Ex-ante α-core with communication system in a KT-logic model

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Abstract

We consider a strategic form game with asymmetric information where each players' information are given by communication system, derive a non-transferable utility game (NTU-game) from it, and define the solution concept of α-core in the NTU-game. Our main purpose is to prove the non-emptiness of the α-core dependent on the nestedness of the communication system, even if neither the axiom of transparency nor the axiom of wisdom is satisfied.

Keywords: NTU-game derived from strategic form game, communication system, ex-ante α-core, nestedness, axiom of transparency, axiom of wisdom, non-partitional information

JEL classification: C79, D82, D85

1 Introduction

This paper is in the literature of the researches on the core of games and on bounded rationality. We define a strategic form game with asymmetric information where players' information are given by communication system. Communication system gives each player a non-partitional information structure[1] which satisfies axiom of knowledge, and neither axiom of transparency nor axiom of

[1]Most of the literature assume that each player's information is a partition on the state space.

wisdom. We derive a non-transferable utility game (NTU-game) from the strategic form game above and define the solution concept, ex-ante α-core with communication system (simply, the core) as the set of all payoff profiles which can not be improved by any coalition. Our main purpose is to examine the non-emptiness of the core when the communication system is nested [2].

The literature is started with the research on the game with symmetric information. Scarf (1971) defines the core of the NTU-game derived from a strategic form game. Scarf (1971) also suggests a sufficient condition for the non-emptiness of the core in the general settings. Wilson (1978) considers an NTU-game derived from an exchange economy with asymmetric information and defines two types of the core concepts according to the players' information exchange patterns. One is based on the idea the players in a coalition can use only the common information among them, while the other is based on the idea they can make use of the members' information freely. Wilson (1978) also proves the non-emptiness of the coarse core in the general settings.[3] Yannelis (1991) considers the case where each player uses his or her private information independent of the coalition structure and proves the non-emptiness of the core concept.

Maus (2003) uses the communication system which exogenously defines each player's information dependent on the coalition he or she belongs to. Maus (2003) derives the NTU-game from an exchange economy and shows that the core concept is nonempty if the communication system is nested. Hirase and Utsumi (2005) derive the NTU-game from a strategic form game and show that the core concept is nonempty provided the communication system is nested.

All the researches in the literature above assume that each player's information is a partition on the state space. Rubinstein (1998) shows that player's information is partition on state space if and only if it satisfies the axiom of knowledge[4], the axiom of transparency[5], and the axiom of wisdom[6]. In modal logic, the model without the axiom of wisdom is called S-4 logic model. And as the literature of bounded rationality, Hirase (2016) extends Hirase and Utsumi (2005) and proves the non-emptiness of the core concept in an S-4 logic model[7].

In this paper, we would also like to relax the axiom of transparency and extend the resurlt of

[2]We call communication is nested if the players in the larger coalition is provided the richer information by the communication system.

[3]Note that Wilson (1978) considers interim decision making while this paper considers ex-ante decision making.

[4]player's knowledge is always true.

[5]player knows his or her knowledge.

[6]player knows his or her ignorance.

[7]In the same way, Samet (1991) proves the agreement theorem in an S-4 logic model and extends Aumann (1976).

Hirase (2016). In modal logic, the model with neither the axiom of transparency nor the axiom of wisdom is called KT-logic. Our main purpose is to prove the non-emptiness of the core concept of the NTU-game derived from a strategic form game in a KT-logic model.

The rest of the work is organized as follows. In Section 2, we define the game, the properties of the communication system, and the core concept. We discuss the players' information structure in Section 3. Section 4 provides the main theorem, that is, a sufficient condition for non-emptiness of the core concept. We discuss the concluding remarks and the future problems in Section 5.

2 The Model

The basic definition of the game is based on Hirase (2016). This section is constructed as follows. First, we define the strategic form game with communication system where players can exchange their information. Second, we define the NTU-game derived from the strategic form game with communication system. Third, we define the ex-ante α-core of the NTU-game above.

2.1 The Strategic Form Game

We start with the definition of the strategic form game with communication system.

Let $N = \{1, ..., n\}$ be a nonempty finite set of *players*. A *coalition* S is a non-empty subset of N. We denote the finite state space as Ω. We define the communication system which determines each player's information dependent on a coalition he or she belongs to as follows.

Definition 1. $\{\mathcal{P}_i^S\}_{i \in S, S \subset N}$ is the *communication system* where $\mathcal{P}_i^S$, which we call the information (structure) of player i, is a function Ω to $2^{\Omega} \setminus \{\emptyset\}$ for all $S \subset N$ and $i \in S$.

As in Section 1, each player's information is assumed to be a partition on the state space in most researches in the literature. We discuss the properties of the players' information in detail in Section 3.

Definition 2. We call a list of data $(N, \Omega, \{A_i, u_i\}_{i \in N}, \{\mathcal{P}_i^S\}_{i \in N, S \subset N})$ a *strategic form game with communication system* Γ where

- player i's action set $A_i \subset \Re^{m_i}$ is non-empty, convex, and compact for all $i \in S$ and
- player i's payoff function $u_i : \prod_{i \in N}(A_i^{\Omega}) \to \Re$ is continuous and quasi-concave on $\prod_{i \in N}(A_i^{\Omega})$.

We consider the concept of the strategy of the game. According to the information which is given to the player, the set of actions he or she can choose as a strategy is restricted to some extent. We would like to use the notations and the definitions as follows. Let A_i^Ω be the set of functions from Ω to A_i and Σ_i denote A_i^Ω which we call *the universal strategy set for player i.*

Definition 3. We define the set of *player i's strategies for information* $\mathcal{P}_i^S$ as follows.

$$\Sigma_i^S := \{\sigma_i^S \in \Sigma_i | \sigma_i^S \text{ is } \mathcal{P}_i^S\text{-measurable}\}.$$

The measurability condition above can be interpreted as follows. If a player can not distinct a state ω from another state ω' with the information $\mathcal{P}_i^S$, he or she can not take different actions at ω and ω'.

The set of the joint strategies of the coalition S is described as $\Sigma^S = \prod_{i\in S}\Sigma_i^S$. We denote σ^S as a typical element of Σ^S. For any set of the players $R(\subset N)$, a partition on R is interpreted as a *coalition structure of R* . $P(R)$ denotes the set of all coalition structure of R. That means, for $R \subset N$, $P(R)$ is defined by

$$P(R) := \left\{ \{S_1, ..., S_L\} \;\middle|\; \bigcup_{l=1}^{L} S_l = R \text{ and } S_l \cap S_m = \emptyset, \text{ for all } l, m \in \{1, ..., L\} \text{ s.t. } l \neq m \right\}.$$

2.2 The NTU-Game

Using the notations and the definitions in previous subsection, we can derive an NTU-game from the strategic form game with communication system Γ as follows.

Definition 4. We define *the NTU characteristic function V derived from the strategic form game with communication system* Γ as follows[8].

$$V(S) := \bigcup_{\sigma^S\in\Sigma^S} \bigcap_{Q\in P(N\setminus S)} \bigcap_{\substack{(\sigma^T)_{T\in Q}\\ \in(\Sigma^T)_{T\in Q}}} \left\{ (u_1, ..., u_n) \in \Re^n \;\middle|\; \forall i \in S,\ u_i \leq u_i(\sigma^S, (\sigma^T)_{T\in Q}) \right\} \text{for all} S \in \mathcal{N}.$$

Note that the set of the payoff profile of the players in S depends not only on the strategies of S

[8] The definition is same as Hirase and Utsumi (2005) and Hirase (2016).

but also on the coalition structure and the strategies of $N \setminus S$. $V(S)$ means the set of payoff profiles, which the players in a coalition S can gain at least, even if the worst situation (coalition structure and strategies of $N \setminus S$) for S occurs. This α-concept is suggested by Aumann and Peleg (1960).

2.3 The Core

Before the definition of the core concept of the NTU-game, we define the improvement concept.

Definition 5. We say that *a coalition S improves upon the payoff vector u in $\Re^n$* if

$$u \in \mathrm{int}V(S).$$

We can interpret the definition of the improvement concept as follows. If a coalition S improves upon the payoff vector u, then all members in the coalition S have the incentive to deviate from the situation where the players gain the payoff profiles u. All members of S can be better off by the deviation. We can not say u is stable in this sense, and u can not be a candidate for a solution concept of the NTU-game. We would like to define the solution concept of the NTU-game which is robust for such a deviation.

Definition 6. For an NTU characteristic function V, we define *the ex-ante α-core with communication system $C(V)$* as

$$C(V) := V(N) \setminus \bigcup_{T \subset N} \mathrm{int}V(T).$$

The ex-ante α-core with communication system is the set of the payoff profiles that are achieved by the grand coalition and are not improved upon by any coalition of players. Without confusion, we use the word just "core" to refer to the ex-ante α-core with communication system.

3 The Information Structure

In this section, we deal with the information structure of the players precisely. Most of the literature which consider the cooperative games with asymmetric information assume that each player's information be a partition on Ω. One aim of this research paper is to relax the assumption and to extend the main theorem of Hirase (2016). For this aim, we discuss three axioms of the players information as follows. We omit the index of the players and coalitions in this section, that is, $\mathcal{P}_i^S$ is simply denoted by $\mathcal{P}$.

- **P-1** (axiom of knowledge) : $\omega \in \mathcal{P}(\omega)$ for all ω in Ω.

This property means that a player never excludes the real state. When the real state is ω, the player recognizes that ω may have occurred.

- **P-2** (axiom of transparency) : $\omega' \in \mathcal{P}(\omega)$ implies $\mathcal{P}(\omega') \subset \mathcal{P}(\omega)$ for all ω in Ω.

To explain the meaning of P-2, we consider its contraposition: If there is a state z such that $z \notin \mathcal{P}(\omega)$ and $z \in \mathcal{P}(\omega')$, then $\omega' \notin \mathcal{P}(\omega)$. We can interpret the situation as follows. If a player at ω knows that the state z is impossible and that if the real state is ω', then z is probable, then he or she infers that ω' is not the real state. P-2 says that a player can use this kind of inference. We will consider the situation where P-2 is not necessarily satisfied.

- **P-3** (axiom of wisdom) : $\omega' \in \mathcal{P}(\omega)$ implies $\mathcal{P}(\omega') \supset \mathcal{P}(\omega)$ for all ω in Ω.

To clarify the meaning of P-3, we consider its contraposition again: If there is a state z such that $z \in \mathcal{P}(\omega)$ and $z \notin \mathcal{P}(\omega')$, then $\omega' \notin \mathcal{P}(\omega)$. This means that if a player at ω knows that the state z is probable and that if the real state is ω', then z is impossible, then he or she infers that ω' is not the real state. This work and Hirase (2016) consider the situation where P-3 is not necessarily satisfied.

Note that the following remark holds.

Remark 1. If P-1 and P-3 are satisfied, then P-2 is also satisfied.[9]

We discuss the relation between these axioms and players' (partitional or non-partitional) information. We say $\mathcal{P}$ is *partitional* if there is a partition on Ω such that for any $\omega \in \Omega$ the set $\mathcal{P}(\omega)$ is equal to the element of the partition that contains ω. As for the relation between the axioms above and the information structure, the following proposition holds.

Proposition 1

An information $\mathcal{P}$ is partitional if and only if it satisfies P-1, P-2, and P-3.[10]

[9] See Hirase (2016).
[10] See Rubinstein (1998) and Hirase (2016).

This proposition implies that most of the literature assume players' information satisfies P-1, P-2, and P-3. However, if we consider more bounded rational players, three properties of the information are not necessarily satisfied. Some partitional information structure and non-partitional information structure are displayed in the following examples.

Example. Let the state space Ω be $\{\omega_1, \omega_2, \omega_3\}$.

Suppose $\mathcal{P}$ is as follows. $\mathcal{P}(\omega_1) = \{\omega_1\}$ and $\mathcal{P}(\omega_2) = \mathcal{P}(\omega_3) = \{\omega_2, \omega_3\}$. In this case, all the axioms, P-1, P-2, and P-3 are satisfied and the information structure is partitional as described in Figure1. Note that Hirase and Utsumi (2005) proves the non-emptiness of the core concept in this situation.

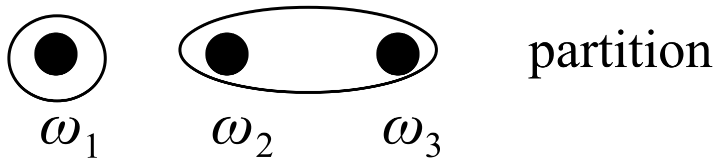

Figure 1: Partitional Information Structure

Suppose $\mathcal{P}$ is as follows. $\mathcal{P}(\omega_1) = \{\omega_1\}, \mathcal{P}(\omega_2) = \{\omega_2\}$, and $\mathcal{P}(\omega_3) = \{\omega_2, \omega_3\}$. In this case, we can check P-1 and P-2 are satisfied and the information structure is non-partitional as described in Figure 2. Note that Hirase (2016) proves the non-emptiness of the core concept in this situation.

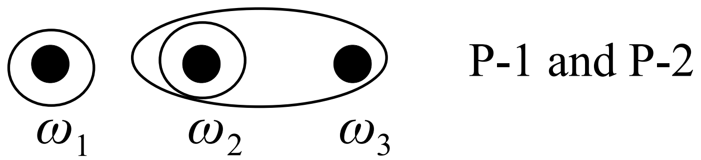

Figure 2: Non-Partitional Information Structure

Suppose $\mathcal{P}$ is as follows. $\mathcal{P}(\omega_1) = \{\omega_1, \omega_2\}$ and $\mathcal{P}(\omega_2) = \mathcal{P}(\omega_3) = \{\omega_2, \omega_3\}$. In this case, we can check only P-1 is satisfied and the information structure is partitional as described in Figure1. Note that we would like to prove the non-emptiness of the core concept in this situation in the next section.

Suppose $\mathcal{P}$ is as follows. $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_2) = \mathcal{P}(\omega_3) = \{\omega_2, \omega_3\}$. In this case, we can check P-1 is not

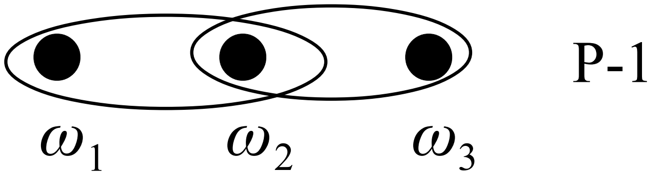

Figure 3: Non-Partitional Information Structure

satisfied as described in Figure 4. Note that we would like to exclude this kind of situation.

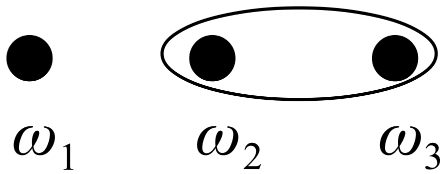

Figure 4: P-1 is not satisfied

$\diamondsuit$

Example. Let the size of the state space Ω be 6 and suppose that $\mathcal{P}'$ is given as follows.

$\mathcal{P}'(\omega_1) = \mathcal{P}'(\omega_2) = \{\omega_1, \omega_2\}$, $\mathcal{P}'(\omega_3) = \mathcal{P}'(\omega_4) = \{\omega_1, \omega_3, \omega_3, \omega_4\}$, and $\mathcal{P}'(\omega_5) = \mathcal{P}'(\omega_6) = \{\omega_5, \omega_5, \omega_6\}$.

This case is described as in Figure 5.

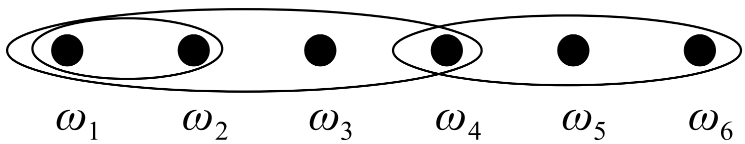

Figure 5: P-1 is satisfied

We can check that P-1 is satisfied, that is, for all $\omega \in \Omega, \omega \in \mathcal{P}'(\omega)$.

We can check that P-2 is not satisfied. because we can find that $\omega_3 \in \mathcal{P}'(\omega_2)$ but $\mathcal{P}'(\omega_3) \not\subset \mathcal{P}'(\omega_2)$.

We can also check that P-3 is not satisfied. because we can find that $\omega_4 \in \mathcal{P}'(\omega_5)$ but $\mathcal{P}'(\omega_4) \not\supset \mathcal{P}'(\omega_5)$.

$\diamondsuit$

We prove the non-emptiness of the core in relation to the axioms of the information and the properties of the communication system in the next section.

4 The Non-emptiness of the Core

We provide the definitions of the nestedness and the boundedness of the communication system. Both of the definitions are originally defined by Maus (2003)

Definition 7. A communication system $\{\mathcal{P}_i^S\}_{i\in S, S\subset N}$ is *nested* if for all S and $T \subset \mathcal{N}$ such that $S \subset T$,

$$\mathcal{P}_i^S(\omega) \supset \mathcal{P}_i^T(\omega) \quad \text{for all } i \in S \text{ and } \omega \in \Omega.$$

The nestedness of the communication system can be interpreted as each player in the larger coalition has the richer information.

Definition 8. A communication system $\{\mathcal{P}_i^S\}_{i\in S, S\subset N}$ is *bounded* if for all S,

$$\mathcal{P}_i^S(\omega) \supset \mathcal{P}_i^N(\omega) \quad \text{for all } i \in N \text{ and } \omega \in \Omega.$$

The boundedness can be interpreted as all player have the richest information when they form the grand coalition.

Remark 2. We can check that the nestedness of a communication system implies the boundedness of the communication system.

Hirase and Utsumi (2005) show the nestedness or the boundedness of the communication system ensures the non-emptiness of the core when each player's information structure is partitional. From this fact, Remark 1, and Remark 2, we can obtain the following proposition.[11]

Proposition 2

The ex-ante α-core with communication system of V is non-empty if the communication system is bounded and satisfies P-1 and P-3.

From Hirase (2016), we can obtain a similar proposition.

[11] We can use the same definitions of the nestedness and the boundedness when we relax P-2 and P-3.

Proposition 3

The ex-ante α-core with communication system of V is non-empty if the communication system is bounded and satisfies P-1 and P-2.

Combining these two propositions, we can say either P-2 or P-3 is not necessary to ensure the non-emptiness of the core if the communication system is nested or bounded. However, our result in this paper can say neither of them is necessary! The main theorem is following.

Proposition 4

The ex-ante α-core with communication system of V is non-empty if the communication system is bounded and satisfies P-1.

Therefore, we extend the results in the literature and we can say that the core non-emptiness theorem holds even if we consider more bounded rational players.

5 Concluding Remarks

We can interpret our main result as the extensions of some seminal works in the literature. Our model is an asymmetric information version of Scarf (1971) and our framework allows other non-partitional information structure than those of Hirase (2016).

We can consider the following points as future problems.

First, we would like to extend the results of Hirase (2009 and 2015) that examine the partial cooperation situation by using the network. Their models consider that players can cooperate and exchange information only through the network.

Second, we would like to explain how the coalition of the players are formed, that is, to consider the models in which the coalition structure (or the network of the players) is endogenously determined. Considering asymmetric information versions of the models by Hirase (2012 and 2013), Jackson and Wolinsky (1996) and etc. can be the case.

Third, we would like to explain how the information exchange occurs, that is, to consider the model where (the property of) the communication system is endogenously determined. We would also like to consider incentive compatibility of the information exchange like Sun et al. (2012).

At last, applying our model to the situation which has more economic implication (for instance, oligopoly and etc.) can also be a future problem.

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