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Tran, Truong X; Biancalana, Fabio

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Truong X. Tran[1,2] and Fabio Biancalana[1,3]

[1]*Max Planck Institute for the Science of Light, Günther-Scharowsky Strasse 1, 91058 Erlangen, Germany*
[2]*Department of Physics, Le Quy Don University, 236 Hoang Quoc Viet Street, 10000 Hanoi, Vietnam*
[3]*School of Engineering and Physical Sciences, Heriot-Watt University, EH14 4AS Edinburgh, United Kingdom*
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We study analytically and numerically the diffractive resonant radiation emitted by spatial solitons, which is generated in waveguide arrays with Kerr nonlinearity. The phase matching condition between solitons and radiation is derived and studied for the first time and agrees well with direct pulse propagation simulations. The folded dispersion due to the Brillouin zone leads to a peculiar anomalous soliton recoil that we describe in detail.

Introduction.—Waveguide arrays (WAs) consisting of identical, equally spaced waveguides, present a unique discrete platform to explore many interesting fundamental phenomena such as discrete diffraction [1], discrete solitons [2,3], and photonic Bloch oscillations [4–7]. In applications, 2D networks of nonlinear waveguides with discrete solitons may be useful for designing signal-processing circuits [8]. Recently, WAs have also been used to mimic relativistic phenomena typical of quantum field theory, such as *Zitterbewegung* [9], Klein paradox [10], fermion pair production [11], and the Dirac equation [12].

The concept of dispersive resonant radiation (DisRR), which emerges due to higher-order dispersion (HOD) terms, has been well studied in the last decade in the temporal case for optical fibers [13–17]. When an ultra-short pulse is launched into optical fibers, a DisRR due to the phase matching between the fiber and the soliton group velocity dispersion generates one or more new frequencies [16,17]. This DisRR, together with other nonlinear effects such as self- and cross-phase modulation, soliton fission [18], and stimulated Raman scattering [19], are the main ingredients of the so-called supercontinuum generation [20,21], particularly in highly nonlinear photonic crystal fibers [22]. Supercontinuum generation is among the most important phenomena in nonlinear fiber optics which has led to a number of important technological advances in various fields, such as spectroscopy and medical imaging [23], metrology [24], and the realization of broadband sources [25].

Inspired by advances in DisRR studies in the last decade, in this Letter we describe the dynamics of a kind of resonant radiation—namely, the *diffractive resonant radiation* (DifRR), which occurs when a continuous-wave (CW) beam or a relatively spatially long pulse is launched into WAs. Although clues of the existence of such radiation is somehow implicit in several previous numerical works on WAs based on the discrete nonlinear Schrödinger equation, there is still no systematic description of the details of the emission of this radiation in the literature. Similarities and differences between DisRRs and DifRRs are analyzed. We show that when the phase matching condition is satisfied, a spatial soliton emits DifRR with a new well-defined direction, i.e., a transverse wave number. Moreover, due to the periodicity of discrete systems, and thus the existence of a Brillouin zone, unusual effects which cannot exist in continuous media can now occur. One of these is the anomalous solitonic recoil described in this Letter for the first time.

Phase matching condition for the diffractive resonant radiation.—Light propagation in a discrete, periodic array of Kerr nonlinear waveguides can be described, in the cw regime, by the following well-known dimensionless set of ordinary differential equations [3,7,21]:

$$i\frac{da_n(z)}{dz} + c[a_{n+1}(z) + a_{n-1}(z)] + |a_n(z)|^2 a_n(z) = 0, \quad (1)$$

where a_n is the electric field amplitude in the nth waveguide, $n = \{1, \ldots, N\}$, N is the total number of waveguides, z is the longitudinal spatial coordinate, $c \equiv C/(\gamma P)$, where C is the coupling coefficient (in units of a wavenumber) resulting from the field overlap between neighboring waveguides, γ is the nonlinear coefficient of a single waveguide, and P is the input beam peak power. Each waveguide which composes the array is nondiffracting and supports a single fundamental mode. Therefore, it is not necessary to include intrinsic spatial diffraction via a transverse Laplacian in Eq. (1)—discrete diffraction occurs only via the coupling between neighboring waveguides. The main assumptions on Eq. (1) are (i) that the electric field is monochromatic (of frequency and wave number ω_0 and k_0, respectively) and slowly varying in space with respect to a spatial scale k_0^{-1}, so that a very precise reduction of the full Maxwell equations is possible, and (ii) that the most important interaction between different waveguides is of the nearest-neighbor type. Both these assumptions are extremely accurate and it has been proved in many works that Eq. (1) lead to very reliable predictions, and are suitable for important analytical manipulations [3,7].

By using the stationary discrete plane wave solution for the nth waveguide $a_n(z) = a_0 \exp[i(nk_x d + \kappa_z z)]$ one arrives, in the linear case, at the well-known dispersion relation between κ_z and k_x [1]:

$$\kappa_z(k_x) = 2c \cos(k_x d), \qquad (2)$$

where d is the center-to-center spacing between two adjacent waveguides, and k_x is the transverse wave number; see solid blue line of Fig. 1(a). It is clear from Eq. (2) that κ_z is periodic in $\kappa \equiv k_x d$, which represents the phase difference between adjacent waveguides. Thus, within the coupled mode approximation, it suffices to investigate the first Brillouin zone of the folded dispersion, $-\pi \leq \kappa \leq \pi$.

Since a typical input beam has a finite width covering several waveguides, its Fourier spectrum has a certain bandwidth with a central transverse wave number κ_0, which is fixed by the input angle of incidence of the exciting beam. We can then use a Taylor expansion of Eq. (2) as follows:

$$\kappa_z(\kappa) = \kappa_z(\kappa_0) + \sum_{m \geq 1} \frac{D_m}{m!} \Delta \kappa^m, \qquad (3)$$

where $\Delta \kappa \equiv \kappa - \kappa_0$, and $D_m \equiv (d^m \kappa_z / d\kappa^m)|_{\kappa_0}$ is the mth order diffractive Taylor coefficient [thus, $D_1 = -2c \sin(\kappa_0)$, $D_2 = -2c \cos(\kappa_0)$; etc., all the derivatives can obviously be calculated explicitly]. In Fig. 1(a) we plot a typical curve for $D_2(\kappa)$ (dashed red line), showing the existence of two zero-diffraction points located at $\kappa = \pm \pi/2$. In a sense, this shape of D_2 is analogous to the
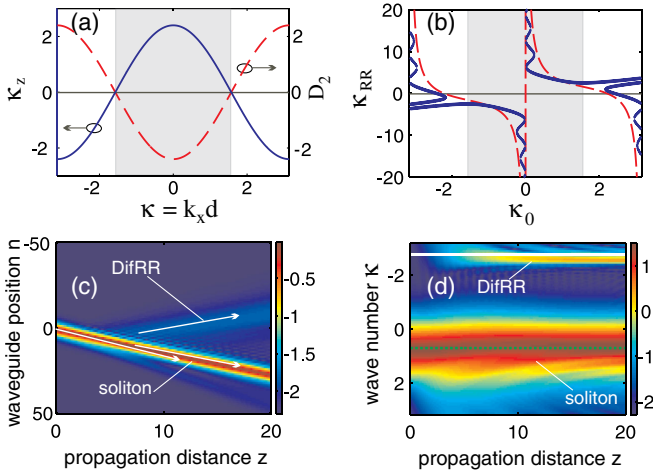

FIG. 1 (color online). (a) Solid blue line: WA dispersion κ_z vs κ. Dashed red line: D_2 vs κ, showing the two zero-diffraction points located at $\kappa = \pm \pi/2$. (b) Wave number κ_{RR} of the generated DifRR, as a function of the input soliton wave number κ_0. The red dashed line indicates the approximated formula $\kappa_{RR} \simeq \kappa_0 + 3/\tan(\kappa_0)$ for the position of the resonant radiation, while the blue solid line is the result of the exact implicit formula given by Eqs. (6) and (7). In both (a),(b) the gray shaded area indicates the region where bright solitons can propagate. (c),(d) Beam propagation in the (n, z) plane (c) and (κ, z) plane. (d) Parameters are $A_0 = 0.8$, $c = 1.2$, $\kappa_0 = 0.7$, $N = 100$.

group velocity dispersion of photonic crystal fibers in the temporal case [22]. In this Letter, we make full use of this analogy when we shall describe the dynamics and the formation of the DifRR.

Following Refs. [7,26], we now approximate the discrete variable n with a continuous one. This is justified since we shall use pulses and solitons that extend for several waveguides (typically 5 waveguides or more are enough for the continuous model to give excellent results), and this approximation will be fully vindicated by our numerical simulations in the next section. Defining n as a continuous variable of the distributed amplitude function $\Psi(n, z) = a_{n,z} \exp(-i\kappa_0 n)$, we eliminate the zeroth order term $\kappa_z(\kappa_0)$, which is responsible for a general phase evolution through the substitution $\Psi(n, z) \rightarrow \Psi(n, z) \exp[i\kappa_z(\kappa_0)z]$. The first order term, $-iD_1 \partial_n$, takes into account the transverse velocity and can also be eliminated by introducing a comoving frame, $n \rightarrow n + D_1 z$. After dropping these two low-order terms one arrives at the following equation:

$$\left[i\partial_z - \frac{D_2}{2} \partial_n^2 + \sum_{m \geq 3} \frac{D_m}{m!} (-i\partial_n)^m + |\Psi(n,z)|^2 \right] \Psi(n,z) = 0. \qquad (4)$$

Equation (4) is formally identical to the well-known generalized nonlinear Schrödinger equation (GNLSE), which describes the evolution of pulses in a single optical fiber, plus HOD terms [25]. In Eq. (4) we have the transverse spatial variable n instead of the temporal variable t of the conventional GNLSE. Unlike the temporal GNLSE, where a Taylor series for the fiber dispersion can usually be expanded up to a small number of terms (because HOD coefficients become rapidly very small), in Eqs. (3) and (4) many higher-order diffraction terms $D_{m \geq 2}$ should be taken into account, since their absolute values will be either $|2c \sin(\kappa_0)|$, or $|2c \cos(\kappa_0)|$, and the sum only converges due to the factorial in the denominator.

In the temporal version of the GNLSE, it is well known that a temporal soliton propagating in a fiber emits small amplitude, dispersive, and quasimonochromatic waves at well-defined frequencies (the DisRR) when the linear fiber dispersion and the nonlinear soliton dispersion (which is constant and proportional to its peak power) are matched [16,17]. It is thus natural to conjecture that, in a WA, a *spatial* soliton, which in the continuous variable approximation extends over several waveguides, emits during the propagation a similar kind of small-amplitude diffractive radiation, within a narrow wave number range, due to the phase matching between the spatial soliton nonlinear dispersion and the linear array dispersion given by Eq. (2). By using the perturbation approach, which was developed for DisRRs in Ref. [16], here we derive the phase-matching condition for the DifRR in a similar way. We first find the unperturbed soliton solution of Eq. (4) where all diffractive terms $D_{m \geq 3}$ are dropped. Under these conditions, the soliton solution is given by

$$a_{\text{sol}}(z, n) = A_0 \text{sech}\left(\frac{nA_0}{\sqrt{2c\cos(\kappa_0)}}\right)\exp(ik_{\text{sol}}z), \quad (5)$$

where $k_{\text{sol}} = A_0^2/2$ is the spatial soliton longitudinal wave number, identical to its temporal counterpart. The bright soliton solution (5) only exists when $2c\cos(\kappa_0) > 0$, i.e., only in half of the Brillouin zone, where $-\pi/2 < \kappa_0 < \pi/2$. Now we look for the linearized dispersion relation of plane wave solutions of Eq. (4), by substituting $\exp[i(k_{\text{lin}}z + \Delta\kappa n)]$ into Eq. (4) and using Eq. (3). We obtain

$$k_{\text{lin}}(\Delta\kappa) \equiv \sum_{m\geq 2} \frac{D_m}{m!}\Delta\kappa^m$$
$$= 2c[\cos(\kappa) - \cos(\kappa_0) + \sin(\kappa_0)\Delta\kappa]. \quad (6)$$

In Eq. (6), κ_0 is the central wave number (which is related to the incident angle) of the incident beam, while $\Delta\kappa$ is the detuning from κ_0, and $\kappa = \kappa_0 + \Delta\kappa$. Energy exchange between radiation and solitons is possible for those values of $\Delta\kappa$ that satisfy

$$k_{\text{lin}}(\Delta\kappa) = k_{\text{sol}}, \quad (7)$$

where k_{sol} is constant and has been defined above. This phase matching condition, an implicit equation for the radiation wave number detuning $\Delta\kappa$, is the central result of this Letter. Even though there are hints in previous works on the existence of the DifRR in WAs, this phase-matching condition is given for the first time here and it leads to a complete understanding of the dynamics of such radiation. It is important to note that although the phase-matching condition expressed in Eq. (7) has been derived from the continuous model of Eq. (4), such a formula very accurately predicts the DifRR wave number in the full original discrete model of Eq. (1), as we shall show below.

Incidentally, if we would have followed what is commonly done for optical fibers, i.e., taking into account only D_2 and D_3 in Eq. (6), and ignoring the power dependence ($k_{\text{sol}} \rightarrow 0$), one can easily get the approximate DifRR wave number in the form $\kappa_{\text{RR}} \simeq \kappa'_{\text{RR}} \equiv \kappa_0 + 3/\tan(\kappa_0)$. Such approximations are perfectly fine in fiber optics when dealing with DisRR—they lead to very accurate predictions of the DisRR frequency. However, the same approximation is not good enough for the case of WAs, since, as explained above, the coefficients ($D_m/m!$) decay not as rapidly as in the temporal case, and a large number of orders must be taken into account, as we show explicitly in the next section. However, even if not explicit as in the case of the temporal DisRR, Eq. (7) is exact and can be easily solved numerically.

Emission of DifRR and soliton anomalous recoil.—We now prove numerically the formation of DifRR in the full discrete model of Eq. (1), and the accuracy of the predictions made by the phase-matching condition Eq. (7).

In Fig. 1(b) we show the DifRR wave number $\kappa_{\text{RR}} \equiv \kappa_0 + \Delta\kappa$ as a function of the input soliton wave number (which is related to the angle of incidence) κ_0. The blue solid curve is obtained by finding numerically the roots $\Delta\kappa$ of Eq. (7), while the dashed red curve shows the approximated analytical expression given in the previous section. It is clear that κ'_{RR} is not accurate enough to be used in practice, when compared to the solid line, which shows a complexity that goes beyond any truncation of the Taylor expansion in Eq. (3), especially when the power dependence is included via the right-hand side of Eq. (7).

In Fig. 1(b) we depict the full range of the first Brillouin zone for completeness, but only the interval $-\pi/2 < \kappa_0 < \pi/2$ (indicated by a gray shaded area), in which pulses experience "anomalous" diffraction (i.e., $D_2 < 0$), should be considered, since this is the only region where solitons can form in the WA (for focusing nonlinearity), analogously to the anomalous dispersion frequency range of optical fibers. Parameters used in Fig. 1 are $A_0 = 0.8$, $c = 1.2$, and $\kappa_0 = 0.7$. For these parameters, in the range $0.235 < |\kappa_0| < \pi/2$, one can find only one solution for κ_{RR}, but when $0 < |\kappa_0| \leq 0.235$, Eq. (7) shows several roots [see the solid blue curve in Fig. 1(b)]. Thus, one should expect to simultaneously generate several DifRRs with different wave numbers in the latter interval. However, full numerical simulations of Eq. (1) show that only the solution corresponding to the branch that is the closest to the central horizontal axis (i.e., the axis $\kappa_{\text{RR}} = 0$) can be generated and observed, and all other DifRR waves corresponding to roots from other branches are too weak to be seen numerically, since the overlap between the soliton spectral tail and the radiation wave numbers becomes exponentially small. When $\kappa_0 = 0$, i.e., for a normal incidence of the input CW beam, there is no solution for Eq. (7), regardless of the parameters used. This is also confirmed by the direct simulation of Eq. (1).

The evolution of a CW beam along z according to Eq. (1) is shown in Fig. 1(c), for an input beam $a_{\text{in}}(n) = A_0\text{sech}[nA_0/\sqrt{2c\cos(\kappa_0)}]e^{i\kappa_0 n}$ (i.e., the approximate soliton solution in the continuous limit), and for a WA made of $N = 100$ waveguides. After some propagation, around $z \simeq 3$, a DifRR is emitted by the soliton. The evolution of the Fourier transform of the field $a(n)$ of Fig. 1(c) along z is shown in Fig. 1(d). The dashed green horizontal line represents the input wave number ($\kappa_0 = 0.7$), while the solid white line is obtained by solving Eq. (7) numerically, showing excellent agreement with the pulse propagation. In Fig. 1(b) one can notice that the soliton emits the DifRR with a positive detuning $\Delta\kappa$ when $0 < \kappa_0 < \pi/2$. For instance, when $\kappa_0 = 0.7$, then from Eq. (7) one gets $\kappa_{\text{RR}} \equiv \kappa_0 + \Delta\kappa \simeq 3.53$. However, since the Brillouin zone has a limited extension, when $2\pi > \kappa_{\text{RR}} > \pi$ the DifRR will be emitted with a negative detuning due to the folding of the band structure. In the example shown in Fig. 1(d) the effective DifRR wave number will be equal to

$\kappa_{RR} - 2\pi \simeq -2.75$ [see the white solid line in Fig. 1(d)]. This means that in real space the soliton, instead of recoiling in an opposite direction than the DifRR, will recoil *towards* the DifRR itself, see the white arrows in Fig. 1(c). The same phenomenon occurs in the wave number space: the soliton spectral momentum, instead of recoiling away from the radiation, moves slightly towards it [see Fig. 1(d)]. We call this unique effect (which cannot be found in continuous media such as fibers due to the lack of a Brillouin zone) *anomalous recoil*.

Conclusions.—We have numerically demonstrated the existence and studied the properties of diffractive resonant radiation emitted by discrete spatial solitons in waveguide arrays. Because of the periodicity, several new effects can take place, which have no counterpart in continuous systems, such as the anomalous soliton recoil. The analysis of these unusual phenomena are applicable to virtually any nonlinear discrete periodic system supporting solitons, therefore making our results very general and of relevance for a number of very diverse communities.

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