0.62361	0.0002
G	40	25.6	0.6235	0.0034
	70	**36.2**	**0.62354**	**0.0002**
	100	47.2	0.62358	0.0001

Table 4.7: Genetic algorithm parameter comparison (Hang Seng data set)

It is important to note that all three different algorithms give us the same value of the objective function. This fact verifies the solution obtained with the proposed solution approaches and confirms the accuracy of the implementation of the prospect theory model into heuristic approaches.

We notice that the value of criterion ξ for the genetic algorithm is slightly worse than the results achieved when testing the differential evolution algorithm. At the same time the CPU time of the GA is much less which gives a benefit compared to the DE. This benefit defines the choice of this solution approach for further computational study for this research.

4.2 Comparative analysis of the performance of the models

4.2.1 Connection to previous research

We would like to distinguish two empirical studies in the literature which contributes to the development and application of the behaviourally based models.

Maringer [49] studied PT investor's risk aversion and loss aversion using higher order moments such as skewness and kurtosis. He found and proved empirically that "higher level of risk aversion might lead to an investment with more, not less volatility" [49]. It can be explained by sensitiveness towards increasing positive skewness and decreasing kurtosis. The more loss aversion increases, the more risk seeking appears and the more aggressive the portfolio that is chosen when behavioural investors face losses.

The preferences of some assets under the CPT assumptions in terms of specific characteristics of the assets was researched by Barberis and Huang in 2008 [9]. They found that positively skewed securities are more preferable in the CPT optimal portfolios in comparison with the MV model. They proposed that this fact is the effect of the probability weighting function [9]. The same result was obtained by Bernard and Ghossoub in 2010 [14].

4.2.2 Comparative analysis of the basic models

The summary of the mean variance, prospect theory and cumulative prospect theory basic models performance in-sample is displayed in Table 4.8. This table shows the ratio $\bar{r}/\sigma$, mean portfolio $\bar{r}$, standard deviation σ, VaR and CVaR as well as the number of assets n in the optimal portfolio and CPU time in seconds (CPU) which is significant for the heuristic approaches.

As was mentioned before (in Section 2.4) the VaR and the CVaR are often used together and the application of this combination of two risk measures can be beneficial to the risk estimation for non-symmetric loss distributions which are characteristics for real market conditions.

Obviously, computational time for the heuristic approach used for behaviourally based models is much higher than for quadratic linear programming which is applied for the mean variance model. Computational complexity of cumulative prospect theory makes it even worse. The CPU time is triple that for the prospect theory.

Usually the quality of the heuristic approach for smaller problems can be measured in deviation of the heuristic solution from the optimal solution [87]. However, the efficient frontier for prospect theory portfolio optimisation problem is unknown. As can be noticed in Table 4.8 it is very difficult to compare the portfolios average return and risk using different models because the PT model's behaviour is more aggressive in terms of mean return and respectively gets higher risk. In this case we suggest to use the ratio $\bar{r}/\sigma$ as a unified measure of performance of the portfolio which includes the mean portfolio and a risk measure. The larger the value of $\bar{r}/\sigma$ the more efficient the portfolio.

We proposed previously that the mean variance model will be used as a benchmark in this research. It is justified by the fact that the mean variance model provides the optimal solution in terms of return and variance. One can see that in each data set the ratio $\bar{r}/\sigma$ of the MV model is the best among the others. Only in the Nikkei 225 data set the PT model did achieve a higher ratio because it found the best portfolio with an extremely high mean return and it is hard to compare the results in this case. If we set this high level of portfolio mean return as d in the constraint for expected return for the mean variance model the ratio $\bar{r}/\sigma$ is higher than achieved for the prospect theory model due to the smaller σ.

As was mentioned previously we treat the index as a normal asset and allow it to be chosen as an asset in an optimal portfolio in order to check its attractiveness for the investigated models. We suppose that the index should be an efficient asset. According to our experiments only cumulative prospect theory chooses the index in two data sets out of five. In spite of the poor results in most of the parameters, CPT shows a better CVaR value compared to the PT model. This model shows mostly conservative investment behaviour according to the risk measure σ and CVaR when compared to the PT model.

It is easy to see that according to the in-sample computational results the prospect theory model achieved higher mean portfolio return in each data set (especially in the Nikkei 225 data set) than MV and CPT models. In spite of this PT and CPT model got mostly the best value of the VaR parameter which indicates good downside protection and agrees with the theoretical concept of BPT. However, PT and CPT models demonstrate less diversification of their portfolios in comparison with the mean variance model.

Data set	Model	CPU	n	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR
Hang Seng	MV	0.015	9	**0.3926**	0.0118	**0.0301**	-0.0373	**-0.0644**
d=0.0118	PT	36.2	8	0.3922	**0.0131**	0.0335	**-0.0371**	-0.0727
	CPT	104	5	0.3616	0.0130	0.0359	-0.0500	-0.0668
DAX 100	MV	0.031	16	**0.4683**	0.0060	**0.0128**	**-0.0141**	**-0.0197**
d=0.006	PT	550	12	0.4529	**0.0083**	0.0183	-0.0145	-0.0248
	CPT	1790	7	0.4369	0.0080	0.0183	-0.0144	-0.0206
FTSE 100	MV	0.031	14	**0.5636**	0.0077	**0.0137**	-0.0121	-0.0178
d=0.0077	PT	630	17	0.4797	**0.0090**	0.0188	**-0.0114**	-0.0272
	CPT	1904	22	0.4933	0.0085	0.0171	-0.0153	**-0.0163**
S&P 100	MV	0.046	11	**0.5115**	**0.0109**	**0.0213**	-0.0279	-0.0328
d=0.0109	PT	721	6	0.4940	**0.0109**	0.0221	-0.0267	-0.0391
	CPT	1994	7	0.4717	0.0105	0.0222	**-0.0265**	**-0.0265**
Nikkei 225	MV	0.14	13	0.0159	0.0005	**0.0196**	-0.0349	-0.0395
d=0.0005	PT	1179	4	0.1434	0.0034	0.0238	-0.0338	-0.0384
	CPT	4862	4	**0.1598**	**0.0039**	0.0246	**-0.0325**	**-0.0326**

TABLE 4.8: Comparative analysis of basic models (in-sample). Summary

In summary, the main findings are:

- The PT model, mostly, is more aggressive than MV and CPT because it chooses portfolios with higher level of returns. Probably, the reference point forces this model to focus more on the assets with high returns. In spite of the similarity of the PT and CPT models the cumulative prospect theory is not so aggressive because of the probability weight function which prevents the appearance of high risk in the optimal portfolio.
- The PT model is more efficient than CPT according to the $\bar{r}/\sigma$ indicator in the main.
- Behaviourally based models are more beneficial in terms of VaR in comparison with the MV in most of the data sets. This fact reflects the nature of the PT model which focuses on downside protection.
- Behaviourally based models provide portfolios which are normally less diversified than the mean variance model.

- The index as an asset, generally, was not attractive for all the three models in our data sets. In a volatile market, the index returns are not attractive as an investment for portfolio selection models giving less benefits than ordinary assets.

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR
Hang Seng	MV	**0.6844**	**0.0014**	**0.0020**	**-0.0018**	**-0.0028**
	PT	0.5388	0.0012	0.0021	-0.0023	-0.0031
	CPT	0.4260	0.0009	0.0021	-0.0026	-0.0035
DAX 100	MV	1.9785	0.0024	**0.0012**	0.0004	-0.0001
	PT	1.7766	0.0024	0.0013	0.0005	-0.0005
	CPT	**2.4833**	**0.0033**	0.0013	**0.0012**	**0.0006**
FTSE 100	MV	1.1103	0.0016	0.0014	-0.0008	-0.0014
	PT	1.6570	0.0023	0.0014	0.0000	-0.0006
	CPT	**1.8717**	**0.0024**	**0.0013**	**0.0002**	**-0.0003**
S&P 100	MV	0.7232	0.0013	**0.0019**	-0.0017	-0.0024
	PT	0.8441	0.0016	**0.0019**	-0.0015	-0.0023
	CPT	**0.9175**	**0.0017**	**0.0019**	**-0.0012**	**-0.0021**
Nikkei 225	MV	0.3317	0.0005	**0.0016**	-0.0021	-0.0029
	PT	0.9804	**0.0019**	0.0020	-0.0014	**-0.0022**
	CPT	**0.9960**	**0.0019**	0.0020	**-0.0013**	**-0.0022**

Table 4.9: Comparative analysis of basic models (out-of-sample: bootstrap). Summary

We now investigate the performance and behaviour of the models for out-of-sample tests. We applied bootstraping, using the data sets with the time periods from 101 to 290. We randomly choose observations from the specified range to obtain out-of-sample data set. We repeat this iteration 1000 times and statistically obtain portfolio characteristics in the form of mean return, risk, VaR and CVaR. We would like to draw the riders attention that here and further on values of VaR and CVaR are calculated with respect to return (instead of loss). It means that starting from Table 4.9 the higher the values or these risk measures the better.

As shown in Table 4.9, behaviourally based models maintain the leading position in terms of portfolio returns in most of the data sets while the MV is better in the risk parameter σ. In looking at the ratio $\bar{r}/\sigma$, one can notice that the cumulative

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR
Hang Seng	MV	**22.3694**	**0.0795**	**0.0036**	**0.0736**	0.0722
	PT	19.5623	0.0792	0.0040	0.0727	**0.0724**
	CPT	15.0646	0.0794	0.0053	0.0705	0.0686
DAX 100	MV	**31.5966**	0.1582	**0.0050**	**0.1501**	**0.1478**
	PT	22.6324	0.1583	0.0070	0.1471	0.1443
	CPT	21.0312	**0.1584**	0.0075	0.1458	0.1430
FTSE 100	MV	22.9827	**0.1189**	0.0052	0.1106	0.1084
	PT	27.4194	0.1187	0.0043	0.1115	0.1099
	CPT	**30.1336**	0.1188	**0.0039**	**0.1120**	**0.1103**
S&P 100	MV	23.7931	0.0986	**0.0041**	0.0918	0.0901
	PT	22.1445	**0.0991**	0.0045	0.0917	0.0894
	CPT	**24.0876**	**0.0991**	**0.0041**	**0.0924**	**0.0904**
Nikkei 225	MV	**26.3628**	0.1385	**0.0053**	0.1263	0.1236
	PT	18.3664	**0.1388**	0.0076	**0.1301**	**0.1272**
	CPT	17.5419	0.1382	0.0079	0.1248	0.1217

TABLE 4.10: Comparative analysis of basic models (out-of-sample: simulation of bullish market). Summary

prospect theory is better compared to the other models. Also this model shows better performance according to the VaR and CVaR parameters.

To investigate further, we extended our out-of-sample tests to look at the models performance for both bullish and bearish market data. In out-of-sample simulation of bullish market tests based on a distribution which is typical for increasing market in the period of economic growth, the results are shown in Table 4.10.

According to Table 4.10, behaviourally based models, especially the PT model, shows best results in terms of mean return of the portfolio as well as VaR and CVaR parameters. The cumulative prospect theory model also shows better VaR and CVaR statistics compared to MV. At the same time, the MV model is more beneficial from the risk parameter (σ) point of view and also better in the ratio $\bar{r}/\sigma$.

Our out-of-sample simulation of bearish market tests based on a distribution which is typical for a decreasing market in the period of economic crisis. As one can see in Table 4.11, surprisingly, behaviourally based models, especially the PT model, look mostly better in terms of VaR and CVaR but are worse in σ and the

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR
Hang Seng	MV	0.2580	**0.0004**	**0.0015**	**-0.0022**	**-0.0026**
	PT	0.2187	0.0003	0.0016	**-0.0022**	-0.0030
	CPT	**0.2659**	**0.0004**	0.0016	-0.0023	-0.0030
DAX 100	MV	-0.1797	-0.0007	**0.0036**	-0.0067	-0.0081
	PT	-0.1460	**-0.0006**	0.0039	**-0.0066**	**-0.0075**
	CPT	**-0.1912**	-0.0008	0.0040	-0.0074	-0.0088
FTSE 100	MV	**-0.0235**	**-0.0001**	0.0035	-0.0061	-0.0075
	PT	-0.1712	-0.0004	**0.0026**	**-0.0047**	**-0.0057**
	CPT	-0.0619	-0.0002	0.0031	-0.0052	-0.0064
S&P 100	MV	-0.1088	**-0.0003**	0.0031	**-0.0054**	-0.0066
	PT	**-0.0956**	**-0.0003**	**0.0030**	**-0.0054**	**-0.0065**
	CPT	-0.2678	-0.0008	0.0032	-0.0060	-0.0074
Nikkei 225	MV	-0.0658	-0.0002	0.0038	-0.0064	-0.0079
	PT	**0.0420**	**0.0002**	**0.0037**	**-0.0062**	**-0.0074**
	CPT	-0.2317	-0.0009	0.0039	-0.0069	-0.0083

Table 4.11: Comparative analysis of basic models (out-of-sample: simulation of bearish market). Summary

$\bar{r}/\sigma$ indicator in comparison with the MV model. According to the ratio $\bar{r}/\sigma$, VaR and CVaR parameters, the PT is more beneficial than the CPT in the bearish market. Generally, the cumulative prospect theory model demonstrates the worst results in these tests, especially in the ratio $\bar{r}/\sigma$, VaR and CVaR parameters. Also it is hard to say which model performed better in terms of mean returns because the results for this parameter fluctuates between different models.

Tests on simulated bearish market data show the benefits of PT model in terms of VaR and CVaR risk measures which are significantly important in decreasing market conditions. It means that the behavioural component of PT model provides better downside protection in critical market situations in comparison with the traditional mean variance approach.

The results for all tests with higher order moments values (skewness and kurtosis indicators) can be found in Appendix C.

4.2.3 Comparative analysis of the models with cardinality constraint

As was proposed previously we investigate the performance of a cardinality constrained mean variance and prospect theory models in this section. In-sample results for these models are presented in Table 4.12. We noticed that CPU time for cardinality constrained prospect theory model is slightly less than for the basic version. The genetic algorithm works faster in this case because the parameter K restricts the searching space. It is interesting to see that the PT model does not reach the maximum of the allowed number of assets in the portfolio in 3 sets while the MV model takes the opportunity to include as many assets as is allowed.

Similar to in-sample results for the basic models the cardinality constrained MV model showed better values of σ, the ratio $\bar{r}/\sigma$ and CVaR while the PT model was better in mean portfolio return and VaR parameters. We can conclude that the behavior of the prospect theory model does not change much with the additional cardinality constraint and it demonstrates the aggressive portfolio choice.

Data set	Model	CPU	K	n	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR
Hang Seng	MV	0.09	7	7	**0.3919**	0.0118	**0.0301**	**-0.0363**	**-0.0643**
d=0.0118	PT	37	7	6	0.3915	**0.0132**	0.0338	**-0.0363**	-0.0731
DAX 100	MV	0.25	10	10	**0.4604**	0.0060	**0.0130**	-0.0159	**-0.0191**
d=0.006	PT	520	10	8	0.4484	**0.0080**	0.0179	**-0.0141**	-0.0250
FTSE 100	MV	0.11	10	10	**0.5631**	0.0077	**0.0137**	-0.0122	**-0.0180**
d=0077	PT	600	10	8	0.5218	**0.0096**	0.0184	**-0.0107**	-0.0269
S&P 100	MV	0.14	5	5	**0.4911**	0.0109	**0.0222**	-0.0263	**-0.0371**
d=0.0109	PT	690	5	5	0.4729	**0.0120**	0.0253	**-0.0255**	-0.0413
Nikkei 225	MV	0.89	3	3	0.0023	0.0000	**0.0209**	-0.0381	-0.0439
d=0.0005	PT	1105	3	3	**0.1420**	**0.0034**	0.0239	**-0.0327**	**-0.0369**

TABLE 4.12: Comparative analysis of cardinality constrained models (in-sample). Summary

The performance of the models out-of-sample using the bootstrap method are presented in Table 4.13. The prospect theory model again is better in terms of portfolio mean return and VaR as compared to the MV while the mean variance model shows benefit in σ and mostly in CVaR parameters. At the same time the resulting values of the ratio $\bar{r}/\sigma$ are difficult to analyse due to the ambiguity of the results.

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR
Hang Seng	MV	**0.5302**	**0.0011**	**0.0021**	**-0.0023**	**-0.0023**
	PT	0.4391	0.0009	0.0022	-0.0027	-0.0035
DAX 100	MV	**2.2284**	0.0027	**0.0012**	0.0007	**0.0002**
	PT	2.1879	**0.0028**	0.0013	**0.0008**	0.0001
FTSE 100	MV	1.2833	0.0017	**0.0014**	-0.0005	-0.0011
	PT	**1.6132**	**0.0025**	0.0015	**0.0000**	**-0.0006**
S&P 100	MV	0.8257	0.0016	**0.0020**	**-0.0017**	**-0.0024**
	PT	**0.9538**	**0.0021**	0.0022	**-0.0017**	-0.0035
Nikkei 225	MV	0.1783	0.0003	0.0019	-0.0027	-0.0035
	PT	**0.8366**	**0.0015**	**0.0018**	**-0.0015**	**-0.0022**

Table 4.13: Comparative analysis of cardinality constrained models (out-of-sample: bootstrap). Summary

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR
Hang Seng	MV	20.2696	**0.0792**	**0.0039**	**0.0729**	0.0715
	PT	**20.2971**	**0.0792**	**0.0039**	0.0727	**0.0716**
DAX 100	MV	**27.3371**	**0.1585**	**0.0058**	**0.1491**	0.1463
	PT	23.1736	0.1582	0.0068	0.1469	**0.1465**
FTSE 100	MV	**22.9842**	0.1187	**0.0052**	**0.1103**	**0.1085**
	PT	22.8369	**0.1191**	**0.0052**	**0.1103**	0.1088
S&P 100	MV	**20.9687**	0.0993	**0.0047**	**0.0910**	0.0890
	PT	20.4622	**0.0994**	0.0049	**0.0910**	**0.0892**
Nikkei 225	MV	**16.6797**	**0.1393**	**0.0083**	**0.1246**	0.1210
	PT	16.1642	0.1387	0.0086	0.1241	**0.1214**

Table 4.14: Comparative analysis of cardinality constrained models (out-of-sample: simulation of bullish market). Summary

According to the out-of-sample test results (simulation of bullish and bearish market) which are shown in Table 4.14 and Table 4.15 similar conclusions regarding the behaviour of the two studied models can be made. The mean variance model is mostly better in σ and the ratio $\bar{r}/\sigma$ criteria in a bullish market while in a bearish market it demonstrates the CVaR is slightly worse than the prospect theory. The advantage of the ratio $\bar{r}/\sigma$ and mean return for the models in a bearish market changes from one data set to another which underlines the high volatility of the returns in such type of market situations. However, the prospect

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR
Hang Seng	MV	0.1935	**0.0003**	**0.0015**	**-0.0022**	**-0.0028**
	PT	**0.1988**	**0.0003**	**0.0015**	-0.0023	-0.0030
DAX 100	MV	-0.2595	-0.0010	**0.0037**	-0.0070	-0.0089
	PT	**-0.2159**	**-0.0008**	**0.0037**	**-0.0066**	**-0.0084**
FTSE 100	MV	**0.0603**	**0.0002**	0.0035	**-0.0056**	**-0.0068**
	PT	-0.0764	-0.0002	**0.0033**	**-0.0056**	**-0.0068**
S&P 100	MV	**-0.2076**	**-0.0006**	**0.0031**	**-0.0056**	-0.0069
	PT	-0.4262	-0.0013	**0.0031**	-0.0063	**-0.0068**
Nikkei 225	MV	**-0.2621**	-0.0011	**0.0043**	-0.0082	-0.0100
	PT	-0.0329	**-0.0002**	0.0046	**-0.0076**	**-0.0094**

Table 4.15: Comparative analysis of cardinality constrained models (out-of-sample: simulation of bearish market). Summary

theory model demonstrates the benefit in $\bar{r}$ in the bullish market and slightly better VaR and CVaR in both types of market similar to the performance of the basic model.

4.2.4 Comparative analysis of the models with a CVaR constraint

Analysing the performance of the basic and cardinality constrained models we notice that behaviourally based models are generally better on the CVaR criteria compared to the mean variance model. Thus, it is interesting to see the performance of the prospect theory and mean variance models with a limit on CVaR. In this section we consider the results of the two models (MV and PT models) with a CVaR constraint as formulated in Section 2.4.

As can be seen in Table 4.16 the implemented CVaR constraint increases the CPU time for the prospect theory model dramatically. We can conclude that the target return constraint together with a limit on the CVaR makes the search space for the solution too tight in the region of the intersection of feasible sets. The genetic algorithm requires much more time to overcome local optimum because of less freedom. We also notice that the diversification of the portfolios for the prospect theory model is not changed much when compared to the basic version.

Similar to the previous in-sample results the prospect theory model has larger $\bar{r}$ in the portfolios than the mean variance portfolios losing out in σ, VaR and CVaR parameters due to the fact that z^* (which is point up the CVaR constraint) for PT model is much lower then for the MV one because of feasibility of the solutions. The mean variance model also gains in the ratio $\bar{r}/\sigma$ criteria in each set except the largest one. In addition, for that data set, the PT model shows better diversification as well.

Data set	Model	z^*	CPU	n	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR
Hang Seng	MV	-0.056	0.03	8	0.380	0.011	**0.029**	**-0.038**	**-0.056**
d=0.011	PT	-0.073	40	5	**0.392**	**0.013**	0.034	-0.039	-0.070
DAX 100	MV	-0.019	0.09	14	**0.468**	0.007	**0.015**	**-0.013**	**-0.019**
d=0.007	PT	-0.027	647	5	0.442	**0.009**	0.020	-0.019	-0.024
FTSE 100	MV	-0.014	0.11	15	**0.541**	0.007	**0.013**	**-0.012**	**-0.014**
d=0.007	PT	-0.025	754	18	0.513	**0.009**	0.018	-0.019	-0.025
S&P 100	MV	-0.014	0.12	23	**0.550**	0.006	**0.011**	**-0.012**	**-0.014**
d=0.006	PT	-0.038	685	18	0.494	**0.009**	0.018	-0.023	-0.033
Nikkei 225	MV	-0.036	0.41	13	0.098	0.002	**0.021**	**-0.030**	**-0.036**
d=0.002	PT	-0.049	2553	24	**0.119**	**0.003**	0.023	-0.033	-0.040

TABLE 4.16: Comparative analysis of models with CVaR constraint (in-sample). Summary

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR
Hang Seng	MV	**0.6665**	**0.0013**	**0.0019**	**-0.0021**	**-0.0029**
	PT	0.0760	0.0002	0.0022	-0.0035	-0.0043
DAX 100	MV	**2.2560**	**0.0028**	**0.0013**	**0.0009**	**0.0003**
	PT	1.7529	0.0024	0.0014	0.0001	-0.0004
FTSE 100	MV	1.2400	0.0016	**0.0013**	-0.0006	-0.0011
	PT	**1.7619**	**0.0024**	0.0014	**0.0001**	**-0.0005**
S&P 100	MV	**1.6640**	0.0021	**0.0012**	**0.0000**	**-0.0004**
	PT	1.2857	**0.0025**	0.0019	-0.0007	-0.0015
Nikkei 225	MV	0.5530	0.0009	**0.0016**	-0.0017	-0.0024
	PT	**0.9456**	**0.0015**	**0.0016**	**-0.0012**	**-0.0018**

TABLE 4.17: Comparative analysis of models with CVaR constraint (out-of-sample: bootstrap). Summary

Out-of-sample bootstrap tests results for the models with a CVaR constraint are presented in Table 4.17. The mean variance model demonstrates better values of σ while the prospect theory model holds the leading position with regard to the largest $\bar{r}$ for each set of data which is predictable due to the previous analysis.

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR
Hang Seng	MV	**19.7664**	**0.0793**	**0.0040**	0.0727	0.0712
	PT	17.6371	**0.0793**	0.0045	**0.0730**	**0.0716**
DAX 100	MV	**26.4675**	**0.1582**	**0.0060**	**0.1479**	0.1457
	PT	20.4444	0.1579	0.0077	0.1477	**0.1461**
FTSE 100	MV	25.6109	0.1187	**0.0046**	0.1110	0.1089
	PT	**25.9197**	**0.1189**	**0.0046**	**0.1114**	**0.1091**
S&P 100	MV	**38.2829**	0.0990	**0.0026**	**0.0947**	0.0936
	PT	24.0364	**0.0992**	0.0041	0.0943	**0.0938**
Nikkei 225	MV	25.6196	**0.1387**	0.0054	**0.1298**	0.1271
	PT	**27.5647**	0.1381	**0.0050**	0.1294	**0.1276**

Table 4.18: Comparative analysis of models with CVaR constraint (out-of-sample: simulation of bullish market). Summary

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR
Hang Seng	MV	**0.4767**	**0.0005**	**0.0011**	**-0.0013**	**-0.0017**
	PT	0.3905	0.0004	**0.0011**	-0.0014	**-0.0017**
DAX 100	MV	-0.1402	-0.0005	**0.0037**	**-0.0064**	-0.0079
	PT	**-0.1175**	**-0.0004**	0.0038	-0.0068	**-0.0078**
FTSE 100	MV	**-0.1006**	**-0.0003**	0.0034	-0.0057	-0.0071
	PT	-0.1472	-0.0004	**0.0030**	**-0.0054**	**-0.0067**
S&P 100	MV	**-0.1560**	**-0.0004**	0.0028	**-0.0050**	-0.0062
	PT	-0.3418	-0.0009	**0.0027**	-0.0054	**-0.0060**
Nikkei 225	MV	**-0.1673**	**-0.0007**	0.0042	-0.0077	-0.0090
	PT	-0.1905	**-0.0007**	**0.0039**	**-0.0071**	**-0.0085**

Table 4.19: Comparative analysis of models with CVaR constraint (out-of-sample: simulation of bearish market). Summary

At the same time other criteria can not tell us much about the behaviour of these models. The results are too inconsistent to draw any conclusions.

The other out-of-sample tests which are simulation of bullish and bearish markets are presented in Table 4.18 and Table 4.19, and mostly confirm the findings obtained previously. In a bullish market we notice the trend that the prospect theory model shows better performance in terms of $\bar{r}$ and CVaR while the mean variance model is better according to the ratio $\bar{r}/\sigma$ and the σ criteria. In the bearish market the MV model with CVaR constraint is still the best in the ratio $\bar{r}/\sigma$, however, it mostly loses the advantage in the σ when compared to PT.

It is interesting to see that in the bearish market the prospect theory model with

CVaR constraint shows more benefits in the CVaR criterion than the MV model. In spite of the lower z^* constraint compared to MV model the PT model manages to exceed the results of the traditional approach in terms of CVaR. Surprisingly, behaviourally based models are more useful in the specific crisis market condition in terms of downside protection than traditional portfolio selection approach.

4.3 The index tracking problem and prospect theory model

The index tracking problem usually chooses many assets in the optimal portfolio which is very difficult to manage and rebalance. That is why the IT has a cardinality constraint which then becomes a computationally challenging problem for researchers. In this section we discuss empirical results of in-sample and out-of-sample performance of the IT and PT with index tracking problems (with and without cardinality constraint). As out-of-sample tests we use only simulation of bullish and bearish market. We do not apply bootstrap method because our in-sample tests include all available observations (all 290 time periods of used data sets).

4.3.1 Basic index tracking and prospect theory models

The computational results presented in this section for index tracking problems were obtained using five data sets described earlier but with all 290 time periods. The first asset in each data set is the index and is not included in the investment universe of assets. We also use a methodology described above for simulation of bullish and bearish markets in out-of-sample tests.

We analyse the performance of the results by several criteria such as CPU time, the number of assets in the portfolio n, tracking error TE, tracking error over the index TE o, tracking error under the index TE u. It should be noted that we use absolute values of TE, TE o, TE u for our analysis. Table 4.20 reflects the empirical results of the experiment for the used sets of data.

Data set	Model	CPU time	n	TE	TE o	TE u
Hang Seng	IT	0.047	30	0.4290	0.2444	0.1845
	PT+IT	70	20	0.8420	0.5690	0.2730
DAX 100	IT	0.109	69	0.3354	0.1835	0.1519
	PT+IT	242	51	1.1763	0.7336	0.4427
FTSE 100	IT	0.141	81	0.2855	0.1657	0.1198
	PT+IT	250	46	1.1463	0.7919	0.3544
S&P 100	IT	0.125	83	0.2682	0.1553	0.1130
	PT+IT	347	67	0.9409	0.5881	0.3529
Nikkei 225	IT	0.266	159	0.1686	0.0921	0.0765
	PT+IT	1803	69	0.9802	0.6300	0.3501

Table 4.20: Comparative analysis of the index tracking and prospect theory with index tracking problem (in-sample)

It is easy to see from the table that the number of assets in the PT with IT optimal portfolios is approximately half those in the IT portfolios. This issue gives a good advantage to the PT with IT in comparison with the IT model because of transaction costs and convenience of portfolio management.

It is obvious that the tracking error of the IT model solution is always less than in PT with IT optimal portfolios but it is still comparable. One can notice that the beneficial difference between parameters TE o for IT and PT with IT models is much greater (in proportion to the tracking error) than between parameters TE u for these models. This means that the PT with IT model chooses assets with higher return than the IT model using the reference point (index) only as a starting point but not as a benchmark. These facts confirm that the PT with IT model focuses more penalty on not achieving the reference point compared with exceeding it.

We test the performance of the two models using out-of-sample simulations and use the same criteria for analysis. Firstly, we simulate on a bullish market. Table 4.21 reflects the empirical results of the experiment.

We should note that the behaviour of the investigated models in the bullish market is very similar to the in-sample performance. According to the tracking

Data set	Model	TE	TE o	TE u
Hang Seng	IT	0.1292	0.1292	0
	PT+IT	0.3589	0.3589	0
DAX 100	IT	0.0934	0.0918	0.0016
	PT+IT	0.5470	0.5470	0
FTSE 100	IT	0.1304	0.1304	0
	PT+IT	0.6335	0.6335	0
S&P 100	IT	0.1271	0.1271	0
	PT+IT	0.4432	0.4432	0
Nikkei 225	IT	0.1225	0.1225	0
	PT+IT	0.5660	0.5660	0

TABLE 4.21: Comparative analysis of the index tracking and prospect theory with index tracking problem (out-of-sample: simulation of bullish market)

error parameter the PT with IT portfolios show smaller value compare to the in-sample results.

We also test the performance of two models using an out-of-sample simulation on a bearish market. It is interesting to explore the performance of the models in opposite conditions. In Table 4.22 one can find the out-of-sample empirical results.

Data set	Model	TE	TE o	TE u
Hang Seng	IT	0.1960	0.1960	0
	PT+IT	0.1806	0.1806	0
DAX 100	IT	0.2991	0.1673	0.1317
	PT+IT	0.2928	0.1481	0.1446
FTSE 100	IT	0.3013	0.1217	0.1795
	PT+IT	0.3136	0.1164	0.1972
S&P 100	IT	0.3026	0.1172	0.1854
	PT+IT	0.2984	0.1085	0.1899
Nikkei 225	IT	0.2750	0.1342	0.1408
	PT+IT	0.3017	0.0880	0.2137

TABLE 4.22: Comparative analysis of the index tracking and prospect theory with index tracking problem (out-of-sample: simulation of bearish market)

In contrast with the previous results, PT with IT model fails to show a good outcome. This model performs worse in each data set for each parameter when compared to the IT. Only tracking error of the prospect theory improved and becomes even less then for IT model portfolios.

Finally, we can conclude that the prospect theory model with index tracking as the reference point is very effective in an increasing market due to its mathematical formulation which makes it desirable to exceed the reference point (in our case it is the index values). In addition it is more beneficial in terms of lower number of assets in the optimal portfolio. However, in a crisis market situation PT with IT model performs worse than IT. Thus, the prospect theory model adjusted for index tracking works well in a stable or increasing market condition.

4.3.2 Cardinality constrained index tracking and prospect theory with index tracking models

The index tracking model with a cardinality constraint is a very computationally challenging problem. On the one hand, the optimal solution is unknown and one should set the termination criteria very carefully to obtain the best results. On the other hand, the CPU time required is significantly large versus the non cardinality constrained model.

For the index tracking and prospect theory with index tracking models with cardinality constraint we used similar asset thresholds $l_i = 0.01$, $u_i = 1$ $(i = 1, \ldots, N)$ as described in Section 4.1.2 and parameter K^* which is the number of assets allowed to be included in the optimal portfolio.

Tables 4.23, 4.24 and 4.25 show the performance of the IT and PT with IT models with the cardinality constraint in-sample, out-of-sample (simulation of bullish market) and out-of-sample (simulation of bearish market) empirical results.

As displayed in the tables the behaviour of the models with the cardinality constraint is completely similar to the behaviour of the non-cardinality constrained IT and PT with IT models in different conditions. It should be noted that CPU time for behavioural models with the additional constraint does not change much and it implies that the genetic algorithm deals well with such type of complex problems. So, the cardinality constrained models results confirms the conclusion about the character of compared models made above.

Data set	Model	CPU time	K^*	n	TE	TE o	TE u
Hang Seng	IT	102	15	15	0.5760	0.3316	0.2448
	PT+IT	74	15	15	1.1871	0.7828	0.4044
DAX 100	IT	200	20	20	0.5889	0.3280	0.2609
	PT+IT	275	20	20	1.3309	0.9616	0.3693
FTSE 100	IT	193	25	25	0.6650	0.3819	0.2831
	PT+IT	323	25	24	1.4432	1.0323	0.4109
S&P 100	IT	176	25	25	0.5555	0.3223	0.2332
	PT+IT	459	25	22	1.2972	0.9111	0.3861
Nikkei 225	IT	612	25	25	0.7211	0.3845	0.3367
	PT+IT	2780	25	25	1.3179	0.9637	0.3542

Table 4.23: Comparative analysis of index tracking and prospect theory with index tracking problem with cardinality constraint (in-sample)

Data set	Model	TE	TE o	TE u
Hang Seng	IT	0.1519	0.1519	0
	PT+IT	0.3915	0.3915	0
DAX 100	IT	0.1202	0.1195	0.0007
	PT+IT	0.7190	0.7190	0
FTSE 100	IT	0.1826	0.1826	0
	PT+IT	0.7285	0.7285	0
S&P 100	IT	0.1674	0.1674	0
	PT+IT	0.6149	0.6149	0
Nikkei 225	IT	0.1296	0.1296	0
	PT+IT	0.6326	0.6326	0

Table 4.24: Comparative analysis of index tracking and prospect theory with index tracking problem with cardinality constraint (out-of-sample: simulation of bullish market)

Summary.

In this chapter the empirical study and analysis are presented. We discuss the parameters of the models and the constraints as well as define parameters for developed heuristic algorithms applied to the prospect theory and cumulative prospect theory model. We mentioned above that using heuristic solution approaches the parameters of these algorithms is very important for an accurate solution.

We also tested MV, PT and CPT basic models as well as with cardinality and CVaR constraints in different market conditions. It is interesting to note that

Data set	Model	TE	TE o	TE u
Hang Seng	IT	0.2484	0.2484	0
	PT+IT	0.1992	0.1992	0
DAX 100	IT	0.2907	0.1761	0.1145
	PT+IT	0.3343	0.1652	0.1691
FTSE 100	IT	0.3248	0.1165	0.2083
	PT+IT	0.3268	0.0954	0.2313
S&P 100	IT	0.2795	0.1208	0.1586
	PT+IT	0.3066	0.0876	0.2189
Nikkei 225	IT	0.2939	0.1720	0.1219
	PT+IT	0.3366	0.0943	0.2423

Table 4.25: Comparative analysis of the index tracking and prospect theory with index tracking problem with cardinality constraint (out-of-sample: simulation of bearish market)

behaviourally based models (with and without additional constraints) mostly were better in terms of returns, VaR and CVaR in all tests. The reference point in these models leads to more aggressive portfolios and higher level of returns. However, CPT is not as aggressive as the PT model because it focuses not only on loss aversion but on transformed probabilities too which take into account the number of returns below and above the reference point. This affects portfolio selection providing the portfolios with good downside protection (see the CVaR criterion of CPT model in-sample and out-of-sample).

We found that even in a bearish market (out-of-sample test) the prospect theory model was more beneficial in terms of the VaR and CVaR than the traditional mean variance model. Significantly this conclusion is valid for the model with CVaR constraint. We can assume that loss aversion and risk aversion which are used in the prospect theory model help to reduce the risk of portfolios in the form of the VaR and CVaR.

In unpredictable market conditions the index tracking portfolio selection problem becomes very popular. We investigated the prospect theory model with the index as the reference point (with and without cardinality) compared to the basic index tracking model. It has been found that PT model is more beneficial in terms of lower number of assets in the portfolio than index tracking (for models without cardinality constraint) that reduces transaction costs and makes rebalancing of the portfolio more convenient. We also noticed that returns of the PT with index tracking model mostly exceed the index returns which confirms our previous conclusion about the impact of the reference point. However, in a bearish market the prospect theory model shows greater losses compared to the index tracking model.

Chapter 5

Conclusion

The behavioural approach to portfolio theory has become very popular in the last decade because the market has demonstrated significant instability. There is much theoretical evidence in the literature that behaviourally based models could help to decrease the risk of the portfolio since they take into account natural loss aversion and risk aversion biases of the investors. However, we found that there is a lack of practical and empirical studies in the literature which could show and prove these benefits and shed light on the performance of these models in different market situations.

5.1 The main contribution

In this research we studied behaviourally based models such as the prospect theory model and its extended version cumulative prospect theory using comparative analysis with the traditional mean variance and index tracking models. In order to investigate the benefits of a behavioural approach we implemented cardinality and CVaR constraints to these models and tested the results out-of-sample using the bootstrap method and simulation of bullish and bearish return distributions. The results were presented for five publicly available data sets which reflect the dynamics of major world markets.

We developed several solution approaches for the prospect theory and cumulative prospect theory models to obtain the accurate solution using heuristics. The differential evolution algorithm and genetic algorithm were implemented in Matlab in order to do this. We also justify the parameter choice for these models using an empirical study due to the importance of the parameters in heuristic algorithms application.

Both applications of the prospect theory model to portfolio optimisation and index tracking problems show the model obtains higher returns in comparison with the mean variance approach and index tracking model. It can be explained by the effect of the reference point. The prospect theory wants to exceed the reference point (for example, risk free rate) as much as possible which reflects the psychological biases. So, this reference point steers the model to choose the assets with higher returns no matter which desired level of return for the whole period is set.

Out-of-sample tests also confirm that application of the prospect theory model in a bullish market is beneficial in terms of returns. At the same time, in a crisis market situation the returns of the PT and CPT models are worse in contrast with the mean variance but not significantly.

The main finding here is that behaviourally based models (with and without proposed constraints) outperform traditional portfolio optimisation model in terms of VaR and CVaR for almost all out-of-sample tests. We showed empirically that the psychological biases used in these models provide secure downside risk protection and leads to a better VaR and CVaR as measures that better captures "tail risk" compared to variance.

In this thesis the prospect theory with index tracking has also been investigated. We can conclude that prospect theory optimal portfolios performed better in terms of returns than index tracking model and the index itself in-sample and in a bullish market. However, the PT model was slightly worse in a bearish trend compared to the index tracking model. At the same time it has been found that the PT with IT model is normally less diversified than the IT model which is a benefit in terms of transaction costs and portfolio management issues.

The main contributions in this thesis are:

1. The heuristic solution approaches (the differential evolution (basic and with spline interpolation) and the genetic algorithm) which are developed particularly for the behaviourally based models and can deal with a large universe of assets.

2. Empirical evidence of VaR and CVaR benefits of behaviourally based models is compared to the mean variance model. The prospect theory model then can be considered as a proxy of mean-CVaR model.

3. Diversification benefit of the prospect theory with index tracking model compared to the traditional index tracking model has been empirically verified.

4. In-sample and out-of-sample results show that the prospect theory with index tracking model has better returns then the index tracking model (with and without cardinality constraint). We can conclude that the prospect theory with index tracking model is a proxy for enhanced index tracking model.

We would like to point that, in this thesis prospect theory was applied to a large universe of assets. Previously, only small experiments were presented in the literature (for example 2-3 assets). Thus, this empirical study aims to encourage the use of prospect theory in practice along with mean variance and index tracking models for specific real market conditions.

5.2 Ideas for the future research

It should be noted that the problem of portfolio optimisation using a behavioural approach is very challenging. There are different ways to investigate its solution and performance.

As was proposed in this research we developed several heuristic solution approaches for the prospect theory model taking into account the specific features of the model. As an idea for future work, one can bring more intelligent choice of the assets in the portfolio into the breeding stage of the genetic algorithm based on the observations and preferences of the studied model. In each generation one distinguishes the assets which is included in the best portfolio and use this information for the breeding stage in the next generation. Instead of checking all assets in the data the algorithm could faster find the preferable one using the information about frequency of appearance of assets in previous best portfolios. It could help to decrease the CPU time for this algorithm by reducing the search space of suitable assets for the best portfolios and decreasing the number of generations.

In this thesis we used coefficients of risk aversion, loss aversion and for probability weighting function obtained by Tversky and Kahneman in [79]. In our tests we noticed that the results of the prospect theory and cumulative prospect theory models are very sensitive to the values of α, β, λ, γ and δ. We propose that these parameters will change depending on real market conditions. Previously we discussed irrationality of the investor because he/she becomes risk seeking (not risk aversive) when faces losses. Therefore, risk aversion coefficient in a bullish market tend to be greater than that in a bearish market. It is interesting to test the prospect theory model with different values of these parameters in order to define appropriate values for different market conditions.

Also we investigated the performance of the prospect theory model with several constraints and in different conditions. However, it could be interesting to add a non-risky asset (i.e., cash account) in the data set to research its influence on the behaviour of the model's choice. As for the out-of-sample testing it would be useful to obtain and analyse the performance of the prospect theory model in simulation trading which includes holding periods and rebalacing thereafter. One can run the model using $1, \ldots, 100$ time periods and obtain the best portfolio. Then another run using $10, \ldots, 110$ should be made which imitates rebalacing stage. Sequential repetition of these steps provide the performance of the model obtained by technique which is close to real market trading including revision of chosen portfolio in several steps.

Appendix A

Implementation of the solution approaches

A.1 Differential evolution algorithm

Pseudo-code of the differential evolution algorithm for the prospect theory portfolio optimisation problem with cardinality constraint is given below.

```
Generate initial population $v_i \in D_K$, $i = 1, \dots, P^2$,
cycle of $G$ generations
    for each $v_i$ in population $P$
    choose 3 random vectors $v_a \neq v_b \neq v_c \neq v_i$
        for each component $j$ of $v_i$ do
            with probability $\pi_1 : z_1 \leftarrow N(0, \sigma_1)$, else $z_1 = 0$
            with probability $\pi_2 : z_2 \leftarrow N(0, \sigma_2)$, else $z_2 = 0$
            pick $u_j \sim U(0, 1)$
            if $u_j < \mathrm{CR}$ or $j = R$
                then $\tilde{v}_{ij} = v_{aj} + (F + z_1)(v_{bj} - v_{cj} + z_2)$
```

`else` $\tilde{v}_{ij} = v_{ij}$

`if` $PT(\tilde{v}_i) > PT(v_i)$

`then` $x_i = \tilde{v}_i$

`else` $x_i = v_i$

`In` $g = G$ `find` $x_i^* = \{x_i | \max\{\mathrm{PT_{cc}}(x_1), \ldots, \mathrm{PT_{cc}}(x_{P^2})\},\ E(\max\ \mathrm{PT_{cc}}(x_i)) \geq d\}$.

In the beginning of the script file we set the CPU time counter. Then all the prescribed parameters and constants are introduced. We use the matrix of asset returns as input data, so, we load a file with the necessary data set, count return means for each asset and define the size of the vector of assets. We also calculate d for a specific data set using a range of possible returns for this set. The main script file follows the steps of the differential evolution algorithm invoking developed functions for generating an initial population, breeding and mutation, calculating the objective function and expected returns.

The first function generates an initial population (matrix $N \times P^2$) with limit on the number of assets K which are chosen for each vector randomly from $[3, K]$. Each element of this matrix is generated randomly following the rule $0.1 + 0.9 * \nu$, where $\nu \in U(0, 1)$ in order to reduce the appearance of small values. Then we normalise each vector of the initial population to provide the condition $\sum_{i=1}^{N} \omega_i = 1$ and check the buy-in threshold constraint l_i and u_i.

In each generation of the algorithm we apply differential evolution breeding and mutation. For each vector in the population using the developed function of choice we randomly choose 3 integer numbers of vectors which are distinct from the given one and each other. Then we find z_1 and z_2 with prescribed probabilities and do differential evolution crossover following the rules described in Section 3.1.1. We normalise the absolute value of the obtained vector, check for the buy-in threshold constraint l_i and u_i and the cardinality constraint. After this it is necessary to normalise again. The specifics of the algorithm is that because of the crossover formula the weights are spread all over the set and became too small. That is why we normalise twice, before and after checking the mentioned constraints. Then we choose a more fittest vector according to the maximum objective function criteria. The developed function provides the value of the prospect theory value in accordance with problem formulation (2.18). The new population is then created.

In the last generation $g = G$ we choose the solution from the last population which has the maximum value of the objective function and satisfies the d constraint. We use the developed function for calculating the expected returns for each potential solution.

Remark.

When dealing with large scale data sets, the differential evolution algorithm fails to find a best solution for the chosen d. In this case we use some neighborhood for the parameter d in the form of $d - n$. We start to select the right n from 0.00002 and increase each time by multiplying by 10 if the programme still does not find an optimal solution.

For the differential evolution with spline interpolation solution approach we modify the function which calculates the prospect theory utility. In the new function we identify the values of the utility which are very close to the origin from both sides, negative and positive, and implement the spline interpolation in order to smooth the objective function using the coefficients obtained in (3.3).

It is easy to change the standard prospect theory problem to the prospect theory with index tracking problem. We simply change the scalar value of the reference point r_0 into dynamic values of the index. We also remove the d constraint check from the main script.

The prospect theory with CVaR can be solved using the same main script where the final assessment stage is modified in order to choose the solution which satisfies not only the target return constraint but the CVaR limit as well. We developed the function for calculation of the CVaR of portfolio.

It should be noticed that for the basic prospect theory model (without cardinality constraint) we put $K = N$ thereby making the cardinality constraint redundant.

A.2 Genetic algorithm

Pseudo-code of the genetic algorithm for the prospect theory portfolio optimisation problem with a cardinality constraint is given below.

Generate initial population $x_i \in D_K$, $i = 1, \dots, P^2$,

cycle of G generations

calculate values $\text{PT}_{\text{cc}}(x_1), \dots, \text{PT}_{\text{cc}}(x_{P^2})$

sort $\text{PT}_{\text{cc}}(x_{m_1}) \geq \dots \geq \text{PT}_{\text{cc}}(x_{m_{P^2}})$

save $\max\ \text{PT}_{\text{cc}}(x_i)$

$x_{m_1}, \dots, x_{m_{2P}} = y_1, \dots, y_{2P}$ proceed to the next generation

randomly pick $\tilde{x}_j$ and $\hat{x}_k$ in the set $\{x_{m_{2P}}, \cdots, x_{m_{P^2}}\}$

$\forall\ i, j, k, l,\ i, j, k = 2P + 1, \dots, P^2$, $l = 1, \dots, N$

if $\tilde{x}_{jl} = \omega_j$ and $\hat{x}_{kl} = \omega_k$

then $a_{il} = \chi \cdot \omega_j + (1 - \chi) \cdot \omega_k$, $\chi \in U(0, 1)$

else if $\tilde{x}_{jl} = 0$ and $\hat{x}_{kl} = 0$

then $a_{il} = 0$

else if $\tilde{x}_{jl} = \omega_j$ and $\hat{x}_{kl} = 0$

then with π $a_{il} = \omega_j$

with mutation probability $\zeta > 0$

$a_{il} \leftarrow \hat{a}_{ij}$, $\hat{a}_{il} \in U(0, 1)$

choose $\max\text{PT}_{\text{cc}}(y_i) = \max\{(a_i, \tilde{x}_j, \hat{x}_k)\}$

find $\text{PT}_{\text{cc}}(y_i) = \max\{\text{PT}_{\text{cc}}(y_1), \dots, \text{PT}_{\text{cc}}(y_{P^2})\}$

`choose` $\mathrm{PT_{cc}}(y_i^*) = \max\{\max\ \mathrm{PT_{cc}}(y_i), \max\ \mathrm{PT_{cc}}(x_i)\}$

y^*i `is an optimal solution`

For this algorithm we develop four functions in order to create an initial population, to calculate the objective function (the prospect theory utility), to breed and mutate the elements of the new population and to calculate the expected return of each vector of asset weights. All these functions are used in the main script which follows the steps of the genetic algorithm discribed in Section 3.1.3. As all functions except for the breeding and mutation are similar to the differential evolution algorithm we will now describe only the second function which is specific for the genetic algorithm.

The function of breeding and mutation was developed in order to create a new population from the $2P$ best elements of the initial population which are not changed and from the $P^2 - 2P$ elements which are used for breeding. In the case when an element is in both parents we randomly choose $\chi \in U(0,1)$ and apply the formula $a_{il} = \chi \cdot \omega_j + (1-\chi) \cdot \omega_k$ in order to decide which weight this asset should take. In our opinion, it must be different from the parents' weights. In the case if the element is not in both parents we simply put it 0 for the child. If it is only in one parent, we include this element with its weight to the child vector with the chosen probability π. The probability $\pi = 10\%$ was chosen because of the asset selection feature of this particular solution approach. It converges to a solution with a low number of assets with high returns. A higher level of probability here might increase the CPU time of the algorithm without any improvement in the solution.

In this function we also implement mutation. It should be noted that we mutate only zeros elements. Then we check the cardinality condition in two ways. If the number of non zero elements n in the vector is greater then K we define the difference $\delta = n - K$ and in a cycle of δ repetitions we randomly choose the element in this vector and if it is greater then zero make it zero. If $n < 3$ (because we are not interesting in portfolios of 2 elements) we randomly choose a zero element and generate the value using the rule $0.1 + 0.9 * \nu$, where $\nu \in U(0,1)$

as in the first function. Then we normalise the child vectors and check the buy-in threshold constraint. It is necessary to define which vector, mum, dad or child, will go to the new population. We choose by the maximum value of the objective function.

All extensions for prospect theory models (index tracking, cumulative prospect theory utility and CVaR constraint) can be made using modified functions for calculating the prospect theory utility or applying the developed function for the CVaR constraint to the main script similar as described in the previous section. Also the cardinality constraint can be removed by making $K = N$.

A.3 Other implementations

Apart form the methods described above we tried other implementations. For example, we started with the genetic algorithm built in solver *ga* in Global Optimization Toolbox in Matlab. This solver finds the minimum of a function using a genetic algorithm. In our case it was difficult to define and set up the most important parameters which impact on the solution. The best portfolio obtained using this approach included only one asset which is not a suitable result for this analysis. The same problem appears when we used the *psoptimset* solver (pattern search algorithm) in the Global Optimization Toolbox in Matlab. This function optimises the objective function subject to linear constraints. We were not happy with the diversification of the best portfolio obtained using the *psoptimset* solver.

We also tried to develop the differential evolution algorithm and the genetic algorithm in AMPL but we found that it is difficult to implement some particular stages and specific rules of breeding in this programming language when compared to Matlab.

Appendix B

Parameters G and P of the heuristic approaches

Parameter	Parameter value	CPU time	$PT(x)$	ξ
G	150	444	0.5937	0.0152
	180	**550**	**0.6442**	**0.0002**
	210	652	0.6243	0.0001
P	35	415	0.5694	0.0032
	40	**550**	**0.6442**	**0.0002**
	45	697	0.6437	0.0001

TABLE B.1: Genetic algorithm parameter comparison for the DAX 100 data set

Parameter	Parameter value	CPU time	$PT(x)$	ξ
G	160	532	0.823	0.0043
	185	**630**	**0.8429**	**0.0004**
	220	718	0.8429	0.0002
P	37	479	0.8423	0.0164
	42	**630**	**0.8429**	**0.0004**
	47	755	0.8431	0.0002

TABLE B.2: Genetic algorithm parameter comparison for the FTSE 100 data set

Remark. The parameters of G and P for the Nikkei 225 data set is equal to the S&P 100 data set in our empirical study because specifically for these returns

Parameter	Parameter value	CPU time	$PT(x)$	ξ
G	160	586	0.7353	0.0172
	190	**721**	**0.7822**	**0.0006**
	220	953	0.7853	0.0004
P	40	542	0.7421	0.0043
	45	**721**	**0.7822**	**0.0006**
	50	994	0.7864	0.0003

TABLE B.3: Genetic algorithm parameter comparison for the S&P 100 data set

Parameter	Parameter value	CPU time	$PT(x)$	ξ
G	160	1050	-0.9555	0.0001
	190	**1179**	**-0.9894**	**0**
	220	1547	-0.9894	0
P	40	939	-0.9468	0.0021
	45	**1179**	**-0.9894**	**0**
	50	1486	-0.9894	0

TABLE B.4: Genetic algorithm parameter comparison for the Nikkei 225 data set

(the Nikkei 225 set) the genetic algorithm finds the best solution quickly enough. So, we do not need to increase the number of generation and population size. The resulting portfolio is undiversified compare to the number of assets available in total. The algorithm defines the preferable assets very fast and the rest of time just plays with the weights.

Appendix C

Performance of the models

Analysis of higher order moments.

We notice that the cumulative prospect theory model mostly has greater value of skewness in-sample and out-of-sample tests compared to other studied basic models. In contrast, the mean variance model obtained lower positive skewness and greater negative skewness. This indicates that behaviourally based models have longer and a fatter tail on the right hand side and smaller tail on the left hand side compared to the traditional mean variance model. We can conclude that the PT and CPT models have lower risk in the left tail which leads to the lower CVaR value for these portfolios. Moreover, CPT has lower value of kurtosis which also indicates thinner tails compared to other basic models.

Cardinality and CVaR constrained models demonstrate inconsistent results and we can notice a trend only in-sample. The negative skewness of the prospect theory model is always less than the negative skewness of the MV model. This provides good downside protection of behaviourally based portfolios. It is difficult to draw any conclusions when comparing the kurtosis of the PT and the MV models with additional constraints. The results change from one data set to another. Only for the CVaR constrained models it is most likely that the PT model has slightly lower kurtosis compared to MV model.

C.1 Basic models

Data set	Model	CPU	n	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR	skewness	kurtosis
Hang Seng	MV	0.015	9	0.3926	0.0118	0.0301	-0.0373	-0.0644	-0.7680	4.5608
	PT	36	8	0.3922	0.0131	0.0335	-0.0371	-0.0727	-0.8875	5.5543
	CPT	104	5	0.3616	0.0130	0.0359	-0.0500	-0.0668	-0.0902	3.4314
DAX 100	MV	0.031	16	0.4683	0.0060	0.0128	-0.0141	-0.0197	0.3862	3.9497
	PT	550	12	0.4529	0.0083	0.0183	-0.0145	-0.0248	0.9056	4.9426
	CPT	1790	7	0.4369	0.0080	0.0183	-0.0144	-0.0206	1.2203	5.9422
FTSE 100	MV	0.031	14	0.5636	0.0077	0.0137	-0.0121	-0.0178	0.4545	3.0106
	PT	630	17	0.4797	0.0090	0.0188	-0.0114	-0.0272	0.5584	3.6605
	CPT	1904	22	0.4933	0.0085	0.0171	-0.0153	-0.0163	0.9358	3.8126
S&P 100	MV	0.046	11	0.5115	0.0109	0.0213	-0.0279	-0.0328	-0.1170	2.5519
	PT	721	6	0.4940	0.0109	0.0221	-0.0267	-0.0391	-0.2376	3.1125
	CPT	1994	7	0.4717	0.0105	0.0222	-0.0265	-0.0265	0.1630	2.5377
Nikkei 225	MV	0.14	13	0.0159	0.0003	0.0196	-0.0349	-0.0395	0.2528	3.1046
	PT	1179	4	0.1434	0.0034	0.0238	-0.0338	-0.0384	0.4089	3.1774
	CPT	4862	4	0.1598	0.0039	0.0246	-0.0325	-0.0326	0.4674	2.5090

Table C.1: Comparative analysis of basic models (in-sample)

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR	skewness	kurtosis
Hang Seng	MV	0.6844	0.0014	0.0020	-0.0018	-0.0028	-0.0643	2.8745
	PT	0.5388	0.0012	0.0021	-0.0023	-0.0031	0.0597	2.9079
	CPT	0.4260	0.0009	0.0021	-0.0026	-0.0035	-0.0179	3.0211
DAX 100	MV	1.9785	0.0024	0.0012	0.0004	-0.0001	0.0886	3.1530
	PT	1.7766	0.0024	0.0013	0.0005	-0.0005	-0.1066	3.0137
	CPT	2.4833	0.0033	0.0013	0.0012	0.0006	0.0271	2.9207
FTSE 100	MV	1.1103	0.0016	0.0014	-0.0008	-0.0014	0.0003	3.0648
	PT	1.6570	0.0023	0.0014	0.0000	-0.0006	-0.0860	2.8541
	CPT	1.8717	0.0024	0.0013	0.0002	-0.0003	-0.0985	3.1009
S&P 100	MV	0.7232	0.0013	0.0019	-0.0017	-0.0024	0.0242	3.1616
	PT	0.8441	0.0016	0.0019	-0.0015	-0.0023	0.0090	3.0802
	CPT	0.9175	0.0017	0.0019	-0.0012	-0.0021	0.0520	2.9262
Nikkei 225	MV	0.3317	0.0005	0.0016	-0.0021	-0.0029	-0.1178	3.1631
	PT	0,9804	0,0019	0,0020	-0,0014	-0,0022	-0.0796	3.0106
	CPT	0.9960	0.0019	0.0020	-0.0013	-0.0022	-0.0470	3.2628

Table C.2: Comparative analysis of basic models (out-of-sample: bootstrap)

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR	skewness	kurtosis
Hang Seng	MV	22.3694	0.0795	0.0036	0.0736	0.0722	0.0077	2.9332
	PT	19.5623	0.0792	0.0040	0.0727	0.0724	-0.0733	2.8786
	CPT	15.0646	0.0794	0.0053	0.0705	0.0686	0.0415	3.0163
DAX 100	MV	31.5966	0.1582	0.0050	0.1501	0.1478	-0.1712	3.2582
	PT	22.6324	0.1583	0.0070	0.1471	0.1443	-0.0164	2.6740
	CPT	21.0312	0.1584	0.0075	0.1458	0.1430	0.0159	3.0393
FTSE 100	MV	22.9827	0.1189	0.0052	0.1106	0.1084	0.0645	2.9462
	PT	27.4194	0.1187	0.0043	0.1115	0.1099	0.0363	2.9529
	CPT	30.1336	0.1188	0.0039	0.1120	0.1103	-0.1184	3.0554
S&P 100	MV	23.7931	0.0986	0.0041	0.0918	0.0901	-0.0690	3.1821
	PT	22.1445	0.0991	0.0045	0.0917	0.0894	-0.1353	3.0277
	CPT	24.0876	0.0991	0.0041	0.0924	0.0904	-0.0349	3.0953
Nikkei 225	MV	26.3628	0.1385	0.0053	0.1263	0.1236	-0.1799	3.0711
	PT	18.3664	0.1388	0.0076	0.1301	0.1272	0.0396	2.9534
	CPT	17.5419	0.1382	0.0079	0.1248	0.1217	-0.0604	2.9483

Table C.3: Comparative analysis of basic models (out-of-sample: simulation of bullish market)

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR	skewness	kurtosis
Hang Seng	MV	0.2580	0.0004	0.0015	-0.0022	-0.0026	0.1180	2.9394
	PT	0.2187	0.0003	0.0016	-0.0022	-0.0030	-0.0006	3.1120
	CPT	0.2659	0.0004	0.0016	-0.0023	-0.0030	-0.0475	3.0858
DAX 100	MV	-0.1797	-0.0007	0.0036	-0.0067	-0.0081	0.0224	2.9247
	PT	-0.1460	-0.0006	0.0039	-0.0066	-0.0075	0.0936	3.2210
	CPT	-0.1912	-0.0008	0.0040	-0.0074	-0.0088	0.0864	2.8924
FTSE 100	MV	-0.0235	-0.0001	0.0035	-0.0061	-0.0075	-0.0557	3.1055
	PT	-0.1712	-0.0004	0.0026	-0.0047	-0.0057	0.0261	2.7377
	CPT	-0.0619	-0.0002	0.0031	-0.0052	-0.0064	0.0311	2.8292
S&P 100	MV	-0.1088	-0.0003	0.0031	-0.0054	-0.0066	-0.0098	2.8096
	PT	-0.0956	-0.0003	0.0030	-0.0054	-0.0065	-0.0253	3.0193
	CPT	-0.2678	-0.0008	0.0032	-0.0060	-0.0074	0.1160	3.4702
Nikkei 225	MV	-0.0658	-0.0002	0.0038	-0.0064	-0.0079	0.1018	2.9981
	PT	0.0420	0.0002	0.0037	-0.0062	-0.0074	0.0569	2.9714
	CPT	-0.2317	-0.0009	0.0039	-0.0069	-0.0083	0.2360	2.9105

Table C.4: Comparative analysis of basic models (out-of-sample: simulation of bearish market)

C.2 Cardinality constrained models

Data set	Model	CPU	K	n	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR	skewness	kurtosis
Hang Seng	MV	0.09	7	7	0.3919	0.0118	0.0301	-0.0363	-0.0643	-0.7415	4.5296
	PT	37	7	6	0.3915	0.0132	0.0338	-0.0363	-0.0731	-0.9107	5.7858
DAX 100	MV	0.25	10	10	0.4604	0.0060	0.0130	-0.0159	-0.0191	0.5002	4.2193
	PT	520	10	8	0.4484	0.0080	0.0179	-0.0141	-0.0250	1.0201	5.4878
FTSE 100	MV	0.11	10	10	0.5631	0.0077	0.0137	-0.0122	-0.0180	0.4317	2.9809
	PT	600	10	8	0.5218	0.0096	0.0184	-0.0107	-0.0269	0.2667	3.0293
S&P 100	MV	0.14	5	5	0.4911	0.0109	0.0222	-0.0263	-0.0371	-0.2159	2.8623
	PT	690	5	5	0.4729	0.0120	0.0253	-0.0255	-0.0413	-0.1791	2.8395
Nikkei 225	MV	0.89	3	3	0.0023	0.0000	0.0209	-0.0381	-0.0439	0.2057	3.4939
	PT	2597	3	3	0.1420	0.0034	0.0239	-0.0327	-0.0369	0.4254	2.9207

Table C.5: Comparative analysis of cardinality constrained models (in-sample)

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR	skewness	kurtosis
Hang Seng	MV	0.5302	0.0011	0.0021	-0.0023	-0.0023	-0.0626	3.0993
	PT	0.4391	0.0009	0.0022	-0.0027	-0.0035	00549	2.9751
DAX 100	MV	2.2284	0.0027	0.0012	0.0007	0.0002	0.0580	2.8808
	PT	2.1879	0.0028	0.0013	0.0008	0.0001	0.0533	3.1195
FTSE 100	MV	1.2833	0.0017	0.0014	-0.0005	-0.0011	-0.0076	2.8525
	PT	1.6132	0.0025	0.0015	0.0000	-0.0006	0.0746	3.0577
S&P 100	MV	0.8257	0.0016	0.0020	-0.0017	-0.0024	0.0175	3.0635
	PT	0.9538	0.0021	0.0022	-0.0017	-0.0035	0.0083	2.8979
Nikkei 225	MV	0.1783	0.0003	0.0019	-0.0027	-0.0035	0.0701	3.1857
	PT	0.8366	0.0015	0.0018	-0.0015	-0.0022	-0.0003	3.0719

Table C.6: Comparative analysis of cardinality constrained models (out-of-sample: bootstrap)

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR	skewness	kurtosis
Hang Seng	MV	20.2696	0.0792	0.0039	0.0729	0.0715	0.0667	2.7705
	PT	20.2971	0.0792	0.0039	0.0727	0.0716	0.0255	3.1279
DAX 100	MV	27.3371	0.1585	0.0058	0.1491	0.1463	-0.0617	3.0681
	PT	23.1736	0.1582	0.0068	0.1469	0.1465	-0.0722	3.0509
FTSE 100	MV	22.9842	0.1187	0.0052	0.1103	0.1085	0.0468	2.6911
	PT	22.8369	0.1191	0.0052	0.1103	0.1088	-0.1554	3.0641
S&P 100	MV	20.9687	0.0993	0.0047	0.0910	0.0890	-0.1045	3.0191
	PT	20.4622	0.0994	0.0049	0.0910	0.0892	-0.0721	2.9410
Nikkei 225	MV	16.6797	0.1393	0.0083	0.1246	0.1210	-0.2357	2.9273
	PT	16.1642	0.1387	0.0086	0.1241	0.1214	-0.0476	3.5053

TABLE C.7: Comparative analysis of cardinality constrained models (out-of-sample: simulation of bullish market)

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR	skewness	kurtosis
Hang Seng	MV	0.1935	0.0003	0.0015	-0.0022	-0.0028	0.0173	2.9740
	PT	0.1988	0.0003	0.0015	-0.0023	-0.0030	-0.1112	3.0485
DAX 100	MV	-0.2595	-0.0010	0.0037	-0.0070	-0.0089	-0.0577	3.0591
	PT	-0.2159	-0.0008	0.0037	-0.0066	-0.0084	-0.0563	3.0952
FTSE 100	MV	0.0603	0.0002	0.0035	-0.0056	-0.0068	0.1463	2.9910
	PT	-0.0764	-0.0002	0.0033	-0.0056	-0.0068	0.1090	2.8824
S&P 100	MV	-0.2076	-0.0006	0.0031	-0.0056	-0.0069	0.0738	3.1490
	PT	-0,4262	-0,0013	0,0031	-0,0063	-0,0068	0.0145	3.0402
Nikkei 225	MV	-0.2621	-0.0011	0.0043	-0.0082	-0.0100	0.0001	3.0062
	PT	-0.0329	-0.0002	0.0046	-0.0076	-0.0094	0.1757	3.2297

TABLE C.8: Comparative analysis of cardinality constrained models (out-of-sample: simulation of bearish market)

C.3 Models with the CVaR constraint

Data set	Model	CPU	n	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR	skewness	kurtosis
Hang Seng	MV	0.03	8	0.3803	0.0110	0.0289	-0.0379	-0.0560	-0.4039	3.7601
	PT	40	5	0.3916	0.0134	0.0343	-0.0389	-0.0695	-0.6682	4.9819
DAX 100	MV	0.09	14	0.4681	0.0070	0.0150	-0.0125	-0.0190	0.8177	4.7537
	PT	647	5	0.4422	0.0087	0.0196	-0.0189	-0.0236	1.0310	4.9838
FTSE 100	MV	0.11	15	0.5405	0.0070	0.0129	-0.0121	-0.0140	0.6732	3.2146
	PT	754	18	0.5127	0.0090	0.0176	-0.0193	-0.0246	0.5455	3.4378
S&P 100	MV	0.12	23	0.5502	0.0060	0.0109	-0.0120	-0.0140	-0.1668	2.5543
	PT	685	18	0.4940	0.0091	0.0184	-0.0228	-0.0326	-0.2698	3.2304
Nikkei 225	MV	0.41	13	0.0976	0.0020	0.0205	-0.0303	-0.0359	0.3086	2.8616
	PT	2553	24	0.1192	0.0027	0.0226	-0.0333	-0.0398	0.3268	2.9935

Table C.9: Comparative analysis of models with CVaR constraint (in-sample)

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR	skewness	kurtosis
Hang Seng	MV	0.6665	0.0013	0.0019	-0.0021	-0.0029	-0.1827	3.0386
	PT	0.0760	0.0002	0.0022	-0.0035	-0.0043	0.0276	3.0790
DAX 100	MV	2.2560	0.0028	0.0013	0.0009	0.0003	0.1613	3.4775
	PT	1.7529	0.0024	0.0014	0.0001	-0.0004	-0.0045	2.9947
FTSE 100	MV	1.2400	0.0016	0.0013	-0.0006	-0.0011	-0.1315	2.8959
	PT	1.7619	0.0024	0.0014	0.0001	-0.0005	-0.0070	3.0636
S&P 100	MV	1.6640	0.0021	0.0012	0.0000	-0.0004	0.0636	2.9329
	PT	1.2857	0.0025	0.0019	-0.0007	-0.0015	0.0631	3.0299
Nikkei 225	MV	0.5530	0.0009	0.0016	-0.0017	-0.0024	-0.0866	2.8032
	PT	0.9456	0.0015	0.0016	-0.0012	-0.0018	-0.0131	2.8478

Table C.10: Comparative analysis of models with CVaR constraint (out-of-sample: bootstrap)

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR	skewness	kurtosis
Hang Seng	MV	19.7664	0.0793	0.0040	0.0727	0.0712	0.0963	2.8726
	PT	17.6371	0.0793	0.0045	0.0730	0.0716	-0.0638	2.9289
DAX 100	MV	26.4675	0.1582	0.0060	0.1479	0.1457	0.0123	3.0686
	PT	20.4444	0.1579	0.0077	0.1477	0.1461	0.0799	2.6602
FTSE 100	MV	25.6109	0.1187	0.0046	0.1110	0.1089	-0.0580	3.1282
	PT	25.9197	0.1189	0.0046	0.1114	0.1091	-0.0336	3.0600
S&P 100	MV	38.2829	0.0990	0.0026	0.0947	0.0936	-0.0364	2.8962
	PT	24,0364	0.0992	0.0041	0.0943	0.0938	-0.1241	3.0880
Nikkei 225	MV	25.6196	0.1387	0.0054	0.1298	0.1271	-0.1693	3.0234
	PT	27.5647	0.1381	0.0050	0.1294	0.1276	-0.0939	2.9247

TABLE C.11: Comparative analysis of models with CVaR constraint (out-of-sample: simulation of bullish market)

Data set	Model	$\bar{r}/\sigma$	$\bar{r}$	σ	VaR	CVaR	skewness	kurtosis
Hang Seng	MV	0.4767	0.0005	0.0011	-0.0013	-0.0017	-0.0812	3.1465
	PT	0.3905	0.0004	0.0011	-0.0014	-0.0017	-0.0050	3.1637
DAX 100	MV	-0.1402	-0.0005	0.0037	-0.0064	-0.0079	0.1444	3.5092
	PT	-0.1175	-0.0004	0.0038	-0.0068	-0.0078	0.1031	3.2481
FTSE 100	MV	-0.1006	-0.0003	0.0034	-0.0057	-0.0071	0.2879	3.3762
	PT	-0.1472	-0.0004	0.0030	-0.0054	-0.0067	0.0371	3.2816
S&P 100	MV	-0.1560	-0.0004	0.0028	-0.0050	-0.0062	0.0431	3.1581
	PT	-0.3418	-0.0009	0.0027	-0.0054	-0.0060	0.0466	3.2402
Nikkei 225	MV	-0.1673	-0.0007	0.0042	-0.0077	-0.0090	0.1943	3.2532
	PT	-0.1905	-0.0007	0.0039	-0.0071	-0.0085	0.0980	2.9356

TABLE C.12: Comparative analysis of models with CVaR constraint (out-of-sample: simulation of bearish market)

C.4 Graphs of distributions of portfolio returns (an example)

Graphs C.1, C.2, C.3 and C.4 show the distribution of returns of the mean variance, prospect theory and cumulative prospect theory models in-sample and out-of-sample in the S&P 100 data set as an example. This illustrates the conclusion which is made in the thesis and in the analysis of higher order moments that behaviourally based models have thicker left tails and higher returns compared to the traditional mean variance model out-of-sample. According to graph C.1 MV model has slightly thicker left tail than PT and CPT models, thus, all models demonstrate very similar distribution of portfolio returns. However, in bootstrap test the cumulative prospect theory has an advantage in left tail compared to other models (see graph C.2). At the same time the prospect theory shows benefit in bullish and bearish market in terms of distribution on the left tail which leads to smaller CVaR risk measure (see graphs C.3 and C.4).

MV - the mean variance model;
PT - the prospect theory model;
CPT - the cumulative prospect theory model.

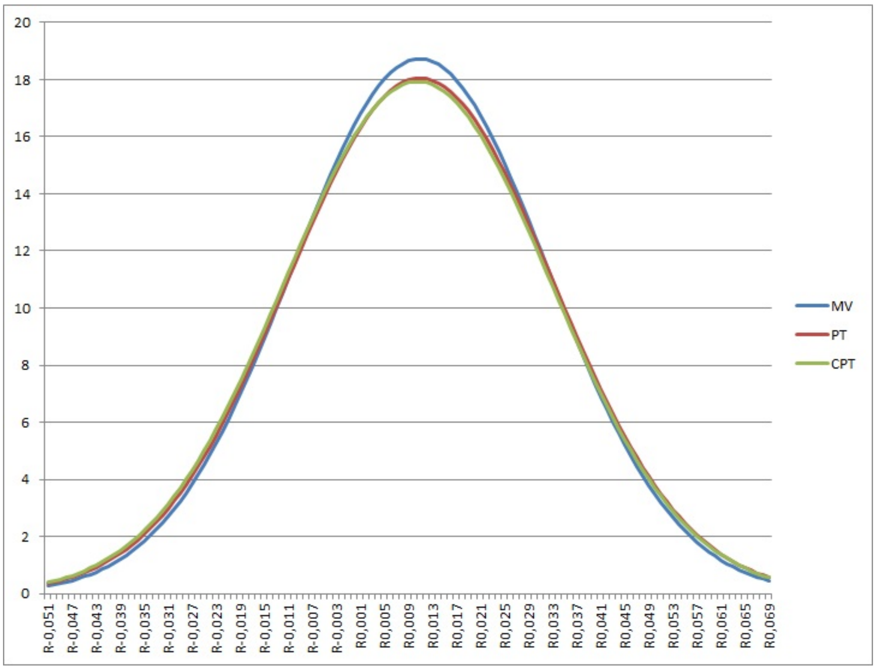

FIGURE C.1: S&P 100. Basic models. In-sample.

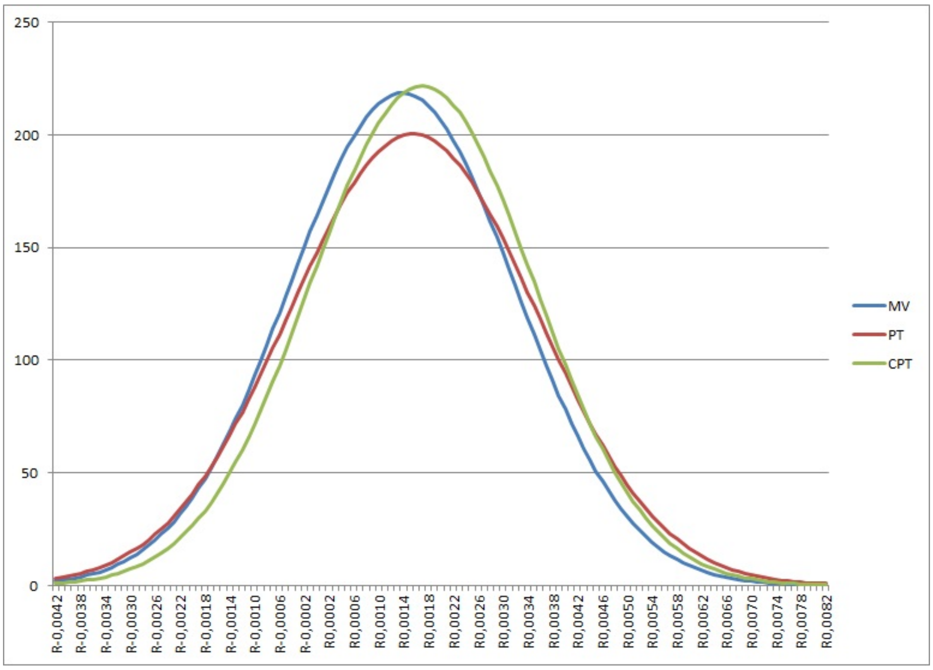

FIGURE C.2: S&P 100. Basic models. Out-of-sample (bootstrap).

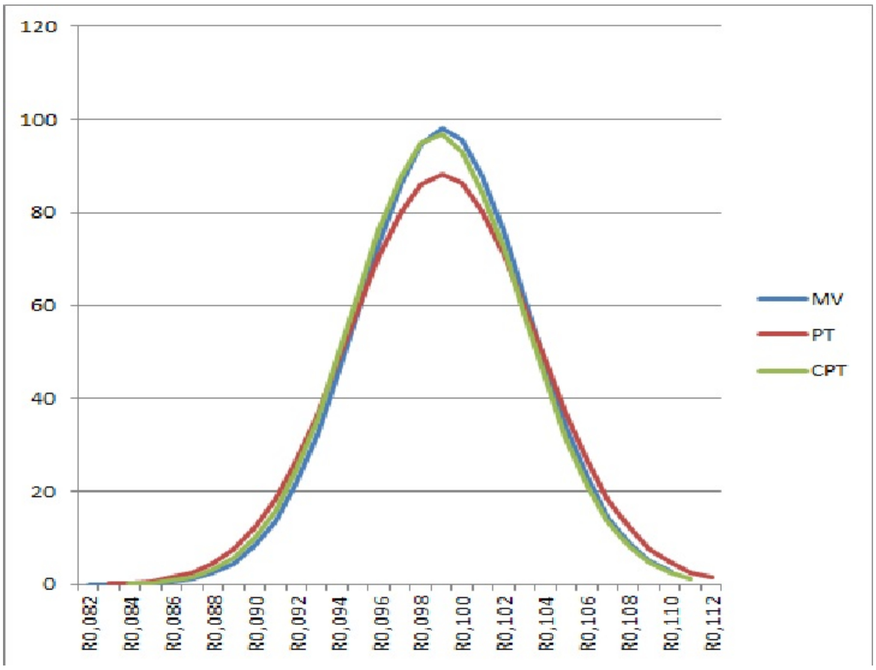

FIGURE C.3: S&P 100. Basic models. Out-of-sample (simulation of bullish market).

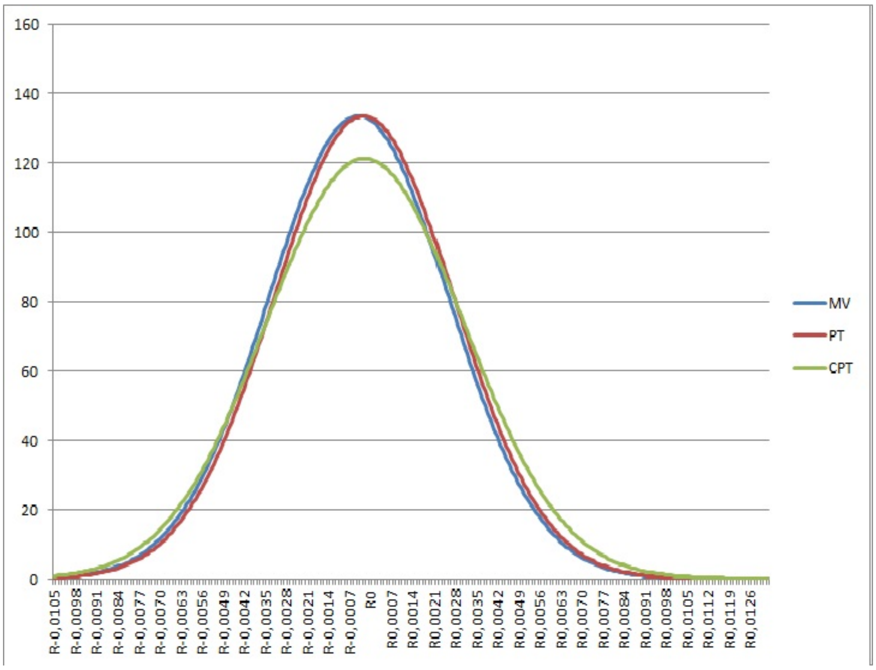

FIGURE C.4: S&P 100. Basic models. Out-of-sample (simulation of bearish market).

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