Random paths to stability in three-sided matching with cyclic preferences

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Random paths to stability in three-sided matching with cyclic preferences

Yusuke Samejima

Abstract

We investigate the three-sided matching model with cyclic preferences: there are three equinumerous disjoint sets of agents, N_1, N_2, and N_3, and each agent $a \in N_n$ has a strict preference relation over $N_{n+1} \cup \{a\}$ with himself ranked last on his preference list. We show that, starting from an arbitrary unstable matching, there exists a finite sequence of successive blockings leading to a stable matching for a three-sided matching problem with cyclic preferences. The result implies that a decentralized process of successive blockings by randomly chosen blocking agents will converge to a stable matching with probability one. When $|N_n| \leq 5$, the result applies for any problem. When $|N_n| = 6$, the result applies for any problem for which there exists a stable matching.

1. Introduction

The three-sided matching model with cyclic preferences, or equivalently *the three-sided cyclic matching model*, is described as follows. There are three disjoint sets of agents: N_1, N_2, and N_3 with $|N_1| = |N_2| = |N_3|$. The set of all agents is denoted by $N \equiv N_1 \cup N_2 \cup N_3$. The word *cyclic* is used because the way how agents have preference relations exhibits a cycle: each agent $a \in N_n$ has a strict preference relation over $N_{n+1} \cup \{a\}$ where N_{n+1} is interpreted as N_1 when $n = 3$. A matching is a bijection $\mu\colon N \to N$ such that, for any $a \in N_n$, we have $\mu(a) \in N_{n+1} \cup \{a\}$ and $\mu(\mu(\mu(a))) = a$. If $\mu(a) \neq a$, agent a is matched at the matching μ. If $\mu(a) = a$, agent a is single at the matching μ.

The major difference between the three-sided *cyclic* matching model and *the three-sided matching model* as described in Alkan [1988] and in Danilov [2003] is the way how agents have preference relations.

In the latter three-sided matching model, each agent $a \in N_n$ has a strict preference relation over $N_{n+1} \times N_{n+2} \cup \{a\}$. So, each agent cares both of his partners in the three-sided matching model while each agent cares only one of his partners in the three-sided cyclic matching model.

Both of the models described above are extensions of the two-sided matching model analyzed by Gale and Shapley [1962]. In the two-sided matching model, there are two kinds of agents, men and women. The sets of men and women are denoted by M and W, respectively. Each agent has a strict preference relation over agents of the opposite sex. A matching is a bijection $\mu\colon M \cup W \to M \cup W$ such that, for any $a \in M$, we have $\mu(a) \in W \cup \{a\}$ and $\mu(\mu(a)) = a$, and for any $a \in W$, we have $\mu(a) \in M \cup \{a\}$ and $\mu(\mu(a)) = a$. So, the two-sided matching model can be regarded as the two-sided cyclic matching model.

Stability has been a key research topic in the field of matching theory. In the two-sided matching model, a matching μ is blocked by a single agent if he prefers being single to being matched to his partner at μ. A matching μ is also blocked by a pair of a man and a woman $(m, w) \in M \times W$ if they prefer each other to their own partners at μ. A matching μ is *stable* if it is not blocked by any single agent or any pair of agents. Gale and Shapley [1962] proved the existence of a stable matching for any two-sided matching problem by presenting an algorithm for finding a stable matching.

As for the three-sided matching model in general, a stable matching does not necessarily exist as Alkan [1988] presented an example. However, if we restrict preference domains of agents in a certain way, there is a possibility that there exists a stable matching for a three-sided matching problem as Danilov [2003] showed. He considered the following restrictions on preference domains. Each agent in N_1 is more interested in agents in N_2 than in agents in N_3 while each agent in N_2 is more interested in agents in N_1 than in agents in N_3. Danilov [2003] presented an algorithm for finding a stable matching for any three-sided matching problem with these restricted preference domains.

Stability has been studied also in the three-sided cyclic matching model. As Biro and McDermid [2010] showed an example with $|N_n| = 6$, a stable matching does not necessarily exist in the three-sided cyclic matching model if there exist some agents who do not rank themselves last on their preference lists, that is, if there exist agents in N_n such that each of them prefers being single to being matched to an unacceptable agent in N_{n+1}.

On the other hand, some researchers have studied stability in the three-sided cyclic matching model, focusing on the case where each agent ranks himself last on his preference list. Boros et al. [2004] proved the existence of a stable matching for the case $|N_n| \leq 3$. Eriksson et al. [2006] proved the existence for the case $|N_n| = 4$. Furthermore, based on computer search, Eriksson et al. [2006] stated a conjecture that there were at least two stable matchings for any three-sided cyclic matching problem when $|N_n| \geq 2$. Woeginger [2013] stated that proving the existence of a stable matching for any three-sided cyclic matching problem was an open problem. Pashkovich and Poirrier [2018] demonstrated that at least two distinct stable matchings exist for the case $|N_n| = 5$. So, the conjecture made by Eriksson et al. [2006] seemed promising. However, Lam and Plaxton [2019] showed that, by utilizing the example with $|N_n| = 6$ presented by Biro and McDermid [2010], a stable matching does not necessarily exist in the three-sided cyclic matching model even if each agent ranks himself last on his preference list. Their result indicates that a stable matching does not exist for some three-sided cyclic matching problems when $|N_n| \geq 6$.

Another key research topic in the field of matching theory is an investigation of random paths to a stable matching. The question here is whether decentralized decision making by each agent leads to stability. This question was answered by Roth and Vande Vate [1990] for the two-sided matching model. They proved that, starting from an arbitrary unstable matching, there exists a finite sequence of successive blockings leading to a stable matching. Their result implies that a decentralized process of successive blockings by randomly chosen blocking agents will converge to a stable matching with probability one. The same question was answered by Samejima [2018] for the three-sided matching model with the restricted preference domains considered by Danilov [2003].

In the present paper, we investigate random paths to a stable matching in the three-sided cyclic matching model for the case where each agent ranks himself last on his preference list. However, as we have noted, the existence of a stable matching for the case $|N_n| \geq 6$ is not guaranteed; a problem may or may not have a stable matching, depending on the preference relations that agents have. So, we focus on a three-sided cyclic matching problem such that, given the preferences of agents, a stable matching exists for the problem itself and for any problem constructed of a subset of the set of all agents. We investigate paths to stability for such a problem. We prove that, starting from an arbitrary unstable matching, there exists a finite sequence of successive blockings leading to a stable matching. Our result implies the following. When $|N_n| \leq 5$, a decentralized process of successive blockings by randomly chosen blocking agents will converge to a stable matching with probability one for any problem. When $|N_n| = 6$, such successive random blockings will lead to stability for any problem for which there exists a stable matching.

The remaining part of this paper is organized as follows. Section 2 explains a model of three-sided cyclic matching model. Section 3 shows our main result. Section 4 provides some concluding remarks.

2. Preliminaries

There are three disjoint sets of agents: N_1, N_2, and N_3. For each agent $a \in N_n$ with $n \in \{1, 2, 3\}$, N_n represents agent a's *gender*. We assume that $|N_1| = |N_2| = |N_3|$. So, each of these sets contains the same number of agents. We denote each of these sets by N_n with $n = 1$, 2, or 3. When $n = 2$, N_{n+2} is interpreted as N_1. When $n = 3$, N_{n+1} and N_{n+2} are interpreted as N_1 and N_2, respectively. The set of all agents is denoted by $N \equiv N_1 \cup N_2 \cup N_3$.

For any $n \in \{1, 2, 3\}$, each agent $a \in N_n$ has a complete, transitive, and strict preference relation $\succsim_a$ over $N_{n+1} \cup \{a\}$. When a prefers $b \in N_{n+1} \cup \{a\}$ to $b' \in N_{n+1} \cup \{a\}$, we write $b \succ_a b'$. When a weakly prefers b to b', that is, b is at least as good as b' for a, we write $b \succsim_a b'$. We assume that each agent ranks himself last on his preference list. That is, for any $n \in \{1, 2, 3\}$, $a \in N_n$, and $b \in N_{n+1}$, we have $b \succ_a a$.

Definition. A *matching* is a bijection μ: $N \to N$ such that for any $n \in \{1, 2, 3\}$ and $a \in N_n$, we have

(i) $\mu(a) \in N_{n+1} \cup \{a\}$, and

(ii) $\mu(\mu(\mu(a))) = a$.

We say that $a \in N$ is *single at* μ if $\mu(a) = a$ while $a \in N$ is *matched at* μ if $\mu(a) \neq a$. The set of single agents at μ is denoted by $S(\mu) \equiv \{a \in N : \mu(a) = a\}$ while the set of matched agents at μ is denoted by $T(\mu) \equiv \{a \in N : \mu(a) \neq a\}$. We also say that a triplet (a, b, c) is *matched at* μ if $(a, b, c) \in N_n \times N_{n+1} \times N_{n+2}$ for some $n \in \{1, 2, 3\}$ and we have $\mu(a) = b$, $\mu(b) = c$, and $\mu(c) = a$. When (a, b, c) is matched at μ, we write $(a, b, c) \in \mu$ by abuse of notation, and we say that b and c are *partners* of a at μ. In this case, we also say that c and a are partners of b at μ as well as a and b are partners of c at μ.

A *three-sided matching problem with cyclic preferences*, or equivalently a *three-sided cyclic matching problem*, is specified by $(N, (\succsim_a)_{a \in N})$.

We now define stability of a matching. A triplet (a, b, c) is called a *blocking triplet for* μ if $(a, b, c) \in N_n \times N_{n+1} \times N_{n+2}$ for some $n \in \{1, 2, 3\}$ and we have $b \succ_a \mu(a)$, $c \succ_b \mu(b)$, and $a \succ_c \mu(c)$. A matching μ is *individually rational* if $\mu(a) \succsim_a a$ for all $a \in N$.

Definition. A matching μ is *stable* if it is individually rational and there is no blocking triplet for μ.

Since we assume that each agent ranks himself last on his preference list, any matching is individually rational in the present paper. Furthermore, no agent is single at a stable matching due to our assumption that $|N_1| = |N_2| = |N_3|$. In other words, if some agents are single at a matching, then the matching is unstable.

3. The Result

This section investigates whether, starting from an arbitrary unstable matching, there exists a finite sequence of successive blockings leading to a stable matching on condition that there exists a stable matching in the three-sided cyclic matching model.

Suppose that a triplet (a, b, c) is a blocking triplet for μ. We say that another matching μ' is obtained from μ by *satisfying* the blocking triplet (a, b, c) for μ if the following three conditions hold.

(i) We have $(a, b, c) \in \mu'$.

(ii) For any $i \in \{a, b, c\}$, if $\mu(i) = j \neq i$ or $\mu(j) = i \neq j$ then $\mu'(j) = j$.

(iii) If $i \notin \{a, b, c, \mu(a), \mu(b), \mu(c)\}$ or $\mu(i) \notin \{a, b, c\}$ then $\mu'(i) = \mu(i)$.

Condition (i) says that the blocking triplet (a, b, c) for μ is newly matched at μ'. Condition (ii) says that the partners of a, b, and c at μ become single at μ'. Condition (iii) says that agents irrelevant to (a, b, c) at μ are unaffected even when μ' is obtained from μ.

Theorem. *Let μ_1 be an arbitrary unstable matching for a three-sided cyclic matching problem, which is specified by $(N, (\succsim_a)_{a \in N})$ with $N \equiv N_1 \cup N_2 \cup N_3$ and $|N_n| = |N|/3$ for $n \in \{1, 2, 3\}$. Suppose that for any $\bar{N}_1 \subset N_1$, $\bar{N}_2 \subset N_2$, and $\bar{N}_3 \subset N_3$ with $|\bar{N}_1| = |\bar{N}_2| = |\bar{N}_3|$, there exists a stable matching for the problem $(\bar{N}, (\succsim_a)_{a \in \bar{N}})$ with $\bar{N} \equiv \bar{N}_1 \cup \bar{N}_2 \cup \bar{N}_3$.*[1] *Then, there exists a finite sequence $\mu_1, \ldots, \mu_L$ of matchings for the problem $(N, (\succsim_a)_{a \in N})$ such that μ_L is stable, and for each $\ell = 1, \ldots, L-1$, there is a blocking triplet for μ_ℓ such that $\mu_{\ell+1}$ is obtained from μ_ℓ by satisfying the blocking triplet.*

Proof. At the beginning, we have an unstable matching μ_1.

Step 1. Suppose that $S(\mu_1) = N$. That is, all agents are single at μ_1. If this is not the case, then we proceed to step 2. Otherwise, we go through the following procedure.

Let μ^* be a stable matching for the problem $(N, (\succsim_a)_{a \in N})$ and let $\{(a_\ell, b_\ell, c_\ell)\}_{\ell=1}^{|N|/3}$ be the collection of triplets that are matched at μ^*, i.e., $(a_\ell, b_\ell, c_\ell) \in \mu^*$ for $\ell = 1, \ldots, |N|/3$. We now obtain a sequence of matchings $\mu_1, \ldots, \mu_{|N/3|+1} = \mu^*$ such that for each $\ell = 1, \ldots, |N|/3$, we obtain $\mu_{\ell+1}$ from μ_ℓ by

[1]When we consider the problem $(\bar{N}, (\succsim_a)_{a \in \bar{N}})$, we must note that the preference relation of agent $a \in \bar{N}_n$, $\succsim_a$, is defined over not $\bar{N}_{n+1} \cup \{a\}$ but $N_{n+1} \cup \{a\}$. However, we use $\succsim_a$ for the specification of the problem since the preference relation over $\bar{N}_{n+1} \cup \{a\}$ can be directly induced from the preference relation over $N_{n+1} \cup \{a\}$.

satisfying the blocking triplet (a_ℓ, b_ℓ, c_ℓ). We note that (a_ℓ, b_ℓ, c_ℓ) is in fact a blocking triplet for μ_ℓ since a_ℓ, b_ℓ, and c_ℓ are single at μ_ℓ.

At the end of step 1, we obtain a stable matching in the sequence of matchings mentioned in the theorem.

Step 2. Let μ_ℓ be the unstable matching that we have at the beginning of this step. Let $|T(\mu_\ell)|$ be the number of matched agents at μ_ℓ. Note that, whenever step 2 is reached, $|T(\mu_\ell)| \geq 3$ and it is clear that $|T(\mu_\ell)|$ is a multiple of three.

As we will see, we may return to step 2 after we go through subsequent steps such as steps 3, 4, 7, and 8. So, we may be in a loop, but this loop is not infinite because every time we return to step 2, the number of matched agents $|T(\mu_\ell)| \geq 3$ is smaller than before.

We now proceed to step 3.

Step 3. Suppose that there exists a blocking triplet (a, b, c) for μ_ℓ such that $a \in T(\mu_\ell)$, $b \in T(\mu_\ell)$, and $c \in T(\mu_\ell)$, that is, these three agents are not single at μ_ℓ.[2] If there does not exist such a blocking triplet, then we proceed to step 4. Otherwise, we go through the following procedure.

We now obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a, b, c) for μ_ℓ. We note that $|T(\mu_{\ell+1})| = |T(\mu_\ell)| - 6$ since the partners of a, b, and c at μ_ℓ become single at $\mu_{\ell+1}$. We also note that $\mu_{\ell+1}$ is unstable since some agents are single at $\mu_{\ell+1}$. We now return to step 2 with the unstable matching $\mu_{\ell+1}$ with the smaller number of matched agents $|T(\mu_{\ell+1})| < |T(\mu_\ell)|$.

Step 4. Suppose that there exists a blocking triplet (a, b, c) for μ_ℓ such that $a \in S(\mu_\ell)$, $b \in T(\mu_\ell)$, and $c \in T(\mu_\ell)$, that is, one of these agents is single and the other two agents are not single at μ_ℓ.[3] If there does not exist such a blocking triplet, then we proceed to step 5. Otherwise, we go through the following procedure.

We now obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a, b, c) for μ_ℓ. We note that $|T(\mu_{\ell+1})| = |T(\mu_\ell)| - 3$ since the partners of b and c at μ_ℓ become single at $\mu_{\ell+1}$ and the formerly single agent a gets matched at $\mu_{\ell+1}$. We also note that $\mu_{\ell+1}$ is unstable since some agents are single at $\mu_{\ell+1}$. We now return to step 2 with the unstable matching $\mu_{\ell+1}$ with the smaller number of matched agents $|T(\mu_{\ell+1})| < |T(\mu_\ell)|$.

[2] We note that these three agents are not partners of each other at μ_ℓ because, if so, they cannot form the blocking triplet for μ_ℓ since every agent in the blocking triplet should be matched to a more preferred agent after blocking. Therefore, if there exists the blocking triplet for μ_ℓ as described in step 3, it must be the case that $|T(\mu_\ell)| \geq 9$.

[3] If there exists the blocking triplet for μ_ℓ as described in step 4, it must be the case that $|T(\mu_\ell)| \geq 6$.

Step 5. We first note that $|S(\mu_\ell)|$ is a multiple of three since both $|N|$ and $|T(\mu_\ell)|$ are multiples of three. We also note that, whenever step 5 is reached, $|S(\mu_\ell)| \geq 3$. This is because, if $|S(\mu_\ell)| = 0$ at the beginning of step 5, the fact that we went through step 3 implies that there exists no blocking triplet for μ_ℓ, which is in contradiction with the instability of μ_ℓ assumed at the beginning of step 2. If $|S(\mu_\ell)| \geq 6$, then we proceed to step 6. If $|S(\mu_\ell)| = 3$, then we go through the following procedure.

Let $S(\mu_\ell)$ be such that $S(\mu_\ell) = \{a, b, c\}$ for which $(a, b, c) \in N_1 \times N_2 \times N_3$. We note that (a, b, c) is in fact a blocking triplet for μ_ℓ since a, b, and c are single at μ_ℓ. We now obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a, b, c) for μ_ℓ.

We claim that $\mu_{\ell+1}$ is a stable matching. Suppose, by way of contradiction, that there exists a blocking triplet (a', b', c') for $\mu_{\ell+1}$. Since we went through steps 3 and 4 before reaching step 5, it must be the case that one member of $\{a', b', c'\}$ is in $T(\mu_\ell)$ and the other two members are in $S(\mu_\ell)$. However, the two members in $S(\mu_\ell)$ are partners of each other at $\mu_{\ell+1}$ and hence they cannot form the blocking triplet for $\mu_{\ell+1}$ since every agent in the blocking triplet should be matched to a more preferred agent after blocking. A contradiction obtains.

At the end of step 5, we obtain a stable matching in the sequence of matchings mentioned in the theorem.

Step 6. Suppose that there does not exist a blocking triplet (a, b, c) for μ_ℓ such that $a \in T(\mu_\ell)$, $b \in S(\mu_\ell)$, and $c \in S(\mu_\ell)$, that is, one of these agents is matched and the other two agents are single at μ_ℓ. If there exists such a blocking triplet, then we proceed to step 7. Otherwise, we go through the following procedure.

Let $\bar{\mu}^*$ be a stable matching for the problem $(S(\mu_\ell), (\succsim_a)_{a \in S(\mu_\ell)})$ and let $\{(a_k, b_k, c_k)\}_{k=1}^{|S(\mu_\ell)|/3}$ be the collection of triplets that are matched at $\bar{\mu}^*$, i.e., $(a_k, b_k, c_k) \in \bar{\mu}^*$ for $k = 1, \ldots, |S(\mu_\ell)|/3$. We now obtain a sequence of matchings $\mu_\ell, \ldots, \mu_{\ell+|S(\mu_\ell)|/3}$ for the problem $(N, (\succsim_a)_{a \in N})$ such that for each $k = 1, \ldots, |S(\mu_\ell)|/3$, we obtain $\mu_{\ell+k}$ from $\mu_{\ell+k-1}$ by satisfying the blocking triplet (a_k, b_k, c_k). We note that (a_k, b_k, c_k) is in fact a blocking triplet for $\mu_{\ell+k-1}$ since a_k, b_k, and c_k are single at $\mu_{\ell+k-1}$.

We claim that $\mu_{\ell+|S(\mu_\ell)|/3}$ is a stable matching. Suppose, by way of contradiction, that there exists a blocking triplet (a', b', c') for $\mu_{\ell+|S(\mu_\ell)|/3}$. Since we went through steps 3 and 4 before reaching step 6, and since we did not proceed to step 7 at the beginning of step 6, it must be the case that all members of $\{a', b', c'\}$ are in $S(\mu_\ell)$. Then, (a', b', c') must be a blocking triplet for $\bar{\mu}^*$ for the problem $(S(\mu_\ell), (\succsim_a)_{a \in S(\mu_\ell)})$, which is in contradiction with the stability of $\bar{\mu}^*$.

At the end of step 6, we obtain a stable matching in the sequence of matchings mentioned in the theorem.

Step 7. We note that, whenever step 7 is reached, $|S(\mu_\ell)| \geq 6$ since we went through step 5. We also note that, since we went through step 6, there exists a blocking triplet (a, b, c) for μ_ℓ such that $a \in T(\mu_\ell)$, $b \in S(\mu_\ell)$, and $c \in S(\mu_\ell)$, that is, one of these agents is matched and the other two agents are single at μ_ℓ. We now investigate whether there exists another blocking triplet (d, e, f) for μ_ℓ such that $d \in T(\mu_\ell) \setminus \{a, \mu_\ell(a), \mu_\ell(\mu_\ell(a))\}$, $e \in S(\mu_\ell)$, and $f \in S(\mu_\ell)$. If there does not exist such a triplet (d, e, f) described above, then we proceed to step 8. Otherwise, one of the following three cases must apply. These three cases are mutually exclusive and collectively exhaustive.

Case 7–1. *Agents a and d are in different genders.*

Without loss of generality, we may assume that $(a, b, c) = (a_1, b_1, c_1) \in N_1 \times N_2 \times N_3$ and $(e, f, d) = (a_2, b_2, c_2) \in N_1 \times N_2 \times N_3$. We note that $a_1 \neq a_2$ and $c_1 \neq c_2$. We consider the following two subcases.

Subcase 7–1–1. We have $c_2 \succ_{b_1} c_1$.

We first obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a_1, b_1, c_1) for μ_ℓ. We note that $|T(\mu_{\ell+1})| = |T(\mu_\ell)|$ since the partners of a_1 at μ_ℓ become single at $\mu_{\ell+1}$ while the formerly single agents b_1 and c_1 get matched at $\mu_{\ell+1}$.

We claim that (a_2, b_1, c_2) is a blocking triplet for $\mu_{\ell+1}$. This is because a_2 is single at $\mu_{\ell+1}$ while b_1 and c_2 prefer c_2 and a_2 to $\mu_{\ell+1}(b_1) = c_1$ and $\mu_{\ell+1}(c_2) = \mu_\ell(c_2)$, respectively. We next obtain another matching $\mu_{\ell+2}$ by satisfying the blocking triplet (a_2, b_1, c_2) for $\mu_{\ell+1}$. We note that $|T(\mu_{\ell+2})| = |T(\mu_{\ell+1})| - 3$ since a_1, c_1, and the partners of c_2 at $\mu_{\ell+1}$ become single at $\mu_{\ell+2}$ while the formerly single agent a_2 gets matched at $\mu_{\ell+2}$. We also note that $\mu_{\ell+2}$ is unstable since some agents are single at $\mu_{\ell+2}$. We now return to step 2 with the unstable matching $\mu_{\ell+2}$ with the smaller number of matched agents $|T(\mu_{\ell+2})| < |T(\mu_\ell)|$.

Subcase 7–1–2. We have $c_1 \succ_{b_1} c_2$.

We note that (a_2, b_1, c_2) is a blocking triplet for μ_ℓ. This is because a_2 and b_1 are single at μ_ℓ while c_2 prefers a_2 to $\mu_\ell(c_2)$. We first obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a_2, b_1, c_2) for μ_ℓ. We note that $|T(\mu_{\ell+1})| = |T(\mu_\ell)|$ since the partners of c_2 at μ_ℓ become single at $\mu_{\ell+1}$ while the formerly single agents a_2 and b_1 get matched at $\mu_{\ell+1}$.

We claim that (a_1, b_1, c_1) is a blocking triplet for $\mu_{\ell+1}$. This is because c_1 is single at $\mu_{\ell+1}$ while a_1 and b_1 prefer b_1 and c_1 to $\mu_{\ell+1}(a_1) = \mu_\ell(a_1)$ and $\mu_{\ell+1}(b_1) = c_2$, respectively. We next obtain another matching $\mu_{\ell+2}$ by satisfying the blocking triplet (a_1, b_1, c_1) for $\mu_{\ell+1}$. We note that $|T(\mu_{\ell+2})| = |T(\mu_{\ell+1})| - 3$ since a_2, c_2, and the partners of a_1 at $\mu_{\ell+1}$ become single at $\mu_{\ell+2}$ while the formerly single agent c_1 gets matched at $\mu_{\ell+2}$. We also note that $\mu_{\ell+2}$ is unstable since some agents are single at $\mu_{\ell+2}$. We now return to step 2 with the unstable matching $\mu_{\ell+2}$ with the smaller number of matched agents $|T(\mu_{\ell+2})| < |T(\mu_\ell)|$.

Case 7–2. *Agents a and d are in the same gender and $b \neq e$.*

Without loss of generality, we may assume that $(a, b, c) = (a_1, b_1, c_1) \in N_1 \times N_2 \times N_3$ and $(d, e, f) = (a_2, b_2, c_2) \in N_1 \times N_2 \times N_3$. We note that $a_1 \neq a_2$ and $b_1 \neq b_2$. We consider the following two subcases.

Subcase 7–2–1. We have $a_2 \succ_{c_1} a_1$.

We first obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a_1, b_1, c_1) for μ_ℓ. We note that $|T(\mu_{\ell+1})| = |T(\mu_\ell)|$ since the partners of a_1 at μ_ℓ become single at $\mu_{\ell+1}$ while the formerly single agents b_1 and c_1 get matched at $\mu_{\ell+1}$.

We claim that (a_2, b_2, c_1) is a blocking triplet for $\mu_{\ell+1}$. This is because b_2 is single at $\mu_{\ell+1}$ while a_2 and c_1 prefer b_2 and a_2 to $\mu_{\ell+1}(a_2) = \mu_\ell(a_2)$ and $\mu_{\ell+1}(c_1) = a_1$, respectively. We next obtain another matching $\mu_{\ell+2}$ by satisfying the blocking triplet (a_2, b_2, c_1) for $\mu_{\ell+1}$. We note that $|T(\mu_{\ell+2})| = |T(\mu_{\ell+1})| - 3$ since a_1, b_1, and the partners of a_2 at $\mu_{\ell+1}$ become single at $\mu_{\ell+2}$ while the formerly single agent b_2 gets matched at $\mu_{\ell+2}$. We also note that $\mu_{\ell+2}$ is unstable since some agents are single at $\mu_{\ell+2}$. We now return to step 2 with the unstable matching $\mu_{\ell+2}$ with the smaller number of matched agents $|T(\mu_{\ell+2})| < |T(\mu_\ell)|$.

Subcase 7–2–2. We have $a_1 \succ_{c_1} a_2$.

We note that (a_2, b_2, c_1) is a blocking triplet for μ_ℓ. This is because b_2 and c_1 are single at μ_ℓ while a_2 prefers b_2 to $\mu_\ell(a_2)$. We first obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a_2, b_2, c_1) for μ_ℓ. We note that $|T(\mu_{\ell+1})| = |T(\mu_\ell)|$ since the partners of a_2 at μ_ℓ become single at $\mu_{\ell+1}$ while the formerly single agents b_2 and c_1 get matched at $\mu_{\ell+1}$.

We claim that (a_1, b_1, c_1) is a blocking triplet for $\mu_{\ell+1}$. This is because b_1 is single at $\mu_{\ell+1}$ while a_1 and c_1 prefer b_1 and a_1 to $\mu_{\ell+1}(a_1) = \mu_\ell(a_1)$ and $\mu_{\ell+1}(c_1) = a_2$, respectively. We next obtain another matching $\mu_{\ell+2}$ by satisfying the blocking triplet (a_1, b_1, c_1) for $\mu_{\ell+1}$. We note that $|T(\mu_{\ell+2})| = |T(\mu_{\ell+1})| - 3$ since a_2, b_2, and the partners of a_1 at $\mu_{\ell+1}$ become single at $\mu_{\ell+2}$ while the formerly single agent b_1 gets matched at $\mu_{\ell+2}$. We also note that $\mu_{\ell+2}$ is unstable since some agents are single at $\mu_{\ell+2}$. We now return to step 2 with the unstable matching $\mu_{\ell+2}$ with the smaller number of matched agents $|T(\mu_{\ell+2})| < |T(\mu_\ell)|$.

Case 7–3. *Agents a and d are in the same gender and $b = e$.*

Without loss of generality, we may assume that $(a, b, c) = (a_1, b_1, c_1) \in N_1 \times N_2 \times N_3$ and $(d, e, f) = (a_2, b_1, c_2) \in N_1 \times N_2 \times N_3$. We note that $a_1 \neq a_2$. Since we went through step 5, we have $|S(\mu_\ell)| \geq 6$

and hence there exists $c_3 \in N_3 \cap S(\mu_\ell)$ such that $c_3 \neq c_1$.[4] We consider the following two subcases.

Subcase 7–3–1. We have $c_3 \succ_{b_1} c_1$.

We first obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a_1, b_1, c_1) for μ_ℓ. We note that $|T(\mu_{\ell+1})| = |T(\mu_\ell)|$ since the partners of a_1 at μ_ℓ become single at $\mu_{\ell+1}$ while the formerly single agents b_1 and c_1 get matched at $\mu_{\ell+1}$.

We claim that (a_2, b_1, c_3) is a blocking triplet for $\mu_{\ell+1}$. This is because c_3 is single at $\mu_{\ell+1}$ while a_2 and b_1 prefer b_1 and c_3 to $\mu_{\ell+1}(a_2) = \mu_\ell(a_2)$ and $\mu_{\ell+1}(b_1) = c_1$, respectively. We next obtain another matching $\mu_{\ell+2}$ by satisfying the blocking triplet (a_2, b_1, c_3) for $\mu_{\ell+1}$. We note that $|T(\mu_{\ell+2})| = |T(\mu_{\ell+1})| - 3$ since a_1, c_1, and the partners of a_2 at $\mu_{\ell+1}$ become single at $\mu_{\ell+2}$ while the formerly single agent c_3 gets matched at $\mu_{\ell+2}$. We also note that $\mu_{\ell+2}$ is unstable since some agents are single at $\mu_{\ell+2}$. We now return to step 2 with the unstable matching $\mu_{\ell+2}$ with the smaller number of matched agents $|T(\mu_{\ell+2})| < |T(\mu_\ell)|$.

Subcase 7–3–2. We have $c_1 \succ_{b_1} c_3$.

We note that (a_2, b_1, c_3) is a blocking triplet for μ_ℓ. This is because b_1 and c_3 are single at μ_ℓ while a_2 prefers b_1 to $\mu_\ell(a_2)$. We first obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a_2, b_1, c_3) for μ_ℓ. We note that $|T(\mu_{\ell+1})| = |T(\mu_\ell)|$ since the partners of a_2 at μ_ℓ become single at $\mu_{\ell+1}$ while the formerly single agents b_1 and c_3 get matched at $\mu_{\ell+1}$.

We claim that (a_1, b_1, c_1) is a blocking triplet for $\mu_{\ell+1}$. This is because c_1 is single at $\mu_{\ell+1}$ while a_1 and b_1 prefer b_1 and c_1 to $\mu_{\ell+1}(a_1) = \mu_\ell(a_1)$ and $\mu_{\ell+1}(b_1) = c_3$, respectively. We next obtain another matching $\mu_{\ell+2}$ by satisfying the blocking triplet (a_1, b_1, c_1) for $\mu_{\ell+1}$. We note that $|T(\mu_{\ell+2})| = |T(\mu_{\ell+1})| - 3$ since a_2, c_3, and the partners of a_1 at $\mu_{\ell+1}$ become single at $\mu_{\ell+2}$ while the formerly single agent c_1 gets matched at $\mu_{\ell+2}$. We also note that $\mu_{\ell+2}$ is unstable since some agents are single at $\mu_{\ell+2}$. We now return to step 2 with the unstable matching $\mu_{\ell+2}$ with the smaller number of matched agents $|T(\mu_{\ell+2})| < |T(\mu_\ell)|$.

Step 8. We note that, whenever step 8 is reached, $|S(\mu_\ell)| \geq 6$ since we went through step 5. We also note that, since we went through steps 6 and 7, there exists a blocking triplet (a, b, c) for μ_ℓ such that $a \in T(\mu_\ell)$, $b \in S(\mu_\ell)$, and $c \in S(\mu_\ell)$, but there does not exist another blocking triplet (d, e, f) for μ_ℓ such that $d \in T(\mu_\ell) \backslash \{a, \mu_\ell(a), \mu_\ell(\mu_\ell(a))\}$, $e \in S(\mu_\ell)$, and $f \in S(\mu_\ell)$. We now investigate whether there exists a pair of agents (i, j) such that $i \in T(\mu_\ell) \setminus \{a, \mu_\ell(a), \mu_\ell(\mu_\ell(a))\}$ and $j \in S(\mu_\ell) \cup \{a, \mu_\ell(a), \mu_\ell(\mu_\ell(a))\}$ and $j \succ_i \mu_\ell(i)$. If there does not exist the pair (i, j) described above, then we proceed to step 9.

[4] It is possible that $c_3 = c_2$, but even so, it does not affect our argument.

Otherwise, one of the following four cases must apply. These four cases are mutually exclusive and collectively exhaustive.

Case 8–1. $j \in S(\mu_\ell)$.

Without loss of generality, we may assume that $(i, j) \in N_1 \times N_2$. Choose any $h \in N_3 \cap S(\mu_\ell)$. We note that (i, j, h) is a blocking triplet for μ_ℓ. This is because j and h are single at μ_ℓ while i prefers j to $\mu_\ell(i)$. However, the existence of the blocking triplet (i, j, h) for μ_ℓ is in contradiction with the assumption of the non-existence of the triplet (d, e, f) described above. Therefore, this case never applies.

Case 8–2. $j = a$.

We note that (i, j, b) is a blocking triplet for μ_ℓ. This is because b is single at μ_ℓ while i and $j = a$ prefer j and b to $\mu_\ell(i)$ and $\mu_\ell(a)$, respectively. We also note that the triplet $(i, j, b) \in T(\mu_\ell) \times T(\mu_\ell) \times S(\mu_\ell)$ is such that one of these agents is single and the other two agents are not single at μ_ℓ. However, the existence of the blocking triplet (i, j, b) for μ_ℓ is in contradiction with the fact that we went through step 4 and proceeded to step 5 before reaching step 8. Therefore, this case never applies.

Case 8–3. $j = \mu_\ell(a)$.

Without loss of generality, we may assume that $(a, \mu_\ell(a), \mu_\ell(\mu_\ell(a))) = (a_1, b_1, c_1) \in N_1 \times N_2 \times N_3$, $(a, b, c) = (a_1, b_2, c_2) \in N_1 \times N_2 \times N_3$, and $(i, j) = (a_0, b_1) \in N_1 \times N_2$. Since $|S(\mu_\ell)| \geq 6$, we can choose any $c_3 \in N_3 \cap S(\mu_\ell)$ such that $c_3 \neq c_2$. We note that (a_1, b_2, c_3) as well as (a_1, b_2, c_2) is a blocking triplet for μ_ℓ. This is because b_2, c_3, and c_2 are single at μ_ℓ while a_1 prefers b_2 to $\mu_\ell(a_1) = b_1$.

Without loss of generality, suppose that $c_2 \succ_{b_2} c_3$. We first obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a_1, b_2, c_3) for μ_ℓ. We note that $|T(\mu_{\ell+1})| = |T(\mu_\ell)|$ since b_1 and c_1, who are the partners of a_1 at μ_ℓ, become single at $\mu_{\ell+1}$ while the formerly single agents b_2 and c_3 get matched at $\mu_{\ell+1}$. Consequently, $|S(\mu_{\ell+1})| = |S(\mu_\ell)| \geq 6$. Choose any $a_2 \in N_1 \cap S(\mu_{\ell+1})$. We note that $(a_0, b_1, c_1) \in T(\mu_{\ell+1}) \times S(\mu_{\ell+1}) \times S(\mu_{\ell+1})$ is a blocking triplet for $\mu_{\ell+1}$ since $a_0 = i$ prefers $b_1 = j$ to $\mu_{\ell+1}(a_0) = \mu_\ell(a_0)$. We also note that $(b_2, c_2, a_2) \in T(\mu_{\ell+1}) \times S(\mu_{\ell+1}) \times S(\mu_{\ell+1})$ is a blocking triplet for $\mu_{\ell+1}$ since b_2 prefers c_2 to $\mu_{\ell+1}(b_2) = c_3$. We now go through the same procedure as step 7 with the unstable matching $\mu_{\ell+1}$. Then, case 7–1 applies and eventually we will return to step 2 with another unstable matching with the smaller number of matched agents.

Case 8–4. $j = \mu_\ell(\mu_\ell(a))$.

Without loss of generality, we may assume that $(a, \mu_\ell(a), \mu_\ell(\mu_\ell(a))) = (a_1, b_1, c_1) \in N_1 \times N_2 \times N_3$, $(a, b, c) = (a_1, b_2, c_2) \in N_1 \times N_2 \times N_3$, and $(i, j) = (b_0, c_1) \in N_2 \times N_3$. Since $|S(\mu_\ell)| \geq 6$, we can choose any $c_3 \in N_3 \cap S(\mu_\ell)$ such that $c_3 \neq c_2$. We note that (a_1, b_2, c_3) as well as (a_1, b_2, c_2) is a blocking triplet for μ_ℓ. This is because b_2, c_3, and c_2 are single at μ_ℓ while a_1 prefers b_2 to $\mu_\ell(a_1) = b_1$.

Without loss of generality, suppose that $c_2 \succ_{b_2} c_3$. We first obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a_1, b_2, c_3) for μ_ℓ. We note that $|T(\mu_{\ell+1})| = |T(\mu_\ell)|$ since b_1 and c_1, who are the partners of a_1 at μ_ℓ, become single at $\mu_{\ell+1}$ while the formerly single agents b_2 and c_3 get matched at $\mu_{\ell+1}$. Consequently, $|S(\mu_{\ell+1})| = |S(\mu_\ell)| \geq 6$. Choose any two different agents $a_2, a_3 \in N_1 \cap S(\mu_{\ell+1})$. We note that $(b_0, c_1, a_3) \in T(\mu_{\ell+1}) \times S(\mu_{\ell+1}) \times S(\mu_{\ell+1})$ is a blocking triplet for $\mu_{\ell+1}$ since $b_0 = i$ prefers $c_1 = j$ to $\mu_{\ell+1}(b_0) = \mu_\ell(b_0)$. We also note that $(b_2, c_2, a_2) \in T(\mu_{\ell+1}) \times S(\mu_{\ell+1}) \times S(\mu_{\ell+1})$ is a blocking triplet for $\mu_{\ell+1}$ since b_2 prefers c_2 to $\mu_{\ell+1}(b_2) = c_3$. We now go through the same procedure as step 7 with the unstable matching $\mu_{\ell+1}$. Then, case 7–2 applies and eventually we will return to step 2 with another unstable matching with the smaller number of matched agents.

Step 9. We note that, whenever step 9 is reached, $|S(\mu_\ell)| \geq 6$ since we went through step 5. We also note that, since we went through steps 6 and 7, there exists a blocking triplet (a, b, c) for μ_ℓ such that $a \in T(\mu_\ell)$, $b \in S(\mu_\ell)$, and $c \in S(\mu_\ell)$. Furthermore, since we went through step 8, for any $i \in T(\mu_\ell) \setminus \{a, \mu_\ell(a), \mu_\ell(\mu_\ell(a))\}$ and $j \in S(\mu_\ell) \cup \{a, \mu_\ell(a), \mu_\ell(\mu_\ell(a))\}$, we have $\mu_\ell(i) \succ_i j$.

Let us define $\tilde{N}$ and $\hat{N}$ by $\tilde{N} \equiv T(\mu_\ell) \setminus \{a, \mu_\ell(a), \mu_\ell(\mu_\ell(a))\}$ and $\hat{N} \equiv S(\mu_\ell) \cup \{a, \mu_\ell(a), \mu_\ell(\mu_\ell(a))\}$, respectively. Without loss of generality, we may assume that $(a, \mu_\ell(a), \mu_\ell(\mu_\ell(a))) = (a_1, b_1, c_1) \in N_1 \times N_2 \times N_3$ and $(a, b, c) = (a_1, b_2, c_2) \in N_1 \times N_2 \times N_3$. Let $\hat{\mu}^*$ be a stable matching for the problem $(\hat{N}, (\succsim_a)_{a \in \hat{N}})$ and let $\{(\hat{a}_k, \hat{b}_k, \hat{c}_k)\}_{k=1}^{|\hat{N}|/3}$ be the collection of triplets that are matched at $\hat{\mu}^*$, i.e., $(\hat{a}_k, \hat{b}_k, \hat{c}_k) \in \hat{\mu}^*$ for $k = 1, \ldots, |\hat{N}|/3$. We now consider the following six cases. These six cases are mutually exclusive and collectively exhaustive.

Case 9–1. $b_1 = \hat{\mu}^*(a_1)$ *and* $c_1 = \hat{\mu}^*(b_1)$.

In this case, we must have $a_1 = \hat{\mu}^*(c_1)$ and $(a_1, b_1, c_1) \in \hat{\mu}^*$. Without loss of generality, we may assume that $(a_1, b_1, c_1) = (\hat{a}_1, \hat{b}_1, \hat{c}_1)$. We now obtain a sequence of matchings $\mu_\ell, \ldots, \mu_{\ell+|\hat{N}|/3-1}$ for the problem $(N, (\succsim_a)_{a \in N})$ such that for each $k = 1, \ldots, |\hat{N}|/3 - 1$, we obtain $\mu_{\ell+k}$ from $\mu_{\ell+k-1}$ by satisfying the blocking triplet $(\hat{a}_{k+1}, \hat{b}_{k+1}, \hat{c}_{k+1})$. We note that $(\hat{a}_{k+1}, \hat{b}_{k+1}, \hat{c}_{k+1})$ is in fact a blocking triplet for $\mu_{\ell+k-1}$ since $\hat{a}_{k+1}$, $\hat{b}_{k+1}$, and $\hat{c}_{k+1}$ are single at $\mu_{\ell+k-1}$.

We claim that $\mu_{\ell+|\hat{N}|/3-1}$ is a stable matching. Suppose, by way of contradiction, that there exists a blocking triplet (a', b', c') for $\mu_{\ell+|\hat{N}|/3-1}$.

Since $\hat{\mu}^*$ is a stable matching for the problem $(\hat{N}, (\succsim_a)_{a \in \hat{N}})$, it cannot be the case that $\{a', b', c'\} \subset \hat{N}$. This is because, considering the fact that $\mu_{\ell+|\hat{N}|/3-1}(i) = \hat{\mu}^*(i)$ for all $i \in \hat{N}$, if $\{a', b', c'\} \subset \hat{N}$ then the triplet (a', b', c') is a blocking triplet for $\hat{\mu}^*$ for the problem $(\hat{N}, (\succsim_a)_{a \in \hat{N}})$, which is in contradiction with the stability of $\hat{\mu}^*$.

Since we went through step 3 before reaching step 9, it cannot be the case that $\{a', b', c'\} \subset \tilde{N}$. This is because, considering the fact that $\mu_{\ell+|\hat{N}|/3-1}(i) = \mu_\ell(i)$ for all $i \in \tilde{N}$, if $\{a', b', c'\} \subset \tilde{N}$ then the triplet (a', b', c') is a blocking triplet for μ_ℓ such that $a' \in T(\mu_\ell)$, $b' \in T(\mu_\ell)$, and $c' \in T(\mu_\ell)$. The existence of the blocking triplet (a', b', c') is in contradiction with the fact that we proceeded to step 4.

Therefore, it must be the case that there exists a pair of agents $(i, j) \in N_n \times N_{n+1}$ for some $n \in \{1, 2, 3\}$ such that $i \in \{a', b', c'\} \cap \tilde{N}$ and $j \in \{a', b', c'\} \cap \hat{N}$. Without loss of generality, we may assume that $(i, j) = (a', b')$. Since we went through step 8 before reaching step 9, we have $\mu_\ell(a') \succ_{a'} b'$. Furthermore, since $a' \in \tilde{N}$ and hence $\mu_{\ell+|\hat{N}|/3-1}(a') = \mu_\ell(a')$, we have $\mu_{\ell+|\hat{N}|/3-1}(a') \succ_{a'} b'$, which is in contradiction with the assumption that (a', b', c') is a blocking triplet for $\mu_{\ell+|\hat{N}|/3-1}$.

Hence, at the end of case 9–1, we obtain a stable matching in the sequence of matchings mentioned in the theorem.

Case 9–2. *$b_1 \neq \hat{\mu}^*(a_1)$ and $c_1 \neq \hat{\mu}^*(b_1)$ and $a_1 \neq \hat{\mu}^*(c_1)$.*

We claim that there exists $h \in \{a_1, b_1, c_1\}$ such that $\hat{\mu}^*(h) \succ_h \mu_\ell(h)$. Suppose, by way of contradiction, that $\mu_\ell(a_1) \succ_{a_1} \hat{\mu}^*(a_1)$ and $\mu_\ell(b_1) \succ_{b_1} \hat{\mu}^*(b_1)$ and $\mu_\ell(c_1) \succ_{c_1} \hat{\mu}^*(c_1)$. Since $(\mu_\ell(c_1), \mu_\ell(a_1), \mu_\ell(b_1)) = (a_1, b_1, c_1)$, the triplet (a_1, b_1, c_1) is a blocking triplet for $\hat{\mu}^*$ for the problem $(\hat{N}, (\succsim_a)_{a\in\hat{N}})$, which is in contradiction with the stability of $\hat{\mu}^*$.

Having proved the claim, we may assume that $h = a_1$, that is, $\hat{\mu}^*(a_1) \succ_{a_1} \mu_\ell(a_1)$ without loss of generality. For notational convenience, we denote $(a_1, \hat{\mu}^*(a_1), \hat{\mu}^*(\hat{\mu}^*(a_1)))$ by (a_1, b_3, c_3).

Since $\hat{N} = S(\mu_\ell) \cup \{a_1, b_1, c_1\}$ and $b_3 \in N_2 \cap \hat{N}$ and $b_3 \neq b_1$, we have $b_3 \in S(\mu_\ell)$. We also note that $c_3 \in S(\mu_\ell)$. This is because, if $c_3 \notin S(\mu_\ell)$ and hence $c_3 = c_1$, we have $\hat{\mu}^*(c_1) = \hat{\mu}^*(c_3) = \hat{\mu}^*(\hat{\mu}^*(\hat{\mu}^*(a_1))) = a_1$, which is in contradiction with the assumption $a_1 \neq \hat{\mu}^*(c_1)$ in case 9–2.

We note that (a_1, b_3, c_3) is a blocking triplet for μ_ℓ since $b_3 \succ_{a_1} \mu_\ell(a_1)$, and b_3 and c_3 are single at μ_ℓ. We now obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a_1, b_3, c_3) for μ_ℓ.

Considering the fact that $\hat{N} = S(\mu_{\ell+1}) \cup \{a_1, b_3, c_3\}$ and $(a_1, b_3, c_3) \in \hat{\mu}^*$, we can apply the same argument to the matching $\mu_{\ell+1}$ as the one applied to the matching μ_ℓ in case 9–1. That is, we may assume that $(a_1, b_3, c_3) = (\hat{a}_1, \hat{b}_1, \hat{c}_1)$ without loss, and we obtain a sequence of matchings $\mu_{\ell+1}, \ldots, \mu_{\ell+|\hat{N}|/3}$, where $\mu_{\ell+|\hat{N}|/3}$ is stable.

Hence, at the end of case 9–2, we obtain a stable matching in the sequence of matchings mentioned in the theorem.

Case 9–3. *Cases 9–1 and 9–2 do not apply, but the following conditions hold: $b_2 = \hat{\mu}^*(a_1)$ and there exists $c' \in S(\mu_\ell)$ such that $c' = \hat{\mu}^*(b_2)$.*

We note that (a_1, b_2, c') is a blocking triplet for μ_ℓ since $b_2 \succ_{a_1} \mu_\ell(a_1)$, and b_2 and c' are single at μ_ℓ. We now obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a_1, b_2, c') for μ_ℓ.

Considering the fact that $\hat{N} = S(\mu_{\ell+1}) \cup \{a_1, b_2, c'\}$ and $(a_1, b_2, c') \in \hat{\mu}^*$, we can apply the same argument to the matching $\mu_{\ell+1}$ as the one applied to the matching μ_ℓ in case 9–1. That is, we may assume that $(a_1, b_2, c') = (\hat{a}_1, \hat{b}_1, \hat{c}_1)$ without loss, and we obtain a sequence of matchings $\mu_{\ell+1}, \ldots, \mu_{\ell+|\hat{N}|/3}$, where $\mu_{\ell+|\hat{N}|/3}$ is stable.

Hence, at the end of case 9–3, we obtain a stable matching in the sequence of matchings mentioned in the theorem.

Case 9–4. *Cases 9–1 and 9–2 do not apply, but the following conditions hold:* $b_2 = \hat{\mu}^*(a_1)$ *and there does not exist* $c' \in S(\mu_\ell)$ *such that* $c' = \hat{\mu}^*(b_2)$.

Since $\hat{N} = S(\mu_\ell) \cup \{a_1, b_1, c_1\}$ and $\hat{\mu}^*(b_2) \in N_3 \cap \hat{N}$ and $\hat{\mu}^*(b_2) \notin S(\mu_\ell)$, we have $\hat{\mu}^*(b_2) = c_1$. Since $|S(\mu_\ell)| \geq 6$, at least two agents are single in each gender. So, we may assume that there exists $\{c_4, c_5\} \subset N_3 \cap S(\mu_\ell)$ such that $c_4 \succ_{b_2} c_5$. Evidently, $c_4 \neq \hat{\mu}^*(b_2)$ and $c_5 \neq \hat{\mu}^*(b_2)$ since $\hat{\mu}^*(b_2) = c_1 \in T(\mu_\ell)$. Furthermore, we can choose $a_4 \in N_1 \cap S(\mu_\ell)$ such that $a_4 \neq \hat{\mu}^*(c_4)$. Evidently, $b_2 \neq \hat{\mu}^*(a_4)$ since $b_2 = \hat{\mu}^*(a_1)$.

We note that (a_1, b_2, c_5) is a blocking triplet for μ_ℓ since $b_2 \succ_{a_1} \mu_\ell(a_1)$, and b_2 and c_5 are single at μ_ℓ. We now obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a_1, b_2, c_5) for μ_ℓ.

We next note that (a_4, b_2, c_4) is a blocking triplet for $\mu_{\ell+1}$ since $c_4 \succ_{b_2} c_5 = \mu_{\ell+1}(b_2)$, and a_4 and c_4 are single at $\mu_{\ell+1}$. We obtain another matching $\mu_{\ell+2}$ by satisfying the blocking triplet (a_4, b_2, c_4) for $\mu_{\ell+1}$.

Considering the fact that $\hat{N} = S(\mu_{\ell+2}) \cup \{a_4, b_2, c_4\}$ and $b_2 \neq \hat{\mu}^*(a_4)$ and $c_4 \neq \hat{\mu}^*(b_2)$ and $a_4 \neq \hat{\mu}^*(c_4)$, we can apply the same argument to the matching $\mu_{\ell+2}$ as the one applied to the matching μ_ℓ in case 9–2.

Eventually, at the end of case 9–4, we obtain a stable matching in the sequence of matchings mentioned in the theorem.

Case 9–5. *Cases 9–1 and 9–2 do not apply, but the following conditions hold:* $b_2 \neq \hat{\mu}^*(a_1)$ *and there exists* $c' \in S(\mu_\ell)$ *such that* $c' \neq \hat{\mu}^*(b_2)$ *and* $a_1 \neq \hat{\mu}^*(c')$.

We note that (a_1, b_2, c') is a blocking triplet for μ_ℓ since $b_2 \succ_{a_1} \mu_\ell(a_1)$, and b_2 and c' are single at μ_ℓ. We now obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a_1, b_2, c') for μ_ℓ.

Considering the fact that $\hat{N} = S(\mu_{\ell+1}) \cup \{a_1, b_2, c'\}$ and $b_2 \neq \hat{\mu}^*(a_1)$ and $c' \neq \hat{\mu}^*(b_2)$ and $a_1 \neq \hat{\mu}^*(c')$, we can apply the same argument to the matching $\mu_{\ell+1}$ as the one applied to the matching μ_ℓ in case 9–2.

Eventually, at the end of case 9–5, we obtain a stable matching in the sequence of matchings mentioned in the theorem.

Case 9–6. *Cases 9–1 and 9–2 do not apply, but the following conditions hold:* $b_2 \neq \hat{\mu}^*(a_1)$ *and there does not exist* $c' \in S(\mu_\ell)$ *such that* $c' \neq \hat{\mu}^*(b_2)$ *and* $a_1 \neq \hat{\mu}^*(c')$.

Since $|S(\mu_\ell)| \geq 6$ and there does not exist $c' \in S(\mu_\ell)$ such that $c' \neq \hat{\mu}^*(b_2)$ and $a_1 \neq \hat{\mu}^*(c')$, it must be the case that $N_3 \cap S(\mu_\ell) = \{c_6, c_7\}$ for some (c_6, c_7) such that $c_6 = \hat{\mu}^*(b_2)$ and $a_1 = \hat{\mu}^*(c_7)$. We note that $c_6 \neq c_7$ because, if $c_6 = c_7$, we have $\hat{\mu}^*(a_1) = \hat{\mu}^*(\hat{\mu}^*(c_7)) = \hat{\mu}^*(\hat{\mu}^*(c_6)) = \hat{\mu}^*(\hat{\mu}^*(\hat{\mu}^*(b_2))) = b_2$, which is in contradiction with the assumption $b_2 \neq \hat{\mu}^*(a_1)$ in case 9–6. Since $c_6 \neq c_7$, we have $c_7 \neq \hat{\mu}^*(b_2)$ and $a_1 \neq \hat{\mu}^*(c_6)$. We consider the following two subcases.

Subcase 9–6–1. We have $c_6 \succ_{b_2} c_7$.

We choose $a_5 \in N_1 \cap \hat{N}$ such that $a_5 = \hat{\mu}^*(c_6)$. Since $a_1 \neq \hat{\mu}^*(c_6)$, we have $a_5 \neq a_1$. Furthermore, since $\hat{N} = S(\mu_\ell) \cup \{a_1, b_1, c_1\}$ and $a_5 \neq a_1$, we have $a_5 \in S(\mu_\ell)$.

We note that (a_1, b_2, c_7) is a blocking triplet for μ_ℓ since $b_2 \succ_{a_1} \mu_\ell(a_1)$, and b_2 and c_7 are single at μ_ℓ. We now obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a_1, b_2, c_7) for μ_ℓ.

We next note that (a_5, b_2, c_6) is a blocking triplet for $\mu_{\ell+1}$ since $c_6 \succ_{b_2} c_7 = \mu_{\ell+1}(b_2)$, and a_5 and c_6 are single at $\mu_{\ell+1}$. We obtain another matching $\mu_{\ell+2}$ by satisfying the blocking triplet (a_5, b_2, c_6) for $\mu_{\ell+1}$.

Considering the fact that $\hat{N} = S(\mu_{\ell+2}) \cup \{a_5, b_2, c_6\}$ and $(a_5, b_2, c_6) \in \hat{\mu}^*$, we can apply the same argument to the matching $\mu_{\ell+2}$ as the one applied to the matching μ_ℓ in case 9–1. That is, we may assume that $(a_5, b_2, c_6) = (\hat{a}_1, \hat{b}_1, \hat{c}_1)$ without loss, and we obtain a sequence of matchings $\mu_{\ell+2}, \ldots, \mu_{\ell+|\hat{N}|/3+1}$, where $\mu_{\ell+|\hat{N}|/3+1}$ is stable.

Hence, at the end of subcase 9–6–1, we obtain a stable matching in the sequence of matchings mentioned in the theorem.

Subcase 9–6–2. We have $c_7 \succ_{b_2} c_6$.

Since $|S(\mu_\ell)| \geq 6$, at least two agents are single in each gender. So, we can choose $a_6 \in N_1 \cap S(\mu_\ell)$ such that $b_2 \neq \hat{\mu}^*(a_6)$. Evidently, $a_6 \neq \hat{\mu}^*(c_7)$ since $\hat{\mu}^*(c_7) = a_1 \in T(\mu_\ell)$.

We note that (a_1, b_2, c_6) is a blocking triplet for μ_ℓ since $b_2 \succ_{a_1} \mu_\ell(a_1)$, and b_2 and c_6 are single at μ_ℓ. We now obtain another matching $\mu_{\ell+1}$ by satisfying the blocking triplet (a_1, b_2, c_6) for μ_ℓ.

We next note that (a_6, b_2, c_7) is a blocking triplet for $\mu_{\ell+1}$ since $c_7 \succ_{b_2} c_6 = \mu_{\ell+1}(b_2)$, and a_6 and c_7 are single at $\mu_{\ell+1}$. We obtain another matching $\mu_{\ell+2}$ by satisfying the blocking triplet (a_6, b_2, c_7) for $\mu_{\ell+1}$.

Considering the fact that $\hat{N} = S(\mu_{\ell+2}) \cup \{a_6, b_2, c_7\}$ together with the fact that $b_2 \neq \hat{\mu}^*(a_6)$ and $c_7 \neq \hat{\mu}^*(b_2)$ and $a_6 \neq \hat{\mu}^*(c_7)$, we can apply the same argument to the matching $\mu_{\ell+2}$ as the one applied to the matching μ_ℓ in case 9–2.

Eventually, at the end of subcase 9–6–2, we obtain a stable matching in the sequence of matchings mentioned in the theorem. □

We summarize the algorithm in our proof. Our process starts with an arbitrary unstable matching, and ends with a stable matching in step 1, 5, 6, or 9. In other steps such as steps 3, 4, 7, and 8, we obtain another unstable matching with the smaller number of matched agents than before, and we return to step 2. The loop starting in step 2 is not infinite because the number of agents is finite. Our process exits from the loop at some point and ends in step 5, 6, or 9.

Our approach toward the construction of a sequence of matchings in the proof could be called as a scrap-and-build approach. In the loop, we prompt matched agents to break bonds and become single. After we increase the number of single agents as much as possible, we prompt single agents to get matched in order to form a stable matching.

We finally state the following corollary as Roth and Vande Vate [1990] have established for two-sided matching problems.

Corollary 1. *Let μ_1 be an arbitrary unstable matching for a three-sided cyclic matching problem, which is specified by $(N, (\succsim_a)_{a\in N})$ with $N \equiv N_1 \cup N_2 \cup N_3$ and $|N_n| = |N|/3$ for $n \in \{1, 2, 3\}$. Suppose that for any $\bar{N}_1 \subset N_1$, $\bar{N}_2 \subset N_2$, and $\bar{N}_3 \subset N_3$ with $|\bar{N}_1| = |\bar{N}_2| = |\bar{N}_3|$, there exists a stable matching for the problem $(\bar{N}, (\succsim_a)_{a\in \bar{N}})$ with $\bar{N} \equiv \bar{N}_1 \cup \bar{N}_2 \cup \bar{N}_3$. Consider a random sequence $R(\mu_1) = (\mu_1, \mu_2, \ldots)$ where each $\mu_{\ell+1}$ is obtained from μ_ℓ by satisfying a blocking triplet that is chosen at random from the blocking triplets for μ_ℓ. If the probability that each blocking triplet for μ_ℓ will be chosen is positive and bounded away from zero, then $R(\mu_1)$ converges to a stable matching with probability one.*

4. Conclusion

In the present paper, we have obtained the following result: for a three-sided cyclic matching problem $(N, (\succsim_a)_{a\in N})$ with $N \equiv N_1 \cup N_2 \cup N_3$ and $|N_n| = |N|/3$ for $n \in \{1, 2, 3\}$, if a stable matching exists for the problem itself and for any problem $(\bar{N}, (\succsim_a)_{a\in \bar{N}})$ constructed of $\bar{N} \equiv \bar{N}_1 \cup \bar{N}_2 \cup \bar{N}_3$ with $\bar{N}_n \subset N_n$ and $|\bar{N}_n| = |\bar{N}|/3$ for $n \in \{1, 2, 3\}$, there exists a finite sequence of successive blockings from an arbitrary unstable matching to a stable matching.

To prove the result, we have used the fact that the problem $(N, (\succsim_a)_{a\in N})$ has a stable matching in step 1. We have also used the fact that the problem $(S(\mu_\ell), (\succsim_a)_{a\in S(\mu_\ell)})$ has a stable matching in step 6 and the fact that the problem $(\hat{N}, (\succsim_a)_{a\in\hat{N}})$ has a stable matching in step 9. We recall that $S(\mu_\ell) \subsetneq N$ in step 6 and $\hat{N} \subseteq N$ in step 9.

Pashkovich and Poirrier [2018] and other pioneers showed that a stable matching exists for the case $|N_n| \leq 5$. When $|N_n| = |N|/3 \leq 5$, we have $|S(\mu_\ell)|/3 \leq 4$ in step 6 and $|\hat{N}|/3 \leq 5$ in step 9, and hence each of the problems, $(N, (\succsim_a)_{a\in N})$ in step 1, $(S(\mu_\ell), (\succsim_a)_{a\in S(\mu_\ell)})$ in step 6, and $(\hat{N}, (\succsim_a)_{a\in\hat{N}})$ in step 9, has a stable matching. Thus, we can state the following corollary.

Corollary 2. *Consider any three-sided cyclic matching problem with* $|N_n| \leq 5$*. Starting from an arbitrary unstable matching, there exists a finite sequence of successive blockings leading to a stable matching.*

Lam and Plaxton [2019] showed that a stable matching does not necessarily exist for the case $|N_n| = 6$ by utilizing the example presented by Biro and McDermid [2010]. When $|N_n| = |N|/3 = 6$, we have $|S(\mu_\ell)|/3 \leq 5$ in step 6, and hence the problem $(S(\mu_\ell), (\succsim_a)_{a\in S(\mu_\ell)})$ in step 6 has a stable matching. The problem $(\hat{N}, (\succsim_a)_{a\in\hat{N}})$ in step 9 has a stable matching if $|\hat{N}|/3 \leq 5$, but it may or may not have a stable matching if $|\hat{N}|/3 = 6$. However, if we assume the existence of a stable matching for the problem $(N, (\succsim_a)_{a\in N})$ when $|N_n| = |N|/3 = 6$, the existence of a stable matching for the problem $(\hat{N}, (\succsim_a)_{a\in\hat{N}})$ in step 9 is guaranteed since both of the problems are identical if $|\hat{N}|/3 = 6$. Thus, we can state the following.

Corollary 3. *Consider any three-sided cyclic matching problem with* $|N_n| = 6$ *for which a stable matching exists. Starting from an arbitrary unstable matching, there exists a finite sequence of successive blockings leading to a stable matching.*

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