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Yusuke Samejima

Abstract

We improve on the result by Samejima [2013] and show that in two-player voluntary contribution games as analyzed by Compte and Jehiel [2003], very small uncertainties about opponent players' valuations of completing a project can cause gradualism in accumulation of contributions. Our study differs from Compte and Jehiel [2003] in that our games are played in incomplete information environments where there is a chance for each player to be either a high valuation type or a low valuation type. In such environments, Samejima [2013] shows that, if the prior probability of the opponent player being a high-type is below a certain *upper bound* for both players, and if players are sufficiently patient, there exists a perfect Bayesian equilibrium in which step-by-step contributions are realized along the equilibrium path. This gradual accumulation of contributions is not observed in Compte and Jehiel's equilibrium in complete information environments. In the present paper, we remove the upper bound condition on the prior probabilities in Samejima [2013]. Our result indicates that very small uncertainties about valuations held by the opponent players can be a source of the gradualism.

1. Introduction

Contribution games have been studied in various aspects in the literature. Admati and Perry [1991] have investigated games in which contributions for a joint project are sunk. They show that there exists a subgame-perfect equilibrium in which contributions are made in small steps along the equilibrium path. Such gradual accumulation of contributions is referred to as *gradualism.* Their result of gradualism is obtained under the assumptions that a cost function for the project is arbitrarily convex and valuations of the project are the same between two players. They have suggested that the sunk character of contributions is a source of the gradualism.

Revisiting Admati and Perry's contribution games, Compte and Jehiel [2003] have pointed out that their result of gradualism depends on the convexity of the cost function and the symmetry of the valuations. Compte and Jehiel have introduced a linear cost function and asymmetric valuations into Admati and Perry's contribution games, and show that there exists a unique subgame-perfect equilibrium, in which at most two large contributions are realized. So, the gradualism observed in Admati and Perry [1991] has disappeared due to the linear cost function and the asymmetric valuations. In the proof of Compte and Jehiel's result, they heavily use the assumption under complete information environments: Each player knows his opponent's valuation of the project.

Seeking after sources of gradualism, Samejima [2013] has introduced uncertainties about valuations into Compte and Jehiel's model.[1] In Samejima's model, there is a chance for each player to be either a high valuation type or a low valuation type. Each player is informed of his own valuation but not of his opponent's valuation: He just knows the prior probability of his opponent being a high-type. Samejima shows that, if this prior probability is below a certain *upper bound* for both players, and if players are sufficiently patient, then there exists a perfect Bayesian equilibrium in which step-by-step contributions are realized along the equilibrium path.

In the present paper, we improve on the result by Samejima [2013]: We remove the upper bound condition on the prior probabilities. We regard this condition as a limitation to some extent because the upper bound becomes lower as the difference between the valuations held by a high-type and a low-type becomes larger.

To illustrate the limitation imposed by the upper bound condition, we briefly discuss the model. Suppose that two agents 1 and 2 want to complete a joint project that requires a total amount K of contributions. On completion of the project, each agent obtains a benefit, which is either a high value H or a low value L. Each agent i knows the prior probability P_j of his opponent j being a high-type. The condition $K < 2L$ is assumed so that completing the project is efficient even if both agents are low-types. Samejima [2013] shows that there is an equilibrium in which two agents contribute alternately in small steps until the project is completed if $P_i < 2L/(H + L)$ for $i = 1, 2$, where the right hand side of the inequality is the upper bound. For example, given $K = 99$, $H = 150$, and $L = 50$, the upper bound is $1/2$, which is in fact a limitation.

However, the present paper has succeeded in removing the upper bound condition. According to

[1] Miyagawa and Samejima [2009] have also introduced uncertainties about valuations into Compte and Jehiel's model. In their model, one player has a chance to be either a *high* valuation type or a *zero* valuation type, and the other player has a chance to be either a *low* valuation type or a *zero* valuation type. So, their way of introducing uncertainties differs from that of Samejima [2013].

our result, even if $P_1 = 0.9999$ and $P_2 = 0.0001$, that is, even if agent 1 is almost a high-type and agent 2 is almost a low-type but there still remain very small uncertainties, and if agents are sufficiently patient, then there exists a perfect Bayesian equilibrium in which step-by-step contributions are realized. Furthermore, for any contribution sequence, we can find an equilibrium that realizes the contribution sequence up to the step just before the last step for completing the project.

Since any contribution sequence can be realized along some equilibrium path in our model, there is a chance that almost equal cost-sharing is achieved. This is a big difference from the result by Compte and Jehiel [2003]: If there are no uncertainties about valuations held by the opponent players, then unfair cost-sharing is realized in most cases. For the above numerical example, if agent 1 is a high-type and agent 2 is a low-type with no uncertainties, then agent 1 bears all the costs K and agent 2 contributes nothing in the unique subgame-perfect equilibrium. This cost-sharing is unfair.

The remaining part of this paper is organized as follows. Section 2 explains a model of two-player contribution games under incomplete information. Section 3 proves that there exists a perfect Bayesian equilibrium in which gradualism is observed. Section 4 provides some concluding remarks.

2. The Model

We investigate two-player voluntary contribution games similar to the ones studied by Compte and Jehiel [2003].

Two agents, agents 1 and 2, are the players of the game. They contribute alternately to complete a project, which costs $K > 0$. Upon an immediate completion of the project, agent i obtains a benefit V_i, which is called agent i's *valuation* of the project. At the beginning of the game, the nature decides whether each agent i's valuation is a high value H or a low value L, that is, $V_i \in \{H, L\}$ where $H > L > 0$. So, agent i's valuation V_i also represents his *type*: Agent i with $V_i = H$ is a *high-type* and agent i with $V_i = L$ is a *low-type*. Let $P_i \in (0, 1)$ denote the *prior probability* that $V_i = H$ is drawn by the nature. We assume that P_1 and P_2 are independent. Furthermore, we assume that P_1 and P_2 are common knowledge while the realized value of V_i is known only to agent i.

The game is played in periods $t = 1, 2, \ldots$, where agent 1 moves in periods with odd numbers while agent 2 moves in periods with even numbers until the project is completed. Let $m(t)$ denote the *mover* in period t, that is, $m(t) = 1$ if t is a positive odd number while $m(t) = 2$ if t is a positive even number. Let $c_i^t \geq 0$ denote the amount of *contribution* by agent i in period t. Since two agents take turns in making contributions, $c_i^t = 0$ if $i \neq m(t)$: This constraint on (c_1^t, c_2^t) together with $c_1^0 = c_2^0 = 0$ for notational convenience is called the *feasibility* for (c_1^t, c_2^t). At the end of period t, (c_1^t, c_2^t) is observed by both agents. Let

$$x^t = K - \sum_{\tau=0}^{t}(c_1^\tau + c_2^\tau)$$

be the *remaining amount* necessary for completion of the project at the end of period t. Note that $x^0 = K$ and $(x^0, x^1, x^2, \ldots)$ is a non-increasing sequence. When the remaining amount reaches 0, the project is completed and the game ends. Let T denote the *period of completion* of the project, that is, T is the least natural number that satisfies the condition $x^T \leq 0$. If the project is not completed forever due to an insufficient amount of contributions, then we let $T = \infty$. We assume that contributions are non-refundable even if the project is not completed. So, contributions become sunk costs for agents.

Let $h^t = (x^0, x^1, \ldots, x^{t-1})$ denote a *history* at the beginning of period t. We denote h^{t+1} by (h^t, x^t). Agent i's behavior *strategy* s_i is a function that specifies a probability distribution over contribution amounts for each type of i and for each history: $s_i(c_i^t|V_i, h^t)$ is the probability of choosing c_i^t given V_i and $h^t = (x^0, x^1, \ldots, x^{t-1})$ with $x^{t-1} > 0$.[2] By the feasibility for (c_1^t, c_2^t), we require that $s_i(0|V_i, h^t) = 1$ if $i \neq m(t)$.

On reaching a history $h^t = (x^0, x^1, \ldots, x^{t-1})$ with $x^{t-1} > 0$, agent i holds a *belief* $p_i(h^t)$, which represents the probability that agent i assigns to the event where his opponent is a high-type given h^t. We call p_i agent i's *belief function*. Given the common prior (P_1, P_2), we assume that $p_1(h^1) = P_2$ and $p_2(h^1) = P_1$.

Both agents discount benefits and contributions using a *discount factor* $\delta \in (0, 1)$. When agent i's type is V_i, his payoff for a contribution sequence $c = \{(c_1^t, c_2^t)\}_{t=0}^{T}$ is given by

$$U_i(V_i, c) = \delta^{T-1} V_i - \sum_{t=0}^{T} \delta^{t-1} c_i^t.$$

We assume that agents maximize expected payoffs. Let $u_i(s|V_i, h^t, p_i)$ be the *expected payoff* of agent i with type V_i under a strategy profile $s = (s_1, s_2)$ on condition that he reaches a history h^t with a belief $p_i(h^t)$.

We look for perfect Bayesian equilibria of the game. We follow Fudenberg and Tirole [1991a, 1991b] for the definition of the equilibria. In the present model, a *perfect Bayesian equilibrium* (s, p) is a pair of a strategy profile $s = (s_1, s_2)$ and a belief function profile $p = (p_1, p_2)$ that satisfies the following two conditions.

Sequential Rationality. For all h^t, $i = m(t)$, $j \neq i$, V_i, and s_i', we have $u_i(s|V_i, h^t, p_i) \geq u_i((s_i', s_j)|V_i, h^t, p_i)$.

[2] For the definitions of strategies and belief functions, we require that $x^{t-1} > 0$ because otherwise, the game must have ended before period t.

Reasonability. Bayes' rule is used to update beliefs whenever possible: For all i, $j \neq i$, $h^t = (x^0, \ldots, x^{t-1})$, and (c_1^t, c_2^t) satisfying the feasibility, if $p_i(h^t)s_j(c_j^t|H, h^t) > 0$ or $(1-p_i(h^t))s_j(c_j^t|L, h^t) > 0$, then

$$p_i((h^t, x^t)) = \frac{p_i(h^t)s_j(c_j^t|H, h^t)}{p_i(h^t)s_j(c_j^t|H, h^t) + (1-p_i(h^t))s_j(c_j^t|L, h^t)}$$

where $x^t = x^{t-1} - (c_1^t + c_2^t) > 0$.

We note that the reasonability condition does not impose any constraint on agent i's belief $p_i((h^t, x^t))$ if $s_j(c_j^t|H, h^t) = s_j(c_j^t|L, h^t) = 0$. That is, if it is agent j that moves at h^t and if j chooses c_j^t that should have zero probability for both types of j according to s_j, then agent i's belief at $h^{t+1} = (h^t, x^t)$ can be completely arbitrary.

3. The Result

Let $P_i \in (0,1)$ be given for $i = 1, 2$. We assume that $K < 2L$, which means that even if both agents are low-types, total benefits exceed the cost. Choose any contribution sequence $\{(\bar{c}_1^t, \bar{c}_2^t)\}_{t=0}^{\bar{t}}$ satisfying the feasibility and the following conditions: $\bar{c}_i^t > 0$ if $i = m(t)$, $\sum_{t=1}^{\bar{t}} \bar{c}_i^t < L$ for all i, and $\sum_{t=0}^{\bar{t}}(\bar{c}_1^t + \bar{c}_2^t) = K$. Define a history $\bar{h}^{\bar{t}+1} = (\bar{x}^0, \bar{x}^1, \ldots, \bar{x}^{\bar{t}})$ corresponding to $\{(\bar{c}_1^t, \bar{c}_2^t)\}_{t=0}^{\bar{t}}$. Since $\bar{c}_1^t + \bar{c}_2^t > 0$ for $t = 1, 2, \ldots, \bar{t}$, we have $K = \bar{x}^0 > \bar{x}^1 > \cdots > \bar{x}^{\bar{t}-1} > \bar{x}^{\bar{t}} = 0$. Furthermore, define a history $\bar{h}^{\bar{t}} = (\bar{x}^0, \bar{x}^1, \ldots, \bar{x}^{\bar{t}-1})$ corresponding to $\{(\bar{c}_1^t, \bar{c}_2^t)\}_{t=0}^{\bar{t}-1}$. Note that the project is not completed on reaching the history $\bar{h}^{\bar{t}}$. We want to show that there exists an equilibrium in which the history $\bar{h}^{\bar{t}}$ is realized along the equilibrium path. That is, we want to show that there exists an equilibrium that realizes the contribution sequence $\{(\bar{c}_1^t, \bar{c}_2^t)\}_{t=0}^{\bar{t}-1}$, which may exhibit gradualism since the choice of $\{(\bar{c}_1^t, \bar{c}_2^t)\}_{t=0}^{\bar{t}}$ is arbitrary to a certain extent.

Theorem. *If $\delta \in (0,1)$ is sufficiently large, then there exists a perfect Bayesian equilibrium (s, p) in which the history $\bar{h}^{\bar{t}}$ is realized along the equilibrium path.*

To prove the theorem, we first choose $\bar{\delta} \in (0,1)$ satisfying the following conditions for all i and V_i.

$$K < \bar{\delta}^2 V_i + \bar{\delta} L + (1-\bar{\delta}^2)(1-\bar{\delta})H, \quad (1)$$

$$\bar{c}_1^{\bar{t}} + \bar{c}_2^{\bar{t}} > (1-\bar{\delta}^2)H, \quad (2)$$

$$\bar{\delta}^{\bar{t}-t+1}(V_i - (1-\bar{\delta})H) - \sum_{\tau=t}^{\bar{t}} \bar{\delta}^{\tau-t}\bar{c}_i^\tau > \bar{\delta}^3(V_i - L) \text{ for all } t = 1, 2, \ldots, \bar{t}, \quad (3)$$

$$\bar{\delta}^{\bar{t}-t+1}(V_i - (1-\bar{\delta})H) - \sum_{\tau=t}^{\bar{t}} \bar{\delta}^{\tau-t}\bar{c}_i^\tau > \bar{\delta}V_i + (1-\bar{\delta})H - \bar{x}^{t-1} \text{ for all } t = 1, 2, \ldots, \bar{t}-1. \quad (4)$$

Lemma 1. *If $\bar{\delta} \in (0,1)$ is sufficiently large, then $\bar{\delta}$ satisfies all the conditions* (1) – (4) for all i and V_i.

Proof. The right hand side (RHS, henceforth) of the inequality (1) converges to $V_i + L$ as $\bar{\delta}$ converges to 1. Since $K < 2L \leq V_i + L$, the condition (1) holds for large $\bar{\delta}$.

As for the inequality (2), RHS converges to 0 as $\bar{\delta}$ converges to 1. Since $\bar{c}_1^{\bar{t}} + \bar{c}_2^{\bar{t}} > 0$ by the choice of $\{(\bar{c}_1^t, \bar{c}_2^t)\}_{t=0}^{\bar{t}}$, the condition (2) holds for large $\bar{\delta}$.

As for the inequality (3), we consider the left hand side (LHS, henceforth) subtracted by RHS. As $\bar{\delta}$ converges to 1, (LHS $-$ RHS) converges to

$$(V_i - \sum_{\tau=t}^{\bar{t}} \bar{c}_i^{\tau}) - (V_i - L) = L - \sum_{\tau=t}^{\bar{t}} \bar{c}_i^{\tau},$$

which is positive since $\sum_{t=1}^{\bar{t}} \bar{c}_i^t < L$ by the choice of $\{(\bar{c}_1^t, \bar{c}_2^t)\}_{t=0}^{\bar{t}}$. So, the condition (3) holds for large $\bar{\delta}$.

As for the inequality (4), we note that as $\bar{\delta}$ converges to 1, (LHS $-$ RHS) converges to the following:

$$\bar{x}^{t-1} - \sum_{\tau=t}^{\bar{t}} \bar{c}_i^{\tau} = (K - \sum_{\tau=0}^{t-1}(\bar{c}_1^{\tau} + \bar{c}_2^{\tau})) - \sum_{\tau=t}^{\bar{t}} \bar{c}_i^{\tau} = (K - \sum_{\tau=0}^{\bar{t}}(\bar{c}_1^{\tau} + \bar{c}_2^{\tau})) + \sum_{\tau=t}^{\bar{t}} \bar{c}_j^{\tau} = \sum_{\tau=t}^{\bar{t}} \bar{c}_j^{\tau}.$$

Since we may assume $t \leq \bar{t} - 1$ for the inequality (4), we have $\sum_{\tau=t}^{\bar{t}} \bar{c}_j^{\tau} > 0$ by the choice of $\{(\bar{c}_1^t, \bar{c}_2^t)\}_{t=0}^{\bar{t}}$. So, the condition (4) holds for large $\bar{\delta}$. □

We next define a strategy profile $s = (s_1, s_2)$ and a belief function profile $p = (p_1, p_2)$. Recall that $s_i(c_i^t | V_i, h^t)$ represents the probability of choosing $c_i^t \geq 0$ given $V_i \in \{H, L\}$ and $h^t = (x^0, x^1, \ldots, x^{t-1})$ with $x^{t-1} > 0$. We also recall that $p_i(h^t)$ represents the probability that agent i assigns to the event where his opponent is a high-type given h^t.

To describe the strategies and the belief functions, let us define a *deviator function* $d(h^t)$ as follows. Given a history $h^t = (x^0, x^1, \ldots, x^{t-1})$ with $x^{t-1} > 0$, if $x^{\tau} = \bar{x}^{\tau}$ for all $\tau = 0, 1, \ldots, t-1$, then let $d(h^t) = 0$; Otherwise, let $d(h^t) = m(\hat{\tau})$ where $\hat{\tau}$ is the least τ that satisfies $x^{\tau} \neq \bar{x}^{\tau}$. If $d(h^t) = 0$, then we say that h^t is *on-path* and there is no deviator; Otherwise, we say that h^t is *off-path* and agent $d(h^t)$ is the *deviator*. If h^t is off-path and $i \neq d(h^t)$, then we call agent i the *punisher*. We note that if h^t is off-path, it must be the case that $t \geq 2$ because h^1 is always on-path due to the fact that $x^0 = \bar{x}^0 = K$.

Strategy s_i for $i = 1, 2$.

Let $V_i \in \{H, L\}$ and $h^t = (x^0, x^1, \ldots, x^{t-1})$ with $x^{t-1} > 0$ be given. When $i \neq m(t)$, we require that $s_i(0|V_i, h^t) = 1$ [3] by the feasibility for (c_1^t, c_2^t). When $i = m(t)$, the following descriptions define $s_i(\cdot|V_i, h^t)$.

The on-path case. If $d(h^t) = 0$, then we define $s_i(\cdot|V_i, h^t)$ as follows.[4]

If $t \leq \bar{t} - 1$, then let $s_i(c_i^t|V_i, h^t) = 1$ for $c_i^t = \bar{c}_i^t$.

If $t = \bar{t}$ and $V_i = H$, then let $s_i(c_i^t|V_i, h^t) = 1$ for $c_i^t = \bar{c}_i^{\bar{t}}$.[5]

If $t = \bar{t}$ and $V_i = L$, then let $s_i(c_i^t|V_i, h^t) = 1$ for $c_i^t = \bar{c}_i^{\bar{t}} - (1 - \bar{\delta})H$.[6]

The deviator's case. If $i = d(h^t)$, then we define $s_i(\cdot|V_i, h^t)$ as follows.

If $V_i = H$ and $x^{t-1} \leq \bar{\delta}^3 L + (1 - \bar{\delta}^3)H$, then let $s_i(c_i^t|V_i, h^t) = 1$ for $c_i^t = x^{t-1}$.

If $V_i = H$ and $x^{t-1} > \bar{\delta}^3 L + (1 - \bar{\delta}^3)H$, then let $s_i(c_i^t|V_i, h^t) = 1$ for $c_i^t = 0$.

If $V_i = L$ and $x^{t-1} \leq (1 - \bar{\delta})L$, then let $s_i(c_i^t|V_i, h^t) = 1$ for $c_i^t = x^{t-1}$.

If $V_i = L$ and $(1 - \bar{\delta})L < x^{t-1} \leq (1 - \bar{\delta})H$, then let $s_i(c_i^t|V_i, h^t) = 1$ for $c_i^t = 0$.

If $V_i = L$ and $(1-\bar{\delta})H < x^{t-1} \leq \bar{\delta}L + (1-\bar{\delta})H$, then let $s_i(c_i^t|V_i, h^t) = 1$ for $c_i^t = x^{t-1} - (1-\bar{\delta})H$.

If $V_i = L$ and $x^{t-1} > \bar{\delta}L + (1 - \bar{\delta})H$, then let $s_i(c_i^t|V_i, h^t) = 1$ for $c_i^t = 0$.

The punisher's case. If h^t is off-path and $i \neq d(h^t)$, then we define $s_i(\cdot|V_i, h^t)$ as follows.

If $x^{t-1} \leq (1 - \bar{\delta})H$, then let $s_i(c_i^t|V_i, h^t) = 1$ for $c_i^t = x^{t-1}$.

If $(1 - \bar{\delta})H < x^{t-1} \leq \bar{\delta}L + (1 - \bar{\delta})H$, then let $s_i(c_i^t|V_i, h^t) = 1$ for $c_i^t = 0$.

If $x^{t-1} > \bar{\delta}L + (1 - \bar{\delta})H$, then let $s_i(c_i^t|V_i, h^t) = 1$ for $c_i^t = x^{t-1} - \bar{\delta}L - (1 - \bar{\delta})H$.

Belief function p_i for $i = 1, 2$.

Let $h^t = (x^0, x^1, \ldots, x^{t-1})$ with $x^{t-1} > 0$ be given. For $t = 1$, let $p_1(h^1) = P_2$ and $p_2(h^1) = P_1$. For $t \geq 2$, we define $p_i(h^t)$ as follows. When $i = m(t-1)$, let $p_i(h^t) = p_i(h^{t-1})$. When $i = m(t)$, the following descriptions define $p_i(h^t)$.

The on-path case. If $d(h^t) = 0$, then let $p_i(h^t) = P_j$ where $j \neq i$.

The deviator's case. If $i = d(h^t)$, then let $p_i(h^t) = P_j$ where $j \neq i$.

The punisher's case. If h^t is off-path and $i \neq d(h^t)$, then define $p_i(h^t)$ as follows.

If $x^{t-1} \leq (1 - \bar{\delta})H$, then let $p_i(h^t) = 0$.

If $(1 - \bar{\delta})H < x^{t-1} \leq \bar{\delta}L + (1 - \bar{\delta})H$, then let $p_i(h^t) = 1$.

If $x^{t-1} > \bar{\delta}L + (1 - \bar{\delta})H$, then let $p_i(h^t) = 0$.

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[3] This expression implicitly states that $s_i(c_i^t|V_i, h^t) = 0$ for $c_i^t > 0$.

[4] The following three cases are exhaustive because if $t > \bar{t}$, then it must be the case that $d(h^t) \neq 0$.

[5] Note that $\bar{c}_i^{\bar{t}} = \bar{x}^{\bar{t}-1}$ by the definitions of $\bar{x}^{\bar{t}-1}$ and $\{(\bar{c}_1^t, \bar{c}_2^t)\}_{t=0}^{\bar{t}}$.

[6] We note that $\bar{c}_i^{\bar{t}} - (1 - \bar{\delta})H > \bar{c}_i^{\bar{t}} - (1 - \bar{\delta}^2)H > 0$ due to the condition (2) in Lemma 1 together with the fact that $\bar{c}_j^{\bar{t}} = 0$ in this case.

We now prove that (s, p) satisfies sequential rationality in the series of lemmas. Take any $i \in \{1, 2\}$, $V_i \in \{H, L\}$, $t \in \{1, 2, \ldots\}$, and $h^t = (x^0, x^1, \ldots, x^{t-1})$ with $x^{t-1} > 0$, and fix them for the arguments in Lemmas 2 through 13. Let $\{(c_1^\tau, c_2^\tau)\}_{\tau=0}^{t-1}$ be the contribution sequence corresponding to h^t. When we consider a contribution sequence for period t and beyond, $\{(c_1^\tau, c_2^\tau)\}_{\tau=t}^{T}$, we denote the combined sequence by $c = \{(c_1^\tau, c_2^\tau)\}_{\tau=0}^{T}$ for notational convenience. We assume that c satisfies the feasibility, and we note that, by its construction, c is consistent with h^t.

Let $\bar{s}_i$ be agent i's optimal strategy on condition that (s_j, V_i, h^t, p_i) is given:

$$\bar{s}_i \in \arg\max_{s_i'} u_i((s_i', s_j)|V_i, h^t, p_i).$$

We want to show that $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$ in exhaustive cases, which implies that $u_i(s|V_i, h^t, p_i) \geq u_i((s_i', s_j)|V_i, h^t, p_i)$ for all s_i'.

Lemma 2. *When agent i moves given an off-path history, that is, when h^t is off-path and $i = m(t)$, and if $x^{t-1} \leq (1 - \bar{\delta})V_i$, then $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$.*

Proof. When agent i follows s_i, he chooses $c_i^t = x^{t-1}$ with probability one in period t regardless of his type and whether he is the deviator or the punisher, and the game ends with his payoff

$$u_i(s|V_i, h^t, p_i) = \bar{\delta}^{t-1}(V_i - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau.$$

Let $c = \{(c_1^\tau, c_2^\tau)\}_{\tau=0}^{T}$ be such that positive probability is assigned to c by the probability distribution generated by $p_i(h^t)$ and $(\bar{s}_i, s_j)$ given h^t. We show that agent i's payoff for c does not exceed $u_i(s|V_i, h^t, p_i)$ in exhaustive cases, which implies that $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$.

If $c_i^t \geq x^{t-1}$, then the game ends in period t according to c, and we have

$$U_i(V_i, c) = \bar{\delta}^{t-1}(V_i - c_i^t) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq \bar{\delta}^{t-1}(V_i - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|V_i, h^t, p_i),$$

where the weak inequality holds since $c_i^t \geq x^{t-1}$.

If $c_i^t < x^{t-1}$, then the game continues with $x^t = x^{t-1} - c_i^t$ according to c. In this case, the most preferred scenario for agent i is that agent j contributes all the remaining amount x^t and completes the project in period $(t + 1)$. Even if this scenario applies to c,

$$U_i(V_i, c) = \bar{\delta}^{t-1}(\bar{\delta} V_i - c_i^t) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq \bar{\delta}^{t-1}(V_i - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|V_i, h^t, p_i),$$

where the weak inequality holds since $c_i^t \geq 0$ and $x^{t-1} \leq (1-\bar{\delta})V_i$. □

Lemma 3. *When agent i moves as the deviator, that is, when $i = m(t) = d(h^t) \neq j$, and if $V_i = H$ and $(1-\bar{\delta})H < x^{t-1} \leq \bar{\delta}L + (1-\bar{\delta})H$, then $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$.*

Proof. When high-type deviator i follows s_i, he chooses $c_i^t = x^{t-1}$ with probability one in period t and the game ends with his payoff

$$u_i(s|H, h^t, p_i) = \bar{\delta}^{t-1}(H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau.$$

Let $c = \{(c_1^\tau, c_2^\tau)\}_{\tau=0}^T$ be such that positive probability is assigned to c by the probability distribution generated by $p_i(h^t) = P_j$ and $(\bar{s}_i, s_j)$ given h^t. We show that deviator i's payoff for c does not exceed $u_i(s|H, h^t, p_i)$ in exhaustive cases, which implies that $u_i(s|H, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|H, h^t, p_i)$.

If the project is not completed according to c, that is, if $\sum_{\tau=0}^T (c_1^\tau + c_2^\tau) < K$, then we have

$$\begin{aligned} U_i(H, c) &= -\sum_{\tau=t}^T \bar{\delta}^{\tau-1} c_i^\tau - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq 0 - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq \bar{\delta}^{t-1}(\bar{\delta}L + (1-\bar{\delta})H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \\ &< \bar{\delta}^{t-1}(H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|H, h^t, p_i), \end{aligned}$$

where the second weak inequality holds since $x^{t-1} \leq \bar{\delta}L + (1-\bar{\delta})H$. So, in the remaining part of the proof of the lemma, we consider the cases where the project is completed according to c.

We next show that $c_i^t \geq x^{t-1} - (1-\bar{\delta})H$. If $x^\tau > (1-\bar{\delta})H$ for any $\tau \geq t$, then we have $x^\tau \leq x^{t-1} \leq \bar{\delta}L + (1-\bar{\delta})H$, and punisher j chooses $c_j^{\tau+1} = 0$ in period $(\tau+1)$ as described in the punisher's case of the strategy because otherwise, zero probability is assigned to c. Since the project is completed according to c while punisher j never contributes as long as the remaining amount exceeds $(1-\bar{\delta})H$, it must be the case that deviator i pays at least the difference between x^{t-1} and $(1-\bar{\delta})H$, possibly in one time or in several times. If deviator i should pay an amount necessary for reaching some remaining amount $\hat{x}$ no more than $(1-\bar{\delta})H$, it is optimal for him to do so in one time, because punisher j's strategy does not depend on the path from x^{t-1} to $\hat{x}$ but on the remaining amount $\hat{x}$ itself, and because delaying completion of the project lowers the discounted benefit. Therefore, we must have $c_i^t \geq x^{t-1} - (1-\bar{\delta})H$.

If $c_i^t \geq x^{t-1}$, then the game ends in period t according to c, and we have

$$U_i(H, c) = \bar{\delta}^{t-1}(H - c_i^t) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq \bar{\delta}^{t-1}(H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|H, h^t, p_i),$$

where the weak inequality holds since $c_i^t \geq x^{t-1}$.

If $x^{t-1} > c_i^t \geq x^{t-1} - (1-\bar{\delta})H$, the game continues with $x^t = x^{t-1} - c_i^t \leq (1-\bar{\delta})H$ according to c. Then, punisher j chooses $c_j^{t+1} = x^t$ in period $(t+1)$ as described in the punisher's case of the strategy because otherwise, zero probability is assigned to c. So, the game ends in period $(t+1)$, and we have

$$U_i(H,c) = \bar{\delta}^{t-1}(\bar{\delta}H - c_i^t) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq \bar{\delta}^{t-1}(H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|H, h^t, p_i),$$

where the weak inequality holds since $c_i^t \geq x^{t-1} - (1-\bar{\delta})H$. □

Lemma 4. *When agent i moves as the deviator, that is, when $i = m(t) = d(h^t) \neq j$, and if $V_i = H$ and $\bar{\delta}L + (1-\bar{\delta})H < x^{t-1} \leq \bar{\delta}^3 L + (1-\bar{\delta}^3)H$, then $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$.*

Proof. When high-type deviator i follows s_i, he chooses $c_i^t = x^{t-1}$ with probability one in period t and the game ends with his payoff

$$u_i(s|H, h^t, p_i) = \bar{\delta}^{t-1}(H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau.$$

Let $c = \{(c_1^\tau, c_2^\tau)\}_{\tau=0}^T$ be such that positive probability is assigned to c by the probability distribution generated by $p_i(h^t) = P_j$ and $(\bar{s}_i, s_j)$ given h^t. We show that deviator i's payoff for c does not exceed $u_i(s|H, h^t, p_i)$ in exhaustive cases, which implies that $u_i(s|H, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|H, h^t, p_i)$.

If the project is not completed according to c, that is, if $\sum_{\tau=0}^T (c_1^\tau + c_2^\tau) < K$, then we have

$$U_i(H,c) = -\sum_{\tau=t}^{T} \bar{\delta}^{\tau-1} c_i^\tau - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq 0 - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau < \bar{\delta}^{t-1}(H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|H, h^t, p_i),$$

where the strict inequality holds since $x^{t-1} \leq \bar{\delta}^3 L + (1-\bar{\delta}^3)H < H$. So, in the remaining part of the proof of the lemma, we consider the cases where the project is completed according to c.

If $c_i^t < x^{t-1} - \bar{\delta}L - (1-\bar{\delta})H$, the game continues with $x^t = x^{t-1} - c_i^t > \bar{\delta}L + (1-\bar{\delta})H$ according to c. Then, punisher j in period $(t+1)$ chooses $c_j^{t+1} = x^t - \bar{\delta}L - (1-\bar{\delta})H$ as described in the punisher's case of the strategy because otherwise, zero probability is assigned to c. According to c, the game further continues with $x^{t+1} = \bar{\delta}L + (1-\bar{\delta})H$. For the continuation game from period $(t+2)$, it is optimal for high-type deviator i to follow s_i by Lemma 3. So, we have

$$U_i(H,c) \leq u_i(s|H, (h^t, x^t, x^{t+1}), p_i) = \bar{\delta}^{t+1}(H - x^{t+1}) - \sum_{\tau=0}^{t+1} \bar{\delta}^{\tau-1} c_i^\tau$$

$$
\begin{aligned}
&= \bar{\delta}^{t+1}(H - \bar{\delta}L - (1-\bar{\delta})H) - c_i^{t+1} - c_i^t - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq \bar{\delta}^{t-1}(\bar{\delta}^3 H - \bar{\delta}^3 L) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \\
&\leq \bar{\delta}^{t-1}(H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|H, h^t, p_i),
\end{aligned}
$$

where the weak inequality on the second line holds since $c_i^t \geq 0$ and $c_i^{t+1} = 0$, and the weak inequality on the third line holds since $x^{t-1} \leq \bar{\delta}^3 L + (1-\bar{\delta}^3)H$.

If $c_i^t \geq x^{t-1} - \bar{\delta}L - (1-\bar{\delta})H$, then the same arguments as in the proof of Lemma 3 can show that $c_i^t \geq x^{t-1} - (1-\bar{\delta})H$. Furthermore, by repeating the arguments in the proof of Lemma 3 for the cases $c_i^t \geq x^{t-1}$ and $x^{t-1} > c_i^t \geq x^{t-1} - (1-\bar{\delta})H$, we obtain $U_i(H, c) \leq u_i(s|H, h^t, p_i)$. □

Lemma 5. *When agent i moves as the deviator, that is, when $i = m(t) = d(h^t) \neq j$, and if $V_i = H$ and $x^{t-1} > \bar{\delta}^3 L + (1-\bar{\delta}^3)H$, then $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$.*

Proof. When high-type deviator i follows s_i, he chooses $c_i^t = 0$ with probability one in period t and the game continues with $x^t = x^{t-1} > \bar{\delta}^3 L + (1-\bar{\delta}^3)H > \bar{\delta}L + (1-\bar{\delta})H$. When punisher j follows s_j given $h^{t+1} = (h^t, x^t)$, he chooses $c_j^{t+1} = x^t - \bar{\delta}L - (1-\bar{\delta})H$ with probability one in period $(t+1)$ as described in the punisher's case of the strategy. The game further continues with $x^{t+1} = \bar{\delta}L + (1-\bar{\delta})H$, and high-type deviator i, following s_i given $h^{t+2} = (h^t, x^t, x^{t+1})$, chooses $c_i^{t+2} = x^{t+1}$ with probability one in period $(t+2)$ and the game ends. So, we have

$$
\begin{aligned}
u_i(s|H, h^t, p_i) &= \bar{\delta}^{t-1}(\bar{\delta}^2 H - \bar{\delta}^2 x^{t+1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \\
&= \bar{\delta}^{t-1}(\bar{\delta}^2 H - \bar{\delta}^2(\bar{\delta}L + (1-\bar{\delta})H)) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = \bar{\delta}^{t-1}(\bar{\delta}^3 H - \bar{\delta}^3 L) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau.
\end{aligned}
$$

Let $c = \{(c_1^\tau, c_2^\tau)\}_{\tau=0}^T$ be such that positive probability is assigned to c by the probability distribution generated by $p_i(h^t) = P_j$ and $(\bar{s}_i, s_j)$ given h^t. We show that deviator i's payoff for c does not exceed $u_i(s|H, h^t, p_i)$ in exhaustive cases, which implies that $u_i(s|H, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|H, h^t, p_i)$.

If the project is not completed according to c, that is, if $\sum_{\tau=0}^T (c_1^\tau + c_2^\tau) < K$, then we have

$$
U_i(H, c) = -\sum_{\tau=t}^{T} \bar{\delta}^{\tau-1} c_i^\tau - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq 0 - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau < \bar{\delta}^{t-1}(\bar{\delta}^3 H - \bar{\delta}^3 L) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|H, h^t, p_i).
$$

So, in the remaining part of the proof of the lemma, we consider the cases where the project is completed according to c.

If $c_i^t < x^{t-1} - \bar{\delta}L - (1-\bar{\delta})H$, then the same arguments as in the proof of Lemma 4 can show that

$x^{t+1} = \bar{\delta}L + (1-\bar{\delta})H$ and

$$\begin{aligned} U_i(H,c) &\leq u_i(s|H,(h^t,x^t,x^{t+1}),p_i) = \bar{\delta}^{t+1}(H - x^{t+1}) - \sum_{\tau=0}^{t+1} \bar{\delta}^{\tau-1} c_i^\tau \\ &= \bar{\delta}^{t+1}(H - \bar{\delta}L - (1-\bar{\delta})H) - c_i^{t+1} - c_i^t - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \\ &\leq \bar{\delta}^{t-1}(\bar{\delta}^3 H - \bar{\delta}^3 L) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|H,h^t,p_i), \end{aligned}$$

where the weak inequality on the third line holds since $c_i^t \geq 0$ and $c_i^{t+1} = 0$.

If $c_i^t \geq x^{t-1} - \bar{\delta}L - (1-\bar{\delta})H$, then the same arguments as in the proof of Lemma 3 can show that $c_i^t \geq x^{t-1} - (1-\bar{\delta})H$. Furthermore, by repeating the arguments in the proof of Lemma 3 for the cases $c_i^t \geq x^{t-1}$ and $x^{t-1} > c_i^t \geq x^{t-1} - (1-\bar{\delta})H$, we obtain

$$U_i(H,c) \leq \bar{\delta}^{t-1}(H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau < \bar{\delta}^{t-1}(\bar{\delta}^3 H - \bar{\delta}^3 L) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|H,h^t,p_i),$$

where the strict inequality holds since $x^{t-1} > \bar{\delta}^3 L + (1-\bar{\delta}^3)H$. □

Lemma 6. *When agent i moves as the deviator, that is, when $i = m(t) = d(h^t) \neq j$, and if $V_i = L$ and $(1-\bar{\delta})L < x^{t-1} \leq (1-\bar{\delta})H$, then $u_i(s|V_i,h^t,p_i) \geq u_i((\bar{s}_i,s_j)|V_i,h^t,p_i)$.*

Proof. When low-type deviator i follows s_i, he chooses $c_i^t = 0$ with probability one in period t and the game continues with $x^t = x^{t-1} \leq (1-\bar{\delta})H$. When punisher j follows s_j given $h^{t+1} = (h^t, x^t)$, he chooses $c_j^{t+1} = x^t$ with probability one in period $(t+1)$ as described in the punisher's case of the strategy, and the game ends. So,

$$u_i(s|L,h^t,p_i) = \bar{\delta}^t L - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau.$$

Let $c = \{(c_1^\tau, c_2^\tau)\}_{\tau=0}^T$ be such that positive probability is assigned to c by the probability distribution generated by $p_i(h^t) = P_j$ and $(\bar{s}_i, s_j)$ given h^t. We show that deviator i's payoff for c does not exceed $u_i(s|L,h^t,p_i)$ in exhaustive cases, which implies that $u_i(s|L,h^t,p_i) \geq u_i((\bar{s}_i,s_j)|L,h^t,p_i)$.

If $c_i^t \geq x^{t-1}$, then the game ends in period t according to c, and we have

$$U_i(L,c) = \bar{\delta}^{t-1}(L - c_i^t) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau < \bar{\delta}^t L - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|L,h^t,p_i),$$

where the strict inequality holds since $(1-\bar{\delta})L < x^{t-1} \leq c_i^t$.

If $c_i^t < x^{t-1}$, the game continues with $x^t = x^{t-1} - c_i^t \leq (1-\bar{\delta})H$ according to c. Then, punisher j in period $(t+1)$ chooses $c_j^{t+1} = x^t$ as described in the punisher's case of the strategy because otherwise, zero probability is assigned to c. So, according to c, the game ends in period $(t+1)$, and we have

$$U_i(L,c) = \bar{\delta}^{t-1}(\bar{\delta}L - c_i^t) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq \bar{\delta}^t L - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|L, h^t, p_i),$$

where the weak inequality holds since $c_i^t \geq 0$. □

Lemma 7. *When agent i moves as the deviator, that is, when $i = m(t) = d(h^t) \neq j$, and if $V_i = L$ and $(1-\bar{\delta})H < x^{t-1} \leq \bar{\delta}L + (1-\bar{\delta})H$, then $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$.*

Proof. When low-type deviator i follows s_i, he chooses $c_i^t = x^{t-1} - (1-\bar{\delta})H$ with probability one in period t and the game continues with $x^t = (1-\bar{\delta})H$. When punisher j follows s_j given $h^{t+1} = (h^t, x^t)$, he chooses $c_j^{t+1} = x^t$ with probability one in period $(t+1)$ as described in the punisher's case of the strategy, and the game ends. So,

$$u_i(s|L, h^t, p_i) = \bar{\delta}^{t-1}(\bar{\delta}L - (x^{t-1} - (1-\bar{\delta})H)) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = \bar{\delta}^{t-1}(\bar{\delta}L + (1-\bar{\delta})H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau.$$

Let $c = \{(c_1^\tau, c_2^\tau)\}_{\tau=0}^T$ be such that positive probability is assigned to c by the probability distribution generated by $p_i(h^t) = P_j$ and $(\bar{s}_i, s_j)$ given h^t. We show that deviator i's payoff for c does not exceed $u_i(s|L, h^t, p_i)$ in exhaustive cases, which implies that $u_i(s|L, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|L, h^t, p_i)$.

If the project is not completed according to c, that is, if $\sum_{\tau=0}^T (c_1^\tau + c_2^\tau) < K$, then we have

$$\begin{aligned} U_i(L,c) &= -\sum_{\tau=t}^{T} \bar{\delta}^{\tau-1} c_i^\tau - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq 0 - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \\ &\leq \bar{\delta}^{t-1}(\bar{\delta}L + (1-\bar{\delta})H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|L, h^t, p_i), \end{aligned}$$

where the weak inequality on the second line holds since $x^{t-1} \leq \bar{\delta}L + (1-\bar{\delta})H$. So, in the remaining part of the proof of the lemma, we consider the cases where the project is completed according to c.

We next show that $c_i^t \geq x^{t-1} - (1-\bar{\delta})H$. If $x^\tau > (1-\bar{\delta})H$ for any $\tau \geq t$, then we have $x^\tau \leq x^{t-1} \leq \bar{\delta}L + (1-\bar{\delta})H$, and punisher j chooses $c_j^{\tau+1} = 0$ in period $(\tau+1)$ as described in the punisher's case of the strategy because otherwise, zero probability is assigned to c. Since the project is completed according to c while punisher j never contributes as long as the remaining amount exceeds $(1-\bar{\delta})H$, it must be the case that deviator i pays at least the difference between x^{t-1} and $(1-\bar{\delta})H$, possibly in one time or in several times. If deviator i should pay an amount necessary for reaching some remaining amount $\hat{x}$ no more than $(1-\bar{\delta})H$, it is optimal for him to do so in one time, because punisher j's strategy does not depend on the path from x^{t-1} to $\hat{x}$ but on the remaining amount $\hat{x}$ itself, and because delaying completion of the project lowers the discounted benefit. Therefore, we must have $c_i^t \geq x^{t-1} - (1-\bar{\delta})H$.

If $c_i^t \geq x^{t-1}$, then the game ends in period t according to c, and we have

$$\begin{aligned} U_i(L,c) &= \bar{\delta}^{t-1}(L - c_i^t) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq \bar{\delta}^{t-1}(L - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \\ &< \bar{\delta}^{t-1}(\bar{\delta}L + (1-\bar{\delta})H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|L, h^t, p_i), \end{aligned}$$

where the weak inequality holds since $c_i^t \geq x^{t-1}$ and the strict inequality holds since $L < \bar{\delta}L + (1-\bar{\delta})H$.

If $x^{t-1} > c_i^t \geq x^{t-1} - (1-\bar{\delta})H$, the game continues with $x^t = x^{t-1} - c_i^t \leq (1-\bar{\delta})H$ according to c. Then, punisher j chooses $c_j^{t+1} = x^t$ in period $(t+1)$ as described in the punisher's case of the strategy because otherwise, zero probability is assigned to c. So, the game ends in period $(t+1)$, and we have

$$U_i(L,c) = \bar{\delta}^{t-1}(\bar{\delta}L - c_i^t) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq \bar{\delta}^{t-1}(\bar{\delta}L + (1-\bar{\delta})H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|L, h^t, p_i),$$

where the weak inequality holds since $c_i^t \geq x^{t-1} - (1-\bar{\delta})H$. □

Lemma 8. *When agent i moves as the deviator, that is, when $i = m(t) = d(h^t) \neq j$, and if $V_i = L$ and $x^{t-1} > \bar{\delta}L + (1-\bar{\delta})H$, then $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$.*

Proof. When low-type deviator i follows s_i, he chooses $c_i^t = 0$ with probability one in period t and the game continues with $x^t = x^{t-1} > \bar{\delta}L + (1-\bar{\delta})H$. When punisher j follows s_j given $h^{t+1} = (h^t, x^t)$, he chooses $c_j^{t+1} = x^t - \bar{\delta}L - (1-\bar{\delta})H$ with probability one in period $(t+1)$ as described in the punisher's case of the strategy. The game further continues with $x^{t+1} = \bar{\delta}L + (1-\bar{\delta})H$, and low-type deviator i, following s_i given $h^{t+2} = (h^t, x^t, x^{t+1})$, chooses $c_i^{t+2} = x^{t+1} - (1-\bar{\delta})H$ with probability one in period $(t+2)$, and the game continues with $x^{t+2} = (1-\bar{\delta})H$. When punisher j follows s_j given $h^{t+3} = (h^t, x^t, x^{t+1}, x^{t+2})$, he chooses $c_j^{t+3} = x^{t+2}$ with probability one in period $(t+3)$ as described in the punisher's case of the strategy, and the game ends. So, we have

$$u_i(s|L, h^t, p_i) = \bar{\delta}^{t-1}(\bar{\delta}^3 L - \bar{\delta}^2(x^{t+1} - (1-\bar{\delta})H)) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau$$

$$= \bar{\delta}^{t-1}(\bar{\delta}^3 L - \bar{\delta}^3 L) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = 0 - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau.$$

Let $c = \{(c_1^\tau, c_2^\tau)\}_{\tau=0}^T$ be such that positive probability is assigned to c by the probability distribution generated by $p_i(h^t) = P_j$ and $(\bar{s}_i, s_j)$ given h^t. We show that deviator i's payoff for c does not exceed $u_i(s|L, h^t, p_i)$ in exhaustive cases, which implies that $u_i(s|L, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|L, h^t, p_i)$.

If the project is not completed according to c, that is, if $\sum_{\tau=0}^T (c_1^\tau + c_2^\tau) < K$, then we have

$$U_i(L, c) = -\sum_{\tau=t}^{T} \bar{\delta}^{\tau-1} c_i^\tau - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq 0 - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|L, h^t, p_i).$$

So, in the remaining part of the proof of the lemma, we consider the cases where the project is completed according to c.

If $c_i^t < x^{t-1} - \bar{\delta}L - (1-\bar{\delta})H$, the game continues with $x^t = x^{t-1} - c_i^t > \bar{\delta}L + (1-\bar{\delta})H$ according to c. Then, punisher j in period $(t+1)$ chooses $c_j^{t+1} = x^t - \bar{\delta}L - (1-\bar{\delta})H$ as described in the punisher's case of the strategy because otherwise, zero probability is assigned to c. According to c, the game further continues with $x^{t+1} = \bar{\delta}L + (1-\bar{\delta})H$. For the continuation game from period $(t+2)$, it is optimal for low-type deviator i to follow s_i by Lemma 7. So, we have

$$\begin{aligned} U_i(L, c) &\leq u_i(s|L, (h^t, x^t, x^{t+1}), p_i) = \bar{\delta}^{t+1}(\bar{\delta}L + (1-\bar{\delta})H - x^{t+1}) - \sum_{\tau=0}^{t+1} \bar{\delta}^{\tau-1} c_i^\tau \\ &= 0 - \sum_{\tau=0}^{t+1} \bar{\delta}^{\tau-1} c_i^\tau \leq 0 - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|L, h^t, p_i), \end{aligned}$$

where the weak inequality on the second line holds since $c_i^t \geq 0$ and $c_i^{t+1} = 0$.

If $c_i^t \geq x^{t-1} - \bar{\delta}L - (1-\bar{\delta})H$, then the same arguments as in the proof of Lemma 7 can show that $c_i^t \geq x^{t-1} - (1-\bar{\delta})H$. Furthermore, by repeating the arguments in the proof of Lemma 7 for the cases $c_i^t \geq x^{t-1}$ and $x^{t-1} > c_i^t \geq x^{t-1} - (1-\bar{\delta})H$, we obtain

$$U_i(L, c) \leq \bar{\delta}^{t-1}(\bar{\delta}L + (1-\bar{\delta})H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau < 0 - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|L, h^t, p_i),$$

where the strict inequality holds since $x^{t-1} > \bar{\delta}L + (1-\bar{\delta})H$. □

Lemma 9. *When agent i moves as the punisher, that is, when $i = m(t) \neq d(h^t) = j$, and if $V_i = L$ and $(1-\bar{\delta})L < x^{t-1} \leq (1-\bar{\delta})H$, then $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$.*

Proof. In this case, $p_i(h^t) = 0$, so punisher i believes that deviator j is a low-type. When punisher i

follows s_i, he chooses $c_i^t = x^{t-1}$ with probability one in period t and the game ends with his payoff

$$u_i(s|L, h^t, p_i) = \bar{\delta}^{t-1}(L - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau.$$

Let $c = \{(c_1^\tau, c_2^\tau)\}_{\tau=0}^T$ be such that positive probability is assigned to c by the probability distribution generated by $p_i(h^t) = 0$ and $(\bar{s}_i, s_j)$ given h^t. We show that punisher i's payoff for c does not exceed $u_i(s|L, h^t, p_i)$ in exhaustive cases, which implies that $u_i(s|L, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|L, h^t, p_i)$.

If the project is not completed according to c, that is, if $\sum_{\tau=0}^T (c_1^\tau + c_2^\tau) < K$, then we have

$$\begin{aligned} U_i(L, c) &= -\sum_{\tau=t}^{T} \bar{\delta}^{\tau-1} c_i^\tau - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq 0 - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau < \bar{\delta}^{t-1}(L - (1-\bar{\delta})H) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \\ &\leq \bar{\delta}^{t-1}(L - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|L, h^t, p_i), \end{aligned}$$

where the strict inequality on the first line holds since $L > \bar{c}_1^{\bar{t}} + \bar{c}_2^{\bar{t}} > (1 - \bar{\delta}^2)H > (1 - \bar{\delta})H$ by the choice of $\{(\bar{c}_1^t, \bar{c}_2^t)\}_{t=0}^{\bar{t}}$ and by the condition (2) in Lemma 1, and the weak inequality on the second line holds since $x^{t-1} \leq (1 - \bar{\delta})H$. So, in the remaining part of the proof of the lemma, we consider the cases where the project is completed according to c.

We next show that $c_i^t \geq x^{t-1} - (1 - \bar{\delta})L$. If $x^\tau > (1 - \bar{\delta})L$ for any $\tau \geq t$, then we have $x^\tau \leq x^{t-1} \leq (1 - \bar{\delta})H$, and low-type deviator j in period $(\tau + 1)$ chooses $c_j^{\tau+1} = 0$ as described in the deviator's case of the strategy because otherwise, zero probability is assigned to c. Since the project is completed according to c while low-type deviator j never contributes as long as the remaining amount exceeds $(1 - \bar{\delta})L$, it must be the case that punisher i pays at least the difference between x^{t-1} and $(1 - \bar{\delta})L$, possibly in one time or in several times. If punisher i should pay an amount necessary for reaching some remaining amount $\hat{x}$ no more than $(1 - \bar{\delta})L$, it is optimal for him to do so in one time, because low-type deviator j's strategy does not depend on the path from x^{t-1} to $\hat{x}$ but on the remaining amount $\hat{x}$ itself, and because delaying completion of the project lowers the discounted benefit. Therefore, we must have $c_i^t \geq x^{t-1} - (1 - \bar{\delta})L$.

If $c_i^t \geq x^{t-1}$, then the game ends in period t according to c, and we have

$$U_i(L, c) = \bar{\delta}^{t-1}(L - c_i^t) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq \bar{\delta}^{t-1}(L - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|L, h^t, p_i),$$

where the weak inequality holds since $c_i^t \geq x^{t-1}$.

If $x^{t-1} > c_i^t \geq x^{t-1} - (1 - \bar{\delta})L$, the game continues with $x^t = x^{t-1} - c_i^t \leq (1 - \bar{\delta})L$ according to c.

Then, low-type deviator j in period $(t+1)$ chooses $c_j^{t+1} = x^t$ as described in the deviator's case of the strategy because otherwise, zero probability is assigned to c. So, the game ends in period $(t+1)$, and we have

$$U_i(L, c) = \bar{\delta}^{t-1}(\bar{\delta}L - c_i^t) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq \bar{\delta}^{t-1}(L - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|L, h^t, p_i),$$

where the weak inequality holds since $c_i^t \geq x^{t-1} - (1-\bar{\delta})L$. □

Lemma 10. *When agent i moves as the punisher, that is, when $i = m(t) \neq d(h^t) = j$, and if $(1-\bar{\delta})H < x^{t-1} \leq \bar{\delta}L + (1-\bar{\delta})H$, then $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$.*

Proof. In this case, $p_i(h^t) = 1$, so punisher i believes that deviator j is a high-type. When punisher i follows s_i, he chooses $c_i^t = 0$ with probability one in period t and the game continues with $x^t = x^{t-1} \leq \bar{\delta}L + (1-\bar{\delta})H < \bar{\delta}^3 L + (1-\bar{\delta}^3)H$. When high-type deviator j follows s_j given $h^{t+1} = (h^t, x^t)$, he chooses $c_j^{t+1} = x^t$ with probability one in period $(t+1)$ as described in the deviator's case of the strategy, and the game ends. So,

$$u_i(s|V_i, h^t, p_i) = \bar{\delta}^t V_i - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau.$$

Let $c = \{(c_1^\tau, c_2^\tau)\}_{\tau=0}^T$ be such that positive probability is assigned to c by the probability distribution generated by $p_i(h^t) = 1$ and $(\bar{s}_i, s_j)$ given h^t. We show that punisher i's payoff for c does not exceed $u_i(s|V_i, h^t, p_i)$ in exhaustive cases, which implies that $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$.

If $c_i^t \geq x^{t-1}$, then the game ends in period t according to c, and we have

$$U_i(V_i, c) = \bar{\delta}^{t-1}(V_i - c_i^t) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau < \bar{\delta}^t V_i - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|V_i, h^t, p_i),$$

where the strict inequality holds since $(1-\bar{\delta})V_i < x^{t-1} \leq c_i^t$.

If $c_i^t < x^{t-1}$, the game continues with $x^t = x^{t-1} - c_i^t \leq \bar{\delta}L + (1-\bar{\delta})H < \bar{\delta}^3 L + (1-\bar{\delta}^3)H$ according to c. Then, high-type deviator j in period $(t+1)$ chooses $c_j^{t+1} = x^t$ as described in the deviator's case of the strategy because otherwise, zero probability is assigned to c. So, according to c, the game ends in period $(t+1)$, and we have

$$U_i(V_i, c) = \bar{\delta}^{t-1}(\bar{\delta}V_i - c_i^t) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq \bar{\delta}^t V_i - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|V_i, h^t, p_i),$$

where the weak inequality holds since $c_i^t \geq 0$. □

Lemma 11. *When agent i moves as the punisher, that is, when $i = m(t) \neq d(h^t) = j$, and if $x^{t-1} > \bar{\delta}L + (1-\bar{\delta})H$, then $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$.*

Proof. In this case, $p_i(h^t) = 0$, so punisher i believes that deviator j is a low-type. When punisher i follows s_i, he chooses $c_i^t = x^{t-1} - \bar{\delta}L - (1-\bar{\delta})H$ with probability one in period t and the game continues with $x^t = \bar{\delta}L + (1-\bar{\delta})H$. When low-type deviator j follows s_j given $h^{t+1} = (h^t, x^t)$, he chooses $c_j^{t+1} = x^t - (1-\bar{\delta})H$ with probability one in period $(t+1)$ as described in the deviator's case of the strategy. The game further continues with $x^{t+1} = (1-\bar{\delta})H$, and punisher i, following s_i given $h^{t+2} = (h^t, x^t, x^{t+1})$, chooses $c_i^{t+2} = x^{t+1}$ with probability one in period $(t+2)$ and the game ends. So, we have

$$\begin{aligned} u_i(s|V_i, h^t, p_i) &= \bar{\delta}^{t-1}(\bar{\delta}^2(V_i - (1-\bar{\delta})H) - (x^{t-1} - \bar{\delta}L - (1-\bar{\delta})H)) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \\ &= \bar{\delta}^{t-1}(\bar{\delta}^2 V_i + \bar{\delta}L + (1-\bar{\delta}^2)(1-\bar{\delta})H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau. \end{aligned}$$

Let $c = \{(c_1^\tau, c_2^\tau)\}_{\tau=0}^T$ be such that positive probability is assigned to c by the probability distribution generated by $p_i(h^t) = 0$ and $(\bar{s}_i, s_j)$ given h^t. We show that punisher i's payoff for c does not exceed $u_i(s|V_i, h^t, p_i)$ in exhaustive cases, which implies that $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$.

If the project is not completed according to c, that is, if $\sum_{\tau=0}^T (c_1^\tau + c_2^\tau) < K$, then we have

$$\begin{aligned} U_i(V_i, c) &= -\sum_{\tau=t}^{T} \bar{\delta}^{\tau-1} c_i^\tau - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq 0 - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \\ &< \bar{\delta}^{t-1}(\bar{\delta}^2 V_i + \bar{\delta}L + (1-\bar{\delta}^2)(1-\bar{\delta})H - K) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \\ &\leq \bar{\delta}^{t-1}(\bar{\delta}^2 V_i + \bar{\delta}L + (1-\bar{\delta}^2)(1-\bar{\delta})H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|V_i, h^t, p_i), \end{aligned}$$

where the strict inequality on the second line is due to the condition (1) in Lemma 1, and the weak inequality on the third line holds since $x^{t-1} \leq K$. So, in the remaining part of the proof of the lemma, we consider the cases where the project is completed according to c.

We next show that $c_i^t \geq x^{t-1} - \bar{\delta}L - (1-\bar{\delta})H$. If $x^\tau > \bar{\delta}L + (1-\bar{\delta})H$ for any $\tau \geq t$, then low-type deviator j chooses $c_j^{\tau+1} = 0$ in period $(\tau+1)$ as described in the deviator's case of the strategy because otherwise, zero probability is assigned to c. Since the project is completed according to c while low-type deviator j never contributes as long as the remaining amount exceeds $(\bar{\delta}L + (1-\bar{\delta})H)$, it must be the case that punisher i pays at least the difference between x^{t-1} and $(\bar{\delta}L + (1-\bar{\delta})H)$, possibly in one time or in several times. If punisher i should pay an amount necessary for reaching some remaining amount $\hat{x}$ no more than $(\bar{\delta}L + (1-\bar{\delta})H)$, it is optimal for him to do so in one time, because low-type deviator j's strategy does not depend on the path from x^{t-1} to $\hat{x}$ but on the remaining amount $\hat{x}$ itself, and because delaying completion of the project lowers the discounted benefit. Therefore, we must have $c_i^t \geq x^{t-1} - \bar{\delta}L - (1-\bar{\delta})H$.

If $x^{t-1} - (1-\bar{\delta})H > c_i^t \geq x^{t-1} - \bar{\delta}L - (1-\bar{\delta})H$, the game continues with $x^t = x^{t-1} - c_i^t$ such that $(1-\bar{\delta})H < x^t \leq \bar{\delta}L + (1-\bar{\delta})H$ according to c. Given $h^{t+1} = (h^t, x^t)$, low-type deviator j in period $(t+1)$ chooses $c_j^{t+1} = x^t - (1-\bar{\delta})H$ as described in the deviator's case of the strategy because otherwise, zero probability is assigned to c. According to c, the game further continues with $x^{t+1} = (1-\bar{\delta})H$. For the continuation game from period $(t+2)$, it is optimal for punisher i to follow s_i by Lemmas 2 and 9. So,

$$\begin{aligned} U_i(V_i, c) &\leq u_i(s|V_i, (h^t, x^t, x^{t+1}), p_i) = \bar{\delta}^{t-1}(\bar{\delta}^2(V_i - (1-\bar{\delta})H) - c_i^t) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \\ &\leq \bar{\delta}^{t-1}(\bar{\delta}^2(V_i - (1-\bar{\delta})H) - x^{t-1} + \bar{\delta}L + (1-\bar{\delta})H) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \\ &= \bar{\delta}^{t-1}(\bar{\delta}^2 V_i + \bar{\delta}L + (1-\bar{\delta}^2)(1-\bar{\delta})H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|V_i, h^t, p_i), \end{aligned}$$

where the weak inequality on the second line holds since $c_i^t \geq x^{t-1} - \bar{\delta}L - (1-\bar{\delta})H$.

If $c_i^t \geq x^{t-1} - (1-\bar{\delta})H$, then the same arguments as in the proof of Lemma 9 can show that $c_i^t \geq x^{t-1} - (1-\bar{\delta})L$. Furthermore, by repeating similar arguments to the ones in the proof of Lemma 9 for the cases $c_i^t \geq x^{t-1}$ and $x^{t-1} > c_i^t \geq x^{t-1} - (1-\bar{\delta})L$, we obtain

$$\begin{aligned} U_i(V_i, c) &\leq \bar{\delta}^{t-1}(V_i - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \leq \bar{\delta}^{t-1}(\bar{\delta}^2 V_i + (1-\bar{\delta}^2)H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \\ &< \bar{\delta}^{t-1}(\bar{\delta}^2 V_i + (1-\bar{\delta}^2)H - x^{t-1} + \bar{\delta}(L - (1-\bar{\delta}^2)H)) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau \\ &= \bar{\delta}^{t-1}(\bar{\delta}^2 V_i + \bar{\delta}L + (1-\bar{\delta}^2)(1-\bar{\delta})H - x^{t-1}) - \sum_{\tau=0}^{t-1} \bar{\delta}^{\tau-1} c_i^\tau = u_i(s|V_i, h^t, p_i), \end{aligned}$$

where the second weak inequality on the first line holds since $V_i \leq H$, and the strict inequality on the second line holds since $L > \bar{c}_1^{\bar{t}} + \bar{c}_2^{\bar{t}} > (1-\bar{\delta}^2)H$ by the choice of $\{(\bar{c}_1^t, \bar{c}_2^t)\}_{t=0}^{\bar{t}}$ and by the condition (2) in Lemma 1. □

Lemma 12. *If agent i moves given an on-path history, that is, if $i = m(t) \neq j$ and $d(h^t) = 0$, then*

$u_i(s|V_i, h^t, p_i)$ *satisfies the following conditions:*

$$\begin{aligned} \textit{If } i = m(\bar{t}), \textit{ then } u_i(s|V_i, h^t, p_i) &= \bar{\delta}^{\bar{t}} V_i + \bar{\delta}^{\bar{t}-1}(1-\bar{\delta})H - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1} \bar{c}_i^{\tau} \\ &> \bar{\delta}^{\bar{t}}(V_i - (1-\bar{\delta})H) - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1} \bar{c}_i^{\tau}. \qquad (5) \\ \textit{If } i \neq m(\bar{t}), \textit{ then } u_i(s|V_i, h^t, p_i) &= P_j \bar{\delta}^{\bar{t}-1} V_i + (1-P_j)\bar{\delta}^{\bar{t}}(V_i - (1-\bar{\delta})H) - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1} \bar{c}_i^{\tau} \\ &> \bar{\delta}^{\bar{t}}(V_i - (1-\bar{\delta})H) - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1} \bar{c}_i^{\tau}. \qquad (6) \end{aligned}$$

Proof. When a history h^t is on-path and agents i and j follow s_i and s_j, respectively, the game continues along the path that realizes $\bar{h}^{\bar{t}} = (\bar{x}^0, \bar{x}^1, \ldots, \bar{x}^{\bar{t}-1})$ until the beginning of period $\bar{t}$. Now we pay attention to the agent who moves in period $\bar{t}$.

First, suppose that $i = m(\bar{t})$. When agent i is a high-type, he chooses $c_i^{\bar{t}} = \bar{c}_i^{\bar{t}}$ in period $\bar{t}$, and the game ends. So, we have

$$u_i(s|H, h^t, p_i) = \bar{\delta}^{\bar{t}-1} H - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1} \bar{c}_i^{\tau} = \bar{\delta}^{\bar{t}} H + \bar{\delta}^{\bar{t}-1}(1-\bar{\delta})H - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1} \bar{c}_i^{\tau},$$

which shows that the equality in the condition (5) holds for this case. When agent i is a low-type, he chooses $c_i^{\bar{t}} = \bar{c}_i^{\bar{t}} - (1-\bar{\delta})H$ in period $\bar{t}$, and punisher j pays the remaining amount $(1-\bar{\delta})H$ in period $(\bar{t}+1)$, and the game ends. So, we have

$$u_i(s|L, h^t, p_i) = \bar{\delta}^{\bar{t}} L - \sum_{\tau=0}^{\bar{t}-1} \bar{\delta}^{\tau-1} \bar{c}_i^{\tau} - \bar{\delta}^{\bar{t}-1}(\bar{c}_i^{\bar{t}} - (1-\bar{\delta})H) = \bar{\delta}^{\bar{t}} L + \bar{\delta}^{\bar{t}-1}(1-\bar{\delta})H - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1} \bar{c}_i^{\tau},$$

which shows that the equality in the condition (5) holds for this case. The inequality in the condition (5) holds since $\bar{\delta}^{\bar{t}-1}(1-\bar{\delta})H > -\bar{\delta}^{\bar{t}}(1-\bar{\delta})H$.

Second, suppose that $i \neq m(\bar{t})$. It is agent j that moves in period $\bar{t}$, so we have $c_i^{\bar{t}} = 0$. If agent j is a high-type, he chooses $c_j^{\bar{t}} = \bar{c}_j^{\bar{t}}$ in period $\bar{t}$, and the game ends. If agent j is a low-type, he chooses $c_j^{\bar{t}} = \bar{c}_j^{\bar{t}} - (1-\bar{\delta})H$ in period $\bar{t}$, and punisher i pays the remaining amount $(1-\bar{\delta})H$ in period $(\bar{t}+1)$, and the game ends. Since agent i holds a belief $p_i(h^t) = P_j$ given the on-path history h^t, we obtain

$$u_i(s|V_i, h^t, p_i) = P_j \bar{\delta}^{\bar{t}-1} V_i + (1-P_j)\bar{\delta}^{\bar{t}}(V_i - (1-\bar{\delta})H) - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1} \bar{c}_i^{\tau},$$

which shows that the equality in the condition (6) holds. The inequality in the condition (6) holds since $\bar{\delta}^{\bar{t}-1}V_i > \bar{\delta}^{\bar{t}}(V_i - (1-\bar{\delta})H)$. □

Lemma 13. *If agent i moves given an on-path history, that is, if $i = m(t) \neq j$ and $d(h^t) = 0$, then $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$.*

Proof. Throughout this proof, suppose that agent j follows s_j. By Lemma 12, we have

$$u_i(s|V_i, h^t, p_i) > \bar{\delta}^{\bar{t}}(V_i - (1-\bar{\delta})H) - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1}\bar{c}_i^{\tau} > \bar{\delta}^3(V_i - L) \geq 0,$$

where the second strict inequality is due to the condition (3) in Lemma 1, evaluated at $t = 1$, together with the fact that $c_i^0 = 0$.

Let $c = \{(c_1^\tau, c_2^\tau)\}_{\tau=0}^{T}$ be such that positive probability is assigned to c by the probability distribution generated by $p_i(h^t) = P_j$ and $(\bar{s}_i, s_j)$ given h^t. We show that agent i's expected payoff for contribution sequences does not exceed $u_i(s|V_i, h^t, p_i)$ in Case 1, and agent i's payoff for the contribution sequence c does not exceed $u_i(s|V_i, h^t, p_i)$ in the other cases; These imply that $u_i(s|V_i, h^t, p_i) \geq u_i((\bar{s}_i, s_j)|V_i, h^t, p_i)$.

If the project is not completed according to c, agent i's payoff does not exceed 0, which is less than $u_i(s|V_i, h^t, p_i)$. So, in the remaining part of the proof of the lemma, we consider the cases where the project is completed according to c.

Case 1: $i \neq m(\bar{t})$ and $c_i^\tau = \bar{c}_i^\tau$ for all $\tau \leq \bar{t}$.

We note that $c_j^\tau = \bar{c}_j^\tau$ for all $\tau \leq \bar{t} - 1$ because otherwise, zero probability is assigned to c. So, according to c, agent j in period $\bar{t}$ moves given the on-path history. If he is a high-type, then $c_j^{\bar{t}} = \bar{c}_j^{\bar{t}}$; Otherwise, $c_j^{\bar{t}} = \bar{c}_j^{\bar{t}} - (1-\bar{\delta})H$. From agent i's viewpoint in period $(\bar{t}-1)$, given the conditions in Case 1, the event $c_j^{\bar{t}} = \bar{c}_j^{\bar{t}}$ occurs with probability $p_i(h^{\bar{t}-1}) = P_j$ and the event $c_j^{\bar{t}} = \bar{c}_j^{\bar{t}} - (1-\bar{\delta})H$ occurs with probability $(1 - P_j)$.

We now consider agent i's expected payoff $\hat{U}_i$ for all the contribution sequences satisfying the conditions in Case 1. If the event $c_j^{\bar{t}} = \bar{c}_j^{\bar{t}}$ occurs, then agent i's payoff is $\bar{\delta}^{\bar{t}-1}V_i - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1}\bar{c}_i^\tau$. If the event $c_j^{\bar{t}} = \bar{c}_j^{\bar{t}} - (1-\bar{\delta})H$ occurs, then the game continues with $x^{\bar{t}+1} = (1-\bar{\delta})H$ and agent i moves as a punisher in period $(\bar{t}+1)$; Since it is optimal for punisher i to follow s_i by Lemmas 2 and 9, agent i's payoff in this event is at most $\bar{\delta}^{\bar{t}}(V_i - (1-\bar{\delta})H) - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1}c_i^\tau$. Therefore, we have

$$\hat{U}_i \leq P_j(\bar{\delta}^{\bar{t}-1}V_i - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1}\bar{c}_i^\tau) + (1-P_j)(\bar{\delta}^{\bar{t}}(V_i - (1-\bar{\delta})H) - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1}c_i^\tau) = u_i(s|V_i, h^t, p_i),$$

where the last equality holds by the condition (6) in Lemma 12.

Case 2: $i = m(\bar{t})$ and $c_i^\tau = \bar{c}_i^\tau$ for all $\tau \le \bar{t}$.

We note that $c_j^\tau = \bar{c}_j^\tau$ for all $\tau \le \bar{t}$ because otherwise, zero probability is assigned to c. In this case, agent i in period $\bar{t}$ chooses $c_i^{\bar{t}} = \bar{c}_i^{\bar{t}}$ and the game ends. So, we have

$$\begin{aligned} U_i(V_i, c) &= \bar{\delta}^{\bar{t}-1} V_i - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1} \bar{c}_i^\tau = \bar{\delta}^{\bar{t}} V_i + \bar{\delta}^{\bar{t}-1}(1-\bar{\delta}) V_i - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1} \bar{c}_i^\tau \\ &\le \bar{\delta}^{\bar{t}} V_i + \bar{\delta}^{\bar{t}-1}(1-\bar{\delta}) H - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1} \bar{c}_i^\tau = u_i(s|V_i, h^t, p_i), \end{aligned}$$

where the last equality holds by the condition (5) in Lemma 12.

Case 3: $c_i^\tau \neq \bar{c}_i^\tau$ for some $\tau \le \bar{t}$.

Let $\hat{\tau}$ be the least τ that satisfies $c_i^\tau \neq \bar{c}_i^\tau$. Note that $t \le \hat{\tau} \le \bar{t}$ and $c_i^\tau = \bar{c}_i^\tau$ for all $\tau \le \hat{\tau} - 1$. We also note that $c_j^\tau = \bar{c}_j^\tau$ for all $\tau \le \hat{\tau} - 1$ because otherwise, zero probability is assigned to c. Let $h^{\hat{\tau}+1} = (x^0, x^1, \ldots, x^{\hat{\tau}})$ denote the history corresponding to $\{(c_1^\tau, c_2^\tau)\}_{\tau=0}^{\hat{\tau}}$. If $x^{\hat{\tau}} > 0$ and hence the game continues, then agent j moves as the punisher in period $(\hat{\tau}+1)$ given $h^{\hat{\tau}+1}$.

First, suppose that $c_i^{\hat{\tau}} < x^{\hat{\tau}-1} - \bar{\delta}L - (1-\bar{\delta})H$.[7] Since $x^{\hat{\tau}} > \bar{\delta}L + (1-\bar{\delta})H$ and agent j moves as the punisher in period $(\hat{\tau}+1)$, we have $x^{\hat{\tau}+1} = \bar{\delta}L + (1-\bar{\delta})H$. Since agent i is the deviator in period $(\hat{\tau}+2)$ and it is optimal for him to follow s_i by Lemmas 3 and 7, deviator i's payoff for c satisfies the following:

$$U_i(H, c) \le \bar{\delta}^{\hat{\tau}+1}(H - \bar{\delta}L - (1-\bar{\delta})H) - \sum_{\tau=0}^{\hat{\tau}+1} \bar{\delta}^{\tau-1} c_i^\tau \quad \text{and} \quad U_i(L, c) \le \bar{\delta}^{\hat{\tau}+2} L - \bar{\delta}^{\hat{\tau}+1}(\bar{\delta}L) - \sum_{\tau=0}^{\hat{\tau}+1} \bar{\delta}^{\tau-1} c_i^\tau.$$

So, we obtain

$$\begin{aligned} U_i(V_i, c) &\le \bar{\delta}^{\hat{\tau}+1}(V_i - \bar{\delta}L - (1-\bar{\delta})V_i) - \sum_{\tau=0}^{\hat{\tau}+1} \bar{\delta}^{\tau-1} c_i^\tau \\ &= \bar{\delta}^{\hat{\tau}+1}(V_i - \bar{\delta}L - (1-\bar{\delta})V_i) - c_i^{\hat{\tau}+1} - c_i^{\hat{\tau}} - \sum_{\tau=0}^{\hat{\tau}-1} \bar{\delta}^{\tau-1} c_i^\tau \\ &\le \bar{\delta}^{\hat{\tau}-1}(\bar{\delta}^3 V_i - \bar{\delta}^3 L) - \sum_{\tau=0}^{\hat{\tau}-1} \bar{\delta}^{\tau-1} \bar{c}_i^\tau \\ &< \bar{\delta}^{\hat{\tau}-1}(\bar{\delta}^{\bar{t}-\hat{\tau}+1}(V_i - (1-\bar{\delta})H) - \sum_{\tau=\hat{\tau}}^{\bar{t}} \bar{\delta}^{\tau-\hat{\tau}} \bar{c}_i^\tau) - \sum_{\tau=0}^{\hat{\tau}-1} \bar{\delta}^{\tau-1} \bar{c}_i^\tau \\ &= \bar{\delta}^{\bar{t}}(V_i - (1-\bar{\delta})H) - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1} \bar{c}_i^\tau \\ &< u_i(s|V_i, h^t, p_i), \end{aligned}$$

[7]When this inequality holds, it must be the case that $x^{\hat{\tau}-1} > \bar{\delta}L + (1-\bar{\delta})H$ since $c_i^{\hat{\tau}} \ge 0$.

where the weak inequality on the third line holds since $c_i^{\hat{\tau}} \geq 0$ and $c_i^{\hat{\tau}+1} = 0$ and $c_i^{\tau} = \bar{c}_i^{\tau}$ for all $\tau \leq \hat{\tau} - 1$, and the strict inequality on the fourth line holds by the condition (3) in Lemma 1, evaluated at $t = \hat{\tau} \leq \bar{t}$, and the strict inequality on the last line holds by Lemma 12.

Second, suppose that $c_i^{\hat{\tau}} \geq x^{\hat{\tau}-1} - \bar{\delta}L - (1-\bar{\delta})H$. Since $x^{\hat{\tau}} \leq \bar{\delta}L + (1-\bar{\delta})H$ and agent j moves as the punisher in period $(\hat{\tau}+1)$, he will pay nothing as long as $x^{\hat{\tau}} > (1-\bar{\delta})H$; But he will pay all the remaining amount if $x^{\hat{\tau}} = (1-\bar{\delta})H$. By applying the same arguments as in the proof of Lemma 7, we can show that agent i's payoff for c does not exceed the value $\tilde{U}_i$ that he obtains by paying[8] $c_i^{\hat{\tau}} = x^{\hat{\tau}-1} - (1-\bar{\delta})H$ in period $\hat{\tau}$:

$$U_i(V_i, c) \leq \tilde{U}_i \equiv \bar{\delta}^{\hat{\tau}} V_i - \bar{\delta}^{\hat{\tau}-1}(x^{\hat{\tau}-1} - (1-\bar{\delta})H) - \sum_{\tau=0}^{\hat{\tau}-1} \bar{\delta}^{\tau-1} \bar{c}_i^{\tau},$$

where we have applied $c_i^{\tau} = \bar{c}_i^{\tau}$ for all $\tau \leq \hat{\tau} - 1$.

If $\hat{\tau} \leq \bar{t} - 1$, then

$$\begin{aligned} \tilde{U}_i &= \bar{\delta}^{\hat{\tau}-1}(\bar{\delta}V_i + (1-\bar{\delta})H - x^{\hat{\tau}-1}) - \sum_{\tau=0}^{\hat{\tau}-1} \bar{\delta}^{\tau-1}\bar{c}_i^{\tau} \\ &< \bar{\delta}^{\hat{\tau}-1}(\bar{\delta}^{\bar{t}-\hat{\tau}+1}(V_i - (1-\bar{\delta})H) - \sum_{\tau=\hat{\tau}}^{\bar{t}} \bar{\delta}^{\tau-\hat{\tau}}\bar{c}_i^{\tau}) - \sum_{\tau=0}^{\hat{\tau}-1} \bar{\delta}^{\tau-1}\bar{c}_i^{\tau} \\ &= \bar{\delta}^{\bar{t}}(V_i - (1-\bar{\delta})H) - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1}\bar{c}_i^{\tau} \\ &< u_i(s|V_i, h^t, p_i), \end{aligned}$$

where the first inequality is due to the condition (4) in Lemma 1, evaluated at $t = \hat{\tau} \leq \bar{t} - 1$, and the last inequality is due to Lemma 12.

If $\hat{\tau} = \bar{t}$, then it must be the case that $i = m(\bar{t})$ since $c_i^{\hat{\tau}} \neq \bar{c}_i^{\hat{\tau}}$ in Case 3. Furthermore, we have

$$x^{\hat{\tau}-1} = K - \sum_{\tau=0}^{\hat{\tau}-1}(c_1^{\tau} + c_2^{\tau}) = K - \sum_{\tau=0}^{\hat{\tau}-1}(\bar{c}_1^{\tau} + \bar{c}_2^{\tau}) = K - \sum_{\tau=0}^{\bar{t}-1}(\bar{c}_1^{\tau} + \bar{c}_2^{\tau}) = \bar{c}_1^{\bar{t}} + \bar{c}_2^{\bar{t}} = \bar{c}_i^{\bar{t}}$$

by the definition of $x^{\hat{\tau}-1}$ and $\{(\bar{c}_1^t, \bar{c}_2^t)\}_{t=0}^{\bar{t}}$. So, we obtain

$$\tilde{U}_i = \bar{\delta}^{\bar{t}} V_i - \bar{\delta}^{\bar{t}-1}(\bar{c}_i^{\bar{t}} - (1-\bar{\delta})H) - \sum_{\tau=0}^{\bar{t}-1} \bar{\delta}^{\tau-1}\bar{c}_i^{\tau}$$

[8]We note that $c_i^{\hat{\tau}} = x^{\hat{\tau}-1} - (1-\bar{\delta})H > 0$ since $\hat{\tau} \leq \bar{t}$ and $x^{\hat{\tau}-1} = \bar{x}^{\hat{\tau}-1} \geq \bar{x}^{\bar{t}-1} = \bar{c}_1^{\bar{t}} + \bar{c}_2^{\bar{t}} > (1-\bar{\delta}^2)H > (1-\bar{\delta})H$ by the condition (2) in Lemma 1.

$$\begin{aligned}
&= \bar{\delta}^{\bar{t}} V_i + \bar{\delta}^{\bar{t}-1}(1-\bar{\delta})H - \sum_{\tau=0}^{\bar{t}} \bar{\delta}^{\tau-1}\bar{c}_i^{\tau} \\
&= u_i(s|V_i, h^t, p_i),
\end{aligned}$$

where the last equality holds by the condition (5) in Lemma 12. □

Proposition 1. *A pair (s,p) satisfies sequential rationality.*

Proof. Lemmas 2 through 13 show the result in exhaustive cases. □

We next prove that (s,p) satisfies reasonability in the series of lemmas. Take any $i \in \{1,2\}$, $t \in \{1,2,\dots\}$, and $h^t = (x^0, x^1, \dots, x^{t-1})$ with $x^{t-1} > 0$, and fix them for the arguments in Lemmas 14 through 17. Let $j \neq i$ and let $\{(c_1^\tau, c_2^\tau)\}_{\tau=0}^{t-1}$ be the contribution sequence corresponding to h^t. For $x^t = x^{t-1} - (c_1^t + c_2^t) > 0$ where (c_1^t, c_2^t) satisfies the feasibility, define a *Bayesian-updated belief*

$$p_i^B((h^t, x^t)) = \frac{p_i(h^t) s_j(c_j^t|H, h^t)}{p_i(h^t) s_j(c_j^t|H, h^t) + (1 - p_i(h^t)) s_j(c_j^t|L, h^t)}$$

which is *well-defined* only if $p_i(h^t)s_j(c_j^t|H,h^t) > 0$ or $(1-p_i(h^t))s_j(c_j^t|L,h^t) > 0$.

Lemma 14. *Suppose that agent i moves in period t, that is, $i = m(t)$. For $x^t = x^{t-1} - (c_1^t + c_2^t) > 0$ where (c_1^t, c_2^t) satisfies the feasibility, we have $p_i^B((h^t, x^t)) = p_i((h^t, x^t))$.*

Proof. Since $i = m(t)$, the feasibility requires that $c_j^t = 0$ and $s_j(c_j^t|H, h^t) = s_j(c_j^t|L, h^t) = 1$. So,

$$p_i^B((h^t, x^t)) = \frac{p_i(h^t)}{p_i(h^t) + (1 - p_i(h^t))} = p_i(h^t).$$

On the other hand, $p_i((h^t, x^t)) = p_i(h^t)$ by the definition of $p_i(h^{t+1})$ in the case where $i = m(t)$. □

Lemma 15. *Suppose that agent j moves given an on-path history h^t, that is, $j = m(t) \neq i$ and $d(h^t) = 0$. For $x^t = x^{t-1} - (c_1^t + c_2^t) > 0$ where (c_1^t, c_2^t) satisfies the feasibility, if $p_i^B((h^t, x^t))$ is well-defined, then $p_i^B((h^t, x^t)) = p_i((h^t, x^t))$.*

Proof. By the definition of the on-path history h^t, we have $x^{t-1} = \bar{x}^{t-1}$.

First, suppose that $t \leq \bar{t} - 1$. If $p_i^B((h^t, x^t))$ is well-defined, then $c_j^t = \bar{c}_j^t$ because otherwise, $s_j(c_j^t|H, h^t) = s_j(c_j^t|L, h^t) = 0$. Given such c_j^t, we have $x^t = \bar{x}^{t-1} - (\bar{c}_1^t + \bar{c}_2^t) = \bar{x}^t$. Since $s_j(c_j^t|H, h^t) = s_j(c_j^t|L, h^t) = 1$, we must have $p_i^B((h^t, x^t)) = p_i(h^t) = P_j$. On the other hand, $p_i((h^t, x^t)) = P_j$ by the definition of the belief function in the on-path case because the continuing history $h^{t+1} = (h^t, x^t)$ is on-path. So, we have $p_i^B((h^t, x^t)) = p_i((h^t, x^t))$.

Second, suppose that $t = \bar{t}$. If $p_i^B((h^t, x^t))$ is well-defined, then we have either $c_j^t = \bar{c}_j^{\bar{t}} = \bar{x}^{\bar{t}-1}$ for the case $V_j = H$ or $c_j^t = \bar{c}_j^{\bar{t}} - (1-\bar{\delta})H$ for the case $V_j = L$ because otherwise, $s_j(c_j^t|H, h^t) = s_j(c_j^t|L, h^t) = 0$. If $c_j^t = \bar{x}^{\bar{t}-1}$, then $x^t = 0$, which is inconsistent with our assumption that $x^t > 0$. So, we have $c_j^t = \bar{c}_j^{\bar{t}} - (1-\bar{\delta})H$, and hence $x^t = (1-\bar{\delta})H$. Since $s_j(c_j^t|H, h^t) = 0$, we must have $p_i^B((h^t, x^t)) = 0$. Furthermore, since $x^t \neq \bar{x}^{\bar{t}} \equiv 0$, agent j is the deviator and agent i is the punisher for the continuing history $h^{t+1} = (h^t, x^t)$. By the definition of the belief function in the punisher's case with $x^t \leq (1-\bar{\delta})H$, we have $p_i((h^t, x^t)) = 0$. So, we have $p_i^B((h^t, x^t)) = p_i((h^t, x^t))$. □

Lemma 16. *Suppose that agent j moves as the deviator given an off-path history h^t, that is, $j = m(t) = d(h^t) \neq i$. For $x^t = x^{t-1} - (c_1^t + c_2^t) > 0$ where (c_1^t, c_2^t) satisfies the feasibility, if $p_i^B((h^t, x^t))$ is well-defined, then $p_i^B((h^t, x^t)) = p_i((h^t, x^t))$.*

Proof. For the continuing history (h^t, x^t), agent i moves as the punisher since $i \neq d(h^t)$.

First, suppose that $x^{t-1} \leq (1-\bar{\delta})L$. If $p_i^B((h^t, x^t))$ is well-defined, then $c_j^t = x^{t-1}$ because otherwise, $s_j(c_j^t|H, h^t) = s_j(c_j^t|L, h^t) = 0$. Given such c_j^t, we have $x^t = 0$, which is inconsistent with our assumption that $x^t > 0$.

Second, suppose that $(1-\bar{\delta})L < x^{t-1} \leq (1-\bar{\delta})H$. If $p_i^B((h^t, x^t))$ is well-defined, then we have either $c_j^t = x^{t-1}$ for the case $V_j = H$ or $c_j^t = 0$ for the case $V_j = L$ because otherwise, $s_j(c_j^t|H, h^t) = s_j(c_j^t|L, h^t) = 0$. If $c_j^t = x^{t-1}$, then $x^t = 0$, which is inconsistent with our assumption that $x^t > 0$. So, we have $c_j^t = 0$, and hence $x^t = x^{t-1} \leq (1-\bar{\delta})H$. Since $s_j(c_j^t|H, h^t) = 0$, we must have $p_i^B((h^t, x^t)) = 0$. On the other hand, $p_i((h^t, x^t)) = 0$ by the definition of the belief function in the punisher's case with $x^t \leq (1-\bar{\delta})H$. So, we have $p_i^B((h^t, x^t)) = p_i((h^t, x^t))$.

Third, suppose that $(1-\bar{\delta})H < x^{t-1} \leq \bar{\delta}L + (1-\bar{\delta})H$. If $p_i^B((h^t, x^t))$ is well-defined, then we have either $c_j^t = x^{t-1}$ for the case $V_j = H$ or $c_j^t = x^{t-1} - (1-\bar{\delta})H$ for the case $V_j = L$ because otherwise, $s_j(c_j^t|H, h^t) = s_j(c_j^t|L, h^t) = 0$. If $c_j^t = x^{t-1}$, then $x^t = 0$, which is inconsistent with our assumption that $x^t > 0$. So, we have $c_j^t = x^{t-1} - (1-\bar{\delta})H$, and hence $x^t = (1-\bar{\delta})H$. Since $s_j(c_j^t|H, h^t) = 0$, we must have $p_i^B((h^t, x^t)) = 0$. On the other hand, $p_i((h^t, x^t)) = 0$ by the definition of the belief function in the punisher's case with $x^t \leq (1-\bar{\delta})H$. So, we have $p_i^B((h^t, x^t)) = p_i((h^t, x^t))$.

Fourth, suppose that $\bar{\delta}L + (1-\bar{\delta})H < x^{t-1} \leq \bar{\delta}^3 L + (1-\bar{\delta}^3)H$. If $p_i^B((h^t, x^t))$ is well-defined, then we have either $c_j^t = x^{t-1}$ for the case $V_j = H$ or $c_j^t = 0$ for the case $V_j = L$ because otherwise, $s_j(c_j^t|H, h^t) = s_j(c_j^t|L, h^t) = 0$. If $c_j^t = x^{t-1}$, then $x^t = 0$, which is inconsistent with our assumption that $x^t > 0$. So, we have $c_j^t = 0$, and hence $x^t = x^{t-1} > \bar{\delta}L + (1-\bar{\delta})H$. Since $s_j(c_j^t|H, h^t) = 0$, we must have $p_i^B((h^t, x^t)) = 0$. On the other hand, $p_i((h^t, x^t)) = 0$ by the definition of the belief function in the punisher's case with $x^t > \bar{\delta}L + (1-\bar{\delta})H$. So, we have $p_i^B((h^t, x^t)) = p_i((h^t, x^t))$.

Finally, suppose that $x^{t-1} > \bar{\delta}^3 L + (1-\bar{\delta}^3)H$. Since deviator j moves in period t given the off-path history h^t, it must be the case that deviator j has deviated for the first time in period $(t-2)$ or before. So, h^{t-1} must be off-path. Since $i = m(t-1)$, we have $p_i(h^t) = p_i(h^{t-1})$. Since h^{t-1} is the off-path history with $x^{t-2} \geq x^{t-1} > \bar{\delta}^3 L + (1-\bar{\delta}^3)H > \bar{\delta}L + (1-\bar{\delta})H$, we have $p_i(h^{t-1}) = 0$ by the definition of the belief function in the punisher's case with $x^{t-2} > \bar{\delta}L + (1-\bar{\delta})H$. Hence, $p_i(h^t) = p_i(h^{t-1}) = 0$. If $p_i^B((h^t, x^t))$ is well-defined, then $(1 - p_i(h^t))s_j(c_j^t|L, h^t) > 0$ since $p_i(h^t)s_j(c_j^t|H, h^t) = 0$. When $(1 - p_i(h^t))s_j(c_j^t|L, h^t) > 0$, we have $c_j^t = 0$ because otherwise, $s_j(c_j^t|L, h^t) = 0$. Therefore, if $p_i^B((h^t, x^t))$ is well-defined, then $x^t = x^{t-1} > \bar{\delta}L + (1-\bar{\delta})H$ and $p_i^B((h^t, x^t)) = 0$. On the other hand, $p_i((h^t, x^t)) = 0$ by the definition of the belief function in the punisher's case with $x^t > \bar{\delta}L + (1-\bar{\delta})H$. So, we have $p_i^B((h^t, x^t)) = p_i((h^t, x^t))$. □

Lemma 17. *Suppose that agent j moves as the punisher given an off-path history h^t, that is, $j = m(t) \neq d(h^t) = i$. For $x^t = x^{t-1} - (c_1^t + c_2^t) > 0$ where (c_1^t, c_2^t) satisfies the feasibility, if $p_i^B((h^t, x^t))$ is well-defined, then $p_i^B((h^t, x^t)) = p_i((h^t, x^t))$.*

Proof. For the continuing history (h^t, x^t), agent i moves as the deviator since $i = d(h^t)$.

First, suppose that $x^{t-1} \leq (1-\bar{\delta})H$. If $p_i^B((h^t, x^t))$ is well-defined, then $c_j^t = x^{t-1}$ because otherwise, $s_j(c_j^t|H, h^t) = s_j(c_j^t|L, h^t) = 0$. Given such c_j^t, we have $x^t = 0$, which is inconsistent with our assumption that $x^t > 0$.

Second, suppose that $x^{t-1} > (1-\bar{\delta})H$. If $p_i^B((h^t, x^t))$ is well-defined, then we must have $c_j^t = \max\{0, x^{t-1} - \bar{\delta}L - (1-\bar{\delta})H\}$ because otherwise, $s_j(c_j^t|H, h^t) = s_j(c_j^t|L, h^t) = 0$ as described in the punisher's case of the strategy. Given such c_j^t, we have $x^t = \min\{x^{t-1}, \bar{\delta}L + (1-\bar{\delta})H\} > 0$. Since $s_j(c_j^t|H, h^t) = s_j(c_j^t|L, h^t) = 1$ for such c_j^t, we must have $p_i^B((h^t, x^t)) = p_i(h^t)$. Furthermore, $p_i(h^t) = p_i(h^{t-1})$ by the definition of $p_i(h^t)$ in the case where $i = m(t-1)$. If h^{t-1} is on-path, then $p_i(h^{t-1}) = P_j$ by the definition of the belief function. If h^{t-1} is off-path and hence agent i is the deviator for h^{t-1}, then $p_i(h^{t-1}) = P_j$ by the definition of the belief function. Therefore, $p_i^B((h^t, x^t)) = P_j$. On the other hand, $p_i((h^t, x^t)) = P_j$ by the definition of the belief function in the deviator's case. So, we have $p_i^B((h^t, x^t)) = p_i((h^t, x^t))$. □

Proposition 2. *A pair (s, p) satisfies reasonability.*

Proof. Lemmas 14 through 17 show the result in exhaustive cases. □

By Propositions 1 and 2, (s, p) is a perfect Bayesian equilibrium. It is clear that the history $\bar{h}^{\bar{t}} = (\bar{x}^0, \bar{x}^1, \ldots, \bar{x}^{\bar{t}-1})$ is realized along the equilibrium path. So, the proof of the theorem is completed.

4. Conclusion

We have investigated two-player contribution games that are similar to the ones studied by Compte and Jehiel [2003] but different in that our games are played in incomplete information environments. We have proved that, if each player's type is unknown to his opponent, and if players are sufficiently patient, then there exists a perfect Bayesian equilibrium in which step-by-step contributions are realized along the equilibrium path. Unlike Samejima [2013], our result holds without the upper bound condition on the prior probabilities. So, as long as the prior probabilities are neither 0 nor 1, that is, as long as each player is not 100% sure about the type of his opponent, our result holds. Our result indicates that *very small* uncertainties about valuations held by the opponent players can be a source of the gradualism.

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