Sharp condition number estimates for the symmetric 2-Lagrange multiplier method

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Abstract Domain decomposition methods are used to find the numerical solution of large boundary value problems in parallel. In optimized domain decomposition methods, one solves a Robin subproblem on each subdomain, where the Robin parameter a must be tuned (or optimized) for good performance. We show that the 2-Lagrange multiplier method can be analyzed using matrix analytical techniques and we produce sharp condition number estimates.

1 Introduction.

Consider the model problem

$$\Delta u = f \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is the domain, f is a given forcing and $u \in H_0^1(\Omega)$ is the unknown solution. In the present paper, we describe a symmetric 2-Lagrange multiplier (S2LM) domain decomposition method to solve elliptic problems such as (1). When we discretize (1) using e.g. piecewise linear finite elements, we obtain a linear system of the form

$$A\mathbf{u} = \mathbf{f}, \tag{2}$$

where $\mathbf{u} \in \mathbb{R}^n$ is the finite element coefficient vector of the approximation to the solution u of (1).

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We now consider the domain decomposition [Toselli and Widlund, 2005] $\Omega = \Gamma \cup \Omega_1 \cup \ldots \cup \Omega_p$, where $\Omega_1, \ldots, \Omega_p$ are the (open, disjoint) "subdomains" and $\Gamma = \Omega \cap \bigcup_{k=1}^{p} \partial\Omega_k$ is the "artificial interface". We introduce the "local problems"

$$\begin{cases} \Delta u_k = f & \text{in } \Omega_k, \quad \text{(PDE)} \\ u_k = 0 & \text{on } \partial\Omega_k \cap \partial\Omega, \quad \text{(natural b.c.)} \\ (a + D_\nu) u_k = \lambda_k & \text{on } \partial\Omega_k \cap \Gamma, \quad \text{(artificial b.c.)} \end{cases} \tag{3}$$

where $a > 0$ is the Robin tuning parameter and $k = 1, \ldots, p$ and D_ν denotes the directional derivative in the outwards pointing normal ν of $\partial\Omega_k$. The interface Γ is artificial in that it is not a natural part of the "physical problem" (1) but instead is introduced purely for the purpose of calculation.

We again discretize the systems (3) using a finite element method. The Robin b.c. in (3) gives rise to a mass matrix on the interface $\Gamma \cap \partial\Omega_k$, which is spectrally equivalent to aI. Hence, after a suitable "mild" change of basis, we obtain the discrete system

$$\begin{bmatrix} A_{IIk} & A_{I\Gamma k} \\ A_{\Gamma I k} & A_{\Gamma\Gamma k} + aI \end{bmatrix} \overbrace{\begin{bmatrix} \mathbf{u}_{Ik} \\ \mathbf{u}_{\Gamma k} \end{bmatrix}}^{\mathbf{u}_k} = \overbrace{\begin{bmatrix} \mathbf{f}_{Ik} \\ \mathbf{f}_{\Gamma k} \end{bmatrix}}^{\mathbf{f}_k} + \begin{bmatrix} 0 \\ \boldsymbol{\lambda}_k \end{bmatrix}. \tag{4}$$

The FETI-2LM algorithm was introduced in [Farhat et al., 2000] for cases without cross-points, while the general case including cross points was introduced and analyzed in [Loisel, 2011a]. The method consists of finding the value of $\boldsymbol{\lambda} = [\boldsymbol{\lambda}_1^T, \ldots, \boldsymbol{\lambda}_p^T]^T$ which yields solutions $\mathbf{u}_1, \ldots, \mathbf{u}_p$ to (4) in such a way that $\mathbf{u}_1, \ldots, \mathbf{u}_p$ meet continuously across Γ and glue together into the unique solution $\mathbf{u}$ of (2).

The main result of the present paper is a new estimate the condition number of FETI-2LM algorithms using matrix analytical techniques. This new idea produces sharp condition number estimates with much more straightforward proof techniques than the techniques used in [Loisel, 2011a] (where the estimates are not sharp). As a result, the present paper is a logical follow-up to [Loisel, 2011a].

The present paper focuses on 1-level algorithms which are known not to scale. Scalable algorithms are considered in [Loisel, 2011b] and [Drury and Loisel, 2011].

Our paper is organized as follows. In Section 2, we give the symmetric 2-Lagrange multiplier method for general domains with cross points. In Section 3, we give spectral estimates including our main result on the condition number of the symmetric 2-Lagrange multiplier system. in Section 4, we verify this Theorem with some numerical experiments.

2 The symmetric 2-Lagrange multiplier method.

We now describe the 2-Lagrange multiplier method that we analyze in the present paper. Consider the local problems (4) and eliminate the interior degrees of freedom to obtain the relation

$$a \overbrace{\begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_p \end{bmatrix}}^{\mathbf{u}_G} = \overbrace{\begin{bmatrix} a(S_1 + aI)^{-1} & & \\ & \ddots & \\ & & a(S_1 + aI)^{-1} \end{bmatrix}}^{Q} \left(\overbrace{\begin{bmatrix} \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_p \end{bmatrix}}^{\mathbf{g}} + \overbrace{\begin{bmatrix} \boldsymbol{\lambda}_1 \\ \vdots \\ \boldsymbol{\lambda}_p \end{bmatrix}}^{\boldsymbol{\lambda}} \right), \tag{5}$$

where

$$S_k = A_{\Gamma\Gamma k} - A_{\Gamma I k} A_{IIk}^{-1} A_{I\Gamma k} \quad \text{and} \quad \mathbf{g}_k = \mathbf{f}_{\Gamma k} - A_{\Gamma I k} A_{IIk}^{-1} \mathbf{f}_{Ik}$$

are the “Dirichlet-to-Neumann maps” and “accumulated right-hand-sides”.

The matrices S_k are symmetric and semidefinite. Since $Q = a(S + aI)^{-1}$, we find that the spectrum $\sigma(Q)$ is contained in the set $[\epsilon, 1 - \epsilon] \cup \{1\}$ for some $\epsilon > 0$. The eigenvalue 1 of Q comes from the kernel of S and hence the kernel of $Q - I$ is spanned by the indicating functions of the subdomains that “float”. We define E to be the orthogonal projection onto the kernel of $Q - I$.

2.1 Relations between (4) and (2) and continuity.

We define the boolean restriction matrix R_k by selecting rows of the $n \times n$ identity matrix corresponding to those vertices of Ω that are in $\bar{\Omega}_k \cap \Omega$. As a result, from a finite element coefficient vector $\mathbf{v}$ corresponding to a finite element function $v \in H_0^1(\Omega)$, we can define a finite element coefficient vector $\mathbf{v}_k = R_k \mathbf{v}$, which corresponds to a finite element function $v \in H^1(\Omega_k) \cap H_0^1(\Omega)$, which is obtained by restricting v to Ω_k.

The identity $\int_\Omega = \sum_{k=1}^p \int_{\Omega_k}$ induces the following relations between (4) and (2):

$$A = \sum_{k=1}^{p} R_k^T \begin{bmatrix} A_{IIk} & A_{I\Gamma k} \\ A_{\Gamma I k} & A_{\Gamma\Gamma k} \end{bmatrix} R_k \quad \text{and} \quad \mathbf{f} = \sum_{k=1}^{p} \mathbf{f}_k. \tag{6}$$

Each interface vertex $\mathbf{x}_i \in \Gamma$ is adjacent to $m_i \geq 2$ subdomains. As a result, the “many-sided trace” $\mathbf{u}_G$ defined by (5) contains m_i entries corresponding to $\mathbf{x}_i$, one per subdomain adjacent to $\mathbf{x}_i$. We define the orthogonal projection matrix K which averages function values for each interface vertex $\mathbf{x}_i$. A many-sided trace $\mathbf{u}_G$ corresponds to local functions $\mathbf{u}_1, \ldots, \mathbf{u}_p$ that meet continuously across Γ if and only if

$$K\mathbf{u}_G = \mathbf{u}_G. \tag{7}$$

2.2 A problem in $\boldsymbol{\lambda}$.

The **symmetric 2-Lagrange multiplier** (S2LM) system is given by

$$(Q-K)\boldsymbol{\lambda} = -Q\mathbf{g}. \tag{8}$$

We further let E be the orthogonal projection onto the kernel of $Q-I$.

Lemma 1. *Assume that $\|EK\| < 1$. The problem (2) is equivalent to (8).*

Proof. In order to solve (2) using local problems (4), one should find Robin boundary values $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_p$ which result in local solutions $\mathbf{u}_1, \ldots, \mathbf{u}_p$ that meet continuously across Γ. As a result, we impose the condition (7), which we multiply by $a > 0$ and convert to an expression in $\boldsymbol{\lambda}$ using (5) to obtain $Ka(S+aI)^{-1}(\boldsymbol{\lambda}+\mathbf{g}) = a(S+aI)^{-1}(\boldsymbol{\lambda}+\mathbf{g})$ or

$$(I-K)Q\boldsymbol{\lambda} = (K-I)Q\mathbf{g} \tag{9}$$

With this continuity condition, there is clearly a unique $\mathbf{u}$ which restricts to the $\mathbf{u}_j$:

$$\mathbf{u}_j = R_j\mathbf{u}, \quad j = 1, \ldots, p. \tag{10}$$

Imposing continuity is not sufficient, we must also ensure that the "fluxes" match. Indeed, if we impose on the solution $\mathbf{u}$ of (10) that the equation (2) should hold, one obtains

$$\mathbf{f} = A\mathbf{u} \overset{(6)}{=} \sum_{j=1}^{p} R_{\Gamma j}^T A_{Nj} R_{\Gamma j}\mathbf{u} \overset{(10)}{=} \sum_{j=1}^{p} R_{\Gamma j}^T A_{Nj}\mathbf{u}_j \tag{11}$$

$$\overset{(4),(6)}{=} \mathbf{f} - \sum_{j=1}^{p} R_j^T \begin{pmatrix} 0 \\ \boldsymbol{\lambda}_j - a\mathbf{u}_{\Gamma j} \end{pmatrix} \tag{12}$$

Canceling the $\mathbf{f}$ terms on each side and multiplying by K, we obtain $K\boldsymbol{\lambda} - Ka\mathbf{u}_G = 0$. Using (5), we obtain

$$K(Q-I)\boldsymbol{\lambda} = -KQ\mathbf{g}. \tag{13}$$

We add (9) and (13) to obtain (8).

To see that the solution of (8) is unique, observe that the ranges of E and K intersect trivially by the hypothesis that $\|EK\| < 1$. As a result, the eigenspace of Q of eigenvalue 1 intersects trivially with the range of K and $Q-K$ is nonsingular. □

We will further discuss the choice of the parameter a in Section 3.1.

3 Spectral estimates.

If we use GMRES or MINRES on the symmetric indefinite system (8), the residual norm can be estimated as a function of the condition number of $Q-K$, cf. [Driscoll et al., 1998]. In order to estimate the condition number of $Q-K$, we begin by giving a canonical form for the pair of projections E and K.

Lemma 2. *Let E and K be orthogonal projections. There is a choice of orthonormal basis that block diagonalizes E and K simultaneously and such that the blocks E_k and K_k of E and K satisfy*

$$E_k \in \left\{0, 1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\} \quad \textit{and} \quad K_k \in \left\{0, 1, \begin{bmatrix} c_k^2 & c_k s_k \\ c_k s_k & s_k^2 \end{bmatrix}\right\}, \tag{14}$$

where $c_k = \cos\theta_k > 0$, $s_k = \sin\theta_k > 0$ and $\theta_k \in (0, \pi/2)$ is a "principal angle" relating E and K.

The canonical form (14) can be obtained from the CS decomposition [Davis and Kahan, 1969] by starting from $E = \operatorname{diag}(I, 0)$ and picking orthonormal bases for the range and kernel of K. Due to space constraints, we omit this argument.

We also give a technical lemma which describes the spectrum of a sum of certain symmetric matrices.

Lemma 3. *Let X, Y be symmetric matrices of dimensions $m \times m$. Let $0 < y_{\min} < y_{\max}$ and assume that $|\sigma(Y)| \subset [y_{\min}, y_{\max}]$. Denote by $\rho(X)$ the spectral radius of X and assume that $\rho(X) < y_{\min}$. Then,*

$$|\sigma(X+Y)| \subset [y_{\min} - \rho(X), y_{\max} + \rho(X)]. \tag{15}$$

Proof. This follows from a Theorem of Weyl [Horn and Johnson, 1990, Theorem 4.3.1, pp 181–182]. □

3.1 Condition number of $Q-K$.

We now come to our main result.

Theorem 1. *Let $\epsilon > 0$. Assume that $\sigma(Q) \subset [\epsilon, 1-\epsilon] \cup \{1\}$. Let E, K be orthogonal projections and assume that $\|EK\| < 1$. Then we have the sharp estimates*

$$|\sigma(Q-K)| \subset \left[\frac{\epsilon + \sqrt{(1+\epsilon)^2 \; - 4\|EK\|^2\epsilon} - 1}{2}, 1\right], \quad \textit{and} \tag{16}$$

$$\kappa(Q-K) \leq \frac{2}{\epsilon + \sqrt{(1+\epsilon)^2 \; - 4\|EK\|^2\epsilon} - 1} = O((1-\|EK\|)^{-1}\epsilon^{-1}). \tag{17}$$

Proof. Let $X = Q - \frac{1}{2}I - \epsilon E$ and $Y = \frac{1}{2}I + \epsilon E - K$. Then, $Q - K = X + Y$ and we are in a position to use Lemma 3. We now estimate the spectral properties of X and Y.

Spectral properties of X: Recall that E projects onto the eigenspace of Q with eigenvalue 1. As a result, after some orthonormal change of basis, we find that $Q = \operatorname{diag}(Q_0, I)$ and $E = \operatorname{diag}(0, I)$ and hence

$$\rho(X) \leq \frac{1}{2} - \epsilon. \tag{18}$$

Spectral properties of Y: Lemma 2 shows that E and K block diagonalize simultaneously and Y is also block diagonal in the same basis. Using (14), we find that the kth block Y_k of Y is given by

$$Y_k = \begin{cases} \frac{1}{2} & \text{if } E_k = K_k = 0, \\ -\frac{1}{2} & \text{if } E_k = 0,\ K_k = 1, \\ \frac{1}{2} + \epsilon & \text{if } E_k = 1,\ K_k = 0, \\ \begin{bmatrix} \frac{1}{2} + \epsilon - c_k^2 & -c_k s_k \\ -c_k s_k & \frac{1}{2} - s_k^2 \end{bmatrix}; \end{cases} \tag{19}$$

where the case $E_k = K_k = 1$ is excluded by the hypothesis that $\|EK\| < 1$. As a result, the eigenvalues of Y_k are in the set $\{\pm\frac{1}{2}, \frac{1}{2} + \epsilon, \lambda_\pm(c_k^2)\}$, where

$$\lambda_\pm(c_k^2) = \frac{\epsilon \pm \sqrt{(1+\epsilon)^2 - 4c_k^2\epsilon}}{2}. \tag{20}$$

Note that $\|EK\| = \sqrt{\rho(EKE)} = c_k$ and that the functions $\lambda_\pm(c_k^2)$ are monotonic in c_k^2. Hence, we find the following bounds for the modulus of an eigenvalue of Y:

$$|\sigma(Y)| \subset \Bigg[\overbrace{\frac{\sqrt{(1+\epsilon)^2 - 4\|EK\|^2\epsilon} - \epsilon}{2}}^{y_{\min}}, \overbrace{\frac{1}{2} + \epsilon}^{y_{\max}}\Bigg]. \tag{21}$$

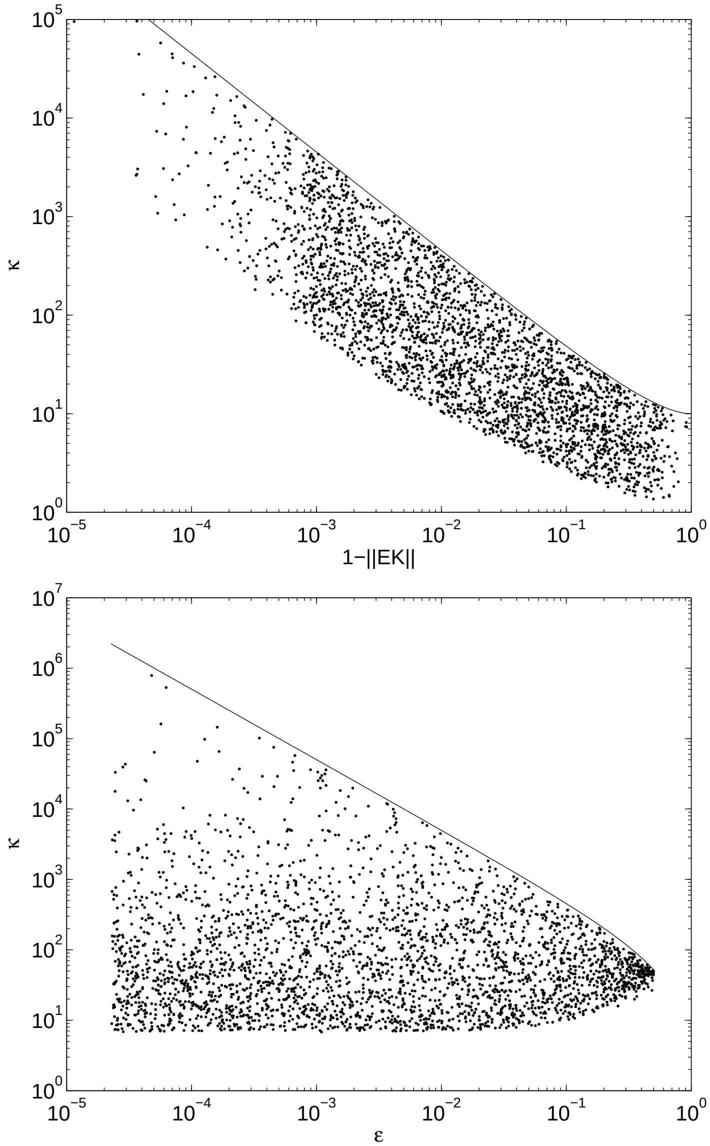

Fig. 1 Comparing random $Q-K$ (points) versus the estimate (17) (solid). Top: $\epsilon = 0.1$, varying $\|EK\|$, 3000 repetitions. Bottom: $\|EK\| = 0.99$, varying ϵ, 3000 repetitions.

Combining (15), (18) and (21) gives (16).

The sharpness of the estimate is shown by considering the example $Q = \operatorname{diag}(1, 1-\epsilon)$ and $K = \begin{bmatrix} c^2 & c\sqrt{1-c^2} \\ c\sqrt{1-c^2} & 1-c^2 \end{bmatrix}$ for $c = 0$ and $c = \|EK\|$. $\square$

In view of Theorem 1, the Robin parameter a should be chosen so as to make ϵ as large as possible. This occurs precisely when a is the geometric mean of the extremal positive eigenvalues of S. More details can be found in [Loisel, 2011a].

4 Numerical verification.

We verify numerically the validity of Theorem 1 by generating random 5×5 matrices Q and E as follows. We set $Q = \text{diag}(\epsilon, q, 1-\epsilon, 1, 1)$ where q is chosen randomly between ϵ and $1-\epsilon$. We generate randomly a 2-dimensional space and set K to be the orthogonal projection onto that space. We compare the resulting condition number $\kappa = \kappa(Q-K)$ against (17), cf. Fig. 1.

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