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DISTINGUISHED MINIMAL TOPOLOGICAL LASSOS*

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Abstract. The ease with which genomic data can now be generated using Next Generation Sequencing technologies combined with a wealth of legacy data holds great promise for exciting new insights into the evolutionary relationships between and within the kingdoms of life. At the subspecies level (e.g., varieties or strains) dendograms, that is, certain edge-weighted rooted trees whose leaves are the elements of a set X of organisms under consideration, are often used to represent those relationships. As is well known, dendrograms can be uniquely reconstructed from distances provided all distances on X are known. More often than not, real biological datasets do not satisfy this assumption, implying that the sought dendrogram need not be uniquely determined by the available distances with regard to topology, edge-weighting, or both. To better understand the structural properties a set $\mathcal{L} \subseteq \binom{X}{2}$ has to satisfy to overcome this problem, various types of lassos have been introduced. Here, we focus on the question of when a lasso uniquely determines the topology of a dendrogram; that is, it is a topological lasso for its underlying tree. We show that any set-inclusion minimal topological lasso for such a tree T can be transformed into a structurally nice minimal topological lasso for T. Calling such a lasso a distinguished minimal topological lasso for T, we characterize it in terms of the novel concept of a cluster marker map for T. In addition, we present novel results concerning the heritability of such lassos in the context of the subtree and supertree problems.

Key words. dendrogram, block graph, claw-free, topological lasso, X-tree

AMS subject classifications. 05C05, 92D15

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1. Introduction.

In many topical studies in computational biology ranging from gene onthology [9] via genome-wide association studies in population genetics [22] to evolutionary genomics [21], the following fundamental mathematical problem is encountered: Given a distance D on some set X of objects, find a dendrogram $\mathcal{D}$ on X (essentially a rooted tree $T = (V, E)$ with no degree-two vertices but possibly the root whose leaf set is X together with an edge-weighting $\omega : E \to \mathbb{R}_{\geq 0}$; see Figure 2 for examples) such that the distance induced by $\mathcal{D}$ on any two of its leaves x and y equals $D(x, y)$. In the ideal case that the distances between any two elements of X are available, it is well understood when such a tree is uniquely determined by them, and fast algorithms for reconstructing it from them are known (see, e.g.,[10, Chapter 9.2] and [28, Chapter 7.2], where dendrograms are considered in the slightly more general forms of dated rooted X-trees and equidistant representations of dissimilarities, respectively, and [2, Chapter 3] as well as the references in all three of these sources for more on this).

The reality, however, tends to be different in many cases in that distances between pairs of objects might be missing or are not sufficiently reliable to warrant inclusion of that distance in an analysis; see, e.g., [25, 26, 29] for more on this topic in an evolutionary genomics context. Exclusion of such a distance might therefore be tempting. Recent studies in [5] and [18] suggest this may, however, have adverse

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effects on the outcomes of a study which raises interesting mathematical, statistical, and algorithmical questions (see, e.g.,[7, 12, 27] for a study concerning the latter and [12, 14, 15, 23] for results concerning its unrooted variant). One such question is the focus of this paper: Calling any subset of a finite set X of size two a *cord* of X and referring to the distance between the two elements of a cord as *distance on a cord*, for what sets $\mathcal{L}$ of cords of X do we need to know the distances so that both the topology of the underlying tree and the edge-weights of the dendrogram on X that induced the distances on the cords in $\mathcal{L}$ are uniquely determined by $\mathcal{L}$?

To help illustrate the intricacies of this question, which is concerned with the structure of the set $\mathcal{L}$ and not so much with the actual distances on the cords in $\mathcal{L}$, denote for any two distinct elements $a, b \in X$ the cord $\{a, b\}$ by ab. Consider the dendrogram $\mathcal{D}$ with leaf set $X = \{a, \dots, e\}$ depicted in Figure 1(i), and assume that the distances on the cords of $\mathcal{L} = \{ac, de, bc, ce, cd\}$ are induced by $\mathcal{D}$; so, for example, the distance on the cord ab is four. Then the dendrogram $\mathcal{D}'$ depicted in Figure 1(ii) induces the same distances on the cords in $\mathcal{L}$ as $\mathcal{D}$, but the topologies of the underlying trees T and T' of $\mathcal{D}$ and $\mathcal{D}'$, respectively, are clearly not the same in the sense that there exists no bijection from $V(T)$ to $V(T')$ that is the identity on $\{a, \dots, e\}$ and induces a rooted graph isomorphism from T to T'. Thus, $\mathcal{L}$ does not uniquely determine T and thus also does not uniquely determine $\mathcal{D}$. However, as can be quickly checked, the situation changes if and only if the cord ab (or a subset of $\binom{X}{2}$ containing that cord) is added to $\mathcal{L}$. To make this more precise, let $\mathcal{L}'$ denote the resulting set of cords on X, and let $\mathcal{D}_1$ denote a dendrogram on X for which the topology of the underlying tree is the same as that of $\mathcal{D}$. If $\mathcal{D}_2$ is a dendrogram on X such that the distances on the cords in $\mathcal{L}'$ induced by $\mathcal{D}_1$ and $\mathcal{D}_2$ coincide, then, as is easy to verify, the topologies of the underlying trees of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, must be the same and so must be their edge-weightings. Thus, $\mathcal{L}'$ uniquely determines $\mathcal{D}$.

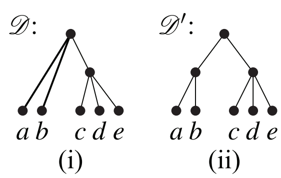

FIG. 1. *For $X = \{a, \dots, e\}$ and $\mathcal{L} = \{ac, de, bc, ce, cd\}$ the dendrograms $\mathcal{D}$ and $\mathcal{D}'$ are depicted in* (i) *and* (ii)*, respectively. Bold edges in $\mathcal{D}$ have weight two, and all other edges as well as all edges in $\mathcal{D}'$ have weight one.*

Although an intriguing question, apart from some recent results in [19], not much is known about it (see [11] and [20] for some partial results in the case of the tree in question being unrooted). By formalizing a dendrogram in terms of a certain edge-weighted X-tree (see the next section for a precise definition of this concept as well as all the other concepts mentioned below) and using the concept of a topological lasso, which was originally introduced for unrooted phylogenetic trees with leaf set X in [11] and extended to X-trees in [19], we study this question in the context of when a set of cords of X is a topological lasso for a given X-tree T. In this context, we are particularly interested in (set-inclusion) minimal topological lassos $\mathcal{L}$ for T for which $\bigcup \mathcal{L} := \bigcup_{A \in \mathcal{L}} A = X$ holds.

For T an X-tree, we show for any such minimal topological lasso $\mathcal{L}$ for T that in case the graph $\Gamma(\mathcal{L})$, whose vertex set is X and any two distinct elements x and

y in X joined by an edge if $xy \in \mathcal{L}$ (see Fig 2(i) for an example of that graph for $\mathcal{L} = \{ab, cd, ef, ac, ce, ea\}$), is a block graph, then the blocks of $\Gamma(\mathcal{L})$ are in one-to-one correspondence with the nonleaf vertices of T (Corollary 4.3). However, it is clearly too much to hope for that $\Gamma(\mathcal{L})$ is a block graph for any minimal topological lasso $\mathcal{L}$, and even if it is, it need not be *claw-free*, that is, contains the complete bipartite graph $K_{1,3}$ as an induced subgraph [17], as is suggested by the example of the minimal topological lasso presented in Figure 2.

Claw-free graphs have been widely studied and shown to enjoy numerous properties relating them to, for example, perfect graphs, perfect matchings, and maximum independent sets (see, e.g., [13] and [6] for overviews). Furthermore, claw-free graphs that are block graphs were related in [4] to k-leaf powers of trees, and their spectrum was studied in [16, 24] (see also [1] for a more general study of the adjacency matrix of such graphs). In Theorem 5.2 we provide a link between this rich body of literature and minimal topological lassos by establishing that any minimal topological lasso $\mathcal{L}$ for T can be transformed into a minimal topological lasso $\mathcal{L}^*$ for T such that $\Gamma(\mathcal{L}^*)$ is a claw-free block graph. Reflecting this, we call a minimal topological lasso $\mathcal{L}$ for T *distinguished* if $\Gamma(\mathcal{L})$ is a claw-free block graph and remark that in [21] we exploit this concept to formulate an efficient algorithm for constructing edge-weighted X-trees from sets $\mathcal{L}$ of cords provided that all the actual distance values for the cords in $\mathcal{L}$ are available. Among a number of attractive properties enjoyed by this algorithm is consistency, by which we mean that in case $\mathcal{L}$ is a distinguished minimal topological lasso, then it will return the unique edge-weighted X-tree that is lassoed by it. In Theorem 7.2 we present a characterization of a distinguished minimal topological lasso for T in terms of the novel concept of a cluster marker map for T. Finally, we characterize when a distinguished minimal topological lasso for T gives rise to a distinguished minimal topological lasso for a subtree of T (Theorem 8.1) and also present a partial answer to the canonical analogue of a question raised for supertrees of unrooted phylogenetic trees in [11].

The paper is organized as follows. In section 2, we introduce relevant terminology surrounding X-trees and lassos. In section 3, we collect first properties of the graph $\Gamma(\mathcal{L})$ associated to a topological lasso $\mathcal{L}$, and in section 4, we establish Corollary 4.3. In section 5, we commence our study of a distinguished minimal topological lasso and establish Theorem 5.2. In section 6, we present a sufficient condition for when a minimal topological lasso is distinguished (Theorem 6.3), and in section 7, we prove Theorem 7.2. We conclude with section 8, where we establish Theorem 8.1 and also outline directions for further research.

2. Basic terminology and assumptions.

In this section, we introduce some relevant basic terminology surrounding X-trees, their edge-weighted counterparts, and lassos. Assume throughout the paper that X is a finite set with at least three elements and that, unless stated otherwise, all sets $\mathcal{L}$ of cords of X considered in this paper satisfy the property that $X = \bigcup \mathcal{L}$.

2.1. X-trees.

A *rooted tree* T is a tree with a unique distinguished vertex called the *root* of T, denoted by ρ_T. Throughout the paper, we assume that the degree of the root of a rooted tree is at least two. A *rooted phylogenetic X-tree*, or *X-tree* for short, is a rooted tree $T = (V, E)$ with no degree-two vertices but possibly the root ρ_T whose leaf set is X. We call an X-tree T a *star-tree on X* if every leaf of T is adjacent with the root of T.

Suppose for the following that T is an X-tree. Then we call a vertex of T that is not a leaf of T an *interior vertex* of T and denote the set of interior vertices of T by

$\mathring{V}(T)$. We call an edge of T that is incident with a leaf of T a *pendant edge* of T and every edge of T that is not a pendant edge an *interior edge* of T. Extending some of the terminology for directed graphs to X-trees, we call for all vertices $v \in V(T) - \{\rho_T\}$ an edge $e \in E(T)$ a *parent edge of* v if e is incident with v and lies on the path from the root ρ_T of T to v. We refer to the vertex incident with e but distinct from v as a *parent* of v.

Suppose for the following that v is an interior vertex of T. If v is not the root of T, then we call an edge $e \in E(T)$ a *child edge of* v if e is incident with v but is not crossed by the path from ρ_T to v. In addition, we call every edge incident with ρ_T a *child edge of* ρ_T. We call the vertex incident with a child edge of an interior vertex w of T but distinct from w a *child of* w and denote the set of all children of v by $ch_T(v)$. We call a vertex $w \in V(T)$ distinct from v a *descendant* of v if either w is a child of v or there exists a path from v to w that crosses a child of v. We denote the set of leaves of T that are also descendants of v by $L_T(v)$. If v is a leaf of T, then we put $L_T(v) := \{v\}$.

We call a nonempty subset $L \subsetneq X$ of leaves of T such that $L = L(v)$ holds for some $v \in \mathring{V}(T)$ a *pseudo-cherry* of T. In that case, we also call v the *parent* of that pseudo-cherry. Note that every X-tree on three or more leaves must contain at least one pseudo-cherry. Also note that a pseudo-cherry of size two is a *cherry* in the usual sense (see, e.g., [28]).

For x and y distinct elements in X, we call the unique vertex of T that simultaneously lies on the path from x to y, on the path from x to ρ_T, and on the path from y to ρ_T the *last common ancestor of* x *and* y, denoted by $lca_T(x, y)$. More generally, for any subset $Y \subseteq X$ of size three or more, we denote the subtree of T with leaf set Y and vertices of degree two suppressed (except the root if there exist $x, y \in Y$ such that ρ_T lies on the path joining x and y) by $T|_Y$ and call the root of $T|_Y$ the *last common ancestor of* Y, denoted by $lca_T(Y)$. If there is no ambiguity as to which X-tree T we are referring to, we simplify our notation by omitting, for all $v \in V(T)$ and all subsets $B \subseteq X$ of size at least three, the index in $ch_T(v)$, $L_T(v)$, and $lca_T(B)$.

Finally, suppose that T' is a further X-tree. Then we say that T and T' are *equivalent* if there exists a bijection $\phi : V(T) \to V(T')$ that extends to a graph isomorphism between T and T' that is the identity on X and maps the root ρ_T of T to the root $\rho_{T'}$ of T'.

2.2. Edge-weighted X-trees and lassos. Suppose for the following again that T is an X-tree. An *edge weighting* ω *of* T is a map $\omega : E(T) \to \mathbb{R}_{\geq 0}$ that maps every edge of T to a nonnegative real. Suppose that ω is an edge weighting for T. Then we call the pair (T, ω) an *edge-weighted* X-tree and ω *proper* if $\omega(e) > 0$ holds for every interior edge e of T. We denote the distance induced by (T, ω) on the vertices of T by $D_{(T,\omega)}$ and call ω *equidistant* if

(i) $D_{(T,\omega)}(x, \rho_T) = D_{(T,\omega)}(y, \rho_T)$ for all $x, y \in X$, and

(ii) $D_{(T,\omega)}(x, u) \geq D_{(T,\omega)}(x, v)$ for all $x \in X$ and all $u, v \in V(T)$ such that u is encountered before v on the path from ρ_T to x.

Note that if ω is an equidistant edge weighting for an X-tree T, then $D_{(T,\omega)}$ is an ultrametric [28, Lemma 7.2.4].

Suppose $\mathcal{L}$ is a set of cords of X. Then we call two edge-weighted X-trees (T_1, ω_1) and (T_2, ω_2) *$\mathcal{L}$-isometric* if $D_{(T_1,\omega_1)}(x, y) = D_{(T_2,\omega_2)}(x, y)$ holds for all cords $xy \in \mathcal{L}$. We say that $\mathcal{L}$ is a *topological lasso* for T if, for every X-tree T' and any equidistant, proper edge weightings ω of T and ω' of T', we have that T and T' are equivalent whenever (T, ω) and (T', ω') are $\mathcal{L}$-isometric. If $\mathcal{L}$ is a topological lasso for T, then we also say that T is *topologically lassoed* by $\mathcal{L}$. Moreover, we say that $\mathcal{L}$ is a *(set-inclusion) minimal topological lasso for* T if $\mathcal{L}$ is a topological lasso for T but no cord $A \in \mathcal{L}$ can be removed from $\mathcal{L}$ such that $\mathcal{L} - \{A\}$ is still a topological lasso for T. For ease of readability, if the X-tree to which a topological lasso $\mathcal{L}$ refers is of no relevance to the discussion, we will simply say that $\mathcal{L}$ is a topological lasso.

To illustrate some of these definitions, let $X = \{a, \dots, f\}$, and let $\mathcal{L}$ be the set of cords such that $\Gamma(\mathcal{L})$ is the graph depicted in Figure 2(i). Using, e.g., [19, Theorem 7.1] (see also Theorem 3.1 below), it is easy to see that the X-trees depicted in Figure 2(ii) and (iii), respectively, are topologically lassoed by $\mathcal{L}$. In fact, $\mathcal{L}$ is a minimal topological lasso for both of them.

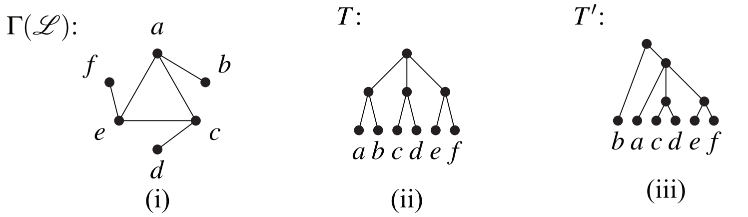

FIG. 2. (i) *The graph* $\Gamma(\mathcal{L})$ *with vertex set* $X = \{a, b, \dots, f\}$ *for the set* $\mathcal{L} = \{ab, cd, ef, ac, ce, ea\}$. (ii)–(iii) *Two nonequivalent X-trees T and T' that are both topologically lassoed by $\mathcal{L}$. In fact, $\mathcal{L}$ is a minimal topological lasso for either one of them.*

3. The graphs $\Gamma(\mathcal{L})$ and $G(\mathcal{L}, v)$. In this section, we investigate properties of the graph $\Gamma(\mathcal{L})$ associated to a set $\mathcal{L}$ of cords of X. We start by remarking that if there is no danger of confusion, we denote an edge $\{a, b\}$ of $\Gamma(\mathcal{L})$ by ab rather than $\{a, b\}$.

To establish our first structural result for $\Gamma(\mathcal{L})$ (see Proposition 3.3), we require further terminology. Suppose T is an X-tree, $v \in \mathring{V}(T)$, and $\mathcal{L}$ is a set of cords of X. Then we call the graph $G_T(\mathcal{L}, v) = (V_{T,v}, E_{T,v})$ with vertex set $V_{T,v}$ the set of all child edges of v and edge set $E_{T,v}$ the set of all $\{e, e'\} \in \binom{V_{T,v}}{2}$ for which there exist leaves $a, b \in X$ such that e and e' are edges on the path from a to b in T and $ab \in \mathcal{L}$ holds the *child-edge graph of v (with respect to T and $\mathcal{L}$)*. Note that when there is no danger of ambiguity regarding the X-tree T to which we refer, we will write $G(\mathcal{L}, v)$ rather than $G_T(\mathcal{L}, v)$ and V_v and E_v rather than $V_{T,v}$ and $E_{T,v}$. The next result, which was originally established in [19, Theorem 7.1], states a crucial property of child-edge graphs.

THEOREM 3.1. *Suppose T is an X-tree and $\mathcal{L}$ is a set of cords of X. Then the following are equivalent:*

(i) *$\mathcal{L}$ is a topological lasso for T.*

(ii) *For every vertex $v \in \mathring{V}(T)$, the graph $G(\mathcal{L}, v)$ is a clique.*

Denoting for an X-tree T, a topological lasso $\mathcal{L}$ for T, and an interior vertex $v \in \mathring{V}(T)$ the set of all cords $ab \in \mathcal{L}$ for which $v = lca(a, b)$ holds by $\mathcal{A}(v)$, Theorem 3.1 readily implies $|\mathcal{A}(v)| \geq \binom{|ch(v)|}{2}$. The next observation is almost trivial yet central to the paper and concerns the special case that $\mathcal{L}$ is a minimal topological lasso for T. Its proof, which combines a straightforward counting argument with Theorem 3.1, is left to the interested reader. To be able to state it, we denote for an interior vertex $v \in \mathring{V}(T)$ and a child edge $e \in E(T)$ of v the child of v incident with e by v_e.

LEMMA 3.2. *Suppose T is an X-tree and $\mathcal{L}$ is a minimal topological lasso for T.*

Then, for all $v \in \mathring{V}(T)$, we have $|\mathcal{A}(v)| = \binom{|ch(v)|}{2}$. In particular, for any two distinct child edges e_1 and e_2 of v there exists precisely one pair $(a_1, a_2) \in L(v_{e_1}) \times L(v_{e_2})$ such that $a_1 a_2 \in \mathcal{L}$.

Note that Lemma 3.2 immediately implies that any two minimal topological lassos for the same X-tree must be of equal size.

To be able to establish Proposition 3.3, we require a further definition. Suppose T is an X-tree and $\mathcal{L}$ is a topological lasso for T. Then for all $v \in V(T)$, we denote by $\Gamma_v(\mathcal{L})$ the subgraph of $\Gamma(\mathcal{L})$ induced by $L(v)$. Note that in case v is a leaf of T and thus an element in X the only vertex in $\Gamma_v(\mathcal{L})$ is v (and $E(\Gamma_v(\mathcal{L})) = \emptyset$).

PROPOSITION 3.3. *Suppose T is an X-tree and $\mathcal{L}$ is a topological lasso for T. Then, for all $v \in V(T)$, the graph $\Gamma_v(\mathcal{L})$ is connected. In particular, $\Gamma(\mathcal{L})$ is connected.*

Proof. Assume for contradiction that there exists some vertex $v \in V(T)$ such that $\Gamma_v(\mathcal{L})$ is not connected. Then v cannot be a leaf of T, and so $v \in \mathring{V}(T)$ must hold. Without loss of generality we may assume that v is such that for all descendants $w \in V(T)$ of v the induced graph $\Gamma_w(\mathcal{L})$ is connected. Since $\mathcal{L}$ is a topological lasso for T and so $G(\mathcal{L}, v)$ is a clique, it follows for any two distinct children $v_1, v_2 \in ch(v)$ that there exists a pair $(x_1, x_2) \in L(v_1) \times L(v_2)$ such that $x_1 x_2 \in \mathcal{L}$. Since the assumption on v implies that the graphs $\Gamma_w(\mathcal{L})$ are connected for all children $w \in ch(v)$, it follows that $\Gamma_v(\mathcal{L})$ is connected, which is impossible. Thus, $\Gamma_v(\mathcal{L})$ is connected for all $v \in V(T)$. That $\Gamma(\mathcal{L})$ is connected is a trivial consequence. ∎

4. The case that $\Gamma(\mathcal{L})$ is a block graph.

To establish a further property of $\Gamma(\mathcal{L})$, which we will do in Proposition 4.1, we require some terminology related to block graphs (see, e.g., [8]). Suppose G is a graph. Then a vertex of G is called a *cut vertex* if its deletion (plus its incident edges) disconnects G. A graph is called a *block* if it has at least one vertex, is connected, and does not contain a cut vertex. A *block of a graph* G is a maximal connected subgraph of G that is a block, and a graph is called a *block graph* if all of its blocks are cliques. For convenience, we refer to a block graph with vertex set X as a *block graph on* X.

As the example of the two minimal topological lassos $\{ab, cd, ef, ac, ce, ea\}$ and $\{ab, bc, cd, de, ef, fa\}$ for the $\{a, \dots, f\}$-tree depicted in Figure 2(ii) indicates, the graph $\Gamma(\mathcal{L})$ associated to a minimal topological lasso $\mathcal{L}$ may be but need not be a block graph. However, if it is, then Lemma 3.2 can be strengthened to the following central result where for all positive integers n we put $\langle n \rangle := \{1, \dots, n\}$ and set $\langle 0 \rangle := \emptyset$.

PROPOSITION 4.1. *Suppose T is an X-tree and $\mathcal{L}$ is a minimal topological lasso for T such that $\Gamma(\mathcal{L})$ is a block graph. Let $v \in \mathring{V}(T)$, and let $v_1, \dots, v_l \in V(T)$ denote the children of v where $l = |ch(v)|$. Then, for all $i \in \langle l \rangle$, there exists a unique leaf $x_i \in L(v_i)$ such that $x_s x_t \in \mathcal{L}$ holds for all $s, t \in \langle l \rangle$ distinct.*

Proof. For all $v \in \mathring{V}(T)$ and all $w \in ch(v)$, put

$$L_w^v := \{x \in L(w) : \text{ there exist } w' \in ch(v) - \{w\} \text{ and } y \in L(w') \text{ such that } xy \in \mathcal{L}\}.$$

We need to show that $|L_w^v| = 1$ holds for all $v \in \mathring{V}(T)$ and all $w \in ch(v)$. To see this, note first that since $G(\mathcal{L}, v)$ is a clique for all $v \in \mathring{V}(T)$, we have, for all $w \in ch(v)$ with $v \in \mathring{V}(T)$, that $L_w^v \neq \emptyset$. Thus, $|L_w^v| \geq 1$ holds for all such v and w.

To establish equality, suppose there exists some interior vertex $v \in \mathring{V}(T)$ and some child $v_1 \in ch(v)$ such that $|L_{v_1}^v| \geq 2$. Choose two distinct leaves x_1 and y_1 of T contained in $L_{v_1}^v$, and denote the parent edge of v_1 by e_1. Note that $v_1 = v_{e_1}$. Since $y_1 \in L_{v_1}^v$, there exists a child edge e_2 of v distinct from e_1 and some $x_2 \in L(v_{e_2})$ such that $y_1x_2 \in \mathcal{L}$. In view of $x_1 \in L^v_{v_1}$, we distinguish between the cases that (i) $x_1z \notin \mathcal{L}$ holds for all $z \in L(v_{e_2})$ and (ii) there exists some $z \in L(v_{e_2})$ such that $x_1z \in \mathcal{L}$.

Assume first that Case (i) holds. Then since $x_1 \in L^v_{v_1}$ there exists a further child edge e_3 of v and some $y_3 \in L(v_{e_3})$ such that $x_1y_3 \in \mathcal{L}$. Since, by Theorem 3.1, $G(\mathcal{L}, v)$ is a clique and so $\{e_2, e_3\}$ is an edge in $G(\mathcal{L}, v)$, there must exist leaves $y_2 \in L(v_{e_2})$ and $x_3 \in L(v_{e_3})$ such that $y_2x_3 \in \mathcal{L}$. By Proposition 3.3, the graphs $\Gamma_{v_{e_i}}(\mathcal{L})$, $i = 2, 3$, are connected and, by definition, clearly do not share a vertex. Hence, there must exist a cycle in $\Gamma(\mathcal{L})$ whose vertex set contains $\bigcup_{j\in\langle 3\rangle}\{x_j, y_j\}$. But then $x_1x_2 \in \mathcal{L}$ must hold since $\Gamma(\mathcal{L})$ is a block graph, and so every block in $\Gamma(\mathcal{L})$ is a clique. By Lemma 3.2 applied to e_1 and e_2, it follows that $x_1 = y_1$ as $x_1, y_1 \in L(v_1)$ and $y_1x_2 \in \mathcal{L}$, which is impossible.

Now assume that case (ii) holds; that is, there exists some $z \in L(v_{e_2})$ such that $x_1z \in \mathcal{L}$. Then Lemma 3.2 applied to e_1 and e_2 implies $x_1 = y_1$ as $y_1x_2 \in \mathcal{L}$ also holds, which is impossible. ∎

To illustrate Proposition 4.1, let T be the X-tree depicted in Figure 2(ii), and let $\mathcal{L}$ be the set of cords of X whose $\Gamma(\mathcal{L})$ graph is pictured in Figure 2(i). Using the notation from Proposition 4.1 and labeling the children of the root of T from left to right by v_1, v_2, and v_3, it is easy to see that Proposition 4.1 holds for $x_1 = a$, $x_2 = c$, and $x_3 = e$.

The next result is the main result of this section and lies at the heart of Corollary 4.3, which provides for an X-tree T and a minimal topological lasso $\mathcal{L}$ for T such that $\Gamma(\mathcal{L})$ is a block graph a close link between the blocks of $\Gamma(\mathcal{L})$, the interior vertices of T, and, for all $v \in \mathring{V}(T)$, the child-edge graphs $G(\mathcal{L}, v)$. To establish it, we denote for all $v \in V(T) - \{\rho_T\}$ the parent edge of v by e_v and the set of blocks of a graph G by $Block(G)$.

THEOREM 4.2. *Suppose T is an X-tree and $\mathcal{L}$ is a minimal topological lasso for T such that $\Gamma(\mathcal{L})$ is a block graph. Then, for all $v \in \mathring{V}(T)$, there exists a unique block $B \in Block(\Gamma(\mathcal{L}))$ such that $v = lca(V(B))$.*

Proof. We first show existence. Suppose $v \in \mathring{V}(T)$. Let $v_1, \dots, v_l \in V(T)$ denote the children of v where $l = |ch(v)|$. By Proposition 4.1, there exists, for all $i \in \langle l\rangle$, a unique leaf $x_i \in L(v_i)$ such that, for all $s, t \in \langle l\rangle$ distinct, we have $x_sx_t \in \mathcal{L}$. Put $A = \{x_1, \dots, x_l\}$. Clearly, $v = lca(A)$, and the graph $G(v)$ with vertex set A and edge set $E = \{\{x, y\} \in \binom{A}{2} : xy \in \mathcal{L}\}$ is a clique. Then, since $\Gamma(\mathcal{L})$ is a block graph, there must exist a block $B \in Block(\Gamma(\mathcal{L}))$ that contains $G(v)$ as an induced subgraph.

We claim that the graphs $G(v)$ and B are equal. In view of the facts that $A \subseteq V(B)$, the blocks in a block graph are cliques, and $G(v)$ is a clique, it suffices to show that $V(B) \subseteq A$. Suppose for contradiction that there exists some $y \in V(B) - A$. Note first that $yx \in \mathcal{L}$ must hold for all $x \in A$. Next note that y cannot be a descendant of v since otherwise there would exist some $i \in \langle l\rangle$ such that $y \in L(v_i)$. Choose some $j \in \langle l\rangle - \{i\}$. Then Lemma 3.2 applied to e_{v_i} and e_{v_j} implies $x_i = y$ as $yx_j, x_ix_j \in \mathcal{L}$, which is impossible.

Choose some $z \in A$, and put $w = lca(z, y)$. Then v is a descendant of w, and $w = lca(x, y)$ holds for all $x \in A$. Let $w_1 \in V(T)$ and $w_2 \in \mathring{V}(T)$ denote two distinct children of w such that $y \in L(w_1)$ and $z \in L(w_2)$. Then Lemma 3.2 applied to e_{w_1} and e_{w_2} implies $x_i = x_j$ for all $i, j \in \langle l\rangle$ distinct since $yx \in \mathcal{L}$ holds for all $x \in A$, which is impossible. Thus, $V(B) \subseteq A$, as required. This concludes the proof of the existence part of the theorem.

We next show uniqueness. Suppose for contradiction that there exists some $v \in \mathring{V}(T)$ and distinct blocks $B, B' \in Block(\Gamma(\mathcal{L}))$ such that $lca(B) = v = lca(B')$.

Since every block of $\Gamma(\mathcal{L})$ contains at least two vertices as $\Gamma(\mathcal{L})$ is connected and $|X| \geq 3$, we may choose distinct vertices $b_1, b_2 \in V(B)$ and $b_1', b_2' \in V(B')$ such that $lca(b_1, b_2) = lca(B) = v = lca(B') = lca(b_1', b_2')$. Note that $b_1 b_2$ and $b_1' b_2'$ must be cords in $\mathcal{L}$ as B and B' are cliques of $\Gamma(\mathcal{L})$. We distinguish between the cases that (i) $\{b_1, b_2\} \cap \{b_1', b_2'\} = \emptyset$ and (ii) $\{b_1, b_2\} \cap \{b_1', b_2'\} \neq \emptyset$.

We first show that case (i) cannot hold. Assume for contradiction that $\{b_1, b_2\} \cap \{b_1', b_2'\} = \emptyset$. We claim that $lca(b_1, b_1') = v$. Assume for contradiction that $w := lca(b_1, b_1') \neq v$. Let $v_1 \in ch(v)$ such that v_1 lies on the path from v to w. If $v \neq lca(b_2, b_2')$, then there exists a descendant $w' \in V(T)$ of v such that $lca(b_2, b_2') = w'$. Let $v_2 \in ch(v)$ such that v_2 lies on the path from v to w'. Then Lemma 3.2 applied to e_{v_1} and e_{v_2} implies $b_1 = b_1'$ and $b_2 = b_2'$ as $b_1 b_2, b_1' b_2' \in \mathcal{L}$, which is impossible. Thus, $lca(b_2, b_2') = v$ must hold. Let $v_2, v_2' \in ch(v)$ such that $b_2 \in L(v_2)$ and $b_2' \in L(v_2')$. Then, since $b_1, b_1' \in L(v_1)$ and $b_1 b_2, b_1' b_2' \in \mathcal{L}$, Proposition 4.1 implies $b_1' = b_1$. Consequently, $\{b_1, b_2\} \cap \{b_1', b_2'\} \neq \emptyset$, which is impossible. Thus, $lca(b_2, b_2') = v$ cannot hold, and so

$$lca(b_1, b_1') = v,$$

as claimed. Swapping the roles of b_1, b_1' and b_2, b_2' in the previous claim implies that $v = lca(b_2, b_2')$ must hold, too. For $i = 1, 2$ let $v_i, v_i' \in ch(v)$ such that $b_i \in L(v_i)$ and $b_i' \in L(v_i')$. Then, by Lemma 3.2, there exist pairs $(c, c') \in L(v_1) \times L(v_1')$ and $(d, d') \in L(v_2) \times L(v_2')$ such that $cc', dd' \in \mathcal{L}$. Since $(b_1, b_2) \in L(v_1) \times L(v_2)$ and $(b_1', b_2') \in L(v_1') \times L(v_2')$ and $b_1 b_2, b_1' b_2' \in \mathcal{L}$, Proposition 4.1 implies that $c = b_1$, $b_2 = d$, $d' = b_2'$, and $c' = b_1'$. But then C: $c' = b_1', b_2' = d', d = b_2, b_1 = c, c'$ is a cycle in $\Gamma(\mathcal{L})$. Since $\Gamma(\mathcal{L})$ is a block graph, it follows that there must exist a block B^C in $\Gamma(\mathcal{L})$ that contains C. Since $\{b_1, b_2\} \subseteq V(B^C) \cap V(B)$ and two distinct blocks of a block graph can share at most one vertex, it follows that B^C and B must coincide. Since $\{b_1', b_2'\} \subseteq V(B^C) \cap V(B')$ holds, too, similar arguments imply that B^C must also coincide with B'. Thus, B and B' must be equal, which is impossible. Hence case (i) cannot hold, as required.

Thus, case (ii) must hold; that is, $\{b_1, b_2\} \cap \{b_1', b_2'\} \neq \emptyset$. Since any two distinct blocks in a block graph can share at most one vertex, it follows that $|\{b_1, b_2\} \cap \{b_1', b_2'\}| = 1$. Without loss of generality, we may assume that $b_1 = b_1'$. We first claim that

$$lca(b_2, b_2') = v.$$

Assume to the contrary that $lca(b_2, b_2') \neq v$. Then there exist distinct children $v_1, v_2 \in ch(v)$ such that $b_1 \in L(v_1)$ and $b_2, b_2' \in L(v_2)$ hold. Since both $b_1 b_2$ and $b_1' b_2' = b_1 b_2'$ are cords in $\mathcal{L}$, Lemma 3.2 applied to e_{v_1} and e_{v_2} implies $b_2' = b_2$. Hence, $|\{b_1, b_2\} \cap \{b_1', b_2'\}| = 2$, which is impossible. Thus, $lca(b_2, b_2') = v$, as claimed.

Let $v_1, v_2, v_2' \in ch(v)$ such that $b_1 \in L(v_1)$, $b_2 \in L(v_2)$, and $b_2' \in L(v_2')$. By Lemma 3.2, there exist some $(c, c') \in L(v_2) \times L(v_2')$ such that $cc' \in \mathcal{L}$. Since we also have $(b_1, b_2) \in L(v_1) \times L(v_2)$ with $b_1 b_2 \in \mathcal{L}$ holding and $(b_1, b_2') \in L(v_1) \times L(v_2')$ with $b_2' b_1 = b_2' b_1' \in \mathcal{L}$ holding, Proposition 4.1 implies that $b_2 = c$ and $b_2' = c'$. Hence, C: $b_1 = b_1', b_2' = c', c = b_2, b_1$ is a cycle in $\Gamma(\mathcal{L})$, and so arguments similar to those in the corresponding subcase for case (i) imply that B and B' must coincide, which is impossible. Thus, $lca(b_2, b_2') = v$ cannot hold, which concludes the discussion of case (ii) and thus the proof of the uniqueness part of the theorem. ∎

In view of Theorem 4.2, we denote, for T an X-tree, a minimal topological lasso $\mathcal{L}$ for T such that $\Gamma(\mathcal{L})$ is a block graph, and a vertex $v \in \mathring{V}(T)$ the unique block B

in $\Gamma(\mathcal{L})$ for which $v = lca(V(B))$ holds by $B_v^{\mathcal{L}}$, or simply by B_v if the set $\mathcal{L}$ of cords is clear from the context. Moreover, we denote for all $x \in L(v)$ the child of v on the path from v to x by v_x.

Corollary 4.3. *Suppose T is an X-tree and $\mathcal{L}$ is a minimal topological lasso for T such that $\Gamma(\mathcal{L})$ is a block graph. Then the map*

$$\psi : \mathring{V}(T) \to Block(\Gamma(\mathcal{L})) : v \mapsto B_v$$

is a bijection with inverse map $\psi^{-1} : Block(\Gamma(\mathcal{L})) \to \mathring{V}(T) :\ B \mapsto lca(V(B))$. Moreover, the map

$$\chi : Block(\Gamma(\mathcal{L})) \to \{G(\mathcal{L}, v)\ :\ v \in \mathring{V}(T)\} : B \mapsto G(\mathcal{L}, \psi^{-1}(B))$$

is bijective, and, for all $B \in Block(\Gamma(\mathcal{L}))$, the map

$$\xi_B : V(B) \to V_{\psi^{-1}(B)} : x \mapsto e_{(\psi^{-1}(B))_x}$$

induces a graph isomorphism between B and the child-edge graph $G(\mathcal{L}, \psi^{-1}(B))$.

Proof. In view of Theorem 4.2, the map ψ is clearly well defined and injective. To see that ψ is surjective, let $B \in Block(\Gamma(\mathcal{L}))$, and put $v_B = lca(V(B))$. Clearly, $v_B \in \mathring{V}(T)$. Since $B_{v_B} = \psi(v_B)$ is a block in $\Gamma(\mathcal{L})$ for which $v_B = lca(V(B_{v_B}))$ also holds, Theorem 4.2 implies that $\psi(v_B)$ and B must coincide. Consequently, ψ must also be surjective and thus bijective. That the map ψ^{-1} is as stated is trivial. Combined with Theorem 3.1, the bijectivity of the map ψ implies in particular that, for all $B \in Block(\Gamma(\mathcal{L}))$, the map $\xi_B : V(B) \to V_{\psi^{-1}(B)}$ from $V(B)$ to the vertex set $V_{\psi^{-1}(B)}$ of the child-edge graph $G(\mathcal{L}, \psi^{-1}(B))$ induces a graph isomorphism between B and $G(\mathcal{L}, \psi^{-1}(B))$.

To see that the map χ is bijective, note first that χ is well defined since $\psi^{-1}(B) \in \mathring{V}(T)$ holds for all blocks $B \in Block(\Gamma(\mathcal{L}))$. To see that χ is injective, assume that there exist blocks $B_1, B_2 \in Block(\Gamma(\mathcal{L}))$ such that $\chi(B_1) = \chi(B_2)$ but B_1 and B_2 are distinct. Then $\psi^{-1}(B_1) \neq \psi^{-1}(B_2)$ as ψ is a bijection from $\mathring{V}(T)$ to $Block(\Gamma(\mathcal{L}))$. Combined with the fact that, for all $B \in Block(\Gamma(\mathcal{L}))$, the map ξ_B induces a graph isomorphism between B and $G(\mathcal{L}, \psi^{-1}(B))$, it follows that $\chi(B_1) = G(\mathcal{L}, \psi^{-1}(B_1)) \neq G(\mathcal{L}, \psi^{-1}(B_2)) = \chi(B_2)$, which is impossible. Thus, χ must be injective. Since $|Block(\Gamma(\mathcal{L}))| = |\mathring{V}(T)| = |\{G(\mathcal{L}, v)\ :\ v \in \mathring{V}(T)\}|$, it follows that χ must also be surjective and thus bijective. ∎

5. A special type of minimal topological lasso.

Returning to the example depicted in Figure 2, it should be noted that, in addition to being a block graph, $\Gamma(\mathcal{L})$ is also claw-free (and thus $\mathcal{L}$ is a distinguished minimal topological lasso). Claw-free block graphs are precisely the line graphs of (unrooted) trees where for any graph G the associated line graph has vertex set $E(G)$ and two vertices $a, b \in E(G)$ are joined by an edge if $a \cap b \neq \emptyset$ [17]. In this section, we relate them with minimal topological lassos in Theorem 5.2 by establishing that for any X-tree T any minimal topological lasso $\mathcal{L}$ for T can be transformed into a distinguished minimal topological lasso $\mathcal{L}^*$ for T via a *repeated application* (i.e., $l \geq 0$ applications) of the following rule:

(R) If $xy, yz \in \mathcal{L}$ and $lca(y, z)$ is a descendant of $lca(x, y)$ in T, then delete xy from the edge set of $\Gamma(\mathcal{L})$, and add the edge xz to it.

Before we make this more precise, which we will do next, we remark that since a topological lasso for a star-tree is in particular a distinguished minimal topological lasso for it, we will for this and the next two sections restrict our attention to *nondegenerate* X-trees, that is, X-trees that are not star-trees on X.

Suppose T is a nondegenerate X-tree and $\mathcal{L}$ is a set of cords of X. Let $\mathring{V}(T)$ denote a set of colors, and let

$$\gamma_{(\mathcal{L},T)} : \mathcal{L} \to \mathring{V}(T) : ab \mapsto lca(a,b)$$

denote an edge coloring of $\Gamma(\mathcal{L})$ in terms of the interior vertices of T. Note that if $\mathcal{L}$ is a topological lasso for T, then Theorem 3.1 implies that $\gamma_{(\mathcal{L},T)}$ is surjective. Returning to rule (R), note that a repeated application of that rule to such a set $\mathcal{L}$ of cords results in a set $\mathcal{L}'$ of cords that is also a topological lasso for T. Furthermore, note that if $\mathcal{L}$ is a minimal topological lasso for T, then $\mathcal{L}'$ is necessarily also a minimal topological lasso for T. Finally, note for all $v \in \mathring{V}(T)$ that $|\gamma^{-1}_{(\mathcal{L},T)}(v)| = 1$ or $|\gamma^{-1}_{(\mathcal{L},T)}(v)| \geq 3$ must hold in this case.

LEMMA 5.1. *Suppose T is a nondegenerate X-tree and $\mathcal{L}$ is a minimal topological lasso for T. Put $\gamma = \gamma_{(\mathcal{L},T)}$, and assume that $v \in \mathring{V}(T)$ such that $|\gamma^{-1}(v)| \geq 3$. Then for any three pairwise distinct cords $c_1, c_2, c_3 \in \gamma^{-1}(v)$, there exists a cycle C_v in $\Gamma(\mathcal{L})$ such that $c_1, c_2, c_3 \in E(C_v)$ and, for all $c \in E(C_v)$, $\gamma(c)$ either equals v or is a descendant of v.*

Proof. Let $v \in \mathring{V}(T)$, and let $c_1 = x_1y_1$, $c_2 = x_2y_2$, and $c_3 = x_3y_3$ denote three pairwise distinct cords in $\gamma^{-1}(v)$. For all $i \in \langle 3 \rangle$, let $v_i \in ch(v)$ such that v_i lies on the path from v to x_i in T, and let $w_i \in ch(v)$ such that w_i lies on the path from v to y_i in T. Then, by Lemma 3.2, there exists unique pairs $(s_1, t_1) \in L(v_1) \times L(v_2)$, $(s_2, t_2) \in L(w_2) \times L(w_3)$, and $(s_3, t_3) \in L(w_1) \times L(v_3)$ such that, for all $i \in \langle 3 \rangle$, we have $s_it_i \in \mathcal{L}$. Since for all such i, we also have that $x_i \in L(v_i)$ and $y_i \in L(w_i)$ and, by Proposition 3.3, the graphs $\Gamma_{v_i}(\mathcal{L})$ and $\Gamma_{w_i}(\mathcal{L})$ are connected, it follows that there exists a cycle C_v in $\Gamma(\mathcal{L})$ that contains, for all $i \in \langle 3 \rangle$, the cords c_i and s_it_i in its edge set.

It remains to show that for every edge $c \in E(C_v)$, we have that $\gamma(c)$ either equals v or is a descendant of v. Suppose $c \in E(C_v)$. If there exists some $i \in \langle 3 \rangle$ such that $c \in \{c_i, s_it_i\}$, then $\gamma(c) = v$ clearly holds. So assume that this is not the case. Without loss of generality, we may assume that c lies on the path P from x_1 to s_1 in C_v that does not cross y_1. Since P is a subgraph of $\Gamma_{v_1}(\mathcal{L})$ and, as implied by Proposition 3.3, every edge in $\Gamma_{v_1}(\mathcal{L})$ is colored via γ with a descendant of v_1, it follows that $\gamma(c)$ is a descendant of v. ∎

To establish Theorem 5.2, we require further terminology. Suppose T is a nondegenerate X-tree, $\mathcal{L}$ is a minimal topological lasso for T, and $v \in \mathring{V}(T)$. Then we denote by $H_{\mathcal{L}}(v)$ the induced subgraph of $\Gamma(\mathcal{L})$ whose vertex set is the set of all $x \in X$ that are incident with some cord $c \in \mathcal{L}$ for which $\gamma_{(\mathcal{L},T)}(c) = v$ holds. Moreover, we denote the set of cut vertices of a connected block graph G by $Cut(G)$. Note that in every connected block graph G there must exist a vertex that is contained in at most one block of G. This last observation is central to the proof of Theorem 5.2(ii).

THEOREM 5.2. *Suppose T is a nondegenerate X-tree and $\mathcal{L}$ is a minimal topological lasso for T. Then there exists an ordering $\sigma : v_0, v_1, \ldots, v_k = \rho_T$, $k = |\mathring{V}(T)|$, of $\mathring{V}(T)$ such that the following hold:*

- (i) *There exists a sequence $\mathcal{L}_{v_0} = \mathcal{L}, \mathcal{L}_{v_1}, \ldots, \mathcal{L}^\dagger = \mathcal{L}_{v_k}$ of minimal topological lassos $\mathcal{L}_{v_i}$ for T, $i \in \langle k \rangle$, such that for all such i, we have the following:*
  - (L1) *$\mathcal{L}_{v_i}$ is obtained from $\mathcal{L}_{v_{i-1}}$ via a repeated application of rule* (R), *and $H_{\mathcal{L}_{v_i}}(v_i)$ is a maximal clique in $\Gamma(\mathcal{L}_{v_i})$.*
  - (L2) *For all $j \in \langle i-1 \rangle$, $H_{\mathcal{L}_{v_i}}(v_j)$ is a maximal clique in $\Gamma(\mathcal{L}_{v_i})$.*

  In particular, $\Gamma(\mathcal{L}^\dagger)$ is a block graph.

(ii) *If* $\Gamma(\mathcal{L})$ *is a block graph, then there exists a sequence* $\mathcal{L}_{v_0} = \mathcal{L}, \mathcal{L}_{v_1}, \ldots, \mathcal{L}^* = \mathcal{L}_{v_k}$ *of minimal topological lassos* $\mathcal{L}_{v_i}$ *for* T*,* $i \in \langle k \rangle$*, such that for all such* i*, we have the following:*

(L1′) $\mathcal{L}_{v_i}$ *is obtained from* $\mathcal{L}_{v_{i-1}}$ *via a repeated application of rule* (R)*, and* $\Gamma(\mathcal{L}_{v_i})$ *is a block graph.*

(L2′) $\Gamma_{v_i}(\mathcal{L}_{v_i})$ *is a claw-free block graph.*

In particular, $\mathcal{L}^*$ *is a distinguished minimal topological lasso for* T*.*

Proof. For all $i \in \langle k \rangle$, put $\mathcal{L}_i = \mathcal{L}_{v_i}$ and $\gamma_i = \gamma_{(\mathcal{L}_i, T)}$. Clearly, if $\mathcal{L}$ is a distinguished minimal topological lasso, then the sequences as described in (i) and (ii) exist. So assume that this is not the case. For all $v \in \mathring{V}(T)$, let $l(v)$ denote the length of the path from the root ρ_T of T to v, and put $h = \max_{v \in \mathring{V}(T)}\{l(v)\}$. Note that $h \geq 1$ as T is nondegenerate. For all $i \in \langle h \rangle$, let $V(i) \subseteq \mathring{V}(T)$ denote the set of all interior vertices v of T such that $l(v) = i$. Let σ denote an ordering of the vertices in $\mathring{V}(T)$ such that the vertices in $V(h)$ come first (in any order), then (again in any order) the vertices in $V(h-1)$, and so on, with the last vertex in that ordering being ρ_T.

(i) Suppose $v \in \mathring{V}(T)$. If $v \in V(h)$, then we may assume without loss of generality that $v = v_1$. Then v_1 is the parent of a pseudo-cherry of T, and so Theorem 3.1 implies that $H_{\mathcal{L}}(v_1)$ is a maximal clique in $\Gamma(\mathcal{L})$. Thus, $\mathcal{L}_1 := \mathcal{L}$ is a minimal topological lasso for T that satisfies properties (L1) and (L2).

So assume that $v \notin V(h)$. Then there exists some $|V(h)| < i \leq k$ such that $v = v_i$. Without loss of generality, we may assume that v_i is such that, for all $j \in \langle i-1 \rangle$, $\mathcal{L}_j$ is a minimal topological lasso for T that satisfies properties (L1) and (L2). If v_i is the parent of a pseudo-cherry of T, then arguments similar to those above imply that $\mathcal{L}_i := \mathcal{L}_{i-1}$ is a minimal topological lasso for T that satisfies properties (L1) and (L2). So assume that v_i is not the parent of a pseudo-cherry of T. We distinguish between the cases that $H_{\mathcal{L}_{i-1}}(v)$ is a maximal clique in $\mathcal{L}_{i-1}$ and that it is not.

Assume first that $H_{\mathcal{L}_{i-1}}(v)$ is a maximal clique in $\mathcal{L}_{i-1}$. Then since $\mathcal{L}_{i-1}$ is a minimal topological lasso for T that satisfies properties (L1) and (L2), it is easy to see that $\mathcal{L}_i := \mathcal{L}_{i-1}$ is also a minimal topological lasso for T that satisfies properties (L1) and (L2). To see that $H_{\mathcal{L}_{i-1}}(v)$ is a maximal clique in $\mathcal{L}_{i-1}$, let $e_1 = x_1y_1$, $e_2 = x_2y_2$, and $e_3 = x_3y_3$ denote three pairwise distinct edges in $H_{\mathcal{L}_{i-1}}(v)$. For all $i \in \langle 3 \rangle$, put $z_i = lca(x_i, y_i)$. By Lemma 5.1 there exists a cycle C_v in $H_{\mathcal{L}_{i-1}}(v)$ that contains $\{e_1, e_2, e_3\}$ in its edge set. A repeated application of rule (R) to $\mathcal{L}_{i-1}$ implies that, for all $i \in \langle 3 \rangle$, we can find elements $x'_i \in L(z_i)$ such that

$$\mathcal{L}'_{i-1} := \mathcal{L}_{i-1} - \{x_1y_1, x_2y_2, x_3y_3\} \cup \{x'_1x'_2, x'_2x'_3, x'_3x'_1\}$$

is a minimal topological lasso for T and the cords $x'_1x'_2$, $x'_2x'_3$, and $x'_3x'_1$ form a 3-clique in $H_{\mathcal{L}'_{i-1}}(v)$. Transforming $\mathcal{L}'_{i-1}$ further by processing any three pairwise distinct edges in $H_{\mathcal{L}'_{i-1}}(v)$ that do not already form a 3-clique in the same way and so on eventually yields a minimal topological lasso $\mathcal{L}_i$ for T such that $H_{\mathcal{L}_i}(v)$ is a maximal clique in $\Gamma(\mathcal{L}_i)$. Thus, property (L1) is satisfied by $\mathcal{L}_i$. Since only edges e of $\Gamma(\mathcal{L}_{i-1})$ have been modified by the above transformation for which $\gamma_{i-1}(e) = v$ holds and, by assumption, $\mathcal{L}_{i-1}$ satisfies property (L2), it follows that $\mathcal{L}_i$ also satisfies that property.

Processing the successor of v_i in σ in the same way and so on yields a minimal topological lasso $\mathcal{L}^\dagger$ for T for which $\Gamma(\mathcal{L}^\dagger)$ is a block graph. This completes the proof of (i).

(ii) For all $i \in \langle k \rangle$ and all vertices $w \in \mathring{V}(T)$, put $B_w^i = B_w^{\mathcal{L}_i}$. Suppose that $v \in \mathring{V}(T)$. If $v \in V(h)$, then we may assume without loss of generality that $v =$

v_1. Then v is the parent of a pseudo-cherry of T, and so $\mathcal{L}_1 := \mathcal{L}$ clearly satisfies properties (L1') and (L2').

So assume that $v \notin V(h)$. Then there exists some $|V(h)| < i \leq k$ such that $v = v_i$. Without loss of generality, we may assume that v_i is minimal; that is, for all $j \in \langle i-1 \rangle$, we have that $\mathcal{L}_j$ is a minimal topological lasso for T that satisfies properties (L1') and (L2'). If v is the parent of a pseudo-cherry of T, then arguments similar to those above imply that $\mathcal{L}_i := \mathcal{L}_{i-1}$ satisfies properties (L1') and (L2'). So assume that v is not the parent of a pseudo-cherry of T. If $\Gamma_v(\mathcal{L}_{i-1})$ is a claw-free block graph, then setting $\mathcal{L}_i := \mathcal{L}_{i-1}$ implies that $\mathcal{L}_i$ satisfies properties (L1') and (L2').

So assume that this is not the case, that is, that there exists a vertex $x \in L(v)$ that, in addition to being a vertex in the block B_v^{i-1} of $\Gamma(\mathcal{L}_{i-1})$ and thus of $\Gamma_v(\mathcal{L}_{i-1})$, is also a vertex in $l \geq 2$ further blocks $B_1, \dots, B_l$ of $\Gamma_v(\mathcal{L}_{i-1})$. Clearly, $B_1, \dots, B_l$ are also blocks in $\Gamma(\mathcal{L})$. Then there exists a path P from v to x in T that contains, for all $l \geq 2$, the vertices $\psi^{-1}(B_1), \dots, \psi^{-1}(B_l)$ in its vertex set, where $\psi : \mathring{V}(T) \to \mathrm{Block}(\Gamma(\mathcal{L}))$ is the map from Corollary 4.3. Let $w \in ch(v)$ denote the child of v that lies on P. Note that since $l \geq 2$, we have $w \in \mathring{V}(T)$. Without loss of generality, we may assume that $w = v_{i-1}$. The fact that $\Gamma_{v_{i-1}}(\mathcal{L}_{i-1})$ is connected combined with the fact that $\Gamma(\mathcal{L}_{i-1})$ is a block graph and so $\Gamma_{v_{i-1}}(\mathcal{L}_{i-1})$ is also a block graph implies, in view of the observation preceding Theorem 5.2, that we may choose some $y \in L(v_{i-1}) - Cut(\Gamma_{v_{i-1}}(\mathcal{L}_{i-1}))$. Then y is a vertex in precisely one block of $\Gamma_{v_{i-1}}(\mathcal{L}_{i-1})$ and thus can be a vertex in at most two blocks of $\Gamma_v(\mathcal{L}_{i-1})$. Consequently, $y \neq x$. Applying rule (R) repeatedly to $\mathcal{L}_{i-1}$, let $\mathcal{L}_i$ denote the set of cords obtained from $\mathcal{L}_{i-1}$ by replacing, for all $i \leq q \leq k$, every cord of $\mathcal{L}_{i-1}$ of the form xa with $a \in V(B_{v_q}^{i-1})$ by the cord ya. Then, by construction, $\mathcal{L}_i$ is a minimal topological lasso for T and $\Gamma(\mathcal{L}_i)$ is a block graph. Hence, $\mathcal{L}_i$ satisfies property (L1'). Moreover, since $\Gamma_{v_{i-1}}(\mathcal{L}_{i-1})$ is claw-free, it follows that $\mathcal{L}_i$ satisfies property (L2'), too.

Applying the above arguments to the successor of v_i in σ and so on eventually yields a minimal topological lasso $\mathcal{L}_k$ for T that satisfies properties (L1') and (L2'). Thus, $\Gamma_{v_k}(\mathcal{L}_k)$ is a claw-free block graph, and so $\mathcal{L}^*$ is a distinguished minimal topological lasso for T. ∎

To illustrate Theorem 5.2, let $X = \{a, \dots, f\}$, and consider the X-tree T' depicted in Figure 2(iii) along with the set $\mathcal{L} = \{ad, ec, fa, ef, cd, bd\}$ of cords of X which we depict in Figure 3(i) in the form of $\Gamma(\mathcal{L})$. Using, for example, Theorem 3.1, it is

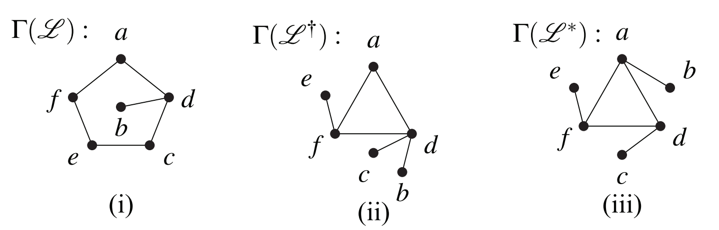

FIG. 3. *For $X = \{a, \dots, f\}$ and the X-tree T' pictured in Figure* 2(iii)*, we depict in* (i) *the minimal topological lasso $\mathcal{L} = \{ad, ec, fa, fe, cd, bd\}$ for T' in the form of $\Gamma(\mathcal{L})$. In the same way as in* (i)*, we depict in* (ii) *the transformed minimal topological lasso $\mathcal{L}^\dagger$ for T' such that $\Gamma(\mathcal{L}^\dagger)$ is a block graph, and we depict in* (iii) *the distinguished minimal topological lasso $\mathcal{L}^*$ for T' obtained from $\mathcal{L}^\dagger$; see text for details.*

straightforward to check that $\mathcal{L}$ is a minimal topological lasso for T', but $\Gamma(\mathcal{L})$ is clearly not a block graph, and so $\mathcal{L}$ is also not distinguished. To transform $\mathcal{L}$ into a distinguished minimal topological lasso $\mathcal{L}^*$ for T' as described in Theorem 5.2, consider the ordering $v_1 = lca(e,f)$, $v_2 = lca(c,d)$, $v_3 = lca(a,d)$, $v_4 = \rho_{T'}$ of the interior vertices of T'. For all $i \in \langle 4 \rangle$, put $\mathcal{L}_i = \mathcal{L}_{v_i}$. Then we first transform $\mathcal{L}$ into a minimal topological lasso $\mathcal{L}^\dagger$ for T' as described in Theorem 5.2(i). For this we have $\mathcal{L} = \mathcal{L}_0 = \mathcal{L}_1 = \mathcal{L}_2$, and $\mathcal{L}_3$ is obtained from $\mathcal{L}_2$ by first applying rule (R) to the cords $ec, cd \in \mathcal{L}_2$, resulting in the deletion of the cord ce from $\mathcal{L}_2$ and the addition of the cord ed to $\mathcal{L}_2$, and then applying rule (R) to the cords $fe, ed \in \mathcal{L}_2$, resulting in the deletion of the cord ed from $\mathcal{L}_2$ and the addition of the cord fd to $\mathcal{L}_2$. The graph $\Gamma(\mathcal{L}_3)$ is depicted in Figure 3(ii). Note that $\mathcal{L}_3 = \mathcal{L}^\dagger$ and that, although $\Gamma(\mathcal{L}^\dagger)$ is clearly a block graph, $\mathcal{L}^\dagger$ is not distinguished.

To transform $\mathcal{L}^\dagger$ into a distinguished minimal topological lasso $\mathcal{L}^*$ for T', we next apply Theorem 5.2(ii). For this, we need only consider the vertex d of $\Gamma(\mathcal{L}^\dagger)$; that is, we have $\mathcal{L}^\dagger = \mathcal{L}_0 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_3$. Since the child of v_4 on the path from v_4 to d is v_3, we may choose a as the element y in $L(v_3) - Cut(\Gamma_{v_3}(\mathcal{L}_3))$. Then applying rule (R) to the cords $bd, da \in \mathcal{L}_3$ implies the deletion of bd from $\mathcal{L}_3$ and the addition of the cord ab to $\mathcal{L}_3$. The resulting minimal topological lasso for T' is $\mathcal{L}^*$, which we depict in Figure 3(iii) in the form of $\Gamma(\mathcal{L}^*)$.

We conclude this section by remarking in passing that, combined with Theorem 3.1, which implies that any minimum-sized topological lasso for an X-tree T must have $\sum_{v\in \mathring{V}(T)} \binom{|ch(v)|}{2}$ cords, Theorem 5.2 and Corollary 4.3 imply that the minimum-sized topological lassos of an X-tree T are precisely the minimal topological lassos of T.

6. A sufficient condition for a minimal topological lasso to be distinguished. In this section, we turn our attention toward presenting a sufficient condition for a minimal topological lasso for some X-tree T to be a distinguished minimal topological lasso for T. In the next section, we will show that this condition is also necessary.

We start our discussion by introducing some more terminology. Suppose T is a nondegenerate X-tree. Put $cl(T) = \{L(v) : v \in \mathring{V}(T) - \{\rho_T\}\}$, and note that $cl(T) \neq \emptyset$. For all $A \in cl(T)$, put $cl_A(T) := \{B \in cl(T) : B \subsetneq A\}$, and note that a vertex $v \in \mathring{V}(T) - \{\rho_T\}$ is the parent of a pseudo-cherry of T if and only if $cl_{L(v)}(T) = \emptyset$. For σ a total ordering of X and $\min_\sigma(C)$ denoting the minimal element of a nonempty subset C of X, we call a map of the form

$$f : cl(T) \to X : A \mapsto \begin{cases} \min_\sigma(A - \{f(B) : B \in cl_A(T)\}) & \text{if } cl_A(T) \neq \emptyset, \\ \min_\sigma(A) & \text{else} \end{cases}$$

a *cluster marker map (for T and σ)*. Note that since $|\mathring{V}(T')| \leq |X| - 1$ holds for all X-trees T' and so $A - \{f(B) : B \in cl_A(T)\} \neq \emptyset$ must hold for all $A \in cl(T)$ with $cl_A(T) \neq \emptyset$, it follows that f is well defined. Also note that if $v \in \mathring{V}(T)$ is the parent of a pseudo-cherry C of T, then $f(L(v)) = f(C) = \min_\sigma(C)$ as $cl_C(T) = \emptyset$ in this case. Finally, note that it is easy to see that a cluster marker map must be injective but need not be surjective.

We are now ready to present a construction of a distinguished minimal topological lasso which underpins the aforementioned sufficient condition that a minimal topological lasso must satisfy to be distinguished. Suppose that T is a nondegenerate X-tree, that σ is a total ordering of X, and that $f : cl(T) \to X$ is a cluster marker map for T and σ. We first associate to every interior vertex $v \in \mathring{V}(T)$ a set $\mathcal{L}_{(T,f)}(v)$

defined as follows. Let $l_1, \dots, l_{k_v}$ denote the children of v that are leaves of T, and let $v_1, \dots, v_{p_v}$ denote the children of v that are also interior vertices of T. Note that $k_v = 0$ or $p_v = 0$ might hold but not both. Put $\binom{\emptyset}{2} = \binom{\langle 1\rangle}{2} = \emptyset$. Then we set

$$\mathcal{L}_{(T,f)}(v) := \bigcup_{\{i,j\}\in\binom{\langle k_v\rangle}{2}} \{l_i l_j\} \cup \bigcup_{\{i,j\}\in\binom{\langle p_v\rangle}{2}} \{f(L(v_i))f(L(v_j))\} \cup \bigcup_{i\in\langle k_v\rangle,\ j\in\langle p_v\rangle} \{l_i f(L(v_j))\}.$$

Note that $|\mathcal{L}_{(T,f)}(v)| \geq 1$ must hold for all $v \in \mathring{V}(T)$. Finally, we set

$$\mathcal{L}_{(T,f)} := \bigcup_{v\in\mathring{V}(T)} \mathcal{L}_{(T,f)}(v).$$

To illustrate these definitions, consider the $X = \{a, \dots, f\}$-tree T' depicted in Figure 2(iii). Let σ denote the lexicographic ordering of the elements in X. Then the map $f : cl(T') \to X$ defined by setting

$$f(\{c, d\}) = c, \quad f(\{e, f\}) = e, \quad \text{and } f(X - \{b\}) = a$$

is a cluster marker map for T' and σ, and $\mathcal{L}_{(T,f)}$ (or more precisely the graph $\Gamma(\mathcal{L}_{(T',f)})$) is depicted in Figure 2(i).

To help establish Theorem 6.3, we require some intermediate results which are of interest in their own right.

LEMMA 6.1. *Suppose T is a nondegenerate X-tree, σ is a total ordering of X, and $f : cl(T) \to X$ is a cluster marker map for T and σ. Then the following hold:*

(i) *$\mathcal{L}_{(T,f)}$ is a minimal topological lasso for T.*

(ii) *$\Gamma(\mathcal{L}_{(T,f)})$ is connected.*

(iii) *If v and w are distinct interior vertices of T, then $|\bigcup\mathcal{L}_{(T,f)}(v) \cap \bigcup\mathcal{L}_{(T,f)}(w)| \leq 1$.*

(iv) *Suppose $x \in X$. Then there exist distinct vertices $v, w \in \mathring{V}(T)$ such that $x \in \bigcup\mathcal{L}_{(T,f)}(v) \cap \bigcup\mathcal{L}_{(T,f)}(w)$ if and only if there exists some $u \in \mathring{V}(T) - \{\rho_T\}$ such that $x = f(L(u))$.*

Proof. For all $v \in \mathring{V}(T)$, set $\mathcal{L}(v) = \mathcal{L}_{(T,f)}(v)$.

(i) This is an immediate consequence of Theorem 3.1 and, for $v \in \mathring{V}(T)$, the respective definitions of the set $\mathcal{L}(v)$ and the graph $G(\mathcal{L}', v)$, where $\mathcal{L}'$ is a set of cords of X.

(ii) This is an immediate consequence of Proposition 3.3 combined with Lemma 6.1(i).

(iii) This is an immediate consequence of the fact that, for all vertices $u \in \mathring{V}(T)$ and all $x, y \in \bigcup\mathcal{L}(u)$ distinct, we have $u = lca(x, y)$.

(iv) Let $x \in X$, and assume for contradiction that there exist distinct vertices $v, w \in \mathring{V}(T)$ such that $x \in \bigcup\mathcal{L}(v) \cap \bigcup\mathcal{L}(w)$ but $x \neq f(L_T(u))$ for all $u \in \mathring{V}(T) - \{\rho_T\}$. Then x must be a leaf of T that is simultaneously adjacent with v and w, which is impossible. Thus, there must exist some $u \in \mathring{V}(T)$ such that $x = f(L(u))$.

Conversely, assume that $x = f(L(u))$ for some $u \in \mathring{V}(T) - \{\rho_T\}$. Then $x \in L(u)$. Let w denote the parent of x on the path from u to x. Then $x \in \bigcup\mathcal{L}(w)$. Let v denote the parent of u in T which exists since $u \neq \rho_T$. Then $x = f(L(u)) \in \bigcup\mathcal{L}(v)$, and so $x \in \bigcup\mathcal{L}(v) \cap \bigcup\mathcal{L}(w)$, as required. ∎

Note that $u \in \{v, w\}$ need not hold for u, v, and w as in the statement of Lemma 6.1(iv). Indeed, suppose T is the $X = \{a, b, c, d\}$-tree with unique cherry $\{a, b\}$ and d adjacent with the root ρ_T of T. Let σ denote the lexicographic ordering of X, and let $f : cl(T) \to X$ be (the unique) cluster marker map for T and σ. Set $x = b$, $v = lca(a,b)$, $w = \rho_T$. Then $x = f(L(u))$, where $u = lca(a,c)$ and $x \in \bigcup \mathcal{L}(v) \cap \bigcup \mathcal{L}(w)$ but $u \notin \{v,w\}$.

PROPOSITION 6.2. *Suppose T is a nondegenerate X-tree, σ is a total ordering of X, and $f : cl(T) \to X$ is a cluster marker map for T and σ. Then $\Gamma(\mathcal{L}_{(T,f)})$ is a connected block graph, and every block is of the form $\Gamma(\mathcal{L}_{(T,f)}(v))$ for some $v \in \mathring{V}(T)$.*

Proof. Put $\mathcal{L} = \mathcal{L}_{(T,f)}$, and, for all $v \in \mathring{V}(T)$, put $\mathcal{L}(v) = \mathcal{L}_{(T,f)}(v)$. We claim that if C is a cycle in $\Gamma(\mathcal{L})$ of length at least three, then there must exist some $v \in \mathring{V}(T)$ such that C is contained in $\Gamma(\mathcal{L}(v))$. Assume to the contrary that this is not the case; that is, there exists some cycle $C : u_1, u_2, \ldots, u_l, u_{l+1} = u_1$, $l \geq 3$, in $\Gamma(\mathcal{L})$ such that, for all $v \in \mathring{V}(T)$, we have that C is not a cycle in $\Gamma(\mathcal{L}(v))$. Without loss of generality, we may assume that C is of minimal length. For all $i \in \langle l \rangle$, put $v_i = lca_T(u_i, u_{i+1})$. Then, by the construction of $\Gamma(\mathcal{L})$, we have for all such i that $u_i u_{i+1}$ is an edge in $\Gamma(\mathcal{L}(v_i))$ and, by the minimality of C, that $v_i \neq v_j$ for all $i, j \in \langle l \rangle$ distinct. Put $Y = V(C)$, and let $T' = T|_Y$ denote the Y-tree obtained by restricting T to Y. Note that $lca_T(u_i, u_{i+1}) = lca_{T'}(u_i, u_{i+1})$ holds for all $i \in \langle l \rangle$. Thus, the map $\phi : E(C) \to \mathring{V}(T')$ defined by putting $u_i u_{i+1} \mapsto lca_T(u_i, u_{i+1})$, $i \in \langle l \rangle$, is well defined. Since $|E(C)| = l$ and for any finite set Z with three or more elements a Z-tree has at most $|Z| - 1$ interior vertices, it follows that there exist $i, j \in \langle l \rangle$ distinct such that $\phi(u_i, u_{i+1}) = \phi(u_j, u_{j+1})$. Consequently, $v_i = lca_T(u_i, u_{i+1}) = lca_T(u_j, u_{j+1}) = v_j$, which is impossible and thus proves the claim. Combined with Lemma 6.1(ii) and (iii), it follows that $\Gamma(\mathcal{L})$ is a connected block graph. That the blocks of $\Gamma(\mathcal{L})$ are of the required form is an immediate consequence of the construction of $\Gamma(\mathcal{L})$. ∎

To be able to establish that $\mathcal{L}_{(T,f)}(v)$ is indeed a distinguished minimal topological lasso for T and f as above, we require a further concept. Suppose $A, B \subseteq X$ are two distinct nonempty subsets of X. Then A and B are said to be *compatible* if $A \cap B \in \{\emptyset, A, B\}$. As is well known (see, e.g., [10, 28]), for any X-tree T' and any two vertices $v, w \in V(T')$ the subsets $L(v)$ and $L(w)$ of X are compatible.

THEOREM 6.3. *Suppose T is a nondegenerate X-tree, σ is a total ordering of X, and $f : cl(T) \to X$ is a cluster marker map for T and σ. Then $\mathcal{L}_{(T,f)}$ is a distinguished minimal topological lasso for T.*

Proof. Put $\mathcal{L} = \mathcal{L}_{(T,f)}$, and, for all $v \in \mathring{V}(T)$, put $\mathcal{L}(v) = \mathcal{L}_{(T,f)}(v)$. In view of Proposition 6.2 and Lemma 6.1(i), it suffices to show that $\Gamma(\mathcal{L})$ is claw-free. Assume to the contrary that this is not the case and that there exists some $x \in X$ that is contained in the vertex set of $m \geq 3$ blocks $A_1, \ldots, A_m$ of $\Gamma(\mathcal{L})$. Then, by Proposition 6.2, there exist distinct interior vertices $v_1, \ldots, v_m$ of T such that, for all $i \in \langle m \rangle$, we have $V(A_i) = \bigcup \mathcal{L}(v_i) \subseteq L(v_i)$. Since for all $v, w \in V(T)$ distinct, the sets $L(v)$ and $L(w)$ are compatible, it follows that there exists a path P from ρ_T to x that contains the vertices $v_1, \ldots, v_m$ in its vertex set. Without loss of generality, we may assume that $m = 3$ and that, starting at ρ_T and moving along P, the vertex v_1 is encountered first, and then v_2 is encountered, followed by v_3. Note that $cl_{L(v_i)}(T) \neq \emptyset$ for $i = 1, 2$. Since T is a tree and so x can be adjacent neither with v_1 nor with v_2, it follows that there must exist for $i = 1, 2$ some $B_i \in cl_{L(v_i)}(T)$ such that $x = f(B_i)$. But this is impossible since $B_2 \in cl_{L(v_1)}(T)$, and so $f(B_1) \neq f(B_2)$ as f is a cluster marker map for T and σ. ∎

7. Characterizing distinguished minimal topological lassos. In this section, we establish the converse of Theorem 6.3 which allows us to characterize distinguished minimal topological lassos of nondegenerate X-trees. We start with a well-known construction for associating an unrooted tree to a connected block graph (see, e.g, [8]). Suppose that G is a connected block graph. Then we denote by T_G the (unrooted) tree associated to G whose vertex set is $Cut(G) \cup Block(G)$ and whose edges are of the form $\{a, B\}$, where $a \in Cut(G)$, $B \in Block(G)$, and $a \in B$. Note that if a vertex $v \in V(T_G)$ is a leaf of T_G, then $(\{v\}, \emptyset) \in Block(G)$.

Suppose T is a nondegenerate X-tree and $\mathcal{L}$ is a distinguished minimal topological lasso for T. Let v denote an interior vertex of T whose children are $v_1 \dots, v_l$, where $l = |ch(v)|$. Then Corollary 4.3 combined with Proposition 4.1 implies that for all $i \in \langle l \rangle$ there exists a unique leaf $x_i \in L(v_i)$ of T such that, for all $i, j \in \langle l \rangle$ distinct, $x_i x_j \in \mathcal{L}$ and $\{x_1, \dots, x_l\} = V(B_v)$. Since $\Gamma(\mathcal{L})$ is claw-free, every vertex of B_v is contained in at most one further block of $\Gamma(\mathcal{L})$. Thus, if there exists some $w \in V(B_v)$ such that $w \in V(B)$ holds too for some block $B \in Block(\Gamma(\mathcal{L}))$ distinct from B_v, then w must be a cut vertex of $\Gamma(\mathcal{L})$. For every vertex $v' \in \mathring{V}(T)$ that is the child of some vertex $v \in \mathring{V}(T)$, we denote the unique element $x \in L(v')$ contained in $V(B_v)$ by $c_{B_{v'}}$ in case $x \in Cut(\Gamma(\mathcal{L}))$. Note that it is not difficult to observe that, in the tree $T_{\Gamma(\mathcal{L})}$, the vertex $c_{B_{v'}}$ is the vertex adjacent with B_v that lies on the path from B_v to $B_{v'}$.

The following result lies at the heart of Theorem 7.2 and establishes a crucial relationship between the nonroot interior vertices of T and the cut vertices of $\Gamma(\mathcal{L})$.

LEMMA 7.1. *Suppose T is an X-tree and $\mathcal{L}$ is a distinguished minimal topological lasso for T. Then the map*

$$\theta : \mathring{V}(T) - \{\rho_T\} \to Cut(\Gamma(\mathcal{L})) : \quad v \mapsto c_{B_v}$$

is bijective.

Proof. Clearly, θ is well defined and injective. To see that θ is bijective, let $T^-_{\Gamma(\mathcal{L})}$ denote the tree obtained from $T_{\Gamma(\mathcal{L})}$ by suppressing all vertices that were contained in $Cut(\Gamma(\mathcal{L}))$. Then $Block(\Gamma(\mathcal{L})) = V(T^-_{\Gamma(\mathcal{L})})$. Corollary 4.3 implies that $|Block(\Gamma(\mathcal{L}))| = |\mathring{V}(T)|$ as $\Gamma(\mathcal{L})$ is a block graph. Since $\Gamma(\mathcal{L})$ is claw-free, we clearly also have $|Cut(\Gamma(\mathcal{L}))| = |E(T^-_{\Gamma(\mathcal{L})})|$. Combined with the fact that $|V(T')| = |E(T')|+1$ holds for every tree T', it follows that $|Cut(\Gamma(\mathcal{L}))| = |Block(\Gamma(\mathcal{L}))|-1 = |\mathring{V}(T)|-1 = |\mathring{V}(T) - \{\rho_T\}|$. Thus, θ is bijective. ∎

Armed with this result, we are now ready to establish the converse of Theorem 6.3, which yields the aforementioned characterization of distinguished minimal topological lassos of nondegenerate X-trees.

THEOREM 7.2. *Suppose T is a nondegenerate X-tree and $\mathcal{L}$ is a set of cords of X. Then $\mathcal{L}$ is a distinguished minimal topological lasso for T if and only if there exists a total ordering σ of X and a cluster marker map f for T and σ such that $\mathcal{L}_{(T,f)} = \mathcal{L}$.*

Proof. Assume first that σ is some total ordering of X and that $f : cl(T) \to X$ is a cluster marker map for T and σ. Then, by Theorem 6.3, $\mathcal{L}_{(T,f)}$ is a distinguished minimal topological lasso for T.

Conversely, assume that $\mathcal{L}$ is a distinguished minimal topological lasso for T, and consider an embedding of T in the plane. By abuse of terminology, we will refer to this embedding of T also as T. We start with defining a total ordering σ of X. To this end, we first define a map $t : \mathring{V}(T) - \{\rho_T\} \to \mathbb{N}$ by setting, for all $v \in \mathring{V}(T) - \{\rho_T\}$, $t(v)$ to be the length of the path from ρ_T and v. Put $h = \max\{t(v) : v \in \mathring{V}(T) - \{\rho_T\}\}$, and note that $h \geq 1$ as T is nondegenerate. Starting at the leftmost interior vertex v of T for which $t(v) = h$ holds and moving, for all $l \in \langle h \rangle$, from left to right, we enumerate all interior vertices of T but the root. We next put $n = |X|$ and $X = \langle n \rangle$ and relabel the elements in X such that when traversing the circular ordering induced by T on $X \cup \{\rho_T\}$ in a counterclockwise fashion we have $\rho_T, 1, 2, 3, \dots, n, \rho_T$. To reflect this with regard to $\mathcal{L}$, we relabel the elements of the cords in $\mathcal{L}$ accordingly and denote the resulting distinguished minimal topological lasso for T also by $\mathcal{L}$.

By Lemma 7.1, the map $\theta : \mathring{V}(T) - \{\rho_T\} \to Cut(\Gamma(\mathcal{L}))$ defined in that lemma is bijective. Put $m = |Cut(\Gamma(\mathcal{L}))|$, and let $v_1, v_2, \dots, v_m$ denote the enumeration of the vertices in $\mathring{V}(T) - \{\rho_T\}$ obtained above. Also, set $Y = X - \{\theta(v_i) : i \in \langle m \rangle\}$. Let $y_1, y_2, \dots, y_l$ denote an arbitrary but fixed total ordering of the elements of Y where $l = |Y|$. Then we define σ to be the total ordering of X given by

$$\sigma : \ \theta(v_1), \theta(v_2), \dots, \theta(v_{i-1}), \theta(v_i), \theta(v_{i+1}), , \dots, \theta(v_m), y_1, y_2, \dots, y_l,$$

where $\theta(v_1)$ is the minimal element and y_l is the maximal element. Note that if $v \in \mathring{V}(T)$ is the parent of a pseudo-cherry C of T, then $\theta(v) = \min_\sigma C$.

We briefly interrupt the proof of the theorem to illustrate these definitions by means of an example. Put $X = \langle 13 \rangle$, and consider the X-tree T depicted in Figure 4(i) (ignoring the labeling of the interior vertices for the moment) and the distinguished minimal topological lasso $\mathcal{L}$ for T pictured in the form of $\Gamma(\mathcal{L})$ in Figure 4(ii). Then the labeling of the interior vertices of T gives the enumeration of those vertices considered in the proof of Theorem 7.2. The total ordering σ of X restricted to the elements in $\{\theta(v_1), \dots, \theta(v_6)\}$ is $3, 5, 12, 1, 10, 7$.

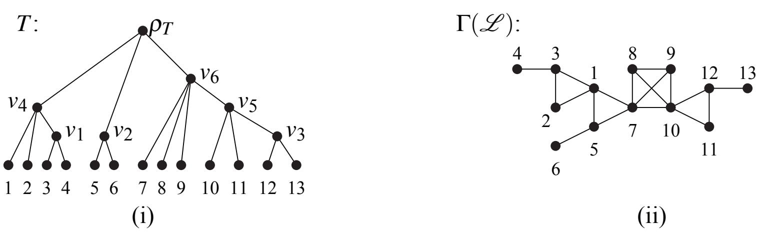

Fig. 4. *For $X = \langle 13 \rangle$ and the depicted X-tree T, the enumeration of the interior vertices of T considered in the proof of Theorem* 7.2 *is indicated in* (i). *With regard to this enumeration and the distinguished minimal topological lasso $\mathcal{L}$ for T pictured in the form of $\Gamma(\mathcal{L})$ in* (ii)*, the total ordering σ of X considered in that proof restricted to the elements in $\{\theta(v_1), \dots, \theta(v_6)\}$ is* $3, 5, 12, 1, 10, 7$.

Returning to the proof of the theorem, we claim that the map $f : cl(T) \to X$ given, for all $A \in cl(T)$, by setting $f(A) = \theta(lca(A))$ is a cluster marker map for T. Indeed, suppose $A \in cl(T)$. Then $\theta(lca(A)) = c_{B_{lca(A)}} \in L(lca(A))$ holds by construction. We distinguish between the cases that $cl_A(T) \neq \emptyset$ and $cl_A(T) = \emptyset$. If $cl_A(T) \neq \emptyset$, then since θ is bijective, it follows that $\theta(lca(A)) \neq \theta(v)$ holds for all descendants $v \in \mathring{V}(T)$ of $lca(A)$. Combined with the definition of σ, we obtain $f(A) = \theta(lca(A)) = \min_\sigma(A - \{\theta(lca(D)) : D \in cl_A(T)\}) = \min_\sigma(A - \{f(D) : D \in cl_A(T)\})$, as required. If $cl_A(T) = \emptyset$, then, as was observed above, $f(A) = \theta(lca(A)) = \min_\sigma A$. Thus, f is a cluster marker map for T and σ, as claimed.

It remains to show that $\mathcal{L}_{(T,f)} = \mathcal{L}$. To see this, note first that, by Theorem 6.3, $\mathcal{L}_{(T,f)}$ is a distinguished minimal topological lasso for T. Since Lemma 3.2 implies that any two minimal topological lassos for T must be of the same size and thus $|\mathcal{L}_{(T,f)}| = |\mathcal{L}|$ holds, it therefore suffices to show that $\mathcal{L} \subseteq \mathcal{L}_{(T,f)}$. Suppose $a, b \in X$ are distinct such that $ab \in \mathcal{L}$. Then there exists some interior vertex $v \in \mathring{V}(T)$ such that $v = lca(a, b)$. Hence, $a, b \in V(B_v)$. We claim that $ab \in \mathcal{L}_{(T,f)}(v)$. To establish this claim, we distinguish between the cases that (i) $a \in ch(v)$ and (ii) $a \notin ch(v)$.

Assume first that case (i) holds, that is, that a is a child of v. If $b \in ch(v)$, then the claim is an immediate consequence of the definition of $\mathcal{L}_{(T,f)}(v)$. So assume that $b \notin ch(v)$. Let $v' \in \mathring{V}(T)$ denote the child of v for which $b \in L(v)$ holds. Then $b = c_{B_{v'}} = \theta(v') = f(L(v'))$ follows by the observation preceding Lemma 7.1 combined with the fact that $b \in V(B_v)$. Hence, $ab = af(L(v')) \in \mathcal{L}_{(T,f)}(v)$, as claimed.

Assume next that case (ii) holds, that is, that a is not a child of v. In view of the previous subcase, it suffices to consider the case that $b \notin ch(v)$. Let $v', v'' \in \mathring{V}(T)$ denote the children of v such that $a \in L(v')$ and $b \in L(v'')$. Then, again by the observation preceding Lemma 7.1 combined with the fact that $a, b \in V(B_v)$, we have $a = c_{B_{v'}} = \theta(v') = f(L(v'))$ and $b = c_{B_{v''}} = \theta(v'') = f(L(v''))$, and so $ab = f(L(v'))f(L(v'')) \in \mathcal{L}_{(T,f)}(v)$ follows, as claimed. This concludes the proof of the claim and thus the proof of the theorem. $\square$

We now take a brief break from our study of distinguished minimal topological lassos to point out a sufficient condition for a set of cords to be a strong lasso for some X-tree which is implied by Theorem 7.2. To make this more precise, we need to introduce some more terminology from [19]. Suppose T is an X-tree and $\mathcal{L}$ is a set of cords of X. Then $\mathcal{L}$ is called an *equidistant lasso* for T if, for all equidistant, proper edge weightings ω and ω' of T, we have that $\omega = \omega'$ holds whenever (T, ω) and (T, ω') are $\mathcal{L}$-isometric. Moreover, $\mathcal{L}$ is called a *strong lasso* for T if $\mathcal{L}$ is simultaneously an equidistant and a topological lasso for T (see [11] for more on such lassos in the unrooted case).

Like a topological lasso for an X-tree T, an equidistant lasso $\mathcal{L}$ for T can also be characterized in terms of a property of the child-edge graph $G(\mathcal{L}, v)$ associated to T and $\mathcal{L}$ where $v \in \mathring{V}(T)$. Namely, a set $\mathcal{L}$ of cords of X is an equidistant lasso for an X-tree T if and only if, for every vertex $v \in \mathring{V}(T)$, the graph $G(\mathcal{L}, v)$ has at least one edge (see [19, Theorem 6.1]). Since for σ some total ordering of X and $f : \mathring{V}(T) - \{\rho_T\} \to X$ a cluster marker map for T and σ the graphs $G(\mathcal{L}_{(T,f)}, v)$ clearly satisfy this property for all $v \in \mathring{V}(T)$, it follows that $\mathcal{L}_{(T,f)}$ is also an equidistant lasso for T and thus a strong lasso for T. Defining a strong lasso $\mathcal{L}$ of an X-tree to be *minimal* in analogy to when a topological lasso is minimal, Theorem 7.2 implies the following corollary.

Corollary 7.3. *Suppose T is a nondegenerate X-tree, $\mathcal{L}$ is a set of cords of X, σ is a total ordering of X, and $f : cl(T) \to X$ is a cluster marker map for T and σ. Then $\mathcal{L}_{(T,f)}$ is a minimal strong lasso for T.*

8. Heredity of distinguished minimal topological lassos. In this section, we turn our attention to the problems of characterizing when a distinguished minimal topological lasso of an X-tree T induces a distinguished minimal topological lasso for a subtree of T and, conversely when distinguished minimal topological lassos of X-trees can be combined to form a distinguished minimal topological lasso of a supertree for those trees (see, e.g., [3] for more on such trees). This will also allow us to partially answer the rooted analogue of a question raised in [11] for supertrees within the unrooted framework. To make this more precise, we require further terminology. Suppose $\mathcal{L}$ is a set of cords of X and $Y \subseteq X$ is a nonempty subset. Then we set

$$\mathcal{L}|_Y = \{ab \in \mathcal{L} : a, b \in Y\}.$$

Clearly, $\Gamma(\mathcal{L}|_Y)$ is the subgraph of $\Gamma(\mathcal{L})$ induced by Y but $Y = \bigcup \mathcal{L}|_Y$ need not hold. Moreover, if $\mathcal{L}$ is a minimal topological lasso for an X-tree T and $|Y| \geq 3$ such that every interior vertex of T is also an interior vertex of $T|_Y$, then Theorem 3.1 implies that $\mathcal{L}|_Y$ is a minimal topological lasso for $T|_Y$. In particular, $\Gamma(\mathcal{L}|_Y)$ must be connected in this case. The next result is a strengthening of this observation.

THEOREM 8.1. *Suppose T is an X-tree, $\mathcal{L}$ is a distinguished minimal topological lasso for T, and $Y \subseteq X$ is a subset of size at least three. Then $\mathcal{L}|_Y$ is a distinguished minimal topological lasso for $T|_Y$ if and only if $\Gamma(\mathcal{L}|_Y)$ is connected.*

Proof. Assume first that $\mathcal{L}|_Y$ is a distinguished minimal topological lasso for $T|_Y$. Then, by Proposition 3.3, $\Gamma(\mathcal{L}|_Y)$ is connected.

Conversely, assume that $\Gamma(\mathcal{L}|_Y)$ is connected. Then the statement clearly holds if T is the star-tree on X. So assume that T is nondegenerate. Let $Y \subseteq X$ be of size at least three, and assume first that $T|_Y$ is the star-tree on Y. We claim that $\Gamma(\mathcal{L}|_Y)$ is a clique. Assume to the contrary that this is not the case, that is, that there exist elements $y, y' \in Y$ distinct such that $yy' \notin \mathcal{L}$. Since $\Gamma(\mathcal{L}|_Y)$ is connected, there must exist a path $P : x_1 = y, x_2, \dots, x_l = y'$, $l \geq 2$, in $\Gamma(\mathcal{L}|_Y)$ from y to y'. Since the vertex set of $\Gamma(\mathcal{L}|_Y)$ is Y, it follows that $X' = \{x_1, x_2, \dots, x_l\} \subseteq Y$. Combined with the fact that $lca_T(x, x') = lca_T(Y)$ holds for all $x, x' \in X'$ distinct as $T|_Y$ is a star-tree on Y, we obtain $X' \subseteq V(B_{lca_T(Y)})$. Thus, $yy' \in \mathcal{L}$, which is impossible and thus proves the claim. That $\mathcal{L}|_Y$ is a distinguished minimal topological lasso for $T|_Y$ is a trivial consequence.

So assume that $T|_Y$ is nondegenerate. Since $\mathcal{L}$ is a distinguished minimal topological lasso for T, Theorem 7.2 implies that there exists a total ordering ω of X and a cluster marker map $f_\omega : cl(T) \to X$ for T and ω such that $\mathcal{L} = \mathcal{L}_{(T, f_\omega)}$. Moreover, Lemma 6.1(iv) implies that the cut vertices of $\Gamma(\mathcal{L})$ are of the form $f_\omega(L_T(v))$, where $v \in \mathring{V}(T)$.

To see that $\mathcal{L}|_Y$ is a distinguished minimal topological lasso for $T|_Y$ and some total ordering of Y, note first that the restriction σ of ω to Y induces a total ordering of Y. Furthermore, the aforementioned form of the cut vertices of $\Gamma(\mathcal{L})$ combined with the assumption that $\Gamma(\mathcal{L}|_Y)$ is connected implies that, for all $A \in cl(T)$ with $A \cap Y \neq \emptyset$, we must have $f_\omega(A) \in Y$. For all $A \in cl(T|_Y)$ denote by A^T the set-inclusion minimal superset of A contained in $cl(T)$. Then, since f_ω is a cluster marker map for T and ω, it follows that the map

$$f_\sigma : cl(T|_Y) \to Y \, : \, A \mapsto f_\omega(A^T)$$

is a cluster marker map for $T|_Y$ and σ. By Theorem 7.2 it now suffices to establish that $\mathcal{L}|_Y = \mathcal{L}_{(T|_Y, f_\sigma)}$. Since both $\mathcal{L}|_Y$ and $\mathcal{L}_{(T|_Y, f_\sigma)}$ are minimal topological lassos for $T|_Y$ and so $|\mathcal{L}|_Y| = |\mathcal{L}_{(T|_Y, f_\sigma)}|$ is implied by Lemma 3.2, it suffices to show that $\mathcal{L}|_Y \subseteq \mathcal{L}_{(T|_Y, f_\sigma)}$.

Suppose $ab \in \mathcal{L}|_Y$; that is, $ab \in \mathcal{L}$ and $a, b \in Y$. Since Y is the leaf set of $T|_Y$, there must exist a vertex $v \in \mathring{V}(T|_Y)$ such that $v = lca_{T|_Y}(a, b)$. Clearly, $v \in \mathring{V}(T)$. If a and b are both adjacent with v in T, then a and b are also adjacent with v in $T|_Y$. Thus $ab \in \mathcal{L}_{(T|_Y, f_\sigma)}(v)$ in this case. So assume that at least one of a and b is not adjacent with v in T. Without loss of generality, let a denote that vertex. Then since $ab \in \mathcal{L} = \mathcal{L}_{(T, f_\omega)}$, it follows that there must exist a unique child $v' \in \mathring{V}(T)$ of v such that $a \in L_T(v')$ and $a = f_\omega(L_T(v'))$. Hence, $a \in V(B_v)$ and a cut vertex of $\Gamma(\mathcal{L})$.

We claim that $v' \in \mathring{V}(T|_Y)$. Assume for contradiction that $v' \notin \mathring{V}(T|_Y)$. Then since f_ω is a cluster marker map for T and ω, it follows that a cannot be a cut vertex in $\Gamma(\mathcal{L}|_Y)$. Since $\Gamma(\mathcal{L})$ is a claw-free block graph, no edge in the unique block $B' \in Block(\Gamma(\mathcal{L})) - \{B_v\}$ that also contains a in its vertex set can therefore be incident with a in $\Gamma(\mathcal{L}|_Y)$. Since $\Gamma(\mathcal{L}|_Y)$ is assumed to be connected, it now suffices to show that there exists some $c \in Y \cap L_T(v')$ distinct from a such that every path from c to b

in $\Gamma(\mathcal{L})$ crosses a. But this is a consequence of the facts that v is not the parent of a in $T|_Y$ and, as implied by Proposition 3.3, that the subgraph $\Gamma_{v'}(\mathcal{L})$ of $\Gamma(\mathcal{L})$ induced by $L_T(v')$ is the connected component of $\Gamma(\mathcal{L})$ containing a obtained from $\Gamma(\mathcal{L})$ by deleting all edges in B_v that are incident with a. This concludes the proof of the claim.

To conclude the proof of the theorem, note that if b is adjacent with v in $T|_Y$, then $ab = f_\omega(L_T(v'))b = f_\omega((L_{T|_Y}(v'))^T)b = f_\sigma(L_{T|_Y}(v'))b \in \mathcal{L}_{(T|_Y,f_\sigma)}(v) \subseteq \mathcal{L}_{(T|_Y,f_\sigma)}$. If b is not adjacent with v in $T|_Y$, then there exists a child $v'' \in \mathring{V}(T)$ of v such that $b = f_\omega(L_T(v''))$. In view of the previous claim, we have $v'' \in \mathring{V}(T|_Y)$. But now arguments similar to those used before imply that $ab \in \mathcal{L}_{(T|_Y,f_\sigma)}(v) \subseteq \mathcal{L}_{(T|_Y,f_\sigma)}$. ◻

We now turn our attention to supertrees, which are formally defined as follows. Suppose $\mathcal{T} = \{T_1, \dots, T_l\}$, $l \geq 1$, is a set of Y_i-trees T_i with $Y_i \subseteq X$ and $|Y_i| \geq 3$, $i \in \langle l \rangle$, and T is an X-tree. Then T is a called a *supertree* of $\mathcal{T}$ if T displays every tree in $\mathcal{T}$ where we say that some X-tree T *displays* some Y-tree T' for $Y \subseteq X$ with $|Y| \geq 3$ if $T|_Y$ and T' are equivalent. More precisely, we have the following result, which relies on the fact that in case $\mathcal{L}$ is a distinguished minimal topological lasso for a *binary* X-tree T, that is, every vertex of T but the leaves has two children, $\Gamma(\mathcal{L})$ must be a path. In particular, $\mathcal{L}$ induces a total ordering of the elements in X in this case. For $Y \subseteq X$ a nonempty subset of X, we denote the maximal and minimal elements in Y with regard to that ordering by $\min_{\mathcal{L}}(Y)$ and $\max_{\mathcal{L}}(Y)$, respectively.

COROLLARY 8.2. *Suppose X' and X'' are two nonempty subsets of X such that $X = X' \cup X''$ and $X' \cap X'' \neq \emptyset$ and T' and T'' are X'-trees and X''-trees, respectively. Suppose also that $\mathcal{L}'$ and $\mathcal{L}''$ are distinguished minimal topological lassos for T' and T'', respectively, such that $\mathcal{L}'|_{X'\cap X''} = \mathcal{L}''|_{X'\cap X''}$ and $\Gamma(\mathcal{L}''|_{X'\cap X''})$ is connected. If T is a binary X-tree that displays both T' and T'', then $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}''$ is a distinguished minimal topological lasso for T if and only if $\min_{\mathcal{L}'}(X' \cap X'') \in \{\min_{\mathcal{L}'}(X'), \min_{\mathcal{L}''}(X'')\}$ and $\max_{\mathcal{L}'}(X' \cap X'') \in \{\max_{\mathcal{L}'}(X'), \max_{\mathcal{L}''}(X'')\}$.*

Continuing with the assumptions of Corollary 8.2, we also have that if $\min_{\mathcal{L}'}(X' \cap X'') \in \{\min_{\mathcal{L}'}(X'), \min_{\mathcal{L}''}(X'')\}$ and $\max_{\mathcal{L}'}(X' \cap X'') \in \{\max_{\mathcal{L}'}(X'), \max_{\mathcal{L}''}(X'')\}$ hold, then $\mathcal{L}' \cup \mathcal{L}''$ is a (minimal) strong lasso for T as every minimal topological lasso for an X-tree is also an equidistant lasso for that tree. However, not all strong lassos for T are of this form. An example for this is furnished for $X' = \{a, c, d\}$ and $X'' = \{a, b, c\}$ by the X'-tree T', the X''-tree T'', and the $X' \cup X''$-tree T depicted in Figure 5 along with the set $\mathcal{L}' = \{cd\}$ and $\mathcal{L}'' = \{ab, bc\}$ of cords of X' and X'', respectively. Clearly, T is a supertree of $\{T', T''\}$, and $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}''$ is a strong lasso for T, but $\mathcal{L}'$ is not even an equidistant lasso for T'. Further investigating the interplay between minimal topological lassos for X-trees and minimal topological lassos for supertrees that display them might therefore be of interest.

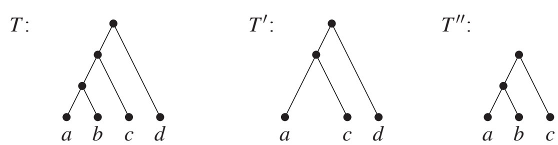

FIG. 5. *For $X' = \{a, c, d\}$ and $X'' = \{a, b, c\}$ the $X' \cup X''$-tree T is a supertree for the depicted X' and X'' trees T' and T'', respectively. Clearly, $\mathcal{L}' = \{cd\}$ and $\mathcal{L}'' = \{ab, bc\}$ are sets of cords of X' and X'', respectively, and $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}''$ is a strong lasso for T, but $\mathcal{L}'$ is not even an equidistant lasso for T'.*

We conclude with returning to Figure 2, which depicts two nonequivalent X-trees that are topologically lassoed by the same set $\mathcal{L}$ of cords of X. In fact, $\mathcal{L}$ is even a minimal topological lasso for both of them. A better understanding of the relationship between X-trees that are topologically lassoed by the same set of cords of X might be an interesting topic of future study.

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