Random paths to stability in Danilov's three-sided matching model

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Yusuke Samejima

Abstract

We investigate three-sided matching problems where three kinds of agents, men, women, and cats are matched. Without any restrictions on preferences of agents, a stable matching does not necessarily exist for a three-sided matching problem. However, Danilov [2003] has proved the existence of a stable matching for any three-sided matching problem if preference domains for men and women are restricted in a certain way. In the present paper, we show that, starting from an arbitrary unstable matching, there exists a finite sequence of successive blockings leading to some stable matching for a three-sided matching problem in Danilov's model, as Roth and Vande Vate [1990] have proved for two-sided matching problems. The result implies that a decentralized process of successive blockings by randomly chosen blocking agents will converge to a stable matching.

1. Introduction

Two-sided matching problems have been extensively studied since they were first analyzed by Gale and Shapley [1962]. There are two kinds of agents, men and women, and the sets of men and women are denoted by M and W, respectively. Each agent has a strict preference relation over those of the opposite sex. A matching μ is a partition of the set of agents $M \cup W$ into non-empty coalitions such that each coalition contains at most one man and at most one woman, and thus at most two agents in total. A matching μ can be blocked by a single man or a single woman if he or she prefers being unmatched to being matched to his or her assignment at μ. A matching μ can also be blocked by a pair $(m, w) \in M \times W$ if they prefer each other to their own assignments at μ. A matching μ is *stable* if it is not blocked by any single agent or any pair of agents. Gale and Shapley [1962] proved the existence of a stable matching for every two-sided matching problem by presenting an algorithm for finding a stable matching.

An extension of the research on two-sided matching problems is to investigate three-sided matching problems. There are three kinds of agents, men, women, and cats. The sets of men, women, and cats are denoted by M, W, and C, respectively. Each man has a strict preference relation over pairs of a woman and a cat, and each woman and each cat also have preference relations over pairs of the other two kinds of agents. A matching μ in this three-sided model is a partition of the set of agents $M \cup W \cup C$ into non-empty coalitions such that each coalition contains at most one man, at most one woman, and at most one cat, and thus at most three agents in total. A matching μ can be blocked by a single agent if he prefers being unmatched to being matched to his assignment at μ. Moreover, a matching μ can be blocked by a pair $(m, w) \in M \times W$ if they prefer forming a coalition of the two without a cat to being matched to their own assignments at μ. A matching μ can be blocked by a pair $(m, c) \in M \times C$ or a pair $(w, c) \in W \times C$ in a similar way. Finally, a matching μ can be blocked by a triplet $(m, w, c) \in M \times W \times C$ if they prefer forming a coalition of the three to being matched to their own assignments at μ. A matching μ is *stable* if it is not blocked by any single agent, any pair of agents, or any triplet of agents. Alkan [1988] has presented an example showing that a stable matching does not necessarily exist for such a three-sided matching problem. However, Danilov [2003] has proved the existence of a stable matching for any three-sided matching problem if preference domains for men and women are restricted in the following way: Each man is interested in women in the first place and cats interest him less while each woman is interested in men in the first place and cats interest her less. Each cat, on the other hand, may have an arbitrary strict preference relation over pairs of a man and a woman. Danilov [2003] has presented an algorithm for finding a stable matching for a three-sided matching problem with these restricted preference domains.

Another extension of the research on two-sided matching problems is to investigate whether decentralized decision making by each agent leads to stability. This question has been answered by Roth and Vande Vate [1990]. They have proved that, starting from an arbitrary unstable matching, there exists a finite sequence of successive blockings leading to some stable matching. Their result implies that a decentralized process of successive blockings by randomly chosen blocking agents will converge to a stable matching with probability one.

In the present paper, we investigate whether decentralized decision making by each agent leads to stability in Danilov's three-sided matching model. The answer is in the affirmative. As Roth and Vande Vate [1990] have proved for two-sided matching problems, we show that, starting from an arbitrary unstable matching, there exists a finite sequence of successive blockings leading to some stable matching for a three-sided matching problem in Danilov's model. So, a decentralized process of successive blockings will converge to a stable matching via random paths in Danilov's three-sided matching model.

There are studies in the literature that show decentralized decision making by each agent leads to stability in a variety of matching models. Among others, Diamantoudi, Miyagawa, and Xue [2004] investigate this issue in roommate problems. Kojima and Unver [2008] investigate this issue in many-to-many matching problems. The present paper is an attempt to do the same analysis for Danilov's three-sided matching problems.

The remaining part of this paper is organized as follows. Section 2 explains a model of three-sided matching and Danilov's restricted preference domains that guarantee the existence of a stable matching. Section 3 shows our main result. Section 4 provides some concluding remarks.

2. The Model

There are a finite number of agents of three kinds: men, women, and cats. The sets of men, women, and cats are denoted by M, W, and C, respectively. We do not assume that $|M| = |W| = |C|$.

In the present paper, a coalition refers to a non-empty set of agents such that it contains at most one man, at most one woman, and at most one cat, and thus at most three agents in total. A matching μ in our three-sided model is a partition of $M \cup W \cup C$ into coalitions. By abuse of terminology, a coalition in the present paper also refers to an ordered list of agents, as the following examples show.

An example of a coalition at a matching is a triplet $(m, w, c) \in M \times W \times C$, which should be expressed as $\{m, w, c\}$ formally. By abuse of notation, we use the two expressions (m, w, c) and $\{m, w, c\}$ interchangeably in the present paper. Another example of a coalition at a matching is a pair $(m, w) \in M \times W$, which should be expressed as $\{m, w\}$ formally. This two-agent coalition is also denoted by $(m, w, *)$, where the element "$*$" in the third argument represents "an empty seat", meaning that there is no cat in this coalition. Yet another example of a coalition at a matching is a single-agent coalition $w \in W$, which should be expressed as $\{w\}$ formally, and which is also denoted by $(*, w, *)$ in the present paper. We define $\bar{M} = M \cup \{*\}$, $\bar{W} = W \cup \{*\}$, and $\bar{C} = C \cup \{*\}$. When we write $m \in \bar{M}$, it is possible that $m = *$. Similarly, when we write $(w, c) \in \bar{W} \times \bar{C}$, it is possible that $(w, c) = (*, *)$.

We denote by $\mu(i)$ the coalition that agent i belongs to at a matching μ. For example, given a coalition $(m, w, c) \in M \times W \times C$ or $\{m, w, c\}$ at a matching μ, the coalition that m, w, and c belong to is denoted by $\mu(m) = \mu(w) = \mu(c) = (m, w, c)$ or $\mu(m) = \mu(w) = \mu(c) = \{m, w, c\}$. Given a coalition

$(m', w', *) \in M \times W \times \bar{C}$ at a matching μ', the coalition that m' and w' belong to is denoted by $\mu'(m') = \mu'(w') = (m', w', *)$ or $\mu'(m') = \mu'(w') = \{m', w'\}$ while $\mu'(*)$ is not defined. We say that agent i is *single at* μ if $\mu(i) = \{i\}$.

Each man $m \in M$ has a complete, transitive, and strict preference relation $\succsim_m$ over $\bar{W} \times \bar{C}$. When m prefers (w, c) to (w', c'), we write $(w, c) \succ_m (w', c')$, or we write $(m, w, c) \succ_m (m, w', c')$ by abuse of notation. When m weakly prefers (w, c) to (w', c'), that is, (w, c) is at least as good as (w', c'), we write $(w, c) \succsim_m (w', c')$ or $(m, w, c) \succsim_m (m, w', c')$. Similarly, each woman $w \in W$ and each cat $c \in C$ have complete, transitive, and strict preference relations $\succsim_w$ and $\succsim_c$ over $\bar{M} \times \bar{C}$ and $\bar{M} \times \bar{W}$, respectively.

A *three-sided matching problem* is specified by $(M, W, C, (\succsim_i)_{i \in M \cup W \cup C})$.

We now define blocking coalitions for a given matching μ. A triplet $(m, w, c) \in M \times W \times C$ is called a *blocking triplet* for μ if $(m, w, c) \succ_i \mu(i)$ for all $i \in \{m, w, c\}$. A pair $(m, w) \in M \times W$ is called a *blocking pair* for μ if $(m, w, *) \succ_i \mu(i)$ for all $i \in \{m, w\}$. Similarly, a pair $(m, c) \in M \times C$ is called a blocking pair for μ if $(m, *, c) \succ_i \mu(i)$ for all $i \in \{m, c\}$ while a pair $(w, c) \in W \times C$ is called a blocking pair for μ if $(*, w, c) \succ_i \mu(i)$ for all $i \in \{w, c\}$. An agent $m \in M$ is called a *blocking single* for μ if $(m, *, *) \succ_m \mu(m)$. Similarly, an agent $w \in W$ is called a blocking single for μ if $(*, w, *) \succ_w \mu(w)$ while an agent $c \in C$ is called a blocking single for μ if $(*, *, c) \succ_c \mu(c)$. A *blocking coalition* refers to either a blocking triplet, a blocking pair, or a blocking single.

Definition. A matching μ is *individually rational* if there are no blocking singles for μ.

Definition. A matching μ is *stable* if there are no blocking coalitions for μ.

Alkan [1988] has presented an example showing that a stable matching does not necessarily exist for a three-sided matching problem. On the other hand, Danilov [2003] has proposed to consider the following restrictions on preference domains for men and women in pursuit of a stable matching for a three-sided matching problem.

Definition. (*Danilov's condition.*) For each $m \in M$, if $(m, w, c) \succ_m (m, w', c)$ for some $(w, w', c) \in \bar{W} \times \bar{W} \times \bar{C}$, then $(m, w, c') \succ_m (m, w', c'')$ for all $(c', c'') \in \bar{C} \times \bar{C}$. For each $w \in W$, if $(m, w, c) \succ_w (m', w, c)$ for some $(m, m', c) \in \bar{M} \times \bar{M} \times \bar{C}$, then $(m, w, c') \succ_w (m', w, c'')$ for all $(c', c'') \in \bar{C} \times \bar{C}$.

Under Danilov's condition on preferences, each man is interested in women in the first place and cats interest him less while each woman is interested in men in the first place and cats interest her less.[1] Danilov's condition does not impose any restrictions on preference domains of cats.

Danilov [2003] has presented an algorithm for finding a stable matching for a three-sided matching problem with these restricted preference domains and proved the following fact.

Fact. A stable matching exists for any three-sided matching problem with strict preferences under Danilov's condition.

3. The Result

This section investigates whether, starting from an arbitrary unstable matching, there exists a finite sequence of successive blockings leading to some stable matching.

We first prove the following two lemmas, in which Danilov's condition and strict preferences play an important role.

Lemma 1. *Let μ be an arbitrary matching for any three-sided matching problem with strict preferences under Danilov's condition, which is specified by $(M, W, C, (\succsim_i)_{i \in M \cup W \cup C})$. If $(m, w, c) \in M \times W \times C$ is a blocking triplet for μ but (m, w) is not a blocking pair for μ, then (m, w) is matched at μ, that is, $\mu(m) = \mu(w) = (m, w, c')$ for some $c' \in \bar{C}$.*

Proof. Since (m, w) is not a blocking pair for μ, we have either $\mu(m) \succsim_m (m, w, *)$ or $\mu(w) \succsim_w (m, w, *)$. Without loss of generality, suppose that $\mu(m) \succsim_m (m, w, *)$ and $\mu(m) = (m, w', c')$ for some $(w', c') \in \bar{W} \times \bar{C}$.

Since (m, w, c) is a blocking triplet for μ, we have $(m, w, c) \succ_m (m, w', c')$. Danilov's condition implies that $(m, w, c) \succsim_m (m, w', c)$, because otherwise we have $(m, w', c') \succ_m (m, w, c)$.

Since $\mu(m) \succsim_m (m, w, *)$, we have $(m, w', c') \succsim_m (m, w, *)$. Danilov's condition implies that $(m, w', c) \succsim_m (m, w, c)$, because otherwise we have $(m, w, *) \succ_m (m, w', c')$.

We now have $(m, w, c) \succsim_m (m, w', c) \succsim_m (m, w, c)$. Since preferences are strict, we conclude that $w' = w$. □

Lemma 2. *Let μ be an arbitrary matching for any three-sided matching problem with strict preferences under Danilov's condition, which is specified by $(M, W, C, (\succsim_i)_{i \in M \cup W \cup C})$. Suppose that μ is individually rational. If $(m, c) \in M \times C$ is a blocking pair for μ, then $\mu(m) = (m, *, c')$ for some $c' \in \bar{C}$. If $(w, c) \in W \times C$ is a blocking pair for μ, then $\mu(w) = (*, w, c')$ for some $c' \in \bar{C}$.*

[1] Preferences under Danilov's condition and lexicographic preferences are similar but slightly different. Under Danilov's condition, it is possible that a man has a preference relation such that $(m, w, c) \succ_m (m, w, c') \succ_m (m, w', c') \succ_m (m, w', c)$. This preference relation is not lexicographic.

Proof. We prove the first "if" statement. The proof of the second "if" statement is similar. Suppose that $\mu(m) = (m, w', c')$ for some $(w', c') \in \bar{W} \times \bar{C}$.

Since (m, c) is a blocking pair for μ, we have $(m, *, c) \succ_m (m, w', c')$. Danilov's condition implies that $(m, *, *) \succsim_m (m, w', *)$, because otherwise we have $(m, w', c') \succ_m (m, *, c)$.

Since μ is individually rational, we have $(m, w', c') \succsim_m (m, *, *)$. Danilov's condition implies that $(m, w', *) \succsim_m (m, *, *)$, because otherwise we have $(m, *, *) \succ_m (m, w', c')$.

We now have $(m, *, *) \succsim_m (m, w', *) \succsim_m (m, *, *)$. Since preferences are strict, we conclude that $w' = *$. □

Suppose that a coalition $(m, w, c) \in \bar{M} \times \bar{W} \times \bar{C}$ with $(m, w, c) \neq (*, *, *)$ is a blocking coalition for a matching μ. We say that another matching μ' is obtained from μ by *satisfying* the blocking coalition (m, w, c) for μ if the following three conditions hold.

(i) If $i \in \{m, w, c\}$ and $i \neq *$, then $\mu'(i) = (m, w, c)$.

(ii) If $i \in M \cup W \cup C \setminus \{m, w, c\}$ and $i \in \mu(j)$ for some $j \in \{m, w, c\}$ with $j \neq *$, then $\mu'(i) = \{i\}$.

(iii) If $i \in M \cup W \cup C \setminus \{m, w, c\}$ and $i \notin \mu(j)$ for any $j \in \{m, w, c\}$ with $j \neq *$, then $\mu'(i) = \mu(i)$.

Condition (i) says that the agents in the blocking coalition for the old matching μ are matched at the new matching μ'. Condition (ii) says that old family members of the agents in the blocking coalition for μ become single at μ'. Condition (iii) says that agents irrelevant to the blocking coalition for μ are matched with the same members between μ and μ'.

Theorem. *Let μ_1 be an arbitrary unstable matching for any three-sided matching problem with strict preferences under Danilov's condition, which is specified by $(M, W, C, (\succsim_i)_{i \in M \cup W \cup C})$. There exists a finite sequence of matchings $\mu_1, \ldots, \mu_N$ such that μ_N is stable, and for each $n = 1, \ldots, N-1$, there is a blocking coalition for μ_n such that μ_{n+1} is obtained from μ_n by satisfying the blocking coalition.*

Proof. We follow four steps.

Step 1. If μ_1 is not individually rational, then we obtain μ_2 from μ_1 by satisfying one of the blocking singles for μ_1. By satisfying all the blocking singles for μ_1 sequentially in this manner, we obtain μ_k, which is individually rational.

Step 2.[2] At the beginning of this step, we have an individually rational matching μ_k. Suppose that there is a blocking pair $(m_k, w_k) \in M \times W$ for μ_k. If there is not such a man-woman blocking

[2]Descriptions of this step closely follow Roth and Vande Vate [1990].

pair for μ_k, then we proceed to step 3.

Let us obtain μ_{k+1} from μ_k by satisfying $(m_k, w_k, *)$ and define the set $A(k) = \{m_k, w_k\}$. We note that μ_{k+1} is individually rational because old family members of m_k and w_k at μ_k have become single at μ_{k+1} as is described by condition (ii) in the definition of satisfaction of the blocking coalition.

Suppose that there is a blocking pair $(m_{k+1}, w_{k+1}) \in M \times W$ for μ_{k+1}. If there is not such a man-woman blocking pair for μ_{k+1}, then we proceed to step 3. We note that $\{m_{k+1}, w_{k+1}\}$ is not contained in $A(k)$, that is, $\{m_{k+1}, w_{k+1}\} \nsubseteq A(k)$. Step 2 will proceed by constructing a sequence of individually rational matchings, in such a way that it can be associated with an increasing sequence of sets $A(q)$ which contain no cats and no man-woman blocking pairs.

We now start the inductive step. Suppose that we have an individually rational matching μ_{q+1}, and we have a set $A(q)$ such that $A(q)$ contains no cats and no man-woman blocking pairs for μ_{q+1}, and μ_{q+1} does not match[3] any agent in $A(q)$ to any agent not in $A(q)$. Suppose also that there is a blocking pair $(m', w') \in M \times W$ for μ_{q+1}. If there is not such a man-woman blocking pair for μ_{q+1}, then we proceed to step 3. Otherwise, we have either or both of $m' \notin A(q)$ and $w' \notin A(q)$. In the following three cases, we desire to show that there exists a sequence of individually rational matchings $\mu_{q+1}, \ldots, \mu_{r+1}$ such that each matching in the sequence is obtained by satisfying a man-woman blocking pair, and we also desire to show that there exists a set $A(r) \supsetneq A(q)$ such that $A(r)$ strictly contains $A(q)$, and $A(r)$ contains no cats and no man-woman blocking pairs for μ_{r+1}, and μ_{r+1} does not match any agent in $A(r)$ to any agent not in $A(r)$.

Case 1. This is the case where there is a blocking pair $(m', w') \in M \times W$ for μ_{q+1} such that $m' \in A(q)$ and $w' \notin A(q)$.

Let $w_{q+1} = w'$, and choose a man $m_{q+1} \in A(q)$ such that (m_{q+1}, w_{q+1}) is a blocking pair for μ_{q+1}, and such that m_{q+1} is w_{q+1}'s most preferred blocking partner among all the blocking partners in $A(q)$.[4] Let us obtain the next individually rational matching μ_{q+2} from μ_{q+1} by satisfying $(m_{q+1}, w_{q+1}, *)$, and define $A(q+1) = A(q) \cup \{w_{q+1}\}$. Clearly, $A(q+1)$ contains no cats and μ_{q+2} does not match any agent in $A(q+1)$ to any agent not in $A(q+1)$.

If m_{q+1} was single at μ_{q+1}, that is, if $\mu_{q+1}(m_{q+1}) = (m_{q+1}, *, *)$, then $A(q+1)$ contains no man-woman blocking pairs for μ_{q+2}. If we define $\mu_{r+1} = \mu_{q+2}$ and $A(r) = A(q+1)$, we obtain the desired result.

[3] For each $i \in A(q)$, we have $\mu_{q+1}(i) \subseteq A(q)$.

[4] That is, $m_{q+1} \in A(q)$ is a man such that (m_{q+1}, w_{q+1}) is a blocking pair for μ_{q+1} and $(m_{q+1}, w_{q+1}, *) \succsim_{w_{q+1}} (m, w_{q+1}, *)$ for all $m \in \{m \in A(q) \colon (m, w_{q+1})$ is a blocking pair for $\mu_{q+1}\}$.

If m_{q+1} was matched to some woman (let us call her w_{q+2}) at μ_{q+1}, that is, if $\mu_{q+1}(m_{q+1}) = (m_{q+1}, w_{q+2}, *)$, then it must be the case that $w_{q+2} \in A(q+1)$ and w_{q+2} has become single at μ_{q+2}. In this situation, w_{q+2} and some man $m'' \in A(q+1)$ may form a blocking pair (m'', w_{q+2}) for μ_{q+2}, in which case we choose a man $m_{q+2} \in A(q+1)$ such that (m_{q+2}, w_{q+2}) is a blocking pair for μ_{q+2}, and such that m_{q+2} is w_{q+2}'s most preferred blocking partner among all the blocking partners in $A(q+1)$. Let us obtain the next individually rational matching μ_{q+3} from μ_{q+2} by satisfying $(m_{q+2}, w_{q+2}, *)$. We repeat this process of satisfying a blocking pair in $A(q+1)$, which will stop in finite times since in each time no man gets worse than before and hence no blocking pair appears twice for the sequence $\mu_{q+2}, \mu_{q+3}, \ldots$. Eventually, we obtain an individually rational matching μ_{r+1} such that $A(q+1)$ contains no man-woman blocking pairs for μ_{r+1}. If we define $A(r) = A(q+1)$, we obtain the desired result.

Case 2. This is the case where there is a blocking pair $(m', w') \in M \times W$ for μ_{q+1} such that $m' \notin A(q)$ and $w' \in A(q)$. By repeating a similar argument as in case 1 with the roles of men and women reversed, we obtain the desired result.

Case 3. This is the case where cases 1 and 2 do not apply but there is a blocking pair $(m', w') \in M \times W$ for μ_{q+1} such that $m' \notin A(q)$ and $w' \notin A(q)$.

Obtain an individually rational matching μ_{q+2} from μ_{q+1} by satisfying $(m', w', *)$, and define $A(q+1) = A(q) \cup \{m', w'\}$. Clearly, $A(q+1)$ contains no cats and μ_{q+2} does not match any agent in $A(q+1)$ to any agent not in $A(q+1)$. Furthermore, $A(q+1)$ contains no man-woman blocking pairs for μ_{q+2}. If we define $\mu_{r+1} = \mu_{q+2}$ and $A(r) = A(q+1)$, we obtain the desired result.

The above arguments complete the proof of the inductive step. When we repeat the inductive step, the cardinality of $A(q)$ strictly increases in each step, but it cannot be greater than the cardinality of $M \cup W$. So, we must eventually reach an individually rational matching μ_{q+1} for which there are no man-woman blocking pairs. We then proceed to step 3.

Step 3. At the beginning of this step, we have an individually rational matching for which there are no man-woman blocking pairs. Let us call this matching μ_ℓ.

Suppose that there is a blocking coalition $(m_\ell, w_\ell, c_\ell) \in \bar{M} \times \bar{W} \times C$ for μ_ℓ.[5] That is, the blocking coalition is containing a cat and it is either a blocking triplet, a man-cat blocking pair, or a woman-cat blocking pair. If there is not such a blocking coalition for μ_ℓ, then we proceed to step 4.

[5]Since μ_ℓ is individually rational, it cannot be the case that $(m_\ell, w_\ell) = (*, *)$.

Since μ_ℓ is an individually rational matching for which there are no man-woman blocking pairs, Lemmas 1 and 2 imply that (m_ℓ, w_ℓ) is matched at μ_ℓ. That is, there exists $c' \in \bar{C}$ such that $\mu_\ell(i) = (m_\ell, w_\ell, c')$ for all $i \in \{m_\ell, w_\ell\} \setminus \{*\}$.

Let us obtain an individually rational matching $\mu_{\ell+1}$ from μ_ℓ by satisfying (m_ℓ, w_ℓ, c_ℓ). If $\mu_\ell(c_\ell) = (\hat{m}, \hat{w}, c_\ell)$ for some $(\hat{m}, \hat{w}) \in M \times W$, then $\hat{m}$ and $\hat{w}$ have become single at $\mu_{\ell+1}$. Since $\hat{m}$ and $\hat{w}$ were previously matched at an individually rational matching μ_ℓ, this man-woman pair $(\hat{m}, \hat{w})$ is a blocking pair for $\mu_{\ell+1}$.[6] Let us obtain the next individually rational matching $\mu_{\ell+2}$ from $\mu_{\ell+1}$ by satisfying $(\hat{m}, \hat{w}, *)$. If $\mu_{\ell+2}$ can be obtained in this manner, then we define $\mu_{L+1} = \mu_{\ell+2}$. Otherwise, we define $\mu_{L+1} = \mu_{\ell+1}$. Moreover, define the set $B(L) = \{m_\ell, w_\ell, c_\ell\} \setminus \{*\}$.[7] Note that, since all the combinations of a man and a woman are the same between μ_ℓ and μ_{L+1},[8] there are no man-woman blocking pairs for μ_{L+1}.[9]

Suppose that there is a blocking coalition $(m_{L+1}, w_{L+1}, c_{L+1}) \in \bar{M} \times \bar{W} \times C$ for μ_{L+1}. If there is not such a blocking coalition for μ_{L+1}, then we proceed to step 4. We note that $\{m_{L+1}, w_{L+1}, c_{L+1}\}$ is not contained in $B(L)$, that is, $\{m_{L+1}, w_{L+1}, c_{L+1}\} \nsubseteq B(L)$. Step 3 will proceed by constructing a sequence of individually rational matchings for which there are no man-woman blocking pairs, in such a way that it can be associated with an increasing sequence of sets $B(s)$ which contain no blocking coalitions.

We now start the inductive step. Suppose that we have an individually rational matching μ_{s+1} for which there are no man-woman blocking pairs, and we have a set $B(s)$ such that $B(s)$ contains no blocking coalitions for μ_{s+1}, and μ_{s+1} does not match any agent in $B(s)$ to any agent not in $B(s)$. Suppose also that there is a blocking coalition $(m', w', c') \in \bar{M} \times \bar{W} \times C$ for μ_{s+1}. If there is not such a blocking coalition for μ_{s+1}, then we proceed to step 4. Otherwise, we have either or both of $\{m', w'\} \nsubseteq B(s) \cup \{*\}$ and $c' \notin B(s)$; Here, we treat (m', w') as a single unit because Lemmas 1 and 2 imply that (m', w') is matched at μ_{s+1}. In the following three cases, we desire to show that there exists a sequence of individually rational matchings $\mu_{s+1}, \ldots, \mu_{t+1}$ such that each matching in the sequence is obtained by satisfying a blocking coalition and there are no man-woman blocking pairs for μ_{t+1}. We also desire to show that there exists a set $B(t) \supsetneq B(s)$ such that $B(t)$ strictly contains $B(s)$, and $B(t)$ contains no blocking coalitions for μ_{t+1}, and μ_{t+1} does not match any agent in $B(t)$ to any agent not in $B(t)$.

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[6] See Appendix for a proof of the statement.
[7] We carefully define $B(L)$ so that "$*$" should not be an element of $B(L)$.
[8] That is, for each $i \in M \cup W$, we have $\mu_\ell(i) \setminus C = \mu_{L+1}(i) \setminus C$.
[9] See Appendix for a proof of the statement.

Case 1. This is the case where there is a blocking coalition $(m', w', c') \in \bar{M} \times \bar{W} \times C$ for μ_{s+1} such that $\{m', w'\} \subseteq B(s) \cup \{*\}$ and $c' \notin B(s)$.

Let $c_{s+1} = c'$, and choose $\{m_{s+1}, w_{s+1}\} \subseteq B(s) \cup \{*\}$ such that $(m_{s+1}, w_{s+1}, c_{s+1})$ is a blocking coalition for μ_{s+1}, and such that (m_{s+1}, w_{s+1}) is c_{s+1}'s most preferred blocking partner(s) among all the blocking partners in $B(s)$.[10] Lemmas 1 and 2 imply that (m_{s+1}, w_{s+1}) is matched at μ_{s+1}. Let us obtain the next individually rational matching μ_{s+2} from μ_{s+1} by satisfying $(m_{s+1}, w_{s+1}, c_{s+1})$. If $\mu_{s+1}(c_{s+1}) = (\hat{m}, \hat{w}, c_{s+1})$ for some $(\hat{m}, \hat{w}) \in M \times W$, then $\hat{m}$ and $\hat{w}$ have become single at μ_{s+2}. Since $\hat{m}$ and $\hat{w}$ were previously matched at an individually rational matching μ_{s+1}, this man-woman pair $(\hat{m}, \hat{w})$ is a blocking pair for μ_{s+2}. Let us obtain the next individually rational matching μ_{s+3} from μ_{s+2} by satisfying $(\hat{m}, \hat{w}, *)$. If μ_{s+3} can be obtained in this manner, then we define $\mu_{S+2} = \mu_{s+3}$. Otherwise, we define $\mu_{S+2} = \mu_{s+2}$. Moreover, define the set $B(S+1) = B(s) \cup \{c_{s+1}\}$. Clearly, μ_{S+2} does not match any agent in $B(S+1)$ to any agent not in $B(S+1)$. Note that, since all the combinations of a man and a woman are the same between μ_{s+1} and μ_{S+2}, there are no man-woman blocking pairs for μ_{S+2}.

If (m_{s+1}, w_{s+1}) was not matched to a cat at μ_{s+1}, that is, if $\mu_{s+1}(i) = (m_{s+1}, w_{s+1}, *)$ for all $i \in \{m_{s+1}, w_{s+1}\} \setminus \{*\}$, then $B(S+1)$ contains no blocking coalitions for μ_{S+2}. If we define $\mu_{t+1} = \mu_{S+2}$ and $B(t) = B(S+1)$, we obtain the desired result.

If (m_{s+1}, w_{s+1}) was matched to some cat (let us call it c_{S+2}) at μ_{s+1}, that is, if $\mu_{s+1}(i) = (m_{s+1}, w_{s+1}, c_{S+2})$ for all $i \in \{m_{s+1}, w_{s+1}\} \setminus \{*\}$, then it must be the case that $c_{S+2} \in B(S+1)$ and c_{S+2} has become single at μ_{S+2}. In this situation, c_{S+2} and some $\{m'', w''\} \subseteq B(S+1) \cup \{*\}$ may form a blocking coalition (m'', w'', c_{S+2}) for μ_{S+2}, in which case we choose $\{m_{S+2}, w_{S+2}\} \subseteq B(S+1) \cup \{*\}$ such that $(m_{S+2}, w_{S+2}, c_{S+2})$ is a blocking coalition for μ_{S+2}, and such that (m_{S+2}, w_{S+2}) is c_{S+2}'s most preferred blocking partner(s) among all the blocking partners in $B(S+1)$. Lemmas 1 and 2 imply that (m_{S+2}, w_{S+2}) is matched at μ_{S+2}. Let us obtain the next individually rational matching μ_{S+3} from μ_{S+2} by satisfying $(m_{S+2}, w_{S+2}, c_{S+2})$. Note that there are no man-woman blocking pairs for μ_{S+3} because all the combinations of a man and a woman are the same between μ_{S+2} and μ_{S+3}. We repeat this process, which will stop in finite times since in each time no man as well as no woman gets worse than before and hence no blocking coalition appears twice for the sequence $\mu_{S+2}, \mu_{S+3}, \ldots$. Eventually, we obtain an individually rational matching μ_{t+1} such that there are no man-woman blocking pairs for μ_{t+1} and $B(S+1)$ contains no blocking coalitions for μ_{t+1}. If we define $B(t) = B(S+1)$,

[10] That is, $\{m_{s+1}, w_{s+1}\} \subseteq B(s) \cup \{*\}$ is such that $(m_{s+1}, w_{s+1}, c_{s+1})$ is a blocking coalition for μ_{s+1} and $(m_{s+1}, w_{s+1}, c_{s+1}) \succsim_{c_{s+1}} (m, w, c_{s+1})$ for all $(m, w) \in \{(m, w)\colon \{m, w\} \subseteq B(s) \cup \{*\}$ and (m, w, c_{s+1}) is a blocking coalition for $\mu_{s+1}\}$.

we obtain the desired result.

Case 2. This is the case where there is a blocking coalition $(m', w', c') \in \bar{M} \times \bar{W} \times C$ for μ_{s+1} such that $\{m', w'\} \nsubseteq B(s) \cup \{*\}$ and $c' \in B(s)$.

Since μ_{s+1} is individually rational, it cannot be the case that $(m', w') = (*, *)$. Without loss of generality, suppose that $m' \notin B(s) \cup \{*\}$.

Let $(m_{s+1}, w_{s+1}) = (m', w')$, and choose $c_{s+1} \in B(s)$ such that $(m_{s+1}, w_{s+1}, c_{s+1})$ is a blocking coalition for μ_{s+1}, and such that c_{s+1} is m_{s+1}'s most preferred blocking cat partner among all the blocking cat partners in $B(s)$.[11] Lemmas 1 and 2 imply that (m_{s+1}, w_{s+1}) is matched at μ_{s+1}. Let us obtain the next individually rational matching μ_{s+2} from μ_{s+1} by satisfying $(m_{s+1}, w_{s+1}, c_{s+1})$. If $\mu_{s+1}(c_{s+1}) = (\hat{m}, \hat{w}, c_{s+1})$ for some $(\hat{m}, \hat{w}) \in M \times W$, then both $\hat{m} \in B(s)$ and $\hat{w} \in B(s)$ have become single at μ_{s+2}. Since $\hat{m}$ and $\hat{w}$ were previously matched at an individually rational matching μ_{s+1}, this man-woman pair $(\hat{m}, \hat{w})$ is a blocking pair for μ_{s+2}. Let us obtain the next individually rational matching μ_{s+3} from μ_{s+2} by satisfying $(\hat{m}, \hat{w}, *)$. If μ_{s+3} can be obtained in this manner, then we define $\mu_{S+2} = \mu_{s+3}$. Otherwise, we define $\mu_{S+2} = \mu_{s+2}$. Moreover, define the set $B(S+1) = B(s) \cup \{m_{s+1}, w_{s+1}\} \setminus \{*\}$. Clearly, μ_{S+2} does not match any agent in $B(S+1)$ to any agent not in $B(S+1)$. Note that, since all the combinations of a man and a woman are the same between μ_{s+1} and μ_{S+2}, there are no man-woman blocking pairs for μ_{S+2}.

If c_{s+1} was single at μ_{s+1}, that is, if $\mu_{s+1}(c_{s+1}) = (*, *, c_{s+1})$, then $B(S+1)$ contains no blocking coalitions for μ_{S+2}. If we define $\mu_{t+1} = \mu_{S+2}$ and $B(t) = B(S+1)$, we obtain the desired result.

If c_{s+1} was not single at μ_{s+1}, then $\mu_{s+1}(c_{s+1}) = (m_{S+2}, w_{S+2}, c_{s+1})$ for some $(m_{S+2}, w_{S+2}) \in \bar{M} \times \bar{W}$ with $(m_{S+2}, w_{S+2}) \neq (*, *)$. Without loss of generality, suppose that $m_{S+2} \neq *$. It must be the case that $\{m_{S+2}, w_{S+2}\} \subseteq B(S+1) \cup \{*\}$ and $\mu_{S+2}(m_{S+2}) = (m_{S+2}, w_{S+2}, *)$. In this situation, (m_{S+2}, w_{S+2}) and some $c'' \in B(S+1)$ may form a blocking coalition (m_{S+2}, w_{S+2}, c'') for μ_{S+2}, in which case we choose $c_{S+2} \in B(S+1)$ such that $(m_{S+2}, w_{S+2}, c_{S+2})$ is a blocking coalition for μ_{S+2}, and such that c_{S+2} is m_{S+2}'s most preferred blocking cat partner among all the blocking cat partners in $B(S+1)$. Let us obtain the next individually rational matching μ_{S+3} from μ_{S+2} by satisfying $(m_{S+2}, w_{S+2}, c_{S+2})$. If $\mu_{S+2}(c_{S+2}) = (\hat{m}, \hat{w}, c_{S+2})$ for some $(\hat{m}, \hat{w}) \in M \times W$, then both $\hat{m} \in B(S+1)$ and $\hat{w} \in B(S+1)$ have become single at μ_{S+3}. Since $\hat{m}$ and $\hat{w}$ were previously matched at an individually rational matching μ_{S+2}, this man-woman pair $(\hat{m}, \hat{w})$ is a blocking pair for μ_{S+3}. Let us obtain the next individually rational matching μ_{S+4} from μ_{S+3} by satisfying $(\hat{m}, \hat{w}, *)$. If μ_{S+4} can

[11] $c_{s+1} \in B(s)$ is such that $(m_{s+1}, w_{s+1}, c_{s+1})$ is a blocking coalition for μ_{s+1} and $(m_{s+1}, w_{s+1}, c_{s+1}) \succsim_{m_{s+1}} (m_{s+1}, w_{s+1}, c)$ for all $c \in \{c \in B(s) \colon (m_{s+1}, w_{s+1}, c)$ is a blocking coalition for $\mu_{s+1}\}$.

be obtained in this manner, then we define $\mu_{\mathbb{S}+3} = \mu_{S+4}$. Otherwise, we define $\mu_{\mathbb{S}+3} = \mu_{S+3}$. Note that there are no man-woman blocking pairs for $\mu_{\mathbb{S}+3}$ because all the combinations of a man and a woman are the same between μ_{S+2} and $\mu_{\mathbb{S}+3}$. We repeat this process of satisfying a blocking coalition in $B(S+1)$ while keeping the combinations of a man and a woman unchanged, which will stop in finite times since in each time no cat gets worse than before and hence no blocking coalition appears twice for the sequence $\mu_{S+2}, \mu_{\mathbb{S}+3}, \ldots$. Eventually, we obtain an individually rational matching μ_{t+1} such that there are no man-woman blocking pairs for μ_{t+1} and $B(S+1)$ contains no blocking coalitions for μ_{t+1}. If we define $B(t) = B(S+1)$, we obtain the desired result.

Case 3. This is the case where cases 1 and 2 do not apply but there is a blocking coalition $(m', w', c') \in \bar{M} \times \bar{W} \times C$ for μ_{s+1} such that $\{m', w'\} \nsubseteq B(s) \cup \{*\}$ and $c' \notin B(s)$.

Lemmas 1 and 2 imply that (m', w') is matched at μ_{s+1}. Let us obtain the next individually rational matching μ_{s+2} from μ_{s+1} by satisfying (m', w', c'). If $\mu_{s+1}(c') = (\hat{m}, \hat{w}, c')$ for some $(\hat{m}, \hat{w}) \in M \times W$, then $\hat{m}$ and $\hat{w}$ have become single at μ_{s+2}. Since $\hat{m}$ and $\hat{w}$ were previously matched at an individually rational matching μ_{s+1}, this man-woman pair $(\hat{m}, \hat{w})$ is a blocking pair for μ_{s+2}. Let us obtain the next individually rational matching μ_{s+3} from μ_{s+2} by satisfying $(\hat{m}, \hat{w}, *)$. If μ_{s+3} can be obtained in this manner, then we define $\mu_{S+2} = \mu_{s+3}$. Otherwise, we define $\mu_{S+2} = \mu_{s+2}$. Moreover, define the set $B(S+1) = B(s) \cup \{m', w', c'\} \setminus \{*\}$. Clearly, μ_{S+2} does not match any agent in $B(S+1)$ to any agent not in $B(S+1)$. Note that, since all the combinations of a man and a woman are the same between μ_{s+1} and μ_{S+2}, there are no man-woman blocking pairs for μ_{S+2}. Furthermore, $B(S+1)$ contains no blocking coalitions for μ_{S+2}. If we define $\mu_{t+1} = \mu_{S+2}$ and $B(t) = B(S+1)$, we obtain the desired result.

The above arguments complete the proof of the inductive step. When we repeat the inductive step, the cardinality of $B(s)$ strictly increases in each step, but it cannot be greater than the cardinality of $M \cup W \cup C$. So, we must eventually reach an individually rational matching μ_{s+1} for which there are no man-woman blocking pairs and there are no blocking coalitions containing a cat in the form of $(m', w', c') \in \bar{M} \times \bar{W} \times C$. We then proceed to step 4.

Step 4. Denote by μ_N the matching that we have at the beginning of this step.

This matching μ_N is stable because it is individually rational, it is not blocked by any man-woman pair, and it is not blocked by any coalition containing a cat. This fact completes the proof of the theorem. $\square$

Finally, we state the following corollary as Roth and Vande Vate [1990] have established for two-sided matching problems.

Corollary. *Let μ_1 be an arbitrary unstable matching for any three-sided matching problem with strict preferences under Danilov's condition. Consider a random sequence $R(\mu_1) = (\mu_1, \mu_2, \ldots)$ where each μ_{n+1} is obtained from μ_n by satisfying a blocking coalition that is chosen at random from the blocking coalitions for μ_n. If the probability that each blocking coalition for μ_n will be chosen is positive and bounded away from zero, then $R(\mu_1)$ converges to a stable matching with probability one.*

4. Conclusion

In the present paper, we have proved that, in Danilov's three-sided matching model, there exists a finite sequence of successive blockings from an arbitrary unstable matching to some stable matching. Our result is an extension of the result by Roth and Vande Vate [1990] for a two-sided matching model to a three-sided matching model.

Danilov's condition and strict preferences have played an important role to obtain our result. It would be desirable if we can drop these assumptions on preferences and still obtain the same result. Of course, a stable matching does not necessarily exist in three-sided matching problems without Danilov's condition. However, it might be possible to prove the existence of random paths to stability in three-sided matching problems for which there exists a stable matching, as Diamantoudi, Miyagawa, and Xue [2004] have proved for roommate problems for which there exists a stable matching. This is an area for our future research.

Appendix

Proof of the statement with footnote 6. If $(\hat{m}, \hat{w})$ is not a blocking pair for $\mu_{\ell+1}$, then we have either $(\hat{m}, *, *) \succsim_{\hat{m}} (\hat{m}, \hat{w}, *)$ or $(*, \hat{w}, *) \succsim_{\hat{w}} (\hat{m}, \hat{w}, *)$. Strict preferences and Danilov's condition imply that we have either $(\hat{m}, *, *) \succ_{\hat{m}} (\hat{m}, \hat{w}, c_\ell)$ or $(*, \hat{w}, *) \succ_{\hat{w}} (\hat{m}, \hat{w}, c_\ell)$, which is in contradiction with individual rationality of μ_ℓ. $\square$

Proof of the statement with footnote 9. Suppose, by way of contradiction, that $(m, w) \in M \times W$ is a blocking pair for μ_{L+1}, that is, $(m, w, *) \succ_m \mu_{L+1}(m) \equiv (m, w', c')$ and $(m, w, *) \succ_w \mu_{L+1}(w) \equiv (m'', w, c'')$. Since all the combinations of a man and a woman are the same between μ_ℓ and μ_{L+1}, we may write $\mu_\ell(m) \equiv (m, w', \grave{c})$ and $\mu_\ell(w) \equiv (m'', w, \acute{c})$.

If $w \neq w'$, then we have $m \neq m''$. In this case, strict preferences and Danilov's condition imply that $(m, w, *) \succ_m (m, w', *)$ and $(m, w, *) \succ_w (m'', w, *)$. Then, Danilov's condition again implies that $(m, w, *) \succ_m$

$(m, w', \grave{c})$ and $(m, w, *) \succ_w (m'', w, \acute{c})$, which is in contradiction with the fact that μ_ℓ is a matching for which there are no man-woman blocking pairs.

If $w = w'$, then we have $m = m''$, and hence we may write $\mu_{L+1}(m) = \mu_{L+1}(w) \equiv (m, w, c')$ and $\mu_\ell(m) = \mu_\ell(w) \equiv (m, w, \grave{c})$. Since μ_ℓ is a matching for which there are no man-woman blocking pairs, it must be the case that $c' \neq \grave{c}$. By the construction of μ_{L+1}, we have either $(m, w) = (m_\ell, w_\ell)$ or $(m, w) = (\hat{m}, \hat{w})$. If $(m, w) = (m_\ell, w_\ell)$, then we must have $(m_\ell, w_\ell, *) \succ_{m_\ell} \mu_{L+1}(m_\ell) \succ_{m_\ell} \mu_\ell(m_\ell)$ and $(m_\ell, w_\ell, *) \succ_{w_\ell} \mu_{L+1}(w_\ell) \succ_{w_\ell} \mu_\ell(w_\ell)$, which is in contradiction with the fact that μ_ℓ is a matching for which there are no man-woman blocking pairs. If $(m, w) = (\hat{m}, \hat{w})$, then we must have $(\hat{m}, \hat{w}, *) \succ_{\hat{m}} \mu_{L+1}(\hat{m}) = (\hat{m}, \hat{w}, *)$, which is a clear contradiction. $\square$

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