# A NOTE ON UNIFORM ASYMPTOTIC WAVE DIFFRACTION BY A WEDGE

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**Summary**

New expressions for asymptotically uniform Green's functions for high-frequency wave diffraction when a plane, cylindrical or point wave field is incident on an ideal wedge are derived. They are useful for deriving a uniform asymptotic expression for the exact solution in terms of the high-frequency diffracted and geometrical optics far field. The present method is simple and consists of differentiating out the singularities of the integral representations and using new representations for trigonometrical sums that arise when the wedge angle is a rational multiple of $\pi$. The new results make explicit the continuity of the fields across shadow and reflection boundaries.

## 1. Introduction

The geometrical theory of diffraction (GTD) was introduced by Keller (**1**) as an asymptotic method for the solution of diffraction problems at high frequencies. Keller's GTD has proved to be very powerful in a wide variety of applications to physical problems. The method uses the saddle point method to derive asymptotic approximations from the exact solution of canonical wedge problems to derive so called "diffraction coefficients". The method gives a useful physical representation of the total far-field in terms of the geometrical optics and a diffracted field term. One drawback of Keller's method is that it breaks down at shadow and reflection boundaries where the diffracted term predicts infinite fields. To overcome this defect, two uniform asymptotic techniques have been developed recently, namely the uniform asymptotic theory (UAT) of edge diffraction and the uniform geometrical theory of diffraction (UTD). A comparison of both these methods is given in the work of Boersma (**2**) where it is shown that the two different methods of asymptotically approximating integrals by pole subtraction or pole factorisation are equivalent if the complete asymptotic expansions are used. These asymptotic expressions are derived from integrals representing the exact solution. The starting point for our analysis is a particular periodic Green's function for the solution of of the ideal wedge problem Rawlins(**3**) . This representation is related to the work of Sommerfeld, Carslaw, Macdonald and others at the beginning of the last century, see Rawlins(**4**) for a detailed history. The Sommerfeld approach was extended non-trivially to absorbing/impedance wedges by Mayuhzinets and Williams in the 1950's, see Babich et al (**5**) for a comprehensive up to date history of general wedge problems solved by the so called Sommerfeld-Malyuzhinets technique. In order to derive useful physical results from these exact solutions the disposition of singularities and saddle points of integrals is crucial for getting accurate uniform results. Oberhettinger (**6**), (**7**) used asymptotic methods to derive results for diffraction by ideal wedges. However his approach which involves adding and subtracting out singularities of the integrand is quite complicated. There are various uniform asymptotic expressions that have been derived for the general wedge angle but these suffer drawbacks. Jones (**8**), (**9**) derived, by an ingenious method, uniform asymptotic expressions for the wedge diffraction integral. However the result is strange in that the residue contribution gives rise to plane waves that do not have the correct phase behavior away from the transition point.There was also a uniform asymptotic result given for plane waves by Wait (**10**) which is based on a work by Felsen (**11**) which does not agree with the present work and does not seem to be uniformly valid. A general drawback of the previous methods is they break down when the incident wave grazes the wedge faces. This is because there is a coalescence of the shadow and specular reflection direction and the wedge boundary. Analytically this amounts to a coalescing of poles of the integrand that also occurs at or near the saddle point. The situation is even more complicated in the case of absorbing/impedance wedges where the coalescing of geometrical optics boundary and surface wave poles may occur near the saddle point. The uniform results of Babich et al (**5**) implicitly assume that such situations will not arise. Ciarkowski et al (**12**), (**13**) have derived a rigorous approach using pole subtraction to obtain uniform asymptotic results for plane wave diffraction by an ideal wedge of any angle; and they show that in the limiting case of a half-plane the results reduce to the well known Sommerfeld solution. However their result does not at the same time reduce to the well known results for a right angled wedge given by Reiche (**14**). Their method would seem to require special consideration of this case. In our approach we obtain a uniform asymptotic results for any wedge which can be expressed as a rational multiple of $\pi$. Since the rational numbers are dense in the real numbers we can approximate any wedge angle to a sufficient accuracy by a rational wedge. In particular our result reduces to the Sommerfeld and Reiche results as two special cases of an infinite number of rational wedge solutions. We also obtain analogous results for cylindrical and spherical source diffraction by a rational wedge. The asymptotic method we use here is a new approach that involves reformulating the integrand and differentiating out the singularities of the integrand and then applying the standard saddle point method on an integrand that has no poles. A further simplification is achieved by using certain trigonometric identities. As an application of these theoretical results a simple Mathematica programme is given that produces graphs of the uniform absolute value of the scattered field for any hard or soft rational wedge for and any angle of incidence. A numerical justification of the convergence of the rational approximations to any given wedge angle is also carried out. Finally some new asymptotic results for plane-wave grazing incidence on a corner are given.

*Geometry of the diffraction problem and the Green's functions*

For the diffraction problems in wedge shaped regions we shall work in Euclidean space of three dimensions with cylindrical polar coordinates $(r, \theta, z)$. In this space we shall assume there is a wedge of open angle $\alpha$ with faces defined by the planes $\theta = 0$, and $\theta = \alpha$ where $0 < r < \infty, -\infty < z < \infty$. For $\alpha = \pi$ it becomes a half space, and for $\alpha = 2\pi$ the wedge becomes a semi-infinite plane. The geometry of the wedge diffraction problem that we are going to analyse is shown in Figure 1. A source of waves at a point $Q(r_0, \theta_0, z_0)$ in space with time harmonic variation $e^{i\omega t}$ †, is incident on an ideal wedge. The total field

† We shall drop this factor in the rest of the paper

$u(P)$ is observed at the point $P(r,\theta,z)$. In the region outside of the wedge defined by $0<r<\infty, 0<\theta<\alpha<2\pi, -\infty<z<\infty$ the total scalar field must satisfy the scalar wave equation:

$$(\Delta+k^2)u(P)=0, \quad 0<r<\infty, 0<\theta<\alpha, P\neq Q; \tag{1.1}$$

and appropriate edge and radiation conditions to ensure uniqueness. For an ideal wedge

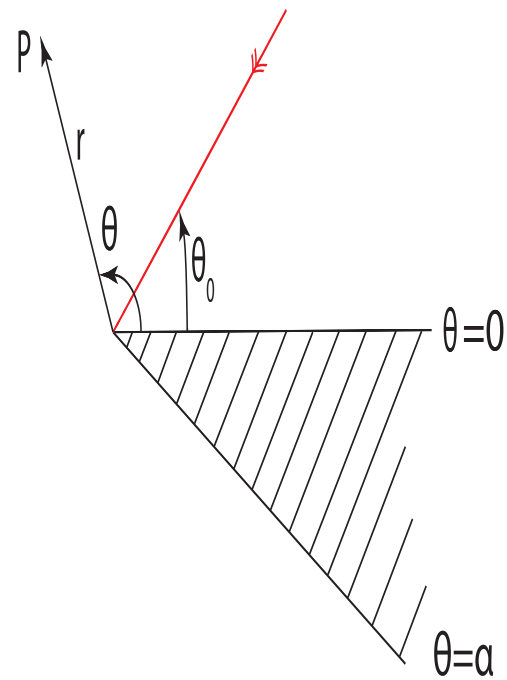


**Fig. 1** Geometry of diffraction by a wedge.

the field will be required to satisfy the Neumann boundary conditions($\frac{\partial u}{\partial\theta}=0$ for $\theta=0$ and $\theta=\alpha$), or the Dirichlet boundary conditions($u=0$ for $\theta=0$ and $\theta=\alpha$), or the Mixed boundary conditions($\frac{\partial u}{\partial\theta}=0$ for $\theta=0$ and $u=0$ for $\theta=\alpha$), for all $0<r<\infty, -\infty<z<\infty$. For an incident plane wave given for $0<\theta_0<\alpha$, by

$$u_0(r,\theta)=e^{ikr\cos(\theta-\theta_0)}, \tag{1.2}$$

then if by symmetry the field is independent of $z$, it has been shown in Rawlins(**4**) that the periodic Greens function for diffraction by a wedge of angle $\alpha$ is given by

$$G_\alpha(r,\theta,\theta_0;k)=\frac{1}{2i\alpha}\int_C e^{ikr\cos\zeta}\left[\frac{\sin(\frac{\pi\zeta}{\alpha})}{\cos(\frac{\pi\zeta}{\alpha})-\cos(\frac{\pi(\theta-\theta_0)}{\alpha})}\right]d\zeta, \tag{1.3}$$

where the open contour of integration $C$ is such that the starting point of integration is given by $i\infty + c_1$, and the termination point is given by $i\infty + c_2$, where $-\pi < c_1 < 0, \pi < c_2 < 2\pi$, see Figure 2.

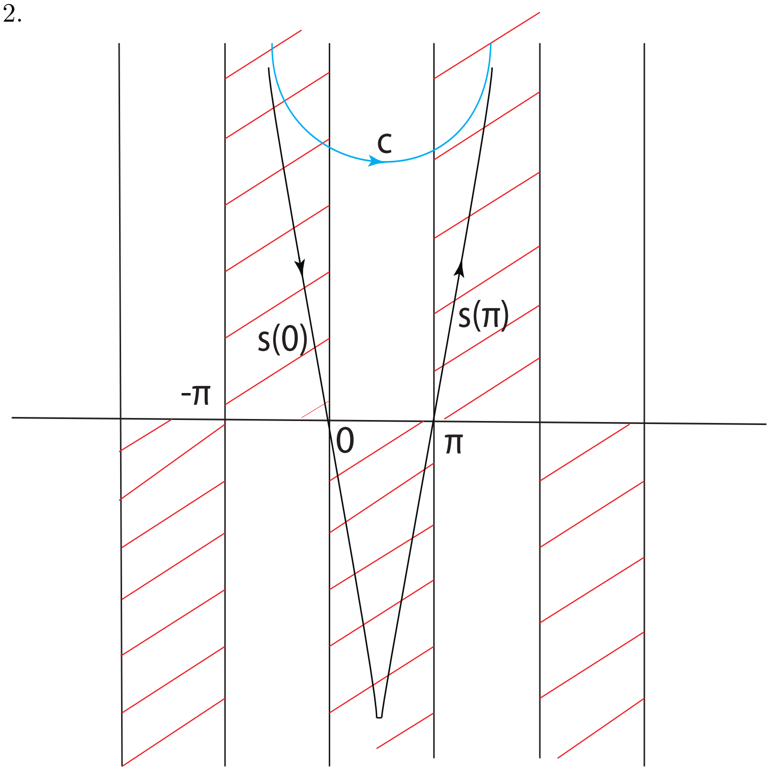


**Fig. 2** Contours of integration $C, S(0)$ and $S(\pi)$.

The Green's function given above enables one to derive solutions to various diffraction problems in ideal wedge shaped regions. The solution for the Neumann, Dirichlet and Mixed problems are given by

$$u_N = G_\alpha(r, \theta, \theta_0; k) + G_\alpha(r, \theta, -\theta_0; k), \tag{1.4}$$

$$u_D = G_\alpha(r, \theta, \theta_0; k) - G_\alpha(r, \theta, -\theta_0; k), \tag{1.5}$$

$$u_{N,D} = G_{2\alpha}(r, \theta, \theta_0; k) + G_{2\alpha}(r, \theta, -\theta_0; k) - G_{2\alpha}(r, \theta, -2\alpha + \theta_0; k) - G_{2\alpha}(r, \theta, 2\alpha - \theta_0; k), \tag{1.6}$$

respectively. If the incident wave is a cylindrical line source;

$$u_0 = H_0^{(2)}[kR(\theta - \theta_0)], R(\xi) = \sqrt{r^2 + r_0^2 - 2rr_0 \cos(\xi)}; \tag{1.7}$$

or a spherical point source:

$$u_0 = e^{[ik\mathbf{R}(\theta-\theta_0)]}/\mathbf{R}(\theta-\theta_0), \mathbf{R}(\xi) = \sqrt{r^2 + r_0^2 + (z-z_0)^2 - 2rr_0\cos(\xi)}; \tag{1.8}$$

the Green's functions in equations (1.4) to (1.6) are replaced by the cylindrical and spherical Green's functions. These cylindrical and spherical Green's functions are given in terms of the plane wave Green's function by:

$$\mathbf{G}_\alpha(r,\theta,\theta_0;k) = \frac{1}{i\pi}\int_{\infty+ia}^{0} e^{-i/2[t+k^2(r^2+r_0^2)/t)]} G_\alpha(r,\theta,\theta_0;\frac{k^2 r_0}{t})\frac{dt}{t}, \tag{1.9}$$

and

$$\mathbf{G}_\alpha(r,\theta,z,r_0,\theta_0,z_0;k) = \frac{-ke^{i\frac{\pi}{4}}}{\sqrt{2\pi}}\int_{\infty+ia}^{0} e^{-i/2[t+k^2(r^2+r_0^2+(z-z_0)^2)/t)]} G_\alpha(r,\theta,\theta_0;\frac{k^2 r_0}{t})\frac{dt}{t^{3/2}}, \tag{1.10}$$

respectively. The contour of integration for the integral occurring in the expressions (1.9) and (1.10) is shown in the figure 3.

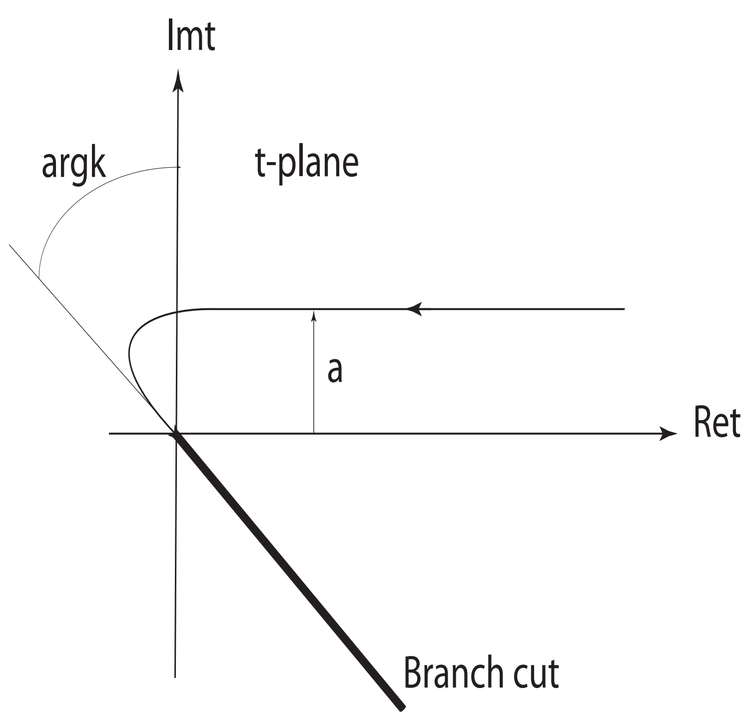


**Fig. 3** The complex contour of integration for the Greens function integrals (1.9)and (1.10).

## 2. Uniform asymptotic representation for the Green's functions for a rational wedge.

If the wedge angle $\alpha$ is a rational multiple of $\pi$, that is $\alpha = p\pi/q$, where $p$ and $q$ are positive integers, the expression (1.3) becomes

$$G_{p\pi/q}(r,\theta,\theta_0;k) = \frac{1}{2\pi ip}\int_C e^{ikr\cos\zeta}\left[\frac{q\sin(\frac{q\zeta}{p})}{\cos(\frac{q\zeta}{p}) - \cos(\frac{q(\theta-\theta_0)}{p})}\right]d\zeta. \tag{2.1}$$

By using the identity

$$\frac{q\sin(q\zeta/p)}{\cos(q\zeta/p) - \cos(q(\theta-\theta_0)/p)} = \sum_{m=0}^{q-1}\frac{\sin(\zeta/p)}{\cos(\zeta/p) - \cos((\theta-\theta_0)/p + 2\pi m/q)}, \tag{2.2}$$

we can rewrite (2.1)in the form

$$G_{p\pi/q}(r,\theta,\theta_0;k) = \sum_{m=0}^{q-1} I_p(kr, \theta-\theta_0 + 2\pi mp/q), \tag{2.3}$$

where

$$I_p(kr,\psi) = \frac{1}{2\pi ip}\int_C e^{ikr\cos\zeta}\left[\frac{\sin(\frac{\zeta}{p})}{\cos(\frac{\zeta}{p}) - \cos(\frac{\psi}{p})}\right]d\zeta. \tag{2.4}$$

We rewrite $I_p$ in the form

$$I_p(kr,\psi) = \frac{1}{2\pi i}\int_C e^{ikr\cos\zeta}\left[\frac{E(\zeta,\psi,p)}{\cos\zeta - \cos\psi}\right]d\zeta, \tag{2.5}$$

where

$$E(\zeta,\psi,p) = \frac{\sin(\zeta/p)(\cos\zeta - \cos\psi)}{p\left(\cos(\frac{\zeta}{p}) - \cos(\frac{\psi}{p})\right)}. \tag{2.6}$$

We note that $E(\zeta,\psi,p)$ is continuous across $\zeta = \psi$ and that $E(0,\psi,p) = 0$. Multiplying across equation (2.5) by $e^{-ikr\cos\psi}$ and then partially differentiating under the integral sign with respects to $kr$ gives:

$$\frac{\partial}{\partial(kr)}[I_p(kr,\psi)e^{-ikr\cos\psi}] = \frac{e^{-ikr\cos\psi}}{2\pi}\int_C e^{ikr\cos\zeta}E(\zeta,\psi,p)d\zeta. \tag{2.7}$$

This operation is allowed since the integral is uniformly convergent before and after the differentiation. We now distort $C$ into the two connected paths of steepest descent $S(0)$ and $S(\pi)$ through the saddle points $\zeta = 0$ and $\zeta = \pi$. Since $E(\zeta,\psi,p)$ has no singularities in the region between $C$ and $S(0) + S(\pi)$ we have by Cauchy's theorem that

$$\frac{\partial}{\partial(kr)}[I_p(kr,\psi)e^{-ikr\cos\psi}] = \frac{e^{-ikr\cos\psi}}{2\pi}\left[\int_{S(0)} + \int_{S(\pi)}\right]e^{ikr\cos\zeta}E(\zeta,\psi,p)d\zeta. \tag{2.8}$$

We now let $kr\to\infty$ and use the standard saddle point method, which means that the dominant contribution for the first integral comes from $\zeta = 0$ and for the second from $\zeta = \pi$. Thus

$$\begin{aligned}\frac{\partial}{\partial(kr)}[I_p(kr,\psi)e^{-ikr\cos\psi}] \quad &= \frac{e^{-ikr\cos\psi}E(0,\psi,p)}{2\pi}\int_{S(0)} e^{ikr\cos\zeta}d\zeta \\ &+\frac{e^{-ikr\cos\psi}E(\pi,\psi,p)}{2\pi}\int_{S(\pi)} e^{ikr\cos\zeta}d\zeta + O\left[\frac{e^{-ikr\cos\psi}}{(kr)^{3/2}}\right], \\ &= \frac{e^{-ikr\cos\psi}E(\pi,\psi,p)}{\sqrt{2\pi kr}}e^{-i(kr-\pi/4)} + O\left[\frac{e^{-ikr\cos\psi}}{(kr)^{3/2}}\right].\end{aligned} \tag{2.9}$$

Hence

$$\frac{\partial}{\partial(kr)}[I_p(kr,\psi)e^{-ikr\cos\psi}] = \frac{e^{-ikr(1+\cos\psi)}E(\pi,\psi,p)}{\sqrt{2\pi kr}}e^{i\pi/4} + O\left[\frac{e^{-ikr\cos\psi}}{(kr)^{3/2}}\right]. \tag{2.10}$$

Integrating the last expression from $kr$ to $\infty$ gives

$$\begin{aligned}[I_p(kr,\psi)e^{-ikr\cos\psi}] \quad &= \lim_{kr\to\infty}[I_p(kr,\psi)e^{-ikr\cos\psi}] \\ &+\frac{e^{i\pi/4}}{\sqrt{2\pi}}E(\pi,\psi,p)\int_\infty^{kr}\frac{e^{-it(1+\cos\psi)}}{\sqrt{t}}dt \\ &+O\left[\frac{1}{(kr)^{3/2}}\right].\end{aligned} \tag{2.11}$$

Notice that in carrying out this integration the order term in the last expression has changed from the previous equation; this can be proved by an application of L'Hopitals rule. The change of integration variable $v = \sqrt{t(1+\cos\psi)} > 0$ in the last expression gives

$$\begin{aligned}[I_p(kr,\psi)e^{-ikr\cos\psi}] \quad &= \lim_{kr\to\infty}[I_p(kr,\psi)e^{-ikr\cos\psi}] \\ &+\frac{\sqrt{2}E(\pi,\psi,p)}{\sqrt{(1+\cos\psi)}}\frac{e^{i\pi/4}}{\sqrt{\pi}}\int_\infty^{\sqrt{kr(1+\cos\psi)}} e^{-iv^2}dv \\ &+O\left[\frac{1}{(kr)^{3/2}}\right].\end{aligned} \tag{2.12}$$

It has been shown in Rawlins(**4**) that

$$\lim_{kr\to\infty}[I_p(kr,\psi)e^{-ikr\cos\psi}] = \sum_N H[\pi - |\psi + 2\pi pN|]. \tag{2.13}$$

Thus by using the Fresnel integral representation

$$F[z] = \frac{e^{i\pi/4}}{\sqrt{\pi}}\int_z^\infty e^{-iv^2}dv, \tag{2.14}$$

we obtain from (2.12)

$$
\begin{aligned}
I_p(kr,\psi) \qquad &= \sum_N H[\pi - |\psi + 2\pi pN|]e^{ikr\cos\psi} \\
&-e^{ikr\cos\psi}\frac{E(\pi,\psi,p)}{|\cos(\psi/2)|}F[\sqrt{2kr}|\cos\psi/2|] + O\left[\frac{1}{(kr)^{3/2}}\right].
\end{aligned}
\tag{2.15}
$$

In the last expression the term $\frac{E(\pi,\psi,p)}{|\cos(\psi/2)|}$ is indeterminate when $\cos(\psi/2) = 0$. This occurs at reflection and shadow boundaries. The indeterminacy can be removed by using the identity[†]:

$$
\frac{\sin(\pi/p)\cos\psi/2}{\left(\cos(\frac{\psi}{p}) - \cos(\frac{\pi}{p})\right)} = \sum_{n=1}^{p-1}\sin\frac{n\pi}{p}\cos\frac{(2n-p)\psi}{2p}, \tag{2.16}
$$

where $p$ is an integer greater than one. Thus we can rewrite the expression (2.15) as

$$
\begin{aligned}
I_p(kr,\psi) \qquad &= \sum_N H[\pi - |\psi + 2\pi pN|]e^{ikr\cos\psi} \\
&-\frac{2}{p}\mathrm{sign}[\cos(\psi/2)]\left(\sum_{n=1}^{p-1}\sin\frac{n\pi}{p}\cos\frac{(2n-p)\psi}{2p}\right)e^{ikr\cos\psi}F[\sqrt{2kr}|\cos\psi/2|] \\
&+O\left[\frac{1}{(kr)^{3/2}}\right];
\end{aligned}
\tag{2.17}
$$

which makes apparent the uniform asymptotic nature of the result. Substituting this last result into the expression (2.3) gives the new Green's function expression

$$
\begin{aligned}
&G_{p\pi/q}(r,\theta,\pm\theta_0;k) = \sum_{m=0}^{q-1}\sum_N H[\pi - |\psi_m^{\pm\theta_0} + 2\pi pN|]e^{ikr\cos\psi_m^{\pm\theta_0}} \\
&-\frac{2}{p}\sum_{m=0}^{q-1}\mathrm{sign}[\cos(\psi_m^{\pm\theta_0}/2)]\left(\sum_{n=1}^{p-1}\sin\frac{n\pi}{p}\cos\frac{(2n-p)\psi_m^{\pm\theta_0}}{2p}\right)e^{ikr\cos\psi_m^{\pm\theta_0}}F[\sqrt{2kr}|\cos\psi_m^{\pm\theta_0}/2|] \\
&\quad +O\left[\frac{1}{(kr)^{3/2}}\right],
\end{aligned}
\tag{2.18}
$$

where $\psi_m^{\pm\theta_0} = \theta \mp \theta_0 + 2\pi mp/q$.
By using the representation (1.9) and the result (2.18) (combined with the result (8) and the method of appendix C in Rawlins (**15**)) it is not difficult to show that for a line source

[†] This can be shown by writing the RHS as a sum of sines and then representing the sines as the imaginary part of exponentials. The resulting geometrical series can be summed to give the LHS.

incidence a new uniformly valid (in $r, \theta, \theta_0$) asymptotic Green's function is given by:

$$G_{p\pi/q}(r,\theta,\theta_0;k) = \sum_{m=0}^{q-1}\sum_N H[\pi - |\psi_m^{\theta_0} + 2\pi p N|]H_0^{(2)}[kR(\psi_m^{\theta_0})]$$
$$+\frac{2i}{\pi p}\sum_{m=0}^{q-1}\text{sign}[\cos(\psi_m^{\theta_0}/2)]\left(\sum_{n=1}^{p-1}\sin\frac{n\pi}{p}\cos\frac{(2n-p)\psi_m^{\theta_0}}{2p}\right)\int_\infty^{\xi(\psi_m^{\theta_0})} e^{-ikR(\psi_m^{\theta_0})cosh\xi}d\xi$$
$$+O\left[\frac{1}{(kR)^{3/2}}\right], \tag{2.19}$$

for $kr \to \infty$, where $\xi(\psi) = \sinh^{-1}[\frac{2\sqrt{rr_0}|\cos(\psi/2)|}{R(\psi)}]$.
Similarly for a point source given by (1.8) the appropriate uniformly valid(in $r, \theta, z, r_0, \theta_0, z_0$) asymptotic Green's function is given by substituting the expression (2.18) into (1.10) (and by using the result (8) and the method of appendix B of Rawlins (**16**)). It can be shown that in this case

$$\mathbf{G}_{p\pi/q}(r,\theta,z,r_0,\theta_0,z_0;k) = \sum_{m=0}^{q-1}\sum_N H[\pi - |\psi_m^{\theta_0} + 2\pi p N|]\frac{e^{ik\mathbf{R}(\psi_m^{\theta_0})}}{\mathbf{R}(\psi_m^{\theta_0})}$$
$$-\frac{ik}{p}\sum_{m=0}^{q-1}\text{sign}[\cos(\psi_m^{\theta_0}/2)]\left(\sum_{n=1}^{p-1}\sin\frac{n\pi}{p}\cos\frac{(2n-p)\psi_m^{\theta_0}}{2p}\right)\int_\infty^{\zeta(\psi_m^{\theta_0})} H_1^{(2)}[k\mathbf{R}(\psi_m^{\theta_0})\cosh\zeta]d\zeta$$
$$+O\left[\frac{1}{(k\mathbf{R})^{3/2}}\right], \tag{2.20}$$

for $kr \to \infty$, where $\zeta(\psi) = \sinh^{-1}[\frac{2\sqrt{rr_0}|\cos(\psi/2)|}{\mathbf{R}(\psi)}]$.

*Numerical results.*

Here we present some simple Mathematica results useful for the uniform numerical computation of the absolute value of the scattered field for plane wave incidence on any ideal wedge whose angle can be represented as a rational multiple of $\pi$. We also support, by some direct numerical computations, that any wedge can be approximated by choosing a sufficiently close rational approximation to the wedge angle. It is not difficult to show that the Mathematica 7.0 coding for the Green's function (2.18) for any rational angle is given by:

```
H[t_]:=If[t==0,1/2,HeavisideTheta[t]];
G[kr_,θ_,θ0_,p_,q_]:=Sum[Exp[IkrCos[θ − θ0 + 2πmp/q]]Sum[H[π − Abs[θ − θ0 + 2πmp/q + 2πp N]],{N,−100,100}],{m,0,q − 1}] − 1/pSum[Sign[Cos[(θ − θ0 + 2πmp/q)/2]]Exp[IkrCos[θ − θ0 + 2πmp/q]]Erfc[Exp[Iπ/4]Sqrt[2kr]Abs[Cos[(θ − θ0 + 2πmp/q)/2]]](Sum[Sin[nPi/p]Cos[(2n − p)(θ − θ0 + 2πmp/q)/(2p)],{n,0,p − 1}]),{m,0,q − 1}];
```

From the above result combined with (1.4) and (1.5), the plot for a hard and soft right angled wedge for plane wave incidence at $\theta_0 = \pi/4$, with $kr = 100, p = 3, q = 2$ is given respectively by PolarPlot[Abs[$N$[$G$[100, $\theta$, Pi/4, 3, 2] + $G$[100, $\theta$, −Pi/4, 3, 2]]], {$\theta$, 0, 3Pi/2}] and PolarPlot[Abs[$N$[$G$[100, $\theta$, Pi/4, 3, 2] − $G$[100, $\theta$, −Pi/4, 3, 2]]], {$\theta$, 0, 3Pi/2}]. These plots are shown in the figure 4 and the figure 5 respectively.

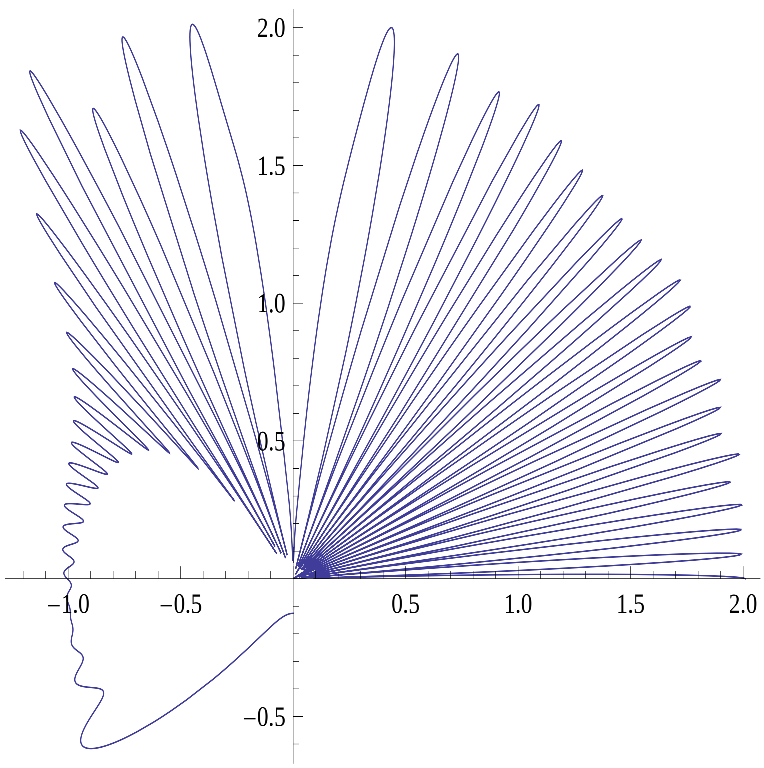


**Fig. 4** The absolute value of the total field when a plane wave at $45^0$ incidence is diffracted by a hard corner with $kr = 100$.

It was not possible to show analytically that uniform asymptotic results for a given arbitrary angled wedge can be obtained by approximating to the given wedge angle by a sufficiently close rational angled wedge. To support this claim as being plausible we carried out a numerical investigation. The rational wedge angles that were considered were of the form $\alpha = (2^n p - 1)\pi/(2^n q)$, and letting $n \to \infty$, in this limit $\alpha \to p\pi/q$. A typical set of examples for a soft and hard corner are shown in figure 6 and figure 7 respectively. The arbitrary wedge angle was taken to be $3\pi/2$ and the rational approximations $5\pi/4, 11\pi/8, 23\pi/16, 47\pi/32$,and $95\pi/64$ were used. The wedge angle concerned is given from the graphs by the extreme right hand intercept of the horizontal axis. To avoid unnecessary clutter from rapid oscillations $kr$ was chosen to equal 10; similar results where obtained for $kr = 100$. Further graphs with closer rational approximations

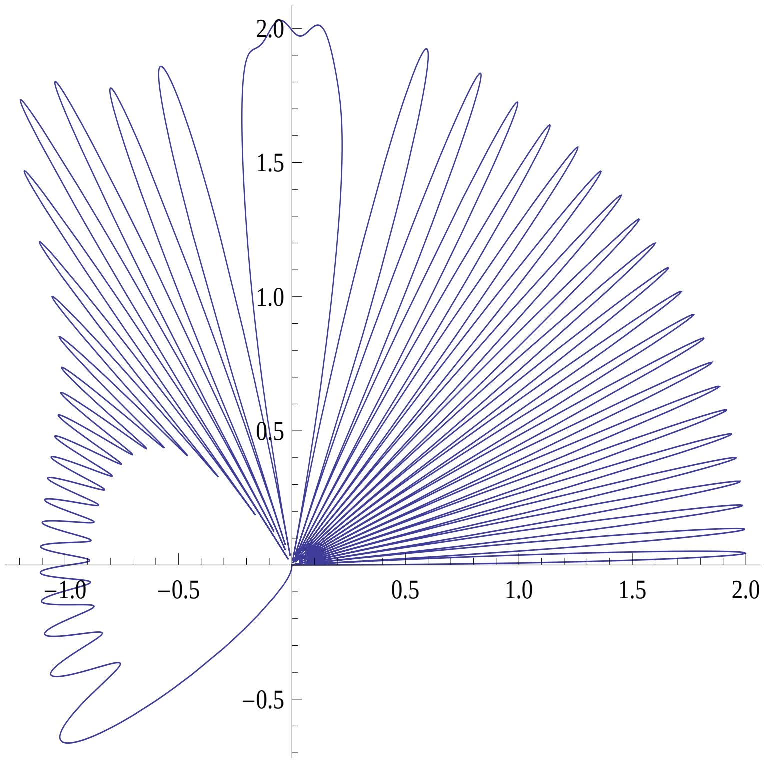


**Fig. 5** The absolute value of the total field when a plane wave at $45^0$ incidence is diffracted by a soft corner with $kr = 100$.

were indistinguishable from the graph for the wedge angle $3\pi/2$. This numerical behaviour was replicated for other wedge angle and various angles of incidence.

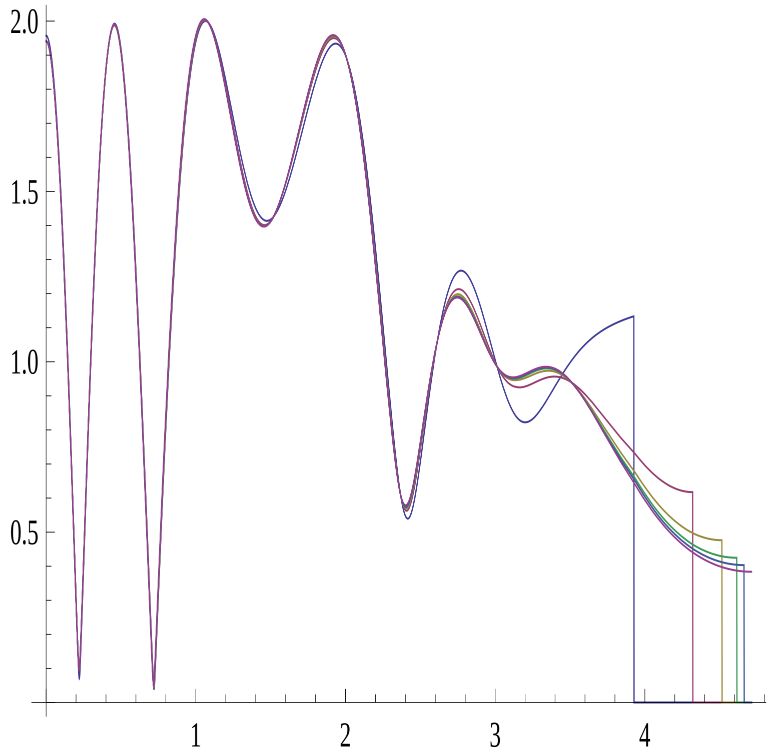


**Fig. 6** The absolute value of the total field $|u_N|$ against $\theta$, when a plane wave at incidence $45^0$ is diffracted by a hard wedge with angles $5\pi/4, 11\pi/8, 23\pi/16, 47\pi/32, 95\pi/64$ and $3\pi/2$ with $kr = 10$.

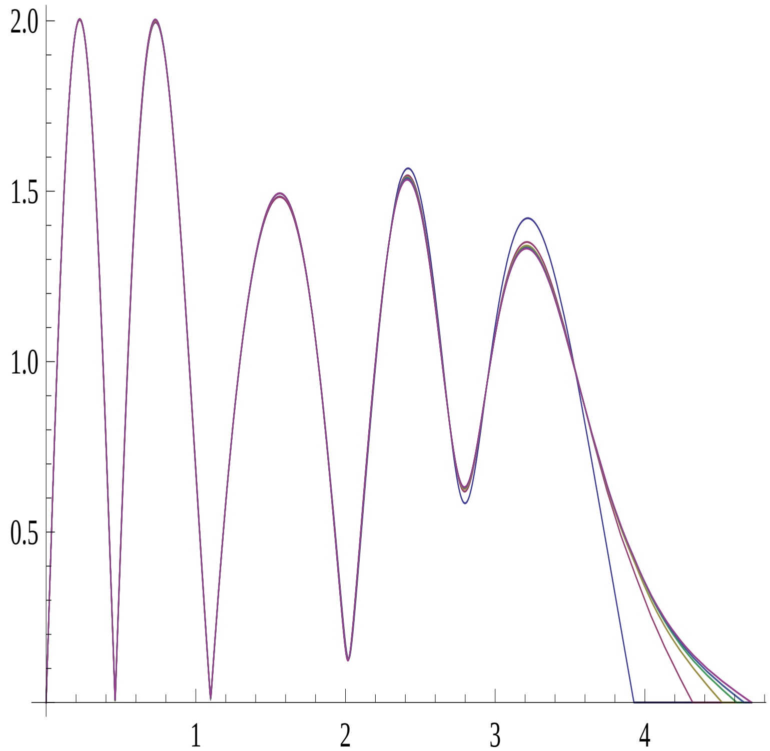


**Fig. 7** The absolute value of the total field $|u_D|$ against $\theta$, when a plane wave at incidence $45^0$ is diffracted by a soft wedge with angles $5\pi/4, 11\pi/8, 23\pi/16, 47\pi/32, 95\pi/64$ and $3\pi/2$ with $kr = 10$.

*Application to plane wave grazing incidence on a corner.*

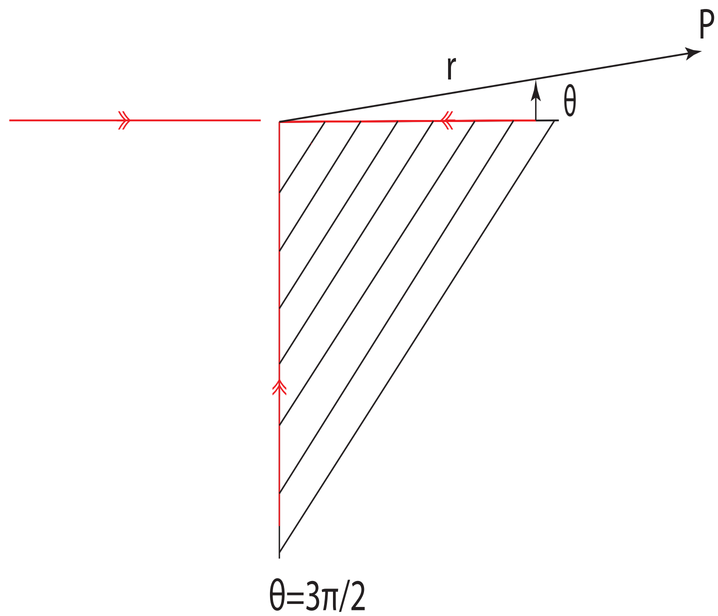


**Fig. 8** Diffraction of an incident Grazing wave by a corner where $\theta \approx 0$.

As an application of the previous analytical results we shall obtain some new expressions for the asymptotic behaviour of the far wave field when a plane wave is incident along one of the faces of an ideal corner and the observation is along the other. Thus for a wedge angle $\alpha = 3\pi/2, p = 3, q = 2$ with angle of observation $\theta \to 0^+$ we shall obtain the Green's functions for angles of incidence $\theta_0 = 0, \pi, 3\pi/2$.These will enable us to derive expressions for the far field at grazing incidence for an acoustically rigid and soft corner. From the expression (2.18) we obtain for $p = 3, q = 2, \theta \to 0^+$

$$
\begin{aligned}
&G_{3\pi/2}(r,\theta,\pm\theta_0;k) = H[\pi - |\theta \mp \theta_0|]e^{ikr\cos(\theta\mp\theta_0)}\\
&-\frac{2}{\sqrt{3}}\cos\frac{(\theta\mp\theta_0)}{6}\text{sign}[\cos\frac{(\theta\mp\theta_0)}{2}]e^{ikr\cos(\theta\mp\theta_0)}F[\sqrt{2kr}|\cos\frac{(\theta\mp\theta_0)}{2}|]\\
&+\frac{2}{\sqrt{3}}\sin\frac{(\theta\mp\theta_0)}{6}\text{sign}[\sin\frac{(\theta\mp\theta_0)}{2}]e^{-ikr\cos(\theta\mp\theta_0)}F[\sqrt{2kr}|\sin\frac{(\theta\mp\theta_0)}{2}|]\\
&+O[(kr)^{-3/2}].
\end{aligned}
\tag{2.21}
$$

*Grazing case:* $\theta_0 = 0, \quad \theta \to 0^+$.

For this case we obtain from the expression (2.18)

$$
G_{3\pi/2}(r,\theta,\pm 0;k) = e^{ikr} - \frac{2}{\sqrt{3}}e^{ikr}F[\sqrt{2kr}] + \frac{\theta}{3\sqrt{3}}e^{-ikr}F\left[\theta\sqrt{\frac{kr}{2}}\right] + O[(kr)^{-3/2}] + O[kr\theta^2],
$$

and by using the results:

$$
F[\sqrt{2kr}] = \frac{e^{-i(2kr+\frac{\pi}{4})}}{2\sqrt{2\pi kr}} + O[(kr)^{-3/2}];\, F\left[\theta\sqrt{\frac{kr}{2}}\right] = \frac{1}{2} - \theta\sqrt{\frac{2\pi}{kr}}e^{i\frac{\pi}{4}} + O[kr\theta^2]; \tag{2.22}
$$

valid for $kr \to \infty$, $\sqrt{(kr)}\theta \to 0^+$ we get:

$$
G_{3\pi/2}(r,\theta,\pm 0;k) = e^{ikr} - \frac{e^{-i(kr+\frac{\pi}{4})}}{\sqrt{6\pi kr}} + \frac{\theta}{6\sqrt{3}}e^{-ikr} + O[(kr)^{-3/2}] + O[kr\theta^2]. \tag{2.23}
$$

Thus the far field for a rigid and soft corner for $\theta_0 = 0, \theta \to 0^+$ is given by

$$
u_h(r,\theta) = 2e^{ikr} - \frac{2e^{-i(kr+\frac{\pi}{4})}}{\sqrt{6\pi kr}} + \frac{\theta}{3\sqrt{3}}e^{-ikr} + O[(kr)^{-3/2}] + O[kr\theta^2], \tag{2.24}
$$

and

$$
u_s(r,\theta) = 0 + O[(kr)^{-3/2}] + O[kr\theta^2] \tag{2.25}
$$

*Grazing case:* $\theta_0 = \pi, \quad \theta \to 0^+$.

For this case we obtain from the expression (2.18)

$$
\begin{aligned}
&G_{3\pi/2}(r,\theta,\pm\pi;k) = H[\pm 1]e^{-ikr} \mp \frac{2}{\sqrt{3}}\cos\frac{(\theta\mp\pi)}{6}e^{-ikr}F\left[\theta\sqrt{\frac{kr}{2}}\right]\\
&\mp\frac{2}{\sqrt{3}}\sin\frac{(\theta\mp\pi)}{6}e^{ikr}F[\sqrt{2kr}] + O[(kr)^{-3/2}] + O[kr\theta^2],
\end{aligned}
\tag{2.26}
$$

and by using the results (2.22) we get:

$$G_{3\pi/2}(r,\theta,\pm 0;k) = H[\pm 1]e^{-ikr} \mp \frac{1}{2}e^{-ikr} - \frac{\theta e^{-ikr}}{12\sqrt{3}} + \frac{e^{-i(kr+\frac{\pi}{4})}}{2\sqrt{6\pi kr}}$$
$$+O[(kr)^{-3/2}] + O[\theta(kr)^{-1/2}] + O[kr\theta^2]. \tag{2.27}$$

Thus the far field for a rigid and soft corner for $\theta_0 = \pi, \theta \to 0^+$ is given by

$$u_h(r,\theta) = e^{-ikr} + \frac{e^{-i(kr+\frac{\pi}{4})}}{\sqrt{6\pi kr}} - \frac{\theta}{6\sqrt{3}}e^{-ikr} \tag{2.28}$$
$$+O[(kr)^{-3/2}] + O[kr\theta^2] + O[\theta(kr)^{-1/2}],$$

and

$$u_s(r,\theta) = 0 + O[(kr)^{-3/2}] + O[kr\theta^2] + O[\theta(kr)^{-1/2}]. \tag{2.29}$$

*Grazing case:* $\theta_0 = 3\pi/2, \quad \theta \to 0^+$.

For this case we obtain from the expression (2.18)

$$G_{3\pi/2}(r,\theta,\pm 3\pi/2;k) = \sqrt{\frac{2}{3}}(1 \pm \frac{\theta}{6})(1 \mp ikr\theta)F\left[\sqrt{kr}\right]$$
$$\mp\sqrt{\frac{2}{3}}(\frac{\theta}{6} \mp 1)(1 \pm ikr\theta)F[\sqrt{kr}] + O[(kr)^{-3/2}] + O[(kr)^{-1/2}\theta^2],$$
$$= 2\sqrt{\frac{2}{3}}F[\sqrt{kr}], \tag{2.30}$$

and by using the results (2.22) we get:

$$G_{3\pi/2}(r,\theta,\pm 0;k) = 2\frac{e^{-i(kr+\frac{\pi}{4})}}{\sqrt{6\pi kr}}$$
$$+O[(kr)^{-3/2}] + O[\theta(kr)^{-1/2}]. \tag{2.31}$$

Thus the far field for a rigid and soft corner for $\theta_0 = 3\pi/2, \theta \to 0^+$ is given by

$$u_h(r,\theta) = 4\frac{e^{-i(kr+\frac{\pi}{4})}}{\sqrt{6\pi kr}} + O[(kr)^{-3/2}] + O[\theta^2(kr)^{-1/2}],$$

and

$$u_s(r,\theta) = 0 + O[(kr)^{-3/2}] + O[\theta^2(kr)^{-1/2}]. \tag{2.32}$$

Finally we remark that the above situations where the values of $\theta$ and $\theta_0$ are interchanged can be dealt with by an application of the reciprocity theorem.

## 3. Conclusions

We have derived new uniformly valid high frequency approximate expressions for the Green's function needed to solve problems of diffraction by plane, cylindrical and spherical wave sources by ideal rational wedges. The method used here can be extended to deal with more complicated boundary conditions when the canonical solution is known as a complex integral; for example the impedance wedge Rawlins(**17**). The method is a considerable improvement on the existing uniform asymptotic methods in that it is more direct and less complicated. It avoids having to produce a special pole addition or subtraction ansatz to deal with each wedge-angle as a special case. As a check on the results some numerical runs where made using Mathematica for the wedges with angle $3\pi/2$ and $\alpha = (2^n 3 - 1)\pi/(2^n 2)$, for n=2,3,4,5 and 6. The results showed that convergence resulted as $n$ increased; giving support to the claim that the uniform results for a any wedge angle can be approximated sufficiently closely by a rational wedge angle. An attempt was made to show this analytically but this proved to be a far from trivial and is another problem altogether. We have also shown how these results will be of use in the practical situations, for example for the numerical computation of the uniform scattered far field; and to obtain uniform results for the highly singular situation of grazing incidence.

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