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Intuitive evidence and formal evidence in proof-formation

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Section I. Introduction.

I would like to discuss “proof formation”

¹ as a general methodology of sciences and philosophy, with a special focus on “intuitive evidence” and “formal deductive evidence”. In my opinion, consideration on the evidence criteria of proof formations is important for consolidating theoretical disciplines. Euclid said at the beginning of the Elements that “Mathematics is Proofs”. What Euclid said would also be the case, in my opinion, for the theoretical aspects of sciences and of philosophy as they require justifications for the claims in a proof form. In fact, when a proof is concerned with foundations/consolidation of scientific methodologies themselves or fundamental metaphysical problems, such a proof should be more than just justification in the usual natural scientific attitude. I would like to take a close look at some issues on proof formation, in particular, on deductive or demonstrative evidence criteria, in this context. We especially compare the similarity and difference between some intuition-based proof models on the one hand and some form-based proof models on the other hand.

Section II. Intuitive evidence-based proof formation.

I first consider a typical intuitive evidence-based model of proof formation. By an intuitive evidence-based proof formation model, I mean the model of deductive proof formation in which validity of each deductive step is guaranteed by intuitive evidence so that intuitive evidence of the starting proposition of a deductive proof derives intuitive evidence of the conclusive proposition of the proof. In this model of proof formation, one always appeals to intuitive evidence at each step or move of proof formation and one presume that the intuitive evidence is “transitive”; A is intuitively evident, the deductive move from A to B is intuitively evident, then B is intuitively evident, and so on.

This model of proof formation may be contrasted to some other typical ones; for example, the truth-based proof formation models. One of the most typical truth based ones is a semantics-based model; for example Tarski style (and Kripke variants) formal semantics-based understanding of proofs, where “truth” of a proposition is defined by means of interpretation of the proposition composed of linguistic elements, and truth of a proposition depends on an interpretation assigned for the linguistic elements in the proposition. Validity of a deductive inference rule means just a preservation of truth in this sense. Another typical truth-based model may be involved with more metaphysical notion of truth of proposition, such as states-of-affairs, truth-maker or trope (although I do not go into the details on this type of models in this note), but still, in my opinion, most of truth-based model of proof formation naively takes the position that the standpoint that truth (in any sense) is “transitive” and that a deductive move from A to B preserves truth. The mathematical theorem to express this is called Soundness

Theorem. Another important proof formation model, which I would like to discuss in comparison with the intuitive evidence-based model is (a few variants of) form-based model.

As we point out in this note, the naïve classifications of types of proof models would not work. For example we discuss how the form-based constructivist proof formation model is also related to the intuitive evidence.

As a starting discussion I would present the intuitive evidence-based model according to a traditional reading of some texts of Descartes (such as Rule III).

Descartes, as well known, inspired from Beeckman, was enthusiastic to study *Mathesis Universalis* for a certain period, without complete success. Instead, he reached to an idea of formulating methods for general problem solving as his “Rules”, although he could not complete it. In “Rules”, he admits only two sources of indisputable knowledge (the knowledge without any possible doubt of error) for us; “intuition” and “deduction” for example Rule III in the edition of Adam and Tannery². The proof formation there consists of (1) the starting proposition(s) called “principles” and (2) a step-by-step deductive inference series (or what he often calls “chain” or “movement” of deductions) from which a conclusive proposition is derived. A conclusive proposition through a proof formation (namely a deductive knowledge) is indisputable when (1) each of the principles (starting propositions) is “intuitively evident” and (2) the deductive movement is deductively evident. Here, what is the criterion of the “deductive evidence” more precisely? In the case of the above reading of Rules a deductive movement is evident when each step of deductive inference is “intuitively evident”. Note that in this model of proof formations, the form of admissible inference rules is not pre-determined. In the traditional model of formal deductive inferences (of the Second Analysis) it is pre-fixed as the syllogism “forms”. In contrast to this traditional model, the Cartesian proof formation model allows any deductive inference move, without any pre-fixed inference form, under the condition that the inference move is intuitively evident. This is, in my opinion, an important difference between the Cartesian style model for the indisputable proof formations and the traditional one such as the Aristotelian one in the First Analysis.

As Rule III remarked, although a short-length proof formation process can be seen with intuitive evidence only, when a proof is long, the original intuitive evidence of each deductive inferential move in the middle of proof, the intuitive evidence of the conclusion would be lost or significantly weakened in the process of the proof formation starting from intuitively evident starting propositions. This is because intuitive evidence requires directness and actuality at least and the long series of deductive inference acts would lose directness and actuality of evidence.

In order to compensate the intuitiveness of the intermediate steps of proof formation, Descartes appeals to reliability of the “memory”. Then the issue remains; reliability of the memory needs to be guaranteed in order to guarantee the indisputability of intuitive evidence-based proofs.

It requires a procedure to retrieve reliability of the memory of the intuitiveness (or clearness and distinctness in his later terminology) of deductive movement. In Rule XI, it is claimed, for example, by means of the movement of the thought, one can “reflect” the relations among the propositions of the deductive movement in the past. In the explanation of Rule XI Descartes explains the words enumeration and induction by saying that certainty of those steps depends on “memory”, (while certainty of the direct deductive move depends only on intuitive evidence)³. The same issue, in my opinion, appears in *Method* and in *Meditations* too in a slight different form; how the proof criteria for existence of God could escape from the difficulty of the above memory issue. Although it is not the main theme of this note how the retrieving argument of memory works in these Cartesian theories we would rather like to discuss the proof criteria of the Cartesian proof formations in comparison with the others.

Section III. Formal deductive evidence in proof formations.

The intuitive evidence-based proof-formations can be well contrasted to the form-based proof-formations, such as Leibniz (who also attempted to study *Mathesis Universalis*, as well known, without success); although he expresses that his agreement with Cartesian notion of proof, Leibniz⁴ criticizes that the “form” of a deductive proof is important while Descartes ignores it; a proof is “solid” when it respects the “logical form”, according to Leibniz. As an example of the use of inference-form, he emphasizes a “hypothetical” use of starting proposition for deductive proof formations; A principle (starting proposition) of a proof formation can be “hypothetical” without knowing that it is intuitively evident. Then, even if the intuitive evidence is not guaranteed for such a starting proposition, say A, one could admit a proof formation of a conclusive proposition, say B; of course, it does not provide us with an indisputable knowledge of the conclusion B, but, with a hypothetical or conditional knowledge, “if A then B” equivalently “A implies B”. This extension of the notion of proof formation is, in fact, important for general theory of deductive proof formations, especially, and Leibniz noticed it very correctly, in my opinion, in his criticism against the Cartesian style intuitive evidence-based proof formations theory. This introduction step of the implicational conclusion is called “introduction rule of implication” or “deduction theorem” nowadays, and it is one of the most essential deductive inference “form” in the modern formal logic.

Although we cannot discuss further examples here from Leibniz, this simple example already shows that a deductive “form” whose reliability is independent of the intuitive evidence of starting principles (namely, independent of “intuitive evidence” of A, one can make a deductive “form” for the proof of a conditional proposition “if A then B” or “A implies B”. In other words, the formal deductive evidence criterion of “A implies B” in this example is provided by the form of proof-formation itself, namely, the type of a proof formation of B from hypothesis A. Of course the proof formations of Leibniz are not always the form-based in the strict sense, but we believe that the above mentioned idea of form-based proof formation model introduced by Leibniz very important. A pure style of form based proof formation models had been introduced in the modern history of logical philosophy and philosophy of mathematics. We shall discuss those in the next section. We also remark that the First Analysis of Aristotle discussed the hypothetical use of the starting proposition in the dialectic proof setting, rather than the deductive formal proof setting. However, Aristotelian hypothetical use is restricted to the dialectic proofs, and the use of implication-inference rules was limited. We shall discuss these issues elsewhere.

The Cartesian style “intuitive evidence-based”-formation of proofs requires the evidence of the starting propositions. For example, Descartes says, in Rule XII of “Rules”, “I exist, therefore God exists” and rephrases it as “starting from the fact that I exist, I can conclude that God exists”. This “therefore” (by means of provability) might look similar to the above mentioned type of hypothetical proof formation of Leibniz at a first glance. But, in fact, it is not the case. Starting the fact “I exist” requires its intuitive evidence or a proof of “I exist” with the intuitive evidence criterion, before moving to prove “God exists”. Descartes tells here about the “order” of the propositions to be proved, but not any conditional-hypothetical propositions to be proved.

For the form-based proof formation, each step of a deductive inference has a form, as we have just seen a typical example about the introduction rule form of logical implication. Since it is a form, each inference is (at least intended) to be applied universally to any domains (any sciences and philosophy), hence aims at making the deductive proof models of universal theoretical sciences. Especially, this model tells that the deductive inference steps could be formal, and combining with postulates/axioms/definitions of a specific domain, the form-based proof model can be adapted to the specific domain. On the other hand, in the Cartesian intuitive evidence based model a deductive inference step could be any particular step under the condition that such a single deductive inference is intuitively evident. But, of course, such a particular inference step, which is evident at a certain particular proof step

in a context/domain, is not applicable to a different context/domain.

(However, Descartes, of course, aimed at consolidating sciences by the Cartesian metaphysical proof formations here.)

Section IV. Crossing point of proof formation and “language”.

In the previous section, I presented the form-based model of proof formations, in contrast to the intuitive-evidence based model. But, a closer look at the former tells us that the situation is more complicated and that one should make a further distinction. The distinction is closely related to their different positions with respect to the underlying “language” framework for the proof formations. I use the words “linguistically closedness” and “linguistic openness” to express the two different positions.

The conception of logical forms emphasized by Leibniz towards his project of *Mathesis Universalis* was further investigated by various logicians, by the early 20th Century, and a logico-mathematical proof theory was partly realized by some logical proof theorists (in a wide sense), especially by the two major logico-mathematical Schools, the Hilbert Formal-Axiomatist School and the Brouwer–Heyting Dutch Constructivist School. The Hilbert Formal Axiomatist School took the linguistically closed model; they set an axiomatic mathematical (or scientific) theory and formulated a framework of the “formal logical language” for a given theory. The Hilbertian Formal Axiomatist presumes that a “closed” formal language framework for each theory in the sense that the vocabularies, definitions and basic inference rules are prefixed in order to axiomatize the formal deductive proof system of a theory, so that the proof formations are carried out inside the pre-fixed formal language. In particular, the whole part of any proof in an axiomatic theory can be written up, i.e., has an “external representation” on papers in principle. On the other hand, the Brouwerian Constructivist School emphasizes that a form-based proof formation is a mental construction and the proof formations are language-independent independent in principle.

One could see the two different positions of the same “form-based” proof formation model by using the above-mentioned example of the construction of the proof of “A implies B” from the proof, say “p” from “A” to “B”. From the point of view of the linguistically closed model of Hilbert, “p” should be inside the linguistically fixed formal proof system, say axiomatic proof system of natural number theory, while from the point of view of the linguistically open model of Brouwer the framework of p is open and as long as p is constructed mentally, say although the vocabularies appearing in A and B belong to a certain theory, such as natural number theory, one might construct p which contains mental constructs corresponding to vocabularies beyond natural number theory. (More generally, p is a “method” of transforming a given proof of A into a new proof of B.)

Then, we ask ourselves as to what would be the criterion for finding a permissible form of mental construction step. It should again be a certain kind of “intuition”. In fact, Brouwer School is called “Intuitionist” as permissibility of the forms of mental construction are based on intuitive evidence. (Brouwer appealed to Kantian pure intuition of space form for the intuitive evidence of certain although we do not go into this topic in this note, while Heyting of the same Intuitionist–Mental Constructivism School appeals to Husserl’s intuition as fulfillment.)

Instead of directly asking what the intuitive evidence based model of the Brouwer–Heyting Constructivist School, we could ask what is the difference of this intuitive evidence-based model from the Cartesian intuitive evidence model the both proof formation criteria are based on intuitive evidence? The characteristic of the Constructivist’s evidence model, in my opinion, lies on the fact that Constructivist’s are concerned with intuitive evidence of the deductive “inference form”, while the Cartesian is concerned with intuitive evidence on particular propositions-level and on relations and movements of particular propositions.

I have remarked that there are two manners, linguistically closed one and open one, for the form-based proofs. Descartes also claims that there are two manners of proof formations, by distinguishing the analytic proof manner

and the synthetic proof manner, in the 2nd Response. This distinction corresponds to the proofs of discovery (he also uses the word invention) and the proofs of justification/verification (i.e., the axiomatic proof manner which is typically used by Greek geometers). And this distinction also corresponds to the distinction between the linguistically open proofs and the linguistically closed proofs in my terminology above.

Descartes prefers to use the analytic proof manner in order to show/teach to the reader the rout to the discovery. But, it does not mean the synthetic proof manner is impossible. On the other hand, the scientific proofs are ideally presented by the synthetic manner. Two important points here are; (1) He admits that the analytic metaphysical proofs could be transformed into the synthetic metaphysical proofs (as he demonstrates, as an example, a part of them at the end of the 2nd Response). (2) He also believes that the scientific deductive knowledge such as geometrical theorems in the Euclid's Elements are originally discovered/invented by the analytic manner, then the proofs are transformed into the axiomatic synthetic proofs for presenting it to others. He says that Greek geometers found a new theorem with the analytic manner secretly before presenting the proof neatly by the synthetic manner (the 2nd Response.)

In fact, Descartes was able to rewrite some metaphysical proofs in the analytic manner into those in the synthetic or axiomatic manner. This is because he already knows the original particular analytic proof with the discovery/invention attitude, hence from that particular proof he knows which vocabularies, definitions and reasonable postulates/axioms are to be chosen in order to re-arrange the original analytic proof into an axiomatic synthetic proof. Hence, his presentation of the axiomatic (synthetic) metaphysical proofs at the end of the 2nd Response is based on the setting of the linguistically closed framework. In fact, the Cartesian re-formulated metaphysical but synthetic proofs can be represented in the modern (predicate) logical language with formal logical rules (although the axioms could be formulated more elegantly). In other words, the Cartesian synthetic metaphysical proof formation belongs to the form-based proofs.

It is also the case for the Brouwer–Heyting Constructivist– Intuitionist proof formations and the Hilbert Axiomatist proof formations. The proof formation of the former has no prefixed linguistic bound and the success of mental proof formation depends only on the intuitive evidence criterion for mental construction of inference form. However, once a theory is developed by this intuitive evidence standpoint, one could list the vocabularies, definitions, inference forms used, and formalize as a formal axiomatist proof system as a linguistically closed proof system. This can be seen as the transformation from the Brouwer intuitionist proof model to the Hilbert one. This transformation also corresponds to the transformation which Descartes explains.

Section V. Reliability vs. creativity in proof formations.

The linguistically closed axiomatic proofs are easy to check reliability in the usual setting once the vocabularies and axioms are chosen suitably. This is what the scientific proofs usually do. The mathematical and physical domains, for example, are explicitly or implicitly axiomatized based on logic (including Russell style set theory). This is because reliability of the logical inference framework itself can be shown independently of the domain-dependent reliability of axioms. Namely, using the notion of semantic truth, one can show that “each logical inference rule-form preserves the semantic truth; if the premises are semantically true, then the conclusion is also true). A proposition A is semantically true when it is true for any “interpretation” of the linguistically represented proposition A. So, this universally (or domain-independently) applicable notion of semantical truth heavily depends on the linguistically closed proofs model, as this notion of semantic truth is based on interpretation of a given language framework. On the other hand, “truth” in the sense of analytic proofs of Descartes is defined in terms of intuitive evidence; (Just after the cogito argument in Method and at the beginning of the 3rd Meditation, he poses the intuitive evidence principle as a general principle saying that everything which is clear and distinct is

true. The situation is similar for the Brouwer intuitionism; the criterion of mental construction step cannot be reduced to the semantic truth but intuitiveness of construction step. As a result, the Brouwer school claims that the proof of Aristotelian “excluded middle” is not constructible, while it is considered true by any standard semantic interpretation.

As suggested by the axiomatically reformulated synthetic metaphysical proofs of Descartes in the 2nd Response, any kind of metaphysical proofs, most plausibly, can be formulated once an “analytic” metaphysical proof is given and the vocabularies and axioms are suitably chosen by seeing the original analytic proof, although it is not always easy to choose a suitable and disputable set axioms for the case of metaphysics, as he remarks. On the other hand, as he points out, the axiomatic, hence formal-logically formalized, synthetic metaphysical proofs are easy to follow without much “attention” once the axioms are accepted. Also, it is difficult to argue against it. That means since the deductive movement are logical (namely, following logical form).

It is exactly the effect of the use of linguistic form for deductions that solves the memory issue of proof formations mentioned at the end of Section II. By checking the truth preservation of fixed deductive inference forms beforehand, one does not have to pay much attention. The trade-off here is that because of the linguistic closedness, one cannot to beyond the prefixed vocabularies and axioms, hence the synthetic proof manner lacks the full power of discovering/inventing new knowledge to and educating the reader how to reach the discovery. On the other hand, the analytic proof manner requires much more careful attention and has difficulty to keep the memory of intuitive evidence but it is suitable for the proof formations towards discoveries.

Section VI. Intuitive evidence of “proof” and intuitive evidence of “proposition”.

In Rules, Descartes mainly discusses on analytic proofs (although he does not talk about metaphysical proofs much in Rules). He faces how to explain the memory issue and the discovery issue. In Rule XI Descartes emphasizes that after having the propositions with intuitive evidence it is important for making the next (deductive or sometime he call it inductive) step to browse (*parcourir*) the movement of the thought on those propositions, to reflect the relations among the propositions, and to conceive many of the propositions at the same time. He explains the two roles of this rule, (1) One role is to stabilize the conclusion of already formed proof. (2) The other is to get to the next step of proof, towards new discoveries.

The above (1) could be considered his tentative answer to the memory issue in Rules mentioned in Section II which necessarily comes up with the Cartesian type of intuitive evidence based proof model. As for (2), I read that he emphasizes “reflection” on the relations among the propositions⁵; reflecting the propositions passed along the proof formation activities (which he sometimes calls enumeration or induction in his particular sense) makes our thought possible to reach a new deductive knowledge. The act of reflection, in my reading, provides relational evidence, in a wider sense than the sense in which intuitive evidence provides a direct evidence for simple proposition or a single deduction step.

For the Brouwer–Heyting constructivist–intuitionist view of proof formations, a proposition cannot be independent of a proof formation. For example, a proposition “A implies B” is asserted with a “proof formation of B from A”, as we discussed. Hence, mental constructability of “proof of B from A”. Heyting suggests to understand the relation between a proposition and its proof formation in the phenomenological intentional structure of Husserl; posing a targeting proposition as “intention” and constructing a proof as “intuitive fulfillment”

Now I claim that this proof-as-intuition” view is important both for the Cartesian analytic proofs and for the form-based constructivist–intuitionist proofs. I would also like to claim further. I remind that the synthetic proofs (well organized rule based, either explicitly or implicitly axiomatized proofs) have the external representations within a linguistically closed framework, hence could escape from the major problems of memory–reliability on the

trade-off with the power of invention. (The linguistically closed framework could provide the guarantee of preservation of truth in a formal linguistic semantics, for example, as long as the inference rules are prefixed, for example.) Once the axioms and definitions are fixed, the proof formations could be carried out even without the contentual meaning of each proposition, in an extreme formalist case. However, in my opinion, even for such a case one could still access to “intuition on the proof”. By getting used to do the non-contentual linguistic game using given rules repeatedly, one gets more and more intuition of this formal activities; the situation is similar in the case of intuition for professional Go-players or Chess players.

Section VII. Towards investigations into interaction between intuition and form, and into compatibility between creativity and reliability.

The relationship between the analytic proofs and the synthetic proofs is further complicated. The role of language for proof formations is also not simple. To end this note of mine I would like to explain this situation by mentioning slightly Husserl’s proof formation theory towards *Mathesis Universalis*.

Husserl himself gave a concrete example of a “part” of *Mathesis Universalis* in the series of manuscripts in winter 1901 (the part included various equational theories and geometry in a general setting.); he explained the importance of developing a theory of multiplicities, or theory of theories, as *Mathesis Universalis*, and some other related unsolved problems, in *Prolegomena*. He could not reach the solution when he prepared *Logical Investigations*, but just after its publication he reached the conviction that he solves all at once in winter 1901. He was strongly influenced by Hilbert’s formal axiomatism at that period, while Husserl influenced the Heyting’s constructivism-intuitionism. As we discussed, Hilbert and Heyting have very different views on proof formations. So, one can expect that the way Husserl considers the proof formations in his *Mathesis Universalis* would be interesting. By a theory he means a formal axiomatic theory in the sense of Hilbert. A “multiplicity” of a theory has the structure of the whole possible proof steps. So, a single axiomatic theory or a single corresponding multiplicity is bounded by a linguistically closed framework. However, although this is classified by the synthetic proof system in our classification, the multiplicity also preserves the intuition-tool. Namely, Husserl explains a (equational) proof progressing by, in my reading, partial-gradual significative fulfillment steps in the same way he explained in the 6th Investigation on the evaluation of arithmetic terms. It is the fulfillment intuition within the linguistic signification side. This explains how linguistic form-based proof notion and the notion of intuition can meet. However, if this is the end of the story, his idea of *Mathesis Universalis* cannot accommodate the discovery-invention steps in proof formations as the linguistic framework of each theory is closed. The important idea of his *Mathesis Universalis* is to allow to introduce new vocabularies, when needed, and to extend the original language and axioms. He gave, in terms of theory of multiplicities, the condition under which one can allow to introduce new vocabularies and new axioms. His condition could be read that one can use the extended proof systems for discovery and return back to the original proof system after discovery.

This presentation of Husserl suggests to us that there could be various dynamic interactions between intuition and form, between the analytic proof manner the synthetic proof manner, between the discovery attitude and formal attitude, and between linguistic openness and closedness. I left this issue as an open question.

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- 1 In this note we do not distinguish the two words, proof and demonstration, and would often use the two words equivalently.
 - 2 Prof. Katsuzo Murakami kindly made a remark that the content of Rule III needs to be examined carefully with a different version of the *Rules* recently discovered.
 - 3 Adam et Tannery, Vrin. See also VII and X on this point.

- 4 e.g., *Méditations sur la connaissance, la vérité, et les idées*, in Vrin 14–15.
- 5 Rule XII. Cf. footnote of Section II above.