α-core of an NTU-game with Directed Link Communication System

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journal or publication title	The Economic Review of Toyo University
volume	41
number	1
page range	159-169
year	2015-12
URL	http://id.nii.ac.jp/1060/00008067/

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Abstract

We consider a non-transferable utility game derived from a strategic form game with asymmetric information where players can exchange their information through the directed link communication system and define the solution concept of α-core in the game. Our main purpose is to prove the nonemptiness of the solution conditional on the property of the directed link communication system.

Keywords: NTU-game derived from strategic form game, directed link communication system, α-core

JEL classification: C79, D82, D85

1 Introduction

This seminal paper is in the literature of the researches on the core of games with asymmetric information. We consider a strategic form game with asymmetric information where players can exchange their information through the directed link communication system. We derive the non-transferable utility game (NTU-game) from the strategic form game above and define the solution concept of core. Our main purpose is to examine the nonemptiness of the core in relation to the property of the directed link communication system.

In the literature, Scarf (1971) proves the nonemptiness of the core of the NTU-game which is derived from the strategic form game in a symmetric information case using the balancedness condition. Wilson (1978) considers the asymmetric information case and defined the cores with two information exchange patterns. The coarse core is based on the idea the members in a coalition can use only the common information among them, while the fine core is based on the idea they can exchange information fully. The nonemptiness of the coarse core is shown in the general settings. Note that Wilson (1978) considers interim decision making in contrast that this paper considers ex-ante decision making. Yannelis (1991) provides the case where each player uses his or her private information independent of the coalition he or she belongs to. The nonemptiness of the private core is shown in his paper.

Maus (2003) proves the nonemptiness of the core concept with a more general information exchange which is called the communication system. It exogenously defines each player's information dependent on the coalition he or she belongs to. In his paper, the NTU-game is derived from an exchange economy, and the nonemptiness of the core concept is proved if a communication system is nested.[1] Hirase and Utsumi (2005) extend the seminal work by Maus (2003). They derive the NTU-game from a strategic form game with communication system and show the nonemptiness of the core concept based on the nestedness.

Hirase (2009) examines the partial cooperation situation by using the network [2]. The model deals with the game where players can cooperate and exchange information only through the network which is called network communication system and given exogenously. It is proved that the core concept is nonempty if network communication system is nested. Hirase (2015) focuses on the properties of the links of the network communication system and examines the nonemptiness of the core concept.

In this seminal work, we would like to consider directed link communication system in contrast that Hirase (2015) considers non-directed links. We examine the properties of the directed links in the communication system, define the core concept with it, and prove its nonemptiness. For this purpose, we define the strategic form game with the "directed" link communication system and derive the NTU-game from it. Then we define the core concept and prove the nonemptiness of it conditional on the properties of the directed link communication system.

In Section 2, we define the game, the properties of the directed link communication system, and the core concept. Section 3 analyzes the core with a directed link communication system and gives a sufficient condition for its nonemptiness. Section 4 presents some examples. In Section 5, we discuss

[1] Nestedness of the communication system means that larger coalition gives members more information.
[2] The idea is based on Myerson (1977).

the concluding remarks and the future problems.

2 The Model

The model is based on Hirase and Utsumi (2005) and Hirase (2015). The difference is that we use "directed " links in contrast to non-directed links in the model of Hirase (2015). This section is organized as follows. We start with the definition of the strategic form game with directed link communication system where players can exchange their information through the direced links. Second, we define the NTU-game derived from the strategic form game with directed link communication system. Third, we define the α-core of the NTU-game above and discuss the properties of the link communication system.

Let $N = \{1, ..., n\}$ be a nonempty finite set of *players*. A *coalition* S is a subset of N. A *directed graph* (or network) g on N is a set of directed links between the players, i. e., $g \subset \Big\{(i, j)|i, j \in N, i \neq j\Big\}$. We consider the situation in which players in a coalition can exchange their information through these directed links. We discuss how the information exchange occurs in Section 3. We call an element of g a *directed link*. (i, j) means the directed link from player i to player j. Through this directed link, player i can tell his or her information to player j. Let G be the set of all directed graphs on N.

Note that every player can make a coalition with any player even if there is not a directed link between them, while they cannot exchange their information without the directed link between them.

Now we define the strategic form game with directed link communication system.

Definition 1. A *strategic form game with directed link communication system* Γ is a following list of data $(N, g, \Omega, \{A_i, u_i\}_{i\in N}, \{\mathcal{P}_i^{S,g}\}_{i\in N, S\subset N})$.

- $N = \{1, ..., n\}$ is the set of players.
- g is a directed graph on N.
- $\Omega = \{\omega_1, ..., \omega_l\}$ is the finite state space.
- $A_i \subset \Re^{m_i}$ is the set of actions for player i. We assume A_i is a non-empty, convex, and compact set.
- $u_i : \prod_{i\in N}(A_i^\Omega) \to \Re$ is player i's payoff function. We assume u_i is a continuous and quasi-concave function on $\prod_{i\in N}(A_i^\Omega)$.

- $\{\mathcal{P}_i^{S,g}\}_{i\in S, S\subset N}$ determines each player's information which we call the *directed link communication system* for g where $\mathcal{P}_i^{S,g}$ is a partition of Ω for all $S \subset N$ and $i \in S$.

A_i^Ω is the set of functions from Ω to A_i. We denote A_i^Ω by Σ_i which we call *the universal strategy set for player i*. $\mathcal{P}_i^{S,g}$ is the information of player i in S when the coalition S is formed.

Definition 2. The set of *player i's strategies for information* $\mathcal{P}_i^{S,g}$ is defined as follows.

$$\Sigma_i^{S,g} := \{\sigma_i^S \in \Sigma_i | \sigma_i^S \text{ is } \mathcal{P}_i^{S,g}\text{-measurable}\}.$$

We require this measurability condition because each player cannot take different actions at ω and ω' if he or she cannot distinct a state ω from another state ω' with the information $\mathcal{P}_i^{S,g}$.

We describe the set of joint strategies of the coalition S as $\Sigma^S = \prod_{i\in S} \Sigma_i^S$. The typical element of Σ^S is denoted as σ^S. For all $R \subset N$, a partition on R is interpreted as a *coalition structure of R* . We denote the set of all coalition structure of R as $P(R)$. That means, for $R \subset N$, $P(R)$ is defined by

$$P(R) := \left\{ \{S_1, ..., S_L\} \;\middle|\; \bigcup_{l=1}^{L} S_l = R \text{ and } S_l \cap S_m = \emptyset, \text{ for all } l, m \in \{1, ..., L\} \right\}.$$

Using these notations and these definitions, we derive an NTU-game from a strategic form game with directed link communication system Γ as follows.

Definition 3. We define *the NTU characteristic function V derived from a strategic form game with directed link communication system* Γ as follows.

$$V(S) := \bigcup_{\sigma^S \in \Sigma^S} \bigcap_{Q \in P(N\setminus S)} \bigcap_{\substack{(\sigma^T)_{T\in Q} \\ \in (\Sigma^T)_{T\in Q}}} \left\{ (u_1, ..., u_n) \in \Re^n \;\middle|\; \forall i \in S,\ u_i \leq u_i(\sigma^S, (\sigma^T)_{T\in Q}) \right\}$$

for all $S \in \mathcal{N}$.

$V(S)$ means the set of payoff profiles, which the players in a coalition S can obtain at least, even if the worst situation (coalition structure and strategies of $N \setminus S$) for S occurs. The idea is suggested by Aumann and Peleg (1960).

Before we define the solution concept of the NTU-game, we give the definition of the improvement. We say that *a coalition S improves upon the payoff vector u in* $\Re^n$ if u is in $\mathrm{int}V(S)$. This means the coalition S has an incentive to deviate from the situation that gives the players the payoff profiles u.

All members of S can be better off by the deviation. We can say u is not stable in this sense, and u can not be a candidate for a solution concept. We define the solution concept of the NTU-game which is robust for these deviations.

Definition 4. For an NTU characteristic function V, we define *the α-core with directed link communication system* $C(V)$ as

$$C(V) := V(N) \setminus \bigcup_{T \subset N} \mathrm{int} V(T).$$

The α-core with directed link communication system is the set of the payoff profiles that is achieved by the grand coalition and is not improved upon by any coalition of the players. Without confusion, we use the word just "core" to refer to the α-core with directed link communication system.

3 The Results

We analyze the relation between the property of the directed link communication system and the nonemptiness of the core in this section.

We provide the definition of the nestedness of the directed link communication system.[3]

Definition 5. A directed link communication system $\{\mathcal{P}_i^{S,g}\}_{i \in S, S \subset N}$ is *nested* if for all S and $T \subset \mathcal{N}$ such that $S \subset T$,

$$\mathcal{P}_i^{S,g}(\omega) \supset \mathcal{P}_i^{T,g}(\omega) \quad \text{for all } i \in S \text{ and } \omega \in \Omega.$$

The nestedness implies that each player in the larger coalition has the richer information. Hirase and Utsumi (2005) prove the nonemptiness of the core dependent on the nestedness of the communication system. By applying their theorem to our model, the following proposition holds.

Proposition 1. The α-core with directed link communication system of V is non-empty if the directed link communication system is nested.

We would like to examine how the nestedness is achieved by focusing the role of the directed links in the communication system. As same as Hirase (2015), we regard $\mathcal{P}_i^{\{i\},g}$ as player i's initial information

[3] Nestedness is originally defined by Maus (2003)

partition and $\mathcal{P}_j^{S,g}$ as the result of the information exchange within the coalition S.

Definition 6. A directed link communication system $\{\mathcal{P}_i^{S,g}\}_{i\in S, S\subset N}$ is *fining* if for all $i \in S$ and $S \subset N$,

$$\mathcal{P}_i^{S,g}(\omega) = \bigvee_{j\in S,(j,i)\in g} \mathcal{P}_j^{\{j\},g} \bigvee \mathcal{P}_i^{\{i\},g}.$$

This property shows the fine case by Wilson (1978). Player i can make use of the information of player j if both players belong to the same coalition and there is a directed link from j to i in S. In this case, the information is conveyed by one link. Hence, this directed link communication system can be interpreted as a result of a short-time communication. We define the case in which information is conveyed by multiple links below.

Suppose that g is in G, i and j are in N, and S is a coalition. *j is connectable from i in S by g* if there is a sequence $(n_0, ..., n_K)$ such that $n_0 = i$, $n_K = j$, and $n_k \in S$ and $(n_{k-1}, n_k) \in g$ for all $k \in \{1, ..., K\}$. We denote the set of the players from whom player i is connectable in S by g as $j(i, S, g)$.

Definition 7. A directed link communication system $\{\mathcal{P}_i^{S,g}\}_{i\in S, S\subset N}$ is *fining through players* if for all $i \in S$ and $S \subset \mathcal{N}$,

$$\mathcal{P}_i^{S,g}(\omega) = \bigvee_{j\in j(i,S,g)} \mathcal{P}_j^{\{j\},g} \bigvee \mathcal{P}_i^{\{i\},g}.$$

This property means that players' information conveyed by directed links go through the players in a coalition. This kind of directed link communication system can be interpreted as a result of a long-time communication within the coalitions.

We can check both fining and fining through players communication systems are nested.

Lemma.

- If a directed link communication system is fining, then it is nested.
- If a directed link communication system is fining through players, then it is nested.

Combining the lemmas and Proposition 1, we can obtain that Proposition 2 and Proposition 3 hold.

Proposition 2. The α-core with directed link communication system of the NTU-game V is non-empty if the link communication system is fining.

Proposition 3. The α-core with directed link communication system of the NTU-game V is nonempty if the link communication system is fining through players.

We can interpret the propositions as follows. Whichever the information exchange within a coalition is short-time or long-time, if the directed links convey the information and every player can make use of it, the communication system is nested, and hence the core is nonempty.

We can define the coarse information exchange which is originally introduced by Wilson (1978).

Definition 8. A directed link communication system $\{\mathcal{P}_i^{S,g}\}_{i\in S, S\subset N}$ is *coarsening* if for all $i \in S$ and $S \subset N$,

$$\mathcal{P}_i^{S,g}(\omega) = \bigwedge_{j\in S,(j,i)\in g} \mathcal{P}_j^{\{j\},g} \bigwedge \mathcal{P}_i^{\{i\},g}.$$

Definition 9. A directed link communication system $\{\mathcal{P}_i^{S,g}\}_{i\in S, S\subset N}$ is *coarsening through players* if for all $i \in S$ and $S \subset \mathcal{N}$,

$$\mathcal{P}_i^{S,g}(\omega) = \bigwedge_{j\in j(i,S,g)} \mathcal{P}_j^{\{j\},g} \bigwedge \mathcal{P}_i^{\{i\},g}.$$

In these cases, we can not necessarily obtain the nonemptiness of the core. Example 5 in the next section shows a counterexample. [4]

4 Examples

Example 1 displays a fining and fining through players link communication system, Example 2 shows a coarsening and coarsening through players link communication system, and Example 3 is a counterexample mentioned in the last paragraph in the previous section.

Example 1.

The following shows a directed link communication system which provides only the singleton coalitions' information.

$$N = \{1,2,3\}, g = \{(1,2),(2,3)\}, \Omega = \{\omega_1,\omega_2,\omega_3\}$$

$$\mathcal{P}_1^{\{1\},g} = \{\{\omega_1\},\{\omega_2,\omega_3\}\}, \mathcal{P}_2^{\{2\},g} = \{\{\omega_2\},\{\omega_1,\omega_3\}\}, \text{and} \mathcal{P}_3^{\{3\},g} = \{\Omega\}.$$

[4] Wilson (1978) proves the nonemptiness of the coarse core considering the interim decision making, while this paper considers the ex-ante decision making.

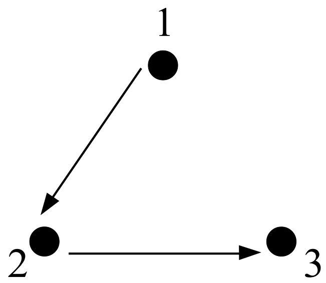

Figure 1:

Figure 1 shows the directed graph of the situation.

Suppose that the directed link communication system is fining. Then the information of the non-singleton coalitions are determined as follows.

$$\mathcal{P}_1^{\{1,2\},g} = \{\{\omega_1\},\{\omega_2,\omega_3\}\}, \quad \mathcal{P}_2^{\{1,2\},g} = \{\{\omega_1\},\{\omega_2\},\{\omega_3\}\},$$
$$\mathcal{P}_2^{\{2,3\},g} = \{\{\omega_2\},\{\omega_1,\omega_3\}\}, \quad \mathcal{P}_3^{\{2,3\},g} = \{\{\omega_2\},\{\omega_1,\omega_3\}\},$$
$$\mathcal{P}_1^{\{1,3\},g} = \{\{\omega_1\},\{\omega_2,\omega_3\}\}, \quad \mathcal{P}_3^{\{1,3\},g} = \{\Omega\},$$
$$\mathcal{P}_1^{N,g} = \{\{\omega_1\},\{\omega_2,\omega_3\}\}, \quad \mathcal{P}_2^{N,g} = \{\{\omega_1\},\{\omega_2\},\{\omega_3\}\}, \quad \text{and} \quad \mathcal{P}_3^{N,g} = \{\{\omega_2\},\{\omega_1,\omega_3\}\}.$$

Player 2 (3) can make use of the information from Player 1 (2) if they are in the same coalition.

Suppose that the directed link communication system is fining through players. Then the information of the non-singleton coalitions are determined as follows.

$$\mathcal{P}_1^{\{1,2\},g} = \{\{\omega_1\},\{\omega_2,\omega_3\}\}, \quad \mathcal{P}_2^{\{1,2\},g} = \{\{\omega_1\},\{\omega_2\},\{\omega_3\}\},$$
$$\mathcal{P}_2^{\{2,3\},g} = \{\{\omega_2\},\{\omega_1,\omega_3\}\}, \quad \mathcal{P}_3^{\{2,3\},g} = \{\{\omega_2\},\{\omega_1,\omega_3\}\},$$
$$\mathcal{P}_1^{\{1,3\},g} = \{\{\omega_1\},\{\omega_2,\omega_3\}\}, \quad \mathcal{P}_3^{\{1,3\},g} = \{\Omega\},$$
$$\mathcal{P}_1^{N,g} = \{\{\omega_1\},\{\omega_2,\omega_3\}\}, \quad \mathcal{P}_2^{N,g} = \{\{\omega_1\},\{\omega_2\},\{\omega_3\}\}, \quad \text{and} \quad \mathcal{P}_3^{N,g} = \{\{\omega_2\},\{\omega_2\},\{\omega_3\}\}.$$

Note that Player 3 in the grand coalition can make use of the information from Player 1 which is conveyed through Player 2. □

Example 2.

The following also shows a directed link communication system which provides only the singleton coalitions' information.

$$N = \{1,2,3\}, g = \{(1,2),(2,3)\}, \Omega = \{\omega_1, \omega_2, \omega_3\}$$
$$\mathcal{P}_1^{\{1\},g} = \{\Omega\}, \mathcal{P}_2^{\{2\},g} = \{\{\omega_1, \omega_2,\}\{\omega_3\}\}, \text{and} \quad \mathcal{P}_3^{\{3\},g} = \{\{\omega_1, \omega_2,\}\{\omega_3\}\}.$$

The directed graph is same as in Figure 1.

Suppose that the directed link communication system is coarsening. Then the information in the non-singleton coalitions are determined as follows.

$$\mathcal{P}_1^{\{1,2\},g} = \{\Omega\}, \quad \mathcal{P}_2^{\{1,2\},g} = \{\Omega\},$$
$$\mathcal{P}_2^{\{2,3\},g} = \{\{\omega_1, \omega_2,\}\{\omega_3\}\}, \mathcal{P}_3^{\{2,3\},g} = \{\{\omega_1, \omega_2,\}\{\omega_3\}\},$$
$$\mathcal{P}_1^{\{1,3\},g} = \{\Omega\}, \quad \mathcal{P}_3^{\{1,3\},g} = \{\{\omega_1, \omega_2,\}\{\omega_3\}\},$$
$$\mathcal{P}_1^{N,g} == \{\Omega\}, \quad \mathcal{P}_2^{N,g} = \{\Omega\}, \quad \text{and} \quad \mathcal{P}_3^{N,g} = \{\{\omega_1, \omega_2,\}\{\omega_3\}\}.$$

Suppose that the directed link communication system is coarsening through players. Then the information of the non-singleton coalitions are determined as follows.

$$\mathcal{P}_1^{\{1,2\},g} = \{\Omega\}, \quad \mathcal{P}_2^{\{1,2\},g} = \{\Omega\},$$
$$\mathcal{P}_2^{\{2,3\},g} = \{\{\omega_1, \omega_2,\}\{\omega_3\}\}, \mathcal{P}_3^{\{2,3\},g} = \{\{\omega_1, \omega_2,\}\{\omega_3\}\},$$
$$\mathcal{P}_1^{\{1,3\},g} = \{\Omega\}, \quad \mathcal{P}_3^{\{1,3\},g} = \{\{\omega_1, \omega_2,\}\{\omega_3\}\},$$
$$\mathcal{P}_1^{N,g} == \{\Omega\}, \quad \mathcal{P}_2^{N,g} = \{\Omega\}, \quad \text{and} \quad \mathcal{P}_3^{N,g} = \{\Omega\}.$$

Note that Player 3 in the grand coalition is confused by the information from Player 1 which is conveyed through Player 2. □

Example 3. The following coarse (through players) link communication system does not ensure the nonemptiness of the core, which is a counterexample mentioned in the last paragraph in the previous section.

$$N = \{1,2\}, g = \{\{1,2\}\}, \Omega = \{\omega_1, \omega_2\}$$
$$\mathcal{P}_1^{\{1\},g} = \{\Omega\}, \mathcal{P}_2^{\{2\},g} = \{\{\omega_1\},\{\omega_2\}, \text{ and,}$$

$$\mathcal{P}_1^N = \mathcal{P}_2^N = \{\Omega\}.$$

Let players' strategy sets and payoffs be defined as the expected payoffs in the following functions.

$$A_1 = A_2 = (0,1)$$

$$u_1(\sigma_1, \sigma_2) = \sigma_1(\omega_1) - \sigma_1(\omega_2), \forall\ \sigma_1 \in \Sigma_1 \text{ and } \sigma_2 \in \Sigma_2$$

$$u_2(\sigma_1, \sigma_2) = \sigma_2(\omega_1) - \sigma_2(\omega_2), \forall\ \sigma_1 \in \Sigma_1 \text{ and } \sigma_2 \in \Sigma_2$$

In this case, both players in the grand coalition obtain the payoff profile $(0,0)$, since each player have to take the same actions at every state.

Note that the singleton coalition $\{1\}$ can improves upon $(0,0)$, because Player 1 obtains the payoff 1 when he or she takes $\sigma_1(\omega_1) = 1$ and $\sigma_1(\omega_2) = 0$. Hence, Player 1 has an incentive to deviate from the grand coalition and the core of the game is empty. □

5 Concluding Remarks

Our main result can be regarded as the extensions of some seminal works in the literature. Our model is an asymmetric information version of Scarf (1971), our framework allows partial communication of players which is not considered by Hirase (2009) and Utsumi (2005), our communication system in this paper is more descriptive than the network communication system of Hirase (2009), and the model by Hirase (2015) is a special case of our model. When the condition if $(i,j) \in g$ then $(j,i) \in g$ for all i and j in N is satisfied, our model is same as the model by Hirase (2015).

We can consider the following points as future problems.

Our directed links of g are given exogenously. We would like to describe how the directed links are formed, that is, to consider the model in which the structure of the links is endogenously determined. Considering asymmetric information versions of the models by Jackson and Wolinsky (1996) and Watts (2001) can be the case.

Information exchange with our directed link communication system does not necessarily satisfy incentive compatible constraints of the players. We would like to define and examine incentive compatible solution concept. Applying the ideas by Forges and Minelli (2001) and Yazar (2001) to our model can be the case.

Our model and the examples are rather abstract. We would also like to apply our model to the situation which has more economic implications, for example, oligopoly market, principal-agent model, and etc.

References

Aumann, R.J., and B. Peleg. [1960], "Von Neumann-Morgenstern solutions to cooperative games without side payments," *Bulletin of the American Mathematical Society* Vol. 66 (3), pp.173–177.

Forges, F., and E. Minelli. [2001], "A Note on the Incentive Compatible Core," *Journal of Economic Theory* Vol. 98 (1), pp.179–188.

Hirase, K., and Y. Utsumi. [2005], "Communication system and Ex Ante α-core," *Mita journal of Economics* Vol. 98 (1), pp.43–53.(in Japanese)

Hirase, K. [2009], "The core with a network communication system," *The Economic Review of Toyo University* Vol. 34, pp.79–88.

Hirase, K. [2015], "The αcore-core of an NTU-game with link communication system," *The Economic Review of Toyo University* Vol. 40 (2), pp.317–326.

Jackson, M. O., and A. Wolinsky. [1996], "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory* Vol. 71(1), pp.44–74.

Jackson, M. O., and A. Watts. [2002], "The Evolution of Social and Economic Networks," *Journal of Economic Theory* Vol. 106 (2), pp.265–295.

Maus, S. [2003], "Balancedness and the core in economies with asymmetric information," *Economic Theory* Vol. 22 (3), pp.613–627.

Myerson, R. [1977], "Graphs and cooperation in games," *Mathematics of Operations Research* Vol. 2 (1), pp.225–229.

Scarf, H. E. [1971], "On the Existence of a Cooperative Solution for a General Class of N-Person Games," *Journal of Economic Theory* Vol. 3 (2), pp.169–181.

Wilson, R. [1978], "Information, Efficiency, and the core of an economy," *Econometrica* Vol. 46, pp. 807–816.

Yannelis, N.C. [1991], "The core of an economy with differential information," *Economic Theory* Vol. 1, pp. 183–198.

Yazar, J. [2001], "Ex ante contracting with endogenously determined communication plans," *Economic Theory* Vol. 18 (2), pp.439–450.