The CAPM and the Single-Index Model--Ex-ante Expectations and Ex-post Tests

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The CAPM and the Single-Index Model
—Ex-ante Expectations and Ex-post Tests—*

Midori Munechika

Contents

Ⅰ. Introduction

The relationship of the risk-return trade-off is the heart of equilibrium asset pricing theories. The capital asset pricing model (CAPM) is a theory of determining equilibrium prices of capital assets, in which a systematic factor plays a key role. Markowitz'[1952] mean-variance analysis of portfolio theory laid the groundwork for the CAPM. It was originally developed independently by Sharp [1964], Lintner [1965] and Mossin [1966] (the Sharp-Lintner-Mossin form of the capital asset pricing model) and has been extended to a variety of forms (often called nonstandard forms of the CAPM) incorporating more realistic phenomena by modifying the stringent assumptions underlying it.

Financial economics is one of the most empirical disciplines in economics. Much of the work in this field approaches theoretical issues in a positive context. The empirical, but nonexperimental nature requires introducing model-based statistical inference to positive analysis. During the past decade the use of econometric methods in finance has dramatically increased,

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paralleling rapid expansion of global financial markets. Financial econometrics is essential for testing theories of determining asset prices.

Another feature of financial economics is that uncertainty plays a crucial role in both theory and its empirical implementation. To understand how the impacts of uncertainty on market prices of assets are involved in the theory and how its uncertainty is reflected in the regression models used to test the theoretical implications is important for an adequate treatment of financial econometrics.

The purpose of this paper is to consider the theoretical implications of the CAPM and examine the issues of testing an ex-ante expectational model by using ex-post data. In Section Ⅱ, mean-variance efficiency of the CAPM is presented on the ground of the Markowitz mean-variance approach to portfolio analysis. In Section Ⅲ, the theoretical implications of the CAPM are examined. In Section Ⅳ, the single-index model as a return generating process is introduced and an estimable theoretical model is derived by incorporating the CAPM and the single-index model. In Section Ⅴ, two types of procedures to test the CAPM are explained. Finally, as concluding remarks of this paper, some of the empirical results are presented and testing problems of the CAPM are pointed out.

Ⅱ. The CAPM and Mean-Variance Efficiency

The CAPM has been developed from the Markowitz mean-variance approach to portfolio analysis. The concept of mean-variance efficiency is the key to considering the CAPM *and its* testable theoretical implications. Mean-variance efficiency stems from the theory of rational choice under uncertainty, that is, the expected utility maxim. How investors construct their optimal portfolios analyzed by the mean-variance approach, which postulates that security returns are normally distributed and investor behavior can be represented by the expected utility function.[1] From the utility function, non-satiation about wealth (i.e. more wealth is preferred to less) and risk-averse investors are assumed. Then, an optimizing behavior of investors is that they prefer a higher expected return to a lower one, other things been equal, and a lower level of risk to a greater level with a given expected rate of return, which is referred to as the dominance principle.

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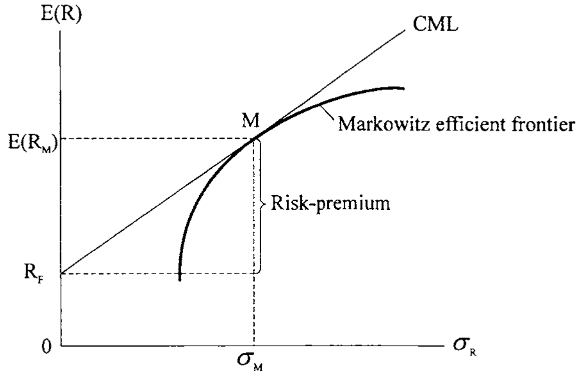
[1] For a more detailed technical discussion of the two fundamental assumptions underlying the mean-variance approach, see Munechika [2002c].

Figure 1 The Markowitz efficient frontier



Portfolios that satisfy the dominance principle are mean-variance efficient. The Markowitz mean-variance efficient frontier is the efficient set of portfolios with risky securities that satisfies the dominance principle, which is depicted as a thick curve in Figure 1.

The CAPM is derived from a linear efficient frontier extended from the Markowitz efficient frontier by allowing riskfree assets to be included in portfolios. By introducing the possibility to hold a riskfree security in portfolio and the assumption of borrowing and lending at the riskfree rate, the new efficient set with a riskfree security becomes a linear efficient frontier, which is called the capital market line (CML).

The CML leads all investors to invest in the same risky asset portfolio of point M in Figure 1. Point M is the point of tangency to the efficient set of risky securities. It provides the investor with the best possible opportunities since it offers the highest ratio of expected excess return on the risky security ($E(R_M)-R_F$) to risk σ_M . The expected excess return on the risky security is known as a risk-premium. This implies that the investor would always choose the risky security of point M. Regardless of the investor's preference, he would never choose any other point on the efficient frontier created by Markowitz diversification. Only one point M of the efficient set remains efficient and the others become inefficient.

In general, the tangency portfolio represented by point M is referred to as the market portfolio. Why is point M the market portfolio? When investors perform portfolio analysis, they must estimate the expected returns and variances for individual securities and the covariances between combinations of securities before calculating the efficient set of risky securities. Although the possibility exists of variation among different investors' estimates, their estimates might not vary much from other investors' estimates. This is because all investors would use the same information to form their expectations in an efficient market.[2] Under such homogeneous expectations, Figure 1 would be the same for all investors and they would hold the portfolio of risky securities represented by point M. The portfolio that all investors hold is a market-valued weighted portfolio of all existing securities, which is called the market portfolio.

Therefore, all risk-averse investors hold combinations of only two portfolios on the CML: the market portfolio and a riskfree asset. This tendency is known as the two mutual fund theorem. It maintains that, in the presence of a riskfree security, the optimal risky portfolio indicated by point M can be uniquely selected without any knowledge of investors preferences. Therefore, investors can separate their decision of selecting the efficient portfolio into two stages. In the first stage is the investor calculates the efficient set of risky securities, depicted by a curved thick line and then determines point M. The second stage is to determine how the investor will combine point M with the riskfree security depending on his risk preference. The two mutual fund theorem is also referred to as the separation theorem because of this division of the investment decision from the financing.[3]

Since all efficient portfolios combining the market portfolio and a riskfree asset lie on the CML, their portfolio returns have perfectly positively correlated systematic fluctuations in the market. That is, portfolio risks presented along the CML only contain market risk. This means that the specific risks of individual securities will be offset by the unique variability of the other assets making up the portfolio, thus the portion of unsystematic (specific) risk has diversified away to zero. This point leads to the CAPM, which provides an explicit formula for the trade-off between the expected return and market (undiversifiable, or systematic) risk.

Ⅲ. The Security Market Line

An investor holding a well-diversified portfolio considers the variance of his portfolio's return as the measure of his portfolio risk. However, he is no longer interested in the variance of each security's return because it can be eliminated through diversification. Now the investor would be interested in the contribution of an individual security to the risk of a well-diversified portfolio, in other words, in the market risk of the individual security. This is measured by beta

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[2] In capital market theory, the financial market is assumed to be efficient in the sense that prices always "fully reflect" available information. The term "fully reflect" means that all the information fully utilized in determining equilibrium prices (or expected returns) on securities. Sharp [1964, p.433] assumed the homogeneity of investor expectations. This assumption is inseparable from the efficient market hypothesis (EMH). See Munechika [2002a, 2002b] for a more detailed discussion about the EMH.

[3] Tobin [1958] first presented this proof for the case in which the riskfree rate is zero (cash).

(β), which represents the sensitivity of a change in the return of an individual security to the change in return of the market portfolio. Beta can be defined as:

$$(1)\quad \beta_i = \frac{Cov(R_i, R_M)}{\sigma_M{}^2} = \frac{\sigma_{iM}}{\sigma_M{}^2}$$

where $Cov(R_i, R_M)$ is the covariance between the return on security i and the return on the market portfolio M, and $\sigma_M{}^2$ is the variance of the market return.

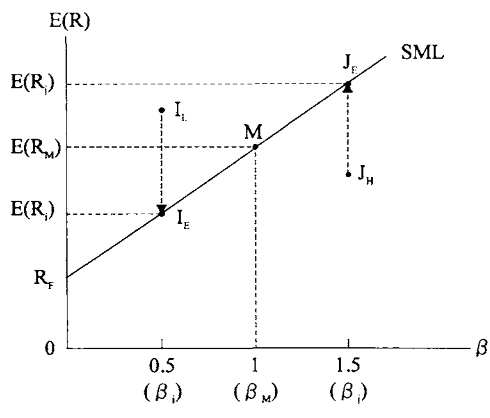
Total portfolio risk consists of diversifiable (specific) risk and undiversifiable (market, or systematic) risk. According to the two mutual fund theorem, everybody will hold a portfolio combining the market portfolio and the riskfree assets. The market portfolio only contains market risk and the riskfree asset does not contain risk (variance of the expected return). Thus, the risk of all portfolios of investments only contains market risk, which is perfectly positively correlated to their expected portfolio return because the CML is depicted as a straight line in Figure 1. That is, all portfolios of investments must lie along a straight line in expected return-beta space as shown in Figure 2.

The straight line can be identified by taking only two points. Under the assumptions of the CAPM, everybody will hold the market portfolio. Thus, we will choose the market portfolio with a beta of one as one point and the intercept as the second point. In general, the equation of a straight line has the form

$$(2)\quad y = a + bx$$

In this case, $y = E(R_i)$ and $x = \beta_i$. One point on the line is the market portfolio whose beta

Figure 2 Expected returns and betas



coefficient is one. Thus,

$$E(R_M) = a + b(1)$$

(3) $b = E(R_M) - a$

Another point on the line is the riskfree asset whose beta coefficient is zero. Thus,

$$E(R_F) = R_F = a + b(0)$$

(4) $a = R_F$

Putting these together and substituting equations (3) and (4) into equation (2) yields

(5) $E(R_i) = R_F + \beta_i[E(R_M) - R_F]$

Equation (5) is the mathematical model of the CAPM, which is depicted as the security market line (SML) in Figure 2. The CAPM is an expectational (ex-ante) model for a single period. It implies that the expected return on security i is linearly related to its beta. Hence, the CAPM demonstrates a positive relation between beta and the expected rate of return, which is required in order to attract investors.

The CAPM can be compactly expressed in terms of expected excess return in lieu of expected return.

(6) $E(R_i) - R_F = \beta_i[E(R_M) - R_F]$

When expected excess return $E(Z_i) = E(R_i) - R_F$, then we get

(7) $E(Z_i) = \beta_{im} E(Z_M)$

where Z_M is the expected excess return on the market portfolio. Therefore, using equation (1), beta can be expressed as

(8) $\beta_{im} = \dfrac{Cov(Z_i, Z_M)}{{\sigma_Z}^2} = \dfrac{\sigma_{iZ}}{{\sigma_Z}^2}$

Equations (1) and (8) are equivalent since the riskfree rate is treated as being nonstochastic.[4]

The SML tells us the relationship between expected return on an individual security and beta of the security in equilibrium. More precisely, it clarifies the relationship between the beta of any asset and its equilibrium expected return. This means that the CAPM expected return-beta relationship applies not only to portfolios but also single assets. To shed light on this point, we suppose two risky securities, i and j, and a portfolio P consisting of securities i and j. In the portfolio P, a proportion α is invested in security i and the remaining proportion $(1-\alpha)$

[4] On the contrary, in empirical implementations, proxies for the riskfree rate R_F are stochastic and thus the beta can differ. Therefore, empirical work often employs excess returns and thus uses equation (8). See Campbell, Lo and MacKinlay [1997], p.182.

is invested in security j.

The return on the portfolio R_P is given by

(9) $$R_P = \alpha R_i + (1-\alpha)R_j$$

By taking expectations of equation (9), we have the expected return on portfolio:

(10) $$E(R_P) = \alpha E(R_i) + (1-\alpha)E(R_j)$$

Since the beta of the portfolio is defined as its covariance with the market portfolio, we can get

(11) $$\beta_P = \frac{Cov(R_P, R_M)}{{\sigma_M}^2} = \frac{Cov(\alpha R_i + (1-\alpha)R_j, R_M)}{{\sigma_M}^2}$$

by using equation (9).

The following property of covariance can be applied to compute equation (11).

$$Cov(aX + bY, Z) = aCov(X, Z) + bCov(Y, Z)$$

In this case, $aX + bY = \alpha R_i + (1-\alpha)R_j$, $Z = R_M$. Therefore,

(12) $$\begin{aligned}\beta_P &= \frac{\alpha Cov(R_i, R_M) + (1-\alpha)Cov(R_j, R_M)}{{\sigma_M}^2} \\ &= \alpha \frac{Cov(R_i, R_M)}{{\sigma_M}^2} + (1-\alpha)\frac{Cov(R_j, R_M)}{{\sigma_M}^2} \\ &= \alpha\beta_i + (1-\alpha)\beta_j\end{aligned}$$

where $a = \alpha$, $X = R_i$, $b = (1-\alpha)$, and $Y = R_j$. As shown in equations (10) and (12), both the expected return and the beta of the portfolio consisting of securities i and j are linear combinations of the expected returns and betas, respectively, of the underlying securities. As a result, the SML in expected return-beta space is depicted as a linear relationship between the beta of any security and its expected return in equilibrium.

The CAPM asserts that all securities must lie on the SML in market equilibrium. This implies that there is no arbitrage opportunity in the market. For example, if security i's expected return lies above the line at I_L, an investor could get a higher expected return at I_L than by holding a mixture with half of the riskfree security and half in the market portfolio at the same level of beta, 0.5. Then, everybody would want to buy security i. Conversely, if security j's expected return lies below the line at J_H, the investor could get a higher expected return on j for the same beta by borrowing 50 cents for every dollar of his own money and investing in the market portfolio. Therefore, there is nobody who wants to hold security j. Security j is priced too high at J_H because its expected return is below the rate of return that investors require to induce them to accept its market risk.[5]

However, the above situation cannot continue for a long time. So long as arbitrage opportunities exist, the price of security i will rise from buying pressure, while the price of security j will fall from selling pressure in the market. These price readjustments lead the expected returns of i and j to their required rate of return positions at point I_E and point J_E on the SML. Thus, each and every security must lie on the SML under no arbitrage condition in equilibrium.[6]

In short, Sharpe [1991, p.499] summarizes the key implications of the CAPM as follows. First, the market portfolio will be an ex-ante mean-variance efficient since it is located on the Markowitz efficient frontier. Second, all efficient portfolios will be equivalent to investment in the market portfolio plus, possibly, lending or borrowing the riskfree asset. Third, there will be a linear relationship between expected return and beta.

As we have already discussed, the assumptions underlying the first implication of the CAPM are the same ones of the mean-variance analysis. The second implication is based on the assumptions of homogeneous expectations, unlimited borrowing and lending at the riskfree rate. The third implication is supported by perfectly competitive capital market, which has no transaction costs.[7]

Ⅳ. The Single-Index Model

The CAPM is an expectational model expressing relationships among expected returns for a single period. However, we can't observe these expectations directly. Theoretically, the value of the beta coefficient is to be interpreted as ex ante value based on probabilistic beliefs about future security returns. Hence, implementation of the CAPM that does not include a time dimension requires adding the assumptions concerning the return generation process (the time-series behavior of returns) and estimating the model over time. Although Sharp [1991, p.497] mentions that there are no assumptions about the return generation process in the CAPM, and thus, its results are completely consistent with any such process, his initial approach to portfolio

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[5] The relationship between the expected return of a security and its market price is given by :

$E(R)$ = (expected capital gain or loss + expected cash dividends)／purchase price at the market

As the market price of the security increases, other things being equal, the expected return decreases, and vice versa.

[6] Black [1972, p. 444] points out that the length of the period for which the model applies is not specified.

[7] More specifically, there are another assumptions such as infinitely divisible assets, the absence of personal income tax, unlimited short sales, and all marketable assets. See Elton and Gruber [1995], p.295.

selection supposed the single index model was a return generating process.[8]

In general, the return on any security R_i consists of two parts: the expected parts of the return $E(R_i)$ and the unexpected part of the return U_i:

(13) $R_i = E(R_i) + U_i$

The unexpected part of the return can be divided into two components: a systematic risk m_i, which is the impact of unanticipated macro events, and specific risk e_i, which is the impact of unanticipated firm-specific events.

(14) $R_i = E(R_i) + m_i + e_i$

The expected values of m_i and e_i are zero since both express the impact of unanticipated events, which by definition must average out to zero.

Different firms can be differently affected by macro events, which implies that they have different sensitivities to macroeconomic events. If we denote the unanticipated components of the macro factor by F and the sensitivity of security i to macro events by beta β_i, then

(15) $R_i = E(R_i) + \beta_i F + e_i$

where $m_i = \beta_i F$. Equation (15) is referred to as a single-factor model.[9]

The unanticipated change in the systematic factor F is a surprise in the return on the market expressed as $R_M - E(R_M)$.

(16) $R_i = E(R_i) + \beta_i[R_M - E(R_M)] + e_i$

$= E(R_i - \beta_i R_M) + \beta_i R_M + e_i$

(17) $R_i = \alpha_i + \beta_i R_M + e_i$

where α_i is an intercept term equal to $E(R_i - \beta_i R_M)$. That is to say, the return on the stock R_i can be divided into three components: a constant α_i, a component proportional to the return on a market index $\beta_i R_M$ and a random and unpredictable component e_i. The intercept term α_i is the expected value of the component of security i's return that is independent of the market's performance. The beta coefficient β_i is specific for each security and measures the security's sensitivity to the market. The random component e_i represents the deviation of the return on the security from its expected value. Equation (17) is the basic equation of the single-index model based on the notion that the correlation structure of security returns is due to a single

[8] The single-index model was originally developed by Sharpe [1963], in which the model was called the diagonal model.

[9] When it uses the market index as a proxy for the only systematic factor, it is called a single-index model. See Bodie, Kane and Marcus [1999], pp.282-283.

common influence or index.[10] It states that security returns are linearly related to the return on a market portfolio.

The assumptions behind the single-index model are as follows. First, the expected values of e_i are zero.

(18) $E(e_i) = 0$

Second, the impacts of unanticipated firm-specific events on the returns on the securities (i.e. specific risk) are independent of the returns on the market. It means that, on average, whether the unpredictable component of the security return is positive or negative is unrelated to whether the return on the market is high or low. This assumption can be expressed in terms of covariances between e_i and R_M.

(19) $Cov(e_i, R_M) = E[(e_i - 0)(R_M - \overline{R_M})] = E[e_i(R_M - \overline{R_M})] = 0$

where $\overline{R_M}$ is the average return on the market. Third, for any two securities i and j, the random and unpredictable components of their returns e_i and e_j are uncorrelated with each other. This is the assumption of no autocorrelation.

(20) $Cov(e_i, e_j) = E[(e_i - 0)(e_j - 0)] = E(e_i, e_j) = 0$

This implies that the error e_i in predicting the returns on security i is independent of the error e_j in predicting the returns on security j, and thus the only reason securities vary together is due to a common co-movement with the market.

The advantage of using the single-index model as a return generating process is to enable investors greatly to relieve the problem of implementation by reducing dramatically the number of parameters they must estimate. This advantage stems from the assumptions of equations (19) and (20) behind the singe-index model.[11]

This advantage of the simplification using the single-index model as a return-generating process is not without cost. The single-index model expressed by equation (17) says that risks of individual securities arise from two sources: market or systematic risk, reflected in $\beta_i R_M$ and firm-specific risk, reflected in e_i. This simple dichotomy may oversimplify factors of real-world uncertainty. For example, it ignores industry events, which affect many firms within a single industry but do not influence the macroeconomy as a whole.

This restriction stems from the assumption of equation (20), which implies that firm-specific

[10] Elton and Gruber [1995] present a more detailed explanation of the single-index model and the problems of estimating beta in chapter 7.

[11] Sharpe [1963] pointed out the advantage of using this model for practical applications of the Markowitz portfolio analysis technique. For a mathematical proof, see Appendix.

risk of each security is uncorrelated with others. A less restrictive form of the single-index model (which lacks the assumption of $Cov(e_i, e_j) = 0$) is known as the market model. The market model is identical to equation (17) except that $Cov(e_i, e_j) = 0$ is not assumed. Now the market model is used extensively in empirical research in finance.[12]

As mentioned earlier, in order to test the empirical performance of the CAPM, we have to obtain the test equation with ex-post data. Taking expected values for equation (17), we obtain

(21) $E(R_i) = \alpha_i + \beta_i E(R_M)$

where α_i and β_i are constant and $E(e_i) = 0$. Subtracting equation (17) from (21), we obtain

$$E(R_i) - R_i = \beta_i E(R_M) - \beta_i R_M - e_i$$

(22) $E(R_i) = R_i + \beta_i E(R_M) - \beta_i R_M - e_i$

Substituting equation (22) into (5), we obtain

$$R_i + \beta_i E(R_M) - \beta_i R_M - e_i = R_F + \beta_i [E(R_M) - R_F]$$

(23) $R_i = R_F + \beta_i (R_M - R_F) + e_i$

This is the model of a form with ex-post data, which has been examined using the empirical tests of the CAPM. Since equation (23) is formulated by combining the CAPM with the single-index model, this model is implicitly based on that assumptions that the CAPM and the single-index model simultaneously hold in every period and that beta is stable over time. Therefore, the hypothesis that should be tested empirically is that beta is positively and linearly related to return.

V. Procedures to Test the CAPM

Basically, there are two types of procedures to test the CAPM. One type is a regression using returns and the other is a regression using excess returns.

The regression based on returns involves a two-step approach. The first step is the time-series regression. For each of N securities included in the sample, the following equation is regressed to estimate security betas.

(24) $R_{it} = \alpha_i + \beta_i R_{Mt} + e_{it}$

where R_{it} and R_{Mt} are the rates of return on security i and on the market portfolio (say, market index such as the S & P 500 or TOPIX) in time period t; α_i is the intercept, β_i is the beta coefficient of security i, and e_{it} are the residuals. The R-squared (R^2) of the regression of equation (24) provides an estimate of the proportion of the risk (variance) of security i that can

[12] As a result, the market model does not have the advantage of the simple expressions of portfolio risk arising under the single-index model. See Elton and Gruber [1995], p.152 and Appendix.

be attributed to market risk. Thus, the specific risk is captured by the balance ($1-R^2$).[13]

According to equation (23), the test equation of the CAPM can be rewritten as

(25) $R_i = R_F(1-\beta_i)+\beta_i R_M + e_i$

A comparison of the estimated values of the intercept $\hat{\alpha}_i$ to $R_F(1-\hat{\beta}_i)$ provides a measure of the security's performance during the period of the regression, relative to the CAPM. When $\hat{\alpha}_i = R_F(1-\hat{\beta}_i)$, security i did as well as expected on the basis of the CAPM during regression period. If $\hat{\alpha}_i > R_F(1-\hat{\beta}_i)$, security i did better than expected. Conversely, if $\hat{\alpha}_i < R_F(1-\hat{\beta}_i)$, security i did worse than expected. The difference between $\hat{\alpha}_i$ and $R_F(1-\hat{\beta}_i)$, given the average market return and the security's beta, is referred to as Jensen's alpha, which is one of the risk-adjusted performance measures.[14]

The second step is the cross-sectional regression. Now we present the following regression model.

(26) $R_{it} = \gamma_{0t} + \gamma_{1t}\beta_i + \gamma_{2t}\beta_i^2 + \gamma_{3t}s_i + \delta_{it}$

This is the model of Fama and MacBeth [1973] which is the first extensive empirical research using a cross-sectional regression methodology. Comparing equation (26) with the test equation (23) of the CAPM, we can regard $\hat{\gamma}_{0t}$ as an estimate of R_F and $\hat{\gamma}_{1t}$ as an estimate of $(R_M - R_F)$, the market risk premium. If the CAPM holds, statistically,

1) $\hat{\gamma}_0 = R_F$

2) $\hat{\gamma}_1 = R_M - R_F$, which should be positive.

3) $\hat{\gamma}_2 = 0$, which is the hypothesis condition of the linear relationship between the expected return on security i and its risk in any efficient portfolio. The variable β_i^2 in equation (26) is included to test this linearity.

4) $\hat{\gamma}_3 = 0$, which is the hypothesis of the condition that β_i is a complete measure of the risk of security i. The variable s_i means some measure of the risk of security i not deterministically related to β_i.

Campbell, Lo and MacKinlay [1997, p.216] point out the usefulness of the Fama-MacBeth approach because it can easily be modified to accommodate additional risk measures beyond the

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[13] The R-squared gives the proportion of the total variation in the dependent variable (R_i) explained by the single explanatory variable (R_M). Total risk of security is divided into two parts: the market risk and the specific risk. The specific risk is a diversifiable risk, and thus, unrewarded in the CAPM.

[14] See Damodaran [1997], p.130.

CAPM beta.[15] In fact, Fama and French [1992] conducted the asset-pricing tests including additional risk measures such as size, book to market equity by using the cross-sectional regression approach of Fama and MacBeth [1973].

Next, another type of procedure to test the CAPM is regression through the origin. As mentioned in Section Ⅲ, the CAPM can be expressed in terms of expected excess return (i.e. the risk-premium).

(7) $E(Z_i) = \beta_{im} E(Z_M)$

For empirical purposes, equation (23) is modified as

$$R_i - R_F = \beta_i (R_M - R_F) + e_i$$

(27) $Z_i = \beta_{im} Z_M + e_i$

and then, the regression equation in the excess-return market model is expressed as

(28) $Z_{it} = \alpha_{im} + \beta_{im} Z_{Mt} + e_{it}$

where Z_{it} and Z_{Mt} are the realized excess returns in time period t for security i and the market portfolio, respectively. When the CAPM holds, the intercept α_{im} should be zero. If α_{im} is greater than zero, security i does better than expected; conversely, if α_{im} is less than zero, it does worse than expected.

Ⅵ. Summary and Empirical Results

In this paper we have considered the CAPM and how to test it empirically. We began with examining the testable theoretical implications of the CAPM and then, introduced the single-index model as a return generation process. Next, in order to formulate an estimable theoretical model, we developed the model of a form with ex-post data by combining the CAPM with the single-index model.

A huge amount of empirical research has been conducted since the CAPM was developed in the 1960s. Empirical results have been controversial from the beginning and summarizing them is one of the most difficult tasks in this field. The test methodology has become more sophisticated with the advance of econometrics. Broadly speaking, the early empirical evidence was largely supportive of the CAPM since it indicated a reliable positive relation between average

[15] Campbell, Lo and MacKinlay [1997, p.216] also mention the two major problems of the Fama-MacBeth methodology. There are the error-in-valiables problem and the unobservability of the market portfolio. The first problem stems from the way in which the regressions is conducted using betas estimated from data since the market betas are not known and thus the Fama-MacBeth methodology can not be directly applied.

return and beta, although there was some evidence against it. After that, less favorable evidence, so-called anomalies, has been presented.

In particular, the paper written by Fama and French [1992] indicating evidence inconsintent with the CAPM attracted a great deal of attention in academic circles. Empirical research conducted by Fama and French [1992, p.459] concluded that a reliable positive relationship between average return and beta for 1941-1990 stocks could not found and the average slope on beta for 1966-1990 stocks was close to zero. Moreover, they suggested two variables having explanatory power regarding returns: size and book value to market value ratio. With the paper of Fama and French [1992] as a start, academic discussions focused on whether beta was dead.

Chan and Lakonishok [1994], whose title is "Are the Reports of Beta's Death Premature?", have drawn two implications from the CAPM for their empirical tests. One is that high-beta stock returns outperform low-beta stock returns, which reveals that beta plays a significant role in stock returns. The other is that the compensation for beta risk is equal to the rate of return on the market less the risk-free rate.

The results of the cross-sectional regressions between stock (portfolio) returns and betas vary considerably over time. During the period of 1932 and 1991 regression results show that high-beta stocks outperformed low-beta stocks although the difference was not as great as the CAPM predicts.[16] Up until 1982, the estimated compensation for beta risk was strikingly close to the realized market premium. However, in the last nine years the gap between them has widened considerably, which means the slope coefficient of the line relating return to beta has been too flat.

More interestingly, by picking up the sub-samples of both the ten largest down and up market months in running the cross-sectional regressions, the results show a close correspondence between the average realized premium and the average slope. These strong results should not be taken as a proof that, on average, high-beta stocks necessarily earn higher returns than low-beta stock. However, to know the close relationship between beta and downside risk can be useful for investors and fund managers because their major concern is downside risk. In this sense the importance of the beta still remains.

Chan and Lakonishok [1994] have accepted that the empirical support for beta was never strong. This is because of the difficulties underlying empirical research such as the influence of "noise" on stock returns, the limitations of the available data, the choice of time period,

[16] The estimated average compensation for beta risk is 0.47% per month and the average excess return on the market is 0.76% per month. See Chan and Lakonishock [1994], p.169.

unobservability of the true market portfolio, and the specific behavioral and institutional factors unrelated to risk.[17] However, they have concluded that sufficient evidence to dump beta could not be obtained from their empirical work.

Appendix

Implementation of the mean-variance approach to portfolio analysis requires investors to calculate portfolio returns and risks, given the expected returns, the variances, and the covariances of the underlying individual securities. In examining any portfolio that consists of n securities, the expected return and variance of any risky portfolio with weights in each security w_i are

(1) $$E(R_p)=\sum_{i=1}^{n} w_i E(R_i)$$

(2) $$\sigma_P{}^2=\sum_{i=1}^{n} w_i{}^2\sigma_i{}^2+\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} w_i w_j \sigma_{ij}$$

The total number of parameters to be calculated is $2n+\frac{n(n-1)}{2}$, because it comprises

n estimates of the expected returns $E(R_i)$,

n estimates of the expected returns $\sigma_i{}^2$, and

$\frac{n(n-1)}{2}$ estimates of covariances σ_{ij} between each pair of underlying security returns.

For instance, when n = 200, the number of parameters to be calculated is 20300. Therefore, the mean-variance approach requires calculating an exceedingly large number of parameters in the case of portfolios including a number of securities.

Now we derive the expected return, the variance and the covariance of securities by using the single-index model as a return generating process. First, the expected return on security i is:

(3) $$\begin{aligned} E(R_i) &= E[\alpha_i+\beta_i R_M+e_i] \\ &= E(\alpha_i)+E(\beta_i R_M)+E(e_i) \\ &= \alpha_i+\beta_i \overline{R_M} \end{aligned}$$

where α_i and β_i are constant, $\overline{R_i}=E(R_i)$ is the average return on security i, and $E(e_i)=0$.

Second, the variance of the return on security i is:

[17] Papers by Roll [1977] and [1978] criticized the usefulness of the CAPM because of its dependence on an unobservable market portfolio of risky assets.

(4) $\sigma_i^2 = E(R_i - \overline{R_i})^2$

Substituting for R_i and $\overline{R_i}$ from equation (4) yields

$$
\begin{aligned}
(5)\quad \sigma_i^2 &= E[\alpha_i + \beta_i R_M + e_i) - (\alpha_i + \beta_i \overline{R_M})]^2 \\
&= E[\beta_i (R_M - \overline{R_M}) + e_i]^2 \\
&= \beta_i^2 E(R_M - \overline{R_M})^2 + 2\beta_i E(R_M - \overline{R_M})e_i + E(e_i)^2
\end{aligned}
$$

By definition, variances of e_i and R_M are

(6) $E(e_i)^2 = \sigma_{e_i}^{\ 2}$ and

(7) $E(R_M - \overline{R_M})^2 = \sigma_M^{\ 2}$

Substituting for equation (6), (7) and $Cov(e_{i,} R_M) = E[e_i (R_M - \overline{R_M})] = 0$ into equation (5),

(8) $\sigma_i^2 = \beta_i^2 \sigma_M^{\ 2} + \sigma_{e_i}^{\ 2}$

Third, the covariance between security i and security j can be expressed as

(9) $\sigma_{ij} = E[(R_i - \overline{R_i})(R_j - \overline{R_j})]$

Substituting for R_i, $\overline{R_i}$, R_j and $\overline{R_j}$ into equation (9) yields

$$
\begin{aligned}
(10)\ \sigma_{ij} &= E[(\alpha_i + \beta_i R_M + e_i) - (\alpha_i + \beta_i \overline{R_M})]\cdot[(\alpha_j + \beta_j R_M + e_j) - (\alpha_j + \beta_j \overline{R_M})] \\
&= E[\beta_i (R_M - \overline{R_M}) + e_i]\cdot[\beta_j (R_M - \overline{R_M}) + e_j] \\
&= \beta_i \beta_j E(R_M - \overline{R_M})^2 + \beta_i E[(R_M - \overline{R_M})e_j] + \beta_j E[(R_M - \overline{R_M})e_i] + E(e_i e_j) \\
&= \beta_i \beta_j \sigma_M^{\ 2}
\end{aligned}
$$

since the last three terms are zero.

The main merit of the single-index model stems from equation (10). Now that we need not directly calculate all the pairs of correlation coefficients between securities, we can calculate them simply as the product of the betas of the securities, multiplied by the variance of the market index. Therefore, the total number of parameters to be calculated is $3n + 2$, because it comprises

n estimates of the expected returns $E(R_i)$,

n estimates of the expected returns $\sigma_i^{\ 2}$,

n estimates of beta coefficients β_i, and

2 estimates of the expected value and the variance of the returns on the market index. In the case of $n = 200$, the number of parameters to be calculated falls from 20300 to 602.

This is the advantage of using the single-index model as a return generating process, which enables investors to relieve the burden of implementation by reducing dramatically the number of parameters they must estimate.

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