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Yusuke Samejima

Abstract

Perez-Castrillo and Wettstein [2004, 2006] have proposed the ordinal Shapley value, a solution concept for pure exchange economies where utility transfers may be impossible. We propose a game form that fully implements the ordinal Shapley value in subgame-perfect equilibrium. Our game form differs from the one proposed by Perez-Castrillo and Wettstein [2005] in that our game form works for economies with three or more agents while their game form works for economies with three or less agents.

1. Introduction

The ordinal Shapley value is a normative solution concept for pure exchange economies where utility transfers may be impossible. The solution concept was proposed by Perez-Castrillo and Wettstein [2004].[1] It is an ordinal solution concept for NTU (non-transferable utility) environments while the Shapley value introduced by Shapley [1953] is a solution concept for TU games. The ordinal Shapley value and the Shapley value share desirable properties such as efficiency, consistency, and fairness as discussed in Perez-Castrillo and Wettstein [2004].

Implementation theory aims to develop a tool for the uninformed social planner who wishes to realize certain desirable allocations. When the social planner, or the society, attempts to realize desirable allocations, he must collect information on the preferences of members in the society. However, it is often the case that the social planner has difficulty in collecting such information while the concerned members share much information on each other. For such circumstances, a game form, also called a

[1] The working paper by Perez-Castrillo and Wettstein [2004] was eventually published in a journal as Perez-Castrillo and Wettstein [2006]. However, proofs of some important results are found only in Perez-Castrillo and Wettstein [2004].

mechanism, can be used as a tool for the social planner. The game form itself is defined independently of the preferences of members in the society. As the literature on implementation theory has proposed, properly designed game forms can realize desirable allocations in equilibrium of the games even if the social planner is given an insufficient amount of information.

In the literature, Perez-Castrillo and Wettstein [2005] have proposed a simple game form that implements the ordinal Shapley value in subgame-perfect equilibrium. However, applicability of their game form is limited in the sense that their game form works for economies with three or *less* agents.[2]

In the present paper, we also propose a game form that fully implements the ordinal Shapley value in subgame-perfect equilibrium. However, our game form differs from the one proposed by Perez-Castrillo and Wettstein [2005] in that our game form works for economies with three or *more* agents. On the other hand, our game form as well as their game form is designed so that every equilibrium allocation possesses properties such as efficiency, consistency, and fairness as the ordinal Shapley value does.

In the field of implementation theory, Moore and Repullo [1988] have developed a canonical game form for subgame-perfect implementation of a social choice correspondence in general environments with three or more agents. They present a sufficient condition, called the condition C^+, for achieving implementation with their canonical game form. The condition C^+ contains the existence of a finite sequence of allocations satisfying certain preference relations. Unfortunately, it is not immediately clear whether such a sequence exists for the ordinal Shapley value, and hence it is not clear whether the ordinal Shapley value can be implemented by their canonical game form. So, we consider that it is worthwhile to design a new game form specifically for the ordinal Shapley value.

The remaining part of this paper is organized as follows. Section 2 explains a model of pure exchange economies where the ordinal Shapley value is defined, and introduces the notion of implementation. Section 3 proposes a game form that implements the ordinal Shapley value in subgame-perfect equilibrium. Section 4 provides some concluding remarks.

2. The Model

Let $N = \{1, 2, \ldots, n\}$ be a set of *agents* with at least three members ($n \geq 3$). A *coalition* is a nonempty subset of N, and it is typically denoted by S. We consider a pure exchange economy with $\ell \geq 2$ commodities. Agent i's initial *endowment bundle* is $\omega_i \in \mathbb{R}^\ell$, where $\mathbb{R}$ is the set of real numbers. An *endowment profile* of a coalition S is denoted by $\omega_S \equiv (\omega_i)_{i \in S}$. Agent i's *consumption bundle* and

[2] Perez-Castrillo and Wettstein [2005] have pointed out that the difficulty for their mechanism to work for three or more agents is due to the transfer paradox as discussed in Safra [1984].

coalition S's *consumption profile* are denoted by $x_i \in \mathbb{R}^\ell$ and $x_S \equiv (x_i)_{i\in S}$, respectively.[3]

We assume that each agent i has a continuous and monotone[4] *preference relation* R_i defined over $\mathbb{R}^\ell$. As usual, $x_i\, R_i\, y_i$ means that x_i is at least as good as y_i for agent i. The associated strict preference relation and indifference relation are denoted by P_i and I_i, respectively. That is, $x_i\, P_i\, y_i$ means that agent i prefers x_i to y_i and $x_i\, I_i\, y_i$ means that agent i is indifferent between x_i and y_i. Let $\mathcal{R}_i$ be the set of *admissible preferences* of agent i. A *preference profile* of a coalition S is denoted by $R_S \equiv (R_i)_{i\in S}$ and the set of admissible preference profiles of a coalition S is denoted by $\mathcal{R}_S \equiv \times_{i\in S}\mathcal{R}_i$.

The set of *feasible allocations* for a coalition S with ω_S is denoted by

$$X_S(\omega_S) \equiv \{x_S \in \mathbb{R}^{\ell|S|} : \sum_{i\in S} x_i \leq \sum_{i\in S} \omega_i\}.$$

A feasible allocation $x_S \in X_S(\omega_S)$ is *efficient* for a coalition S with R_S and ω_S if there is no other feasible allocation $y_S \in X_S(\omega_S)$ such that $y_i\, R_i\, x_i$ for all $i \in S$ and $y_j\, P_j\, x_j$ for some $j \in S$.

Perez-Castrillo and Wettstein [2004] have proposed a solution concept called the *ordinal Shapley value* for pure exchange economies described above. To define the ordinal Shapley value, we use the *base bundle* denoted by $e \equiv (1, \ldots, 1) \in \mathbb{R}^\ell$.

Definition. The *ordinal Shapley value* of an economy of a coalition S with R_S and ω_S is a set-valued function $\varphi(R_S, \omega_S) \subset X_S(\omega_S)$, which is defined recursively as follows.

($|S| = 1$) In the case of an economy with one agent $i \in S = \{i\}$, the ordinal Shapley value is his endowment bundle, i.e., $\varphi(R_i, \omega_i) = \{\omega_i\}$.

For $|S| \geq 2$, suppose that φ has been defined for any economy with $(|S| - 1)$ or less agents.

($|S| \geq 2$) In this case, the ordinal Shapley value is the set of efficient allocations for S with R_S and ω_S such that for each x_S in the ordinal Shapley value, there exists an $|S|$-tuple of *concession vectors*, $(c^i)_{i\in S}$ with $c^i \equiv (c^i_j)_{j\in S\setminus\{i\}} \in \mathbb{R}^{|S|-1}$ for each $i \in S$, and the vectors satisfy the following two properties.

(*Consistency*) For each $i \in S$, there exists $x'_{S\setminus\{i\}} \in \varphi(R_{S\setminus\{i\}}, (\omega_j + c^i_j e)_{j\in S\setminus\{i\}})$ such that $x_j\, I_j\, x'_j$ for all $j \in S \setminus \{i\}$.

(*Fairness*) $\sum_{j\in S\setminus\{i\}} c^i_j = \sum_{j\in S\setminus\{i\}} c^j_i$ for all $i \in S$.

[3] We note that each agent's consumption space is not restricted to the non-negative orthant. We have followed Perez-Castrillo and Wettstein [2004].

[4] The preference relation R_i is *monotone* on $\mathbb{R}^\ell$ if $x_i \gg y_i$ then $x_i\, P_i\, y_i$.

Perez-Castrillo and Wettstein [2004] have proved the following important facts. Fact 1 and Fact 2 are their Theorem 2, and Fact 3 follows from their Lemma 1.

Fact 1. The ordinal Shapley value $\varphi(R_S, \omega_S)$ is non-empty for all $R_S \in \mathcal{R}_S$ and $\omega_S \in \mathbb{R}^{\ell|S|}$.

Fact 2. The ordinal Shapley value satisfies *individual rationality*. That is, for each $x_S \in \varphi(R_S, \omega_S)$, it is the case that $x_i \; R_i \; \omega_i$ for all $i \in S$.

Fact 3. For each $x_S \in \varphi(R_S, \omega_S)$ with $|S| \geq 2$, the associated $|S|$-tuple of concession vectors $(c^i)_{i \in S}$ that appear in the definition of the ordinal Shapley value is unique.

Given R_N, ω_N, and $x_N \in \varphi(R_N, \omega_N)$, we can construct a sequence $Q(R_N, \omega_N, x_N)$ of lists

$$Q(R_N, \omega_N, x_N) \equiv [\ S(t)\ ,\ \omega_{S(t)}(t) \equiv (\omega_i(t))_{i \in S(t)}\ ,\ x_{S(t)}(t) \equiv (x_i(t))_{i \in S(t)}\ ]_{t=1,2,\ldots,n}$$

as follows.

(Step 1) Let $S(1) = N$, $\omega_{S(1)}(1) = \omega_N$, and $x_{S(1)}(1) = x_N$.

(Step t) For $t = 2, \ldots, n$, define the list as follows.

- (Step t–1) Since $x_{S(t-1)}(t-1) \in \varphi(R_{S(t-1)}, \omega_{S(t-1)})$ and since we have Fact 3, there exists a unique $|S(t-1)|$-tuple of concession vectors $(c^i(t))_{i \in S(t-1)}$ that satisfy consistency and fairness in the definition of the ordinal Shapley value.
- (Step t–2) Choose $S(t)$ such that $S(t) \subset S(t-1)$ and $|S(t)| = |S(t-1)| - 1$.
- (Step t–3) Given $i \in S(t-1) \setminus S(t)$ and $c^i(t) \equiv (c^i_j(t))_{j \in S(t)}$, we can define $\omega_{S(t)}(t)$ so that for all $j \in S(t)$, it is the case that $\omega_j(t) = \omega_j(t-1) + c^i_j(t)e$.
- (Step t–4) By the consistency of $(c^i(t))_{i \in S(t-1)}$, we can choose $x_{S(t)}(t) \in \varphi(R_{S(t)}, \omega_{S(t)}(t))$ such that $x_j(t-1) \; I_j \; x_j(t)$ for all $j \in S(t)$. Note that $x_j \; I_j \; x_j(t)$ for all $j \in S(t)$.[5]

The set of sequences that can be constructed as above is denote by $\mathcal{Q}(R_N, \omega_N, x_N)$.

We consider an extensive *game form* as studied in Moore and Repullo [1988]. In the present paper, we define a game form that works for any given endowment profile ω_N of all agents. The game form $\Gamma(\omega_N)$ consists of a game tree with a set of message choices available to agents at each information set, and an outcome function.

Agent i's *strategy* m_i is a function that associates agent i's message choice from available choices with each information set for which he is on the move. We do not consider mixed strategies in the

[5] We have $x_j \; I_j \; x_j(t)$ since $x_j = x_j(1) \; I_j \; x_j(2) \cdots x_j(t-1) \; I_j \; x_j(t)$.

present paper. The set of agent i's strategies is also called agent i's *message space* and denoted by M_i. A *strategy profile* is denoted by $m_N \equiv (m_i)_{i \in N}$. The set of strategy profiles is denoted by $M_N \equiv \times_{i \in N} M_i$.

Given a strategy profile m_N, we can associate with m_N a terminal node, where an *outcome allocation* is realized. So, we can define an *outcome function* $g \colon M_N \to X_N(\omega_N)$ that associates with each strategy profile a feasible allocation for N with ω_N.

A list $(\Gamma(\omega_N), R_N)$ defines an extensive *game*. Agent i's strategy m_i is a *best response* to a strategy profile $m_{N \setminus \{i\}}$ of the other agents if $g(m_N)\ R_i\ g(m'_i, m_{N \setminus \{i\}})$ for all $m'_i \in M_i$. A strategy profile m_N is a *Nash equilibrium* of $(\Gamma(\omega_N), R_N)$ if each agent chooses a best response to a strategy profile of the others at m_N; That is, for all $i \in N$, it is the case that $g(m_N)\ R_i\ g(m'_i, m_{N \setminus \{i\}})$ for all $m'_i \in M_i$. At Nash equilibria, no agent can gain by changing his strategy while the others keep their strategies unchanged. A strategy profile m_N is a *subgame-perfect equilibrium* of $(\Gamma(\omega_N), R_N)$ if in every subgame of $(\Gamma(\omega_N), R_N)$, the strategy profile induced by m_N is a Nash equilibrium. Let $SPE(\Gamma(\omega_N), R_N)$ be the set of *subgame-perfect equilibrium allocations* corresponding to the subgame-perfect equilibria of the game $(\Gamma(\omega_N), R_N)$.[6]

We say that a game form $\Gamma(\omega_N)$ *implements* the ordinal Shapley value φ *in subgame-perfect equilibrium* if for any given endowment profile $\omega_N \in \mathbb{R}^{\ell n}$, it is the case that $SPE(\Gamma(\omega_N), R_N) = \varphi(R_N, \omega_N)$ for all $R_N \in \mathcal{R}_N$. This notion of implementation is called *full implementation* in the literature.

3. The Result

3–1. The game form

This section describes a game form that implements the ordinal Shapley value in subgame-perfect equilibrium. Choose any arbitrary endowment profile $\omega_N \in \mathbb{R}^{\ell n}$ and fix it throughout this section. Formally, the game form $\Gamma(\omega_N)$ is defined as follows.

The game form $\boldsymbol{\Gamma(\omega_N)}$.

The game form $\Gamma(\omega_N)$ consists of a game tree with message spaces M_N and an outcome function g. The following descriptions define the game form $\Gamma(\omega_N)$.

Stage 1. This is a simultaneous-move stage. Each agent $i \in N$ announces

$$(R^i_N, x^i_N, z^i, t^i, j^i, \varepsilon^i) \in \mathcal{R}_N \times X_N(\omega_N) \times \mathbb{N}_0 \times N \times N \times \mathbb{R}_{>0}$$

[6] That is, $SPE(\Gamma(\omega_N), R_N) \equiv \{x_N \in X_N(\omega_N) \colon x_N = g(m_N)$ for some subgame-perfect equilibrium m_N of the game $(\Gamma(\omega_N), R_N)\}$.

where $\mathbb{N}_0$ denotes the set of non-negative integers, $\mathbb{R}_{>0}$ denotes the set of strictly positive real numbers, and it is required that $x^i_N \in \varphi(R^i_N, \omega_N)$. In addition, each agent $i \in N$ also announces simultaneously a sequence

$$Q^i(R^i_N, \omega_N, x^i_N) \equiv [\ S^i(t)\ ,\ \omega^i_{S^i(t)}(t)\ ,\ x^i_{S^i(t)}(t)\ ]_{t=1,2,\ldots,n} \in \mathcal{Q}(R^i_N, \omega_N, x^i_N)$$

together with a feasible allocation for the coalition $S^i(t^i)$ with $\omega^i_{S^i(t^i)}(t^i)$

$$y^i_{S^i(t^i)} \in X_{S^i(t^i)}(\omega^i_{S^i(t^i)}(t^i)).$$

Case 1. If $(R^i_N, x^i_N, z^i) = (\bar{R}_N, \bar{x}_N, 0)$ for all $i \in N$ and for some $(\bar{R}_N, \bar{x}_N)$, then the play stops. This is the case where all agents have announced the same (R^i_N, x^i_N, z^i). At this terminal node, the outcome allocation is $\bar{x}_N$.

Case 2. This is the case where there exists $i^* \in N$ such that $(R^i_N, x^i_N, z^i) = (\bar{R}_N, \bar{x}_N, 0) \neq (R^{i^*}_N, x^{i^*}_N, z^{i^*})$ for all $i \in N \setminus \{i^*\}$ and for some $(\bar{R}_N, \bar{x}_N)$. This case is divided into the following subcases.

Subcase 2–1. If case 2 applies and $(R^{i^*}_N, x^{i^*}_N, z^{i^*}) = (\bar{R}_N, \bar{x}_N, 1)$ and $i^* \notin S^{i^*}(t^{i^*})$ and $j^{i^*} \in S^{i^*}(t^{i^*})$, then the play proceeds to stage 2–1.

Subcase 2–2. If case 2 applies and $(R^{i^*}_N, x^{i^*}_N, z^{i^*}) = (\bar{R}_N, \bar{x}_N, 2)$ and $i^* \in S^{i^*}(t^{i^*})$, then the play proceeds to stage 2–2.

Subcase 2–3. If case 2 applies and $(R^{i^*}_N, x^{i^*}_N, z^{i^*}) = (\bar{R}_N, \bar{x}_N, 3)$ and $i^* \in S^{i^*}(t^{i^*})$, then the play stops. At this terminal node, the outcome allocation x_N is constructed as follows. For agent i^*, the outcome is such that $x_{i^*} = x^{i^*}_{i^*}(t^{i^*})$.[7] Let agent $k \in N \setminus \{i^*\}$ be the agent with the least index number among those other than agent i^*. For each $i \in N \setminus \{i^*, k\}$, the outcome is such that $x_i = (0, \ldots, 0) \in \mathbb{R}^\ell$. For agent k, the outcome is such that $x_k = \sum_{i \in N} \omega_i - \sum_{i \in N \setminus \{k\}} x_i$.

Subcase 2–4. If case 2 applies but none of the above three subcases applies, then the play stops. At this terminal node, the outcome allocation is $\bar{x}_N$.

[7] A consumption bundle $x^{i^*}_{i^*}(t^{i^*})$ is a part of a consumption profile $x^{i^*}_{S^{i^*}(t^{i^*})}(t^{i^*})$ that appears in the sequence $Q^{i^*}(R^{i^*}_N, \omega_N, x^{i^*}_N)$.

Case 3. If neither case 1 nor 2 applies, then the play stops. At this terminal node, the outcome allocation x_N is constructed as follows. Let agent k be the agent with the least index number among those who have announced the largest integer $z^k \in \mathbb{N}_0$ among all agents in stage 1. That is, $k \equiv \min\{i \in N : i \in \arg\max_{j \in N} z^j\}$. For agent k, the outcome is such that $x_k = \omega_k + (n-1)z^k e$. For each $i \in N \setminus \{k\}$, the outcome is such that $x_i = \omega_i - z^k e$.

Stage 2–1. This stage is reached when subcase 2–1 applies in stage 1. In this stage, agent $j^{i^*} \in S^{i^*}(t^{i^*})$ announces $a \in \{0,1\}$, and the play stops. Let agent $k \in N \setminus \{i^*, j^{i^*}\}$ be the agent with the least index number among those other than agents i^* and j^{i^*}.

If $a = 0$, then the outcome allocation x_N at this terminal node is constructed as follows. For agent j^{i^*}, the outcome is such that $x_{j^{i^*}} = x^{i^*}_{j^{i^*}}(t^{i^*}) + \varepsilon^{i^*} e$. For agent i^*, the outcome is such that $x_{i^*} = \bar{x}_{i^*} - \varepsilon^{i^*} e$. If $N \setminus \{i^*, j^{i^*}, k\}$ is non-empty, then for each $i \in N \setminus \{i^*, j^{i^*}, k\}$, the outcome is such that $x_i = (0, \ldots, 0) \in \mathbb{R}^\ell$. For agent k, the outcome is such that $x_k = \sum_{i \in N} \omega_i - \sum_{i \in N \setminus \{k\}} x_i$.

If $a = 1$, then the outcome allocation x_N at this terminal node is constructed as follows. For agent j^{i^*}, the outcome is such that $x_{j^{i^*}} = \bar{x}_{j^{i^*}}$. For agent i^*, the outcome is such that $x_{i^*} = \bar{x}_{i^*} + \varepsilon^{i^*} e$. If $N \setminus \{i^*, j^{i^*}, k\}$ is non-empty, then for each $i \in N \setminus \{i^*, j^{i^*}, k\}$, the outcome is such that $x_i = (0, \ldots, 0) \in \mathbb{R}^\ell$. For agent k, the outcome is such that $x_k = \sum_{i \in N} \omega_i - \sum_{i \in N \setminus \{k\}} x_i$.

Stage 2–2. This stage is reached when subcase 2–2 applies in stage 1. Stage 2–2 consists of $|S^{i^*}(t^{i^*})|$ rounds, and agents in $S^{i^*}(t^{i^*})$ move sequentially. The play starts from round 1 and proceeds, but the play stops before reaching the next round if some agent on the move chooses to do so. These rounds are described as follows.

Round $\boldsymbol{j \in \{1, 2, \ldots, |S^{i^*}(t^{i^*})|\}}$. Let agent $\bar{j}$ be the agent with the j-th least index number among those in the coalition $S^{i^*}(t^{i^*})$. In this round, agent $\bar{j} \in S^{i^*}(t^{i^*})$ announces $b^{\bar{j}} \in \{0,1\}$.

If $b^{\bar{j}} = 0$ and $j < |S^{i^*}(t^{i^*})|$, then the play proceeds to round $j+1$.

If $b^{\bar{j}} = 0$ and $j = |S^{i^*}(t^{i^*})|$, then the play stops and the outcome allocation x_N at this terminal node is constructed as follows. For each $i \in S^{i^*}(t^{i^*})$, the outcome is such that $x_i = y_i^{i^*}$. If $N \setminus S^{i^*}(t^{i^*})$ is non-empty, then let agent $k \in N \setminus S^{i^*}(t^{i^*})$ be the agent with the least index number among those outside the coalition $S^{i^*}(t^{i^*})$. If $N \setminus (S^{i^*}(t^{i^*}) \cup \{k\})$ is non-empty, then for each $i \in N \setminus (S^{i^*}(t^{i^*}) \cup \{k\})$, the outcome is such that $x_i = (0, \ldots, 0) \in \mathbb{R}^\ell$. For agent k, the outcome is such that $x_k = \sum_{i \in N} \omega_i - \sum_{i \in N \setminus \{k\}} x_i$.

If $b^{\bar{j}} = 1$, then the play stops even if there still remain some rounds. At this terminal node, the outcome allocation is $\bar{x}_N$.

Theorem. *The game form* $\Gamma(\omega_N)$ *implements the ordinal Shapley value* φ *in subgame-perfect equilibrium. That is, for any given endowment profile* $\omega_N \in \mathbb{R}^{\ell n}$*, it is the case that* $SPE(\Gamma(\omega_N), R_N) = \varphi(R_N, \omega_N)$ *for all* $R_N \in \mathcal{R}_N$*.*

This theorem is our main result. The following two sections prove the theorem.

3–2. Proof: $SPE(\Gamma(\omega_N), R_N) \subset \varphi(R_N, \omega_N)$ for all $R_N \in \mathcal{R}_N$.

Choose any arbitrary $\omega_N \in \mathbb{R}^{\ell n}$ and $R_N \in \mathcal{R}_N$ and fix them throughout this section. This R_N is regarded as a 'true' preference profile of agents. Furthermore, choose any arbitrary subgame-perfect equilibrium allocation $\bar{x}_N \in SPE(\Gamma(\omega_N), R_N)$. This section proves that $\bar{x}_N \in \varphi(R_N, \omega_N)$.

Lemma 1. *A subgame-perfect equilibrium allocation* $\bar{x}_N \in SPE(\Gamma(\omega_N), R_N)$ *is realized only when case 1 of stage 1 applies on the equilibrium path.*

Proof. Suppose, by way of contradiction, that case 1 of stage 1 does not apply on the equilibrium path. Then, there exist at least $(n-1)$ agents who have an opportunity in stage 1 to change his announcement so that case 3 of stage 1 applies. If such an agent i changes his announcement of z^i so that $z^i > z^j$ for all $j \in N \setminus \{i\}$, then he can obtain the outcome $x_i = \omega_i + (n-1)z^i e$ such that $x_i \; P_i \; \bar{x}_i$ for sufficiently large z^i. That is, this agent can gain by deviating from the equilibrium path. □

By Lemma 1, we may assume that in subgame-perfect equilibrium, all agents in stage 1 announce $(\bar{R}_N, \bar{x}_N, 0)$ such that $\bar{x}_N \in \varphi(\bar{R}_N, \omega_N)$ for some $\bar{R}_N \in \mathcal{R}_N$.

Lemma 2. *Suppose that all agents in stage 1 announce* $(\bar{R}_N, \bar{x}_N, 0)$ *such that* $\bar{x}_N \in \varphi(\bar{R}_N, \omega_N)$ *in subgame-perfect equilibrium. For any sequence*

$$Q(\bar{R}_N, \omega_N, \bar{x}_N) \equiv [\ S(t)\ ,\ \omega_{S(t)}(t)\ ,\ x_{S(t)}(t)\ ]_{t=1,2,\dots,n} \in \mathcal{Q}(\bar{R}_N, \omega_N, \bar{x}_N),$$

for any $t \in N$*, and for any* $i \in S(t)$*, it is the case that* $\bar{x}_i \; R_i \; x_i(t)$*.*

Proof. Suppose, by way of contradiction, that there exists a sequence $Q(\bar{R}_N, \omega_N, \bar{x}_N)$ such that for some $t' \in N$ and some $i' \in S(t')$, it is the case that $x_{i'}(t') \; P_{i'} \; \bar{x}_{i'}$. If agent i' announces $(\bar{R}_N, \bar{x}_N, 3)$ together with t' and $Q(\bar{R}_N, \omega_N, \bar{x}_N)$ in stage 1, then subcase 2–3 applies and agent i' can obtain the outcome $x_{i'}(t')$. That is, agent i' can gain by deviating from the equilibrium path. □

Lemma 3. *Suppose that all agents in stage 1 announce* $(\bar{R}_N, \bar{x}_N, 0)$ *such that* $\bar{x}_N \in \varphi(\bar{R}_N, \omega_N)$ *in* *subgame-perfect equilibrium. For any sequence*

$$Q(\bar{R}_N, \omega_N, \bar{x}_N) \equiv [\ S(t)\ ,\ \omega_{S(t)}(t)\ ,\ x_{S(t)}(t)\ ]_{t=1,2,\ldots,n} \in \mathcal{Q}(\bar{R}_N, \omega_N, \bar{x}_N),$$

for any $t \in N$, and for any $i \in S(t)$, it is the case that $x_i(t)\ R_i\ \bar{x}_i$.

Proof. Suppose, by way of contradiction, that there exists a sequence $Q(\bar{R}_N, \omega_N, \bar{x}_N)$ such that for some $t' \in N$ and some $j' \in S(t')$, it is the case that $\bar{x}_{j'}\ P_{j'}\ x_{j'}(t')$. Since agent j' has a continuous preference relation, there exists a positive real number $\varepsilon' > 0$ such that $\bar{x}_{j'}\ P_{j'}\ (x_{j'}(t') + \varepsilon' e)$.

Since $\bar{x}_{j'} = x_{j'}(1)$ by the construction of the sequence, the fact that $\bar{x}_{j'}\ P_{j'}\ x_{j'}(t')$ implies that $t' \geq 2$. Furthermore, since $t' \geq 2$ and hence $S(t') \subsetneq N$, there exists an agent $i' \notin S(t')$. We will show that agent i' can gain by a unilateral deviation from the equilibrium path.

If agent i' announces $(\bar{R}_N, \bar{x}_N, 1, t', j', \varepsilon')$ together with $Q(\bar{R}_N, \omega_N, \bar{x}_N)$ in stage 1, then subcase 2–1 applies and the play proceeds to stage 2–1. In the subgame to follow, agent j' in stage 2–1 obtains $(x_{j'}(t') + \varepsilon' e)$ if he announces $a = 0$, and he obtains $\bar{x}_{j'}$ if he announces $a = 1$. Since $\bar{x}_{j'}\ P_{j'}\ (x_{j'}(t') + \varepsilon' e)$, agent j' must choose $a = 1$ according to his equilibrium strategy for this subgame.

When agent j' announces $a = 1$ in this subgame, agent i' obtains the outcome $(\bar{x}_{i'} + \varepsilon' e)$. Since $\varepsilon' > 0$ and agent i' has a monotone preference relation, agent i' should prefer this outcome to the equilibrium outcome $\bar{x}_{i'}$. □

Lemma 4. *Suppose that all agents in stage 1 announce $(\bar{R}_N, \bar{x}_N, 0)$ such that $\bar{x}_N \in \varphi(\bar{R}_N, \omega_N)$ in subgame-perfect equilibrium. For any sequence*

$$Q(\bar{R}_N, \omega_N, \bar{x}_N) \equiv [\ S(t)\ ,\ \omega_{S(t)}(t)\ ,\ x_{S(t)}(t)\ ]_{t=1,2,\ldots,n} \in \mathcal{Q}(\bar{R}_N, \omega_N, \bar{x}_N),$$

for any $t \in N$, and for any $i \in S(t)$, it is the case that $\bar{x}_i\ I_i\ x_i(t)$.

Proof. This is a direct implication of Lemmas 2 and 3. □

Lemma 5. *Suppose that all agents in stage 1 announce $(\bar{R}_N, \bar{x}_N, 0)$ such that $\bar{x}_N \in \varphi(\bar{R}_N, \omega_N)$ in subgame-perfect equilibrium. For any sequence*

$$Q(\bar{R}_N, \omega_N, \bar{x}_N) \equiv [\ S(t)\ ,\ \omega_{S(t)}(t)\ ,\ x_{S(t)}(t)\ ]_{t=1,2,\ldots,n} \in \mathcal{Q}(\bar{R}_N, \omega_N, \bar{x}_N)$$

and for any $t \in N$, an allocation $x_{S(t)}(t)$ is efficient for $S(t)$ with $R_{S(t)}(t)$ and $\omega_{S(t)}(t)$.

Proof. Suppose, by way of contradiction, that for some sequence $Q(\bar{R}_N, \omega_N, \bar{x}_N)$ and for some $t' \in N$, there exists an allocation $y_{S(t')} \in X_{S(t')}(\omega_{S(t')})$ such that $y_i \; R_i \; x_i(t')$ for all $i \in S(t')$ and $y_j \; P_j \; x_j(t')$ for some $j \in S(t')$. Then, since each agent has a continuous and monotone preference relation, there exists an allocation $y'_{S(t')} \in X_{S(t')}(\omega_{S(t')})$ such that $y'_i \; P_i \; x_i(t')$ for all $i \in S(t')$.[8] Note that $y'_i \, P_i \, \bar{x}_i$ for all $i \in S(t')$ since $\bar{x}_i \, I_i \, x_i(t')$ for all $i \in S(t')$ by Lemma 4. Consider any agent $i' \in S(t')$. We will show that agent i' can gain by a unilateral deviation from the equilibrium path.

If agent i' announces $(\bar{R}_N, \bar{x}_N, 2)$ together with t' and $Q(\bar{R}_N, \omega_N, \bar{x}_N)$ and $y'_{S(t')}$ in stage 1, then subcase 2–2 applies and the play proceeds to stage 2–2. We use backward induction arguments for the subgame that starts at the beginning of stage 2–2 given the above announcement by agent i'.

First, consider any subgame that starts at the beginning of round $|S(t')|$. In this round, agent i on the move obtains y'_i if he announces $b^i = 0$, and he obtains $\bar{x}_i$ if he announces $b^i = 1$. Since $y'_i \; P_i \; \bar{x}_i$, agent i must choose $b^i = 0$ according to his equilibrium strategy for this subgame.

Next, consider any subgame that starts at the beginning of round k such that $k < |S(t')|$. In this round, if agent i on the move announces $b^i = 0$, then the play proceeds to round $k + 1$ and he obtains y'_i eventually. If he announces $b^i = 1$, then he obtains $\bar{x}_i$. Since $y'_i \; P_i \; \bar{x}_i$, agent i must choose $b^i = 0$ according to his equilibrium strategy for this subgame.

The above arguments show that, by a deviation to a path to stage 2–2, agent i' can obtain the outcome $y'_{i'}$ that he should prefer to the equilibrium outcome $\bar{x}_{i'}$. □

Lemma 6. *Suppose that all agents in stage 1 announce $(\bar{R}_N, \bar{x}_N, 0)$ such that $\bar{x}_N \in \varphi(\bar{R}_N, \omega_N)$ in subgame-perfect equilibrium. For any sequence*

$$Q(\bar{R}_N, \omega_N, \bar{x}_N) \equiv [\ S(t)\ ,\ \omega_{S(t)}(t)\ ,\ x_{S(t)}(t)\ ]_{t=1,2,\ldots,n} \in \mathcal{Q}(\bar{R}_N, \omega_N, \bar{x}_N)$$

and for any $t \in N$, an allocation $x_{S(t)}(t)$ is in $\varphi(R_{S(t)}, \omega_{S(t)}(t))$.

Proof. We use induction arguments to show $x_{S(t)}(t) \in \varphi(R_{S(t)}, \omega_{S(t)}(t))$ for $t = n, n-1, \ldots, 1$.

$(t = n)$ Take any sequence $Q(\bar{R}_N, \omega_N, \bar{x}_N) \in \mathcal{Q}(\bar{R}_N, \omega_N, \bar{x}_N)$. Since $|S(t)| = 1$, it is the case that $\varphi(\bar{R}_{S(t)}, \omega_{S(t)}(t)) = \varphi(R_{S(t)}, \omega_{S(t)}(t)) = \{\omega_{S(t)}(t)\}$ by the definition of the ordinal Shapley value. Since $x_{S(t)}(t) \in \varphi(\bar{R}_{S(t)}, \omega_{S(t)}(t))$ by the construction of the sequence, it is the case that $x_{S(t)}(t) \in$

[8] If $|S(t')| = 1$, then $y_j \; P_j \; x_j(t')$ for $j \in S(t') = \{j\}$. If $|S(t')| \geq 2$, then we can construct an allocation $y'_{S(t')} \in X_{S(t')}(\omega_{S(t')})$ such that $y'_i = y_i + \delta e$ for all $i \in S(t') \setminus \{j\}$ and $y'_j = y_j - (|S(t')| - 1)\delta e$ with very small $\delta > 0$. If δ is sufficiently small, then it is the case that $y'_i \; P_i \; x_i(t')$ for all $i \in S(t')$ since each agent has a continuous and monotone preference relation.

$\varphi(R_{S(t)}, \omega_{S(t)}(t))$.

$(t \leq n-1)$ We assume that it has been shown that $x_{S(t+1)}(t+1) \in \varphi(R_{S(t+1)}, \omega_{S(t+1)}(t+1))$ for all $Q(\bar{R}_N, \omega_N, \bar{x}_N) \in \mathcal{Q}(\bar{R}_N, \omega_N, \bar{x}_N)$.

Take any sequence $Q(\bar{R}_N, \omega_N, \bar{x}_N) \in \mathcal{Q}(\bar{R}_N, \omega_N, \bar{x}_N)$. By Lemma 5, we know that $x_{S(t)}(t)$ is efficient for $S(t)$ with $R_{S(t)}(t)$ and $\omega_{S(t)}(t)$. Since $x_{S(t)}(t) \in \varphi(\bar{R}_{S(t)}, \omega_{S(t)}(t))$ by the construction of the sequence, there exists a unique[9] $|S(t)|$-tuple of concession vectors, $(c^i)_{i \in S(t)}$ with $c^i \equiv (c^i_j)_{j \in S(t) \setminus \{i\}} \in \mathbb{R}^{|S(t)|-1}$ for each $i \in S(t)$, and the concession vectors satisfy the consistency condition with respect to $\bar{R}_{S(t)}(t)$, and the fairness condition, $\sum_{j \in S(t) \setminus \{i\}} c^i_j = \sum_{j \in S(t) \setminus \{i\}} c^j_i$ for all $i \in S(t)$.[10] Considering the definition of the ordinal Shapley value, we are left to show that the concession vectors satisfy the consistency condition with respect to $R_{S(t)}(t)$ in order to prove that $x_{S(t)}(t) \in \varphi(R_{S(t)}, \omega_{S(t)}(t))$. The following claim shows the result and completes the proof of the present lemma.

Claim. *For each $i \in S(t)$, there exists $x'_{S(t) \setminus \{i\}} \in \varphi(R_{S(t) \setminus \{i\}}, (\omega_j(t) + c^i_j e)_{j \in S(t) \setminus \{i\}})$ such that $x_j(t)\, I_j\, x'_j$ for all $j \in S(t) \setminus \{i\}$.*

Proof of Claim. Take any $i' \in S(t)$ and choose a sequence

$$Q'(\bar{R}_N, \omega_N, \bar{x}_N) \equiv [\ S'(t')\ ,\ \omega'_{S'(t')}(t')\ ,\ x'_{S'(t')}(t')\ ]_{t'=1,2,\ldots,n} \in \mathcal{Q}(\bar{R}_N, \omega_N, \bar{x}_N)$$

such that $[S'(t'), \omega'_{S'(t')}(t'), x'_{S'(t')}(t')] = [S(t'), \omega_{S(t')}(t'), x_{S(t')}(t')]$ for all $t' \leq t$ and $S'(t+1) = S'(t) \setminus \{i'\}$. By the construction of this sequence, it is the case that $\omega'_j(t+1) = \omega'_j(t) + c^{i'}_j e$ for all $j \in S'(t+1)$. By the induction assumption, $x'_{S'(t+1)}(t+1) \in \varphi(R_{S'(t+1)}, \omega'_{S'(t+1)}(t+1))$. Lemma 4 implies that $x_j(t)\, I_j\, \bar{x}_j\, I_j\, x'_j(t+1)$ for all $j \in S'(t+1)$. Therefore, we have shown that $x'_{S'(t+1)}(t+1) \in \varphi(R_{S(t) \setminus \{i'\}}, (\omega_j(t) + c^{i'}_j e)_{j \in S(t) \setminus \{i'\}})$ and $x_j(t)\, I_j\, x'_j(t+1)$ for all $j \in S(t) \setminus \{i'\}$. Note that $x'_{S'(t+1)}(t+1)$ can be interpreted as $x'_{S(t) \setminus \{i\}}$ that appears in the statement of the claim. Since the choice of $i' \in S(t)$ is arbitrary, the claim holds. □

Proposition 1. *$SPE(\Gamma(\omega_N), R_N) \subset \varphi(R_N, \omega_N)$ for all $R_N \in \mathcal{R}_N$.*

Proof. Suppose that we are given $R_N \in \mathcal{R}_N$. Choose any arbitrary subgame-perfect equilibrium allocation $\bar{x}_N \in SPE(\Gamma(\omega_N), R_N)$. By Lemma 1, we may assume that all agents in stage 1 announce $(\bar{R}_N, \bar{x}_N, 0)$ such that $\bar{x}_N \in \varphi(\bar{R}_N, \omega_N)$ for some $\bar{R}_N \in \mathcal{R}_N$. Choose any sequence $Q(\bar{R}_N, \omega_N, \bar{x}_N) \equiv [S(t), \omega_{S(t)}(t), x_{S(t)}(t)]_{t=1,2,\ldots,n} \in \mathcal{Q}(\bar{R}_N, \omega_N, \bar{x}_N)$. By Lemma 6, $x_{S(1)}(1) \in \varphi(R_{S(1)}, \omega_{S(1)}(1))$. Since

[9] The uniqueness is due to Fact 3.

[10] The fairness condition is independent of $\bar{R}_{S(t)}(t)$.

$S(1) = N$, $\omega_{S(1)}(1) = \omega_N$, and $x_{S(1)}(1) = \bar{x}_N$ by the construction of the sequence, it is the case that $\bar{x}_N \in \varphi(R_N, \omega_N)$. $\square$

3–3. Proof: $\boldsymbol{\varphi(R_N, \omega_N) \subset SPE(\Gamma(\omega_N), R_N)}$ for all $\boldsymbol{R_N \in \mathcal{R}_N}$.

Choose any arbitrary $\omega_N \in \mathbb{R}^{\ell n}$ and $R_N \in \mathcal{R}_N$ and fix them throughout this section. This R_N is regarded as a 'true' preference profile of agents. Furthermore, choose any $\bar{x}_N \in \varphi(R_N, \omega_N)$. This section proves that $\bar{x}_N \in SPE(\Gamma(\omega_N), R_N)$.

We now define a strategy profile $m_N \in M_N$ that attains $\bar{x}_N$ as a subgame-perfect equilibrium allocation for the game $(\Gamma(\omega_N), R_N)$.

The strategy profile $\boldsymbol{m_N \in M_N}$.

Stage 1. For each agent i, his announcement is such that

$$(R_N^i, x_N^i, z^i, t^i, j^i, \varepsilon^i) = (R_N, \bar{x}_N, 0, 1, 1, 1)$$

together with an arbitrary sequence $Q^i(R_N, \omega_N, \bar{x}_N) \in \mathcal{Q}(R_N, \omega_N, \bar{x}_N)$ and an arbitrary feasible allocation $y_N^i \in X_N(\omega_N)$.

Stage 2–1. For each subgame that starts at the beginning of stage 2–1, agent $j^{i^*} \in S^{i^*}(t^{i^*})$ on the move announces $a = 0$ if $(x_{j^{i^*}}^{i^*}(t^{i^*}) + \varepsilon^{i^*} e)\ P_{j^{i^*}}\ \bar{x}_{j^{i^*}}$, and $a = 1$ otherwise.

Round $\boldsymbol{j \in \{1, 2, \ldots, |S^{i^*}(t^{i^*})|\}}$ in stage 2–2. For each subgame that starts at the beginning of round j in stage 2–2, agent $\bar{j}$ on the move in round j announces $b^{\bar{j}} = 0$ if $y_{\bar{j}}^{i^*}\ P_{\bar{j}}\ \bar{x}_{\bar{j}}$, and $b^{\bar{j}} = 1$ otherwise.

Lemma 7. *For each subgame that starts at the beginning of stage 2–1, the strategy profile induced by m_N is a Nash equilibrium.*

Proof. In this subgame, there is only one agent, agent $j^{i^*} \in S^{i^*}(t^{i^*})$, who is on the move. If he announces $a = 0$, then the outcome for him is $x_{j^{i^*}}^{i^*}(t^{i^*}) + \varepsilon^{i^*} e$. If he announces $a = 1$, then the outcome for him is $\bar{x}_{j^{i^*}}$.

Therefore, if $(x_{j^{i^*}}^{i^*}(t^{i^*}) + \varepsilon^{i^*} e)\ P_{j^{i^*}}\ \bar{x}_{j^{i^*}}$, then it is a best response for him to follow $m_{j^{i^*}}$ and announce $a = 0$. If $\bar{x}_{j^{i^*}}\ R_{j^{i^*}}\ (x_{j^{i^*}}^{i^*}(t^{i^*}) + \varepsilon^{i^*} e)$, then it is a best response for him to follow $m_{j^{i^*}}$ and announce $a = 1$. $\square$

Lemma 8. *For each subgame that starts at the beginning of round* $j \in \{1, 2, \ldots, |S^{i^*}(t^{i^*})|\}$ *in stage 2–2, the strategy profile induced by* m_N *is a Nash equilibrium.*

Proof. Consider any agent $\bar{j}$ who is on the move in any round j' with $j' \geq j$ in this subgame. If he announces $b^{\bar{j}} = 0$, then the outcome for him is either $y_{\bar{j}}^{i^*}$ or $\bar{x}_{\bar{j}}$, depending on the announcements by the agents in the subsequent rounds. If he announces $b^{\bar{j}} = 1$, then the outcome for him is $\bar{x}_{\bar{j}}$.

Therefore, if $y_{\bar{j}}^{i^*} \; P_{\bar{j}} \; \bar{x}_{\bar{j}}$, then it is a best response for him to follow $m_{\bar{j}}$ and announce $b^{\bar{j}} = 0$. If $\bar{x}_{\bar{j}} \; R_{\bar{j}} \; y_{\bar{j}}^{i^*}$, then it is a best response for him to follow $m_{\bar{j}}$ and announce $b^{\bar{j}} = 1$.

Since the above arguments apply for any agent on the move in this subgame, the strategy profile induced by m_N for this subgame is a Nash equilibrium. □

Lemma 9. *The strategy profile* m_N *is a Nash equilibrium of the game* $(\Gamma(\omega_N), R_N)$.

Proof. Note that the outcome allocation for the strategy profile m_N is $\bar{x}_N$ since case 1 applies in stage 1.

Consider any agent $i \in N$. We will prove that m_i is a best response to $m_{N\setminus\{i\}}$ by showing that agent i cannot obtain an outcome better than $\bar{x}_i$ even if he unilaterally quits following m_i and changes his announcements when he is on the move.

By Lemma 8, we know that in any subgame that starts in stage 2–2, agent i's best response is to follow m_i. So, we may assume that agent i moves according to m_i in stage 2–2. Henceforth, we investigate whether agent i can obtain an outcome better than $\bar{x}_i$ by changing his announcements in stage 1. By such a deviation, agent i can become agent i^* and obtain an outcome realized in either subcase 2–1, 2–2, 2–3, or 2–4.

First, suppose that subcase 2–1 applies due to agent i^*'s deviation. In stage 2–1 to follow, agent j^{i^*} moves. We note that $i^* \neq j^{i^*}$ since $i^* \notin S^{i^*}(t^{i^*})$ and $j^{i^*} \in S^{i^*}(t^{i^*})$. By the construction of the sequence $Q^{i^*}(R_N, \omega_N, \bar{x}_N)$ announced by agent i^* in this subcase, it is the case that $\bar{x}_{j^{i^*}} \; I_{j^{i^*}} \; x_{j^{i^*}}^{i^*}(t^{i^*})$. Since $\varepsilon^{i^*} > 0$ and agent j^{i^*} has a monotone preference relation, it is the case that $(x_{j^{i^*}}^{i^*}(t^{i^*}) + \varepsilon^{i^*} e) \, P_{j^{i^*}} \; \bar{x}_{j^{i^*}}$ and hence agent j^{i^*} announces $a = 0$ according to $m_{j^{i^*}}$. Therefore, the outcome for agent i^* is $(\bar{x}_{i^*} - \varepsilon^{i^*} e)$, which is worse than $\bar{x}_{i^*}$ since agent i^* has a monotone preference relation.

Second, suppose that subcase 2–2 applies due to agent i^*'s deviation. In stage 2–2 to follow, agents in $S^{i^*}(t^{i^*})$ move sequentially. We now pay attention to $x_{S^{i^*}(t^{i^*})}^{i^*}(t^{i^*})$ in the sequence $Q^{i^*}(R_N, \omega_N, \bar{x}_N)$ announced by agent i^* in this subcase. Since $x_{S^{i^*}(t^{i^*})}^{i^*}(t^{i^*}) \in \varphi(R_{S^{i^*}(t^{i^*})}, \omega_{S^{i^*}(t^{i^*})}(t^{i^*}))$, the definition of the ordinal Shapley value tells us that $x_{S^{i^*}(t^{i^*})}^{i^*}(t^{i^*})$ is efficient for $S^{i^*}(t^{i^*})$ with $R_{S^{i^*}(t^{i^*})}$ and

$\omega_{S^{i^*}(t^{i^*})}(t^{i^*})$. Therefore, by comparing agent i^*'s announcements of $x^{i^*}_{S^{i^*}(t^{i^*})}(t^{i^*})$ and $y^{i^*}_{S^{i^*}(t^{i^*})} \in X_{S^{i^*}(t^{i^*})}(\omega_{S^{i^*}(t^{i^*})}(t^{i^*}))$, we can say that there exists some agent $j \in S^{i^*}(t^{i^*})$ such that $x^{i^*}_j(t^{i^*})\, R_j\, y^{i^*}_j$. Furthermore, by the construction of the sequence $Q^{i^*}(R_N, \omega_N, \bar{x}_N)$, it is the case that $\bar{x}_j\, I_j\, x^{i^*}_j(t^{i^*})$. Therefore, we can say that there exists some agent $j \in S^{i^*}(t^{i^*})$ such that $\bar{x}_j\, R_j\, y^{i^*}_j$. This agent j announces $b^j = 1$ according to m_j when he is on the move in stage 2–2. So, it cannot happen that every agent j' in $S^{i^*}(t^{i^*})$ announces $b^{j'} = 0$ sequentially in stage 2–2. Therefore, the outcome allocation is $\bar{x}_N$ and the outcome for agent i^* is $\bar{x}_{i^*}$.

Third, suppose that subcase 2–3 applies due to agent i^*'s deviation. In this subcase, the outcome for agent i^* is $x^{i^*}_{i^*}(t^{i^*})$. By the construction of the sequence $Q^{i^*}(R_N, \omega_N, \bar{x}_N)$ announced by agent i^* in this subcase, it is the case that $\bar{x}_{i^*}\, I_{i^*}\, x^{i^*}_{i^*}(t^{i^*})$.

Fourth, suppose that subcase 2–4 applies due to agent i^*'s deviation. In this subcase, the outcome allocation is $\bar{x}_N$ and the outcome for agent i^* is $\bar{x}_{i^*}$.

The above investigations show that agent i^*, namely agent i, cannot obtain an outcome better than $\bar{x}_i$ even if he unilaterally quits following m_i and changes his announcements when he is on the move. Since the above arguments apply not only for agent i but also for any other agent, the strategy profile m_N is a Nash equilibrium of the game $(\Gamma(\omega_N), R_N)$. □

Proposition 2. *$\varphi(R_N, \omega_N) \subset SPE(\Gamma(\omega_N), R_N)$ for all $R_N \in \mathcal{R}_N$.*

Proof. Suppose that we are given $R_N \in \mathcal{R}_N$. Choose any arbitrary ordinal Shapley value allocation $\bar{x}_N \in \varphi(R_N, \omega_N)$. Consider the strategy profile $m_N \in M_N$ defined in this section. Lemmas 7 through 9 show that m_N is a subgame-perfect equilibrium of the game $(\Gamma(\omega_N), R_N)$ and its outcome allocation is $\bar{x}_N$. Therefore, $\bar{x}_N \in SPE(\Gamma(\omega_N), R_N)$. □

Propositions 1 and 2 complete the proof of the theorem.

4. Conclusion

We have proposed a game form that implements the ordinal Shapley value in subgame-perfect equilibrium. Our game form can be used as a tool for the social planner who wishes to realize the ordinal Shapley value but does not possess information on preference relations of agents. Our game form achieves full implementation. That is, not only every allocation in the ordinal Shapley value can be realized as an equilibrium allocation, but also every equilibrium allocation is in fact an allocation in the ordinal Shapley value.

We have closely followed Perez-Castrillo and Wettstein [2004] for modelling pure exchange economies where the ordinal Shapley value is defined. We consider that one drawback of the model is the assumption that each agent's consumption space is not restricted to the non-negative orthant. This assumption seems necessary for proving the existence of the ordinal Shapley value, but the assumption is not standard in most microeconomics textbooks. It would be desirable if we can drop this assumption and restrict each agent's consumption space to the non-negative orthant, and still guarantee the existence of the ordinal Shapley value and design a game form achieving its implementation.

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