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Bringing cost transparency to the life annuity market

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Abstract

The financial industry has recently seen a push away from structured products and towards transparency. The trend is to decompose products, such that customers understand each component as well as its price. Yet the enormous annuity market combining investment and longevity has been almost untouched by this development.

We suggest a simple decomposed annuity structure that enables cost transparency and could be linked to any investment fund. It has several attractive features: (i) it works for any heterogeneous group; (ii) participants can leave before death without financial penalty; and (iii) participants have complete freedom over their own investment strategy.

Keywords: Investment; Pensions; Pooled annuity fund; Lifetime savings; Mutual risk-sharing.

Subject Category and Insurance Branch Category: IM51, IB81.

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1 Introduction

Over 40% of all private industry workers in the U.S. are saving for their retirement through a defined contribution plan (Bureau of Labor Statistics, 2012, Table 2). While the overall value of assets held in these plans is immense, being approximately $10 trillion in the U.S. in 2012 (Towers Watson, 2013), individuals' asset values can be small. For example, the median asset value held by those age 55 or older in funds run by Vanguard, a large mutual fund company, was around $60 000 in 2011 (Vanguard, 2012). A similar magnitude of savings is reported in (Poterba et al., 2011, Table 2) for people age 65 to 69 in the year 2008. With the lifetime guaranteed income offered by Social Security and defined benefit pension plans declining relative to pre-retirement income (Webb, 2011), millions of individuals must maximize their retirement income arising from their defined contribution plan savings. They cannot afford to pay unnecessary charges and fees.

Yet in the life annuity contract, which economic theory recommends as a significant component of the optimal retirement investment strategy (Yaari 1965, Davidoff et al. 2005), costs are hidden from the customer (Blake 1999, Stewart 2007). We argue that cost transparency in life annuities is very important, due to the generally irreversible and very long-term nature of these contracts, which potentially involves all of the life savings of individuals. Consumers have no idea if annuity prices are fair, or if insurance companies are either making excessive profits or are grossly inefficient (Carlin 2009, Del Guercio and Reuter 2013, Glode et al. 2012). We present a solution to these difficulties. We propose a decomposed annuity structure that could be linked to any investment and that enables all costs to be disclosed. Our aim is to improve the transparency of the financial and insurance products that are offered to retirees. Greater transparency may also improve the financial regulation of these products (Kalemli-Ozcan et al., 2013).

In the classical life annuity contract, called a fixed-payout life annuity[1], the annuitant is charged a single (i.e. lump sum) premium and in exchange receives a fixed income stream for life. The anticipated ongoing costs are not disclosed explicitly to the potential annuitant. (Some

[1]More specifically, it is a single premium immediate level annuity written on a single life.

insurance companies may charge explicitly for sales commission and the initial administration costs of setting up the annuity contract.) All the potential annuitant knows is the amount of lifetime income that her lump-sum retirement savings will buy. To evaluate the worth of the annuity compared to other investments, the customer must make a number of sophisticated assumptions and complicated calculations. Generally, it is an irreversible contract, so the customer must trust that the insurance company will continue to pay the income stream over her future lifetime, which may be for decades. It is notable that, worldwide, relatively few people voluntarily annuitize their retirement wealth[2] (Brown 2007, Mitchell and Piggott 2011).

The main reason for the opacity of life annuity contracts is that investment risk is combined with mortality risk and costs are not disclosed by the insurance company. As a consequence, life annuities are not comparable on either an individual risk component basis or on a cost basis. This intransparency has generated a body of literature that questions if annuities offer value-for-money to the annuitant (for example, Mitchell et al. 1999, Cannon and Tonks 2009). Typically, the authors calculate the expected value of a fixed-payout life annuity, using what they believe to be a reasonable calculation basis. Their estimated prices are then compared to those quoted in the market by insurance companies. The difference in the values gives an indication of the amount of costs and profit expected by the insurance companies during the contract period.

Unsurprisingly, given the sensitivity of annuity prices to the mortality and investment return assumptions, there is a wide variation in the results. For example, in Mitchell et al. (1999, Table 3) the annuity prices quoted by insurance companies in the U.S. in 1995 are between 74% and 94% of the authors' calculated expected values. A similar range is observed in the U.K. by Cannon and Tonks (2009). Without more information from insurance companies concerning their annuity calculation basis, we can only hypothesize about the reasons for the range of results. It may be due to the insurance companies assuming a different calculation basis than in the studies. For example, the insurance companies may invest in riskier assets than those assumed in the studies, or they may assume that annuitants live longer. It may be due

[2] A phenomenon referred to as the *annuity puzzle*. Recent reviews of the literature on the annuity puzzle can be found in Brown (2009) and Lown and Robb (2011).

to insurance companies' costs, profit and risk capital requirements, or it may be competitive reasons. Without more information it is difficult to draw strong conclusions concerning the value-for-money of annuities.

The lack of information also means that is not clear if annuity prices quoted by insurance companies are competitive, as they can vary significantly across companies (Mitchell et al. 1999, Cannon and Tonks 2009). Furthermore, even if the annuity market is competitive, it does not follow that consumers have low costs (Orszag and Stiglitz, 2001). For example, in the related mutual fund market, fees can be too high (e.g., see Crespo 2009 for the Spanish mutual fund market, and Gil-Bazo and Ruiz-Verdú 2009 for the U.S. market) and brokers can offer no tangible benefits in exchange for high distribution fees (Bergstresser et al., 2009).

Moreover, the annuity marketplace is not as straightforward as might be imagined. Consider the annuity rate, which is the ratio of the annual income guaranteed for life by the insurance company to the single premium. Typically, the "headline" annuity rates quoted in the popular press are for a single premium of \$100 000. An annuity rate of 5% means that the annuitant receives \$5 000 per annum in exchange for the upfront payment of \$100 000. However, an insurance company that offers the highest headline annuity rate may not offer the highest annuity rate for other amounts of single premium. It may be a tactical decision by the insurance company (Harrison, 2012), or due to fixed costs incurred by selling each annuity contract, or simply a reflection of the fact that annuity rates are not necessarily constant across same sex individuals of the same age; a wealthy man may have a higher expected lifetime than a poor one, resulting in a lower annuity rate for the former.

The need for a transparent annuity market is critical so that individuals can make informed decisions on how to manage their assets. They are required to make very complex decisions on how their retirement will be financed. For example, they have to take account of relatively concrete factors such as Social Security benefits, housing, income from other pension plans, as well as taking a view on unknowns like future inflation, life expectancy and future healthcare costs. There are other considerations regarding the individual's quality of life, as well as the desire to bequeath money to others; see Smith and Keeney (2005) on making decisions about investments in quality of life.

With academic studies able to give only a broad indication if the prices of life annuities are fair, the ability of ordinary consumers to judge their value is likely to be much lower. Many individuals are unaware of basic economics and finance (Lusardi and Mitchell, 2011) and lack confidence in their financial literacy (Graham et al., 2009). Furthermore, the simple life annuity is in competition for retirees' savings with much more complicated structured products. The latter include various financial and insurance options and guarantees, which makes it difficult to ascertain if they offer value-for-money (Carlin et al., 2013). Indeed, attempts to value some of the mortality options in variable annuities are the subject of highly technical academic papers (e.g. see Milevsky and Promislow 2001 and Milevsky and Posner 2001, the latter finding that market prices for insurance risk charges are substantially above their theoretical values). If we can make the basic life annuity contract more transparent, then perhaps we can also improve the transparency of these more complicated products.

We present an annuity overlay fund that enables cost transparency while giving one of the main benefits of the life annuity, namely the pooling of mortality risk across a group of people. It overcomes several disadvantages of the life annuity.

- Cost transparency. Within the proposed annuity overlay fund, costs can be charged to each individual as they occur. As investment risk is separated from mortality risk, costs can be attributed to each source independently. For example, administration costs, investment management fees and sales commission can be charged separately to the consumer. If an individual believes that the investment management fees are too high, then they can switch to another fund manager (Blake et al. 2013, Christoffersen et al. 2013).

- Control over investments. With an annuity overlay fund, each individual retains absolute control over their own investments. They can decide how much to invest and how to allocate those investments among any asset class. They can include their house among the assets while continuing to live there. Contrast this with a life annuity contract, in which the individual no longer has any investments since the underlying assets are held by the insurer.

- Opt in or opt out. An individual can decide to remove the annuity overlay fund from all or some of their assets at any time. For the administrator of the annuity overlay fund, this may

be an incentive to keep the administrative costs low (Bharath et al., 2013). Similarly, the participants can decide to add the overlay to more of their assets at any time. This flexibility does not occur with a life annuity contract which is usually binding until death or, at best, an extremely costly contract to exit.

- Tangible financial gains from pooling mortality risk. Participants in the overlay receive financial payments from the pooling of mortality risk. The payments are in addition to any financial gains and are always nonnegative while the participant is alive.

- Investment framing. The annuity overlay allows the sharing of mortality risk to be evaluated in terms of yield like any other investment decision. It may be a more attractive framing of the financial benefits to be gained from pooling mortality risk than the natural consumption frame of the life annuity (Agnew et al. 2008 and Brown et al. 2008).

The annuity overlay fund enables mortality cross-subsidies, investment returns and costs to be identified individually and communicated to the consumers. Furthermore, the overlay could be managed at a very low cost: as there are no guarantees, there are no reserving requirements.

The annuity overlay fund is fundamentally different to a life annuity: the latter transfers mortality risk to an insurer, whereas the former pools mortality risk among the participants in the structure. Instead, it is a means of sharing the random fluctuations risk of mortality. It does not guarantee an income until death and it does not protect against longevity risk, that is the risk of under-estimating how long you may be expected to live. This means that the annuity overlay fund is not an insurance product.

Even though the annuity overlay fund allows mortality risk to be separated from investment risk, the motivation is not to enable people to trade in the financial market themselves. Trading by individuals in the financial markets is fraught with problems (for example, see Barber and Odean 2000a, Barber and Odean 2000b and Barber and Odean 2000c). Rather, the motivation is to arrive at a transparent market in which people understand what they are paying for and can determine if the costs charged are reasonable, a market in which consumers can more easily compare products between sellers and buy only what they need.

The main purpose of the present paper is to

- explain the structure and operation of the annuity overlay fund,
- show that it can be optimal to join the annuity overlay fund, and
- investigate the trade-off between return and volatility, from both a theoretical and a numerical perspective. We derive rules-of-thumb to explain the trade-off, and find that the spread of the age-wealth profile of the participants is very important.

2 The annuity overlay fund: toy example

We begin by illustrating the annuity overlay fund with a toy example that communicates the basic idea. Note that the toy example is unrealistic, as it assumes that no financial return accumulates on wealth, and it only approximates the risk-sharing mechanism of the proposed fund; the correct, instantaneous approach is detailed in Section 3. Nevertheless, the toy example demonstrates how the proposed fund allows people with very different characteristics to pool their mortality in an actuarially fair way.

In the example, participants in the annuity overlay fund agree to pool their mortality experience together for one month. Each participant has a fixed initial wealth. The wealth of the participants who die during the month is put in a *notional mortality account*. At the end of the month, the money in the notional mortality account is shared among all the participants, including those who just died during the month. The payment that each participant receives from the notional mortality account is proportional to their individual mortality rate and wealth.

The annuity overlay fund has a distinctive feature not shared by either the pooled annuity fundsspace which have been proposed and analyzed before in the literature (e.g., see Donnelly et al. 2013, Piggott et al. 2005, Qiao and Sherris 2013, Richter and Weber 2011, Stamos 2008, Valdez et al. 2006). It allows individuals to exit the fund before death, and to do so without any financial penalty. This is a key feature that distinguishes our pooled fund from all others. The reason why individuals can exit the annuity overlay fund without paying a financial penalty is that it is actuarially-fair *at every instant in time*.

Actuarial fairness is critical, particularly when there is a finite number of heterogeneous

Table 1: Characteristics of Alice and Bob at the start of the month.

Name	Wealth	Probability of dying in the next month
Alice	\$1 000 000	0.2%
Bob	\$50 000	0.1%

members in the group. It means that no single subgroup is subsidizing the remaining members. For example, as shown in Donnelly (2014), the group self-annuitization scheme proposed by Piggott et al. (2005) results in the richer members of the group subsidizing the poorer members.

Our proposed fund differs in another important way from other pooled funds: in the annuity overlay fund, participants have true individual investment freedom. They can decide at any time to change their investment strategy, again without paying any financial penalty. In the other proposed pooled annuity funds, the participants are forced implicitly to follow the same investment strategy as the people with whom they are pooling their mortality risk. The investment freedom becomes apparent when we move to the instantaneous approach in Section 3.

Consider two people, Alice and Bob, with the characteristics shown in Table 1. Alice and Bob agree to enter the annuity overlay fund for one month. There are no other participants. If Alice dies during the month then her wealth is put in a notional mortality account. The same rule applies to Bob if he dies. We assume throughout the paper that deaths occur independently of each other and that there is no uncertainty about the probability of death. At the end of the month, the money in the notional mortality account is shared among Alice and Bob in proportion to their wealth and probability of death.

Suppose Bob is the only one to die during the month. When he dies, his wealth of \$50 000 is put in the notional mortality account. At the end of the month, the money in the account is shared out as follows. Alice gets

$$\$50\,000 \times \frac{\$1\,000\,000 \times 0.2\%}{\$1\,000\,000 \times 0.2\% + \$50\,000 \times 0.1\%} = \$50\,000 \times \frac{40}{41} = \$48\,780.$$

This is Alice's *actuarial gain* from participating in the fund for one month. It is based on Alice's expected wealth at risk due to her death over the month, relative to Bob's expected

wealth at risk. It is a return due to sharing mortality risk. Her wealth at the end of the month is calculated by adding her actuarial gain to her wealth of \$1 000 000, giving her a total wealth of \$1 048 780 at the end of the month.

Meanwhile, Bob gets the \$1 220 that is left in the notional mortality account. This can also be determined by the allocation method:

$$\$50\,000 \times \frac{\$50\,000 \times 0.1\%}{\$1\,000\,000 \times 0.2\% + \$50\,000 \times 0.1\%} = \$50\,000 \times \frac{1}{41} = \$1\,220.$$

Note that to calculate Bob's wealth at the end of the month in the same way as we did for Alice, we determine first his actuarial gain as

$$-\$50\,000 + \$1\,220 = -\$48\,780.$$

His actuarial gain is the sum of the amount of his wealth transferred to the notional mortality account, due to his death, and his share of the notional mortality account at the end of the month. Thus Bob loses \$48 780 as a result of dying. His total wealth at the end of the month is the sum of his actuarial gain and his wealth of \$50 000, giving a total wealth of \$1 220, as before. As Bob is dead, the money is paid to his estate. Although the sum paid to Bob's estate is non-trivial in the toy example, in practice we do not expect that the annuity overlay fund operates for only two people. It is intended to enable a large group of people to pool their mortality risk. From the perspective of the dying members, the annuity overlay fund operates similarly to certain other pooled annuity funds and fixed-payout life annuities. The sum paid to Bob's estate can be thought of as a balancing item to make the annuity overlay fund work for any group of heterogenous participants.

Notice that Alice's actuarial gain of \$48 780 exactly cancels with Bob's actuarial gain of −\$48 780. This is due to the fact that no money is created by pooling mortality risk; the wealth of the dead is simply re-distributed among all the participants.

Repeating the above calculations across all possible scenarios, we obtain Table 2 (the amount of money in the notional mortality account at the end of the month), Table 3 (the actuarial

Table 2: Notional mortality account at the end of month, which depends on who dies during the month.

		Bob	
		alive	dead
Alice	alive	\$0	\$50 000
	dead	\$1 000 000	\$1 050 000

Table 3: Alice's and Bob's actuarial gains at the end of month, which depend on who dies during the month. Alice's actuarial gains are in normal text and *Bob's actuarial gains are in italics.*

		Bob	
		alive	dead
Alice	alive	\$0 *\$0*	+ \$48 780 *−\$48 780*
	dead	−\$24 390 *+\$24 390*	+ \$24 390 *−\$24 390*

gains of Alice and Bob) and Table 4 (the wealth of Alice and Bob at the end of the month).

We see from Table 3 that, as long as Alice survives to the end of the month, her actuarial gains are positive. The same observation holds for Bob and, indeed, holds more generally for any group. It is an important feature of the fund since it is an incentive to join the fund.

At the end of the month, the surviving participants choose whether or not to pool their mortality for another month, and how much wealth they want to pool. This is a highly attractive feature of our fund. It means that individuals can withdraw money according to their needs. For example, they may have long-term care or large medical bills to pay. In comparison, conventional annuities and other pooled annuity funds either do not permit exits for reasons other than death, or they apply a severe financial penalty to any withdrawn funds.

Allowing the participants to leave the fund without financial penalty is a consequence of the expected actuarial gains of Alice over all scenarios being zero, and similarly for Bob. In other

Table 4: Alice's and Bob's wealth at the end of month, which depend on who dies during the month. Alice's wealth is in normal text and *Bob's wealth is in italics.*

		Bob	
		alive	dead
Alice	alive	\$1 000 000 *\$50 000*	\$1 048 780 *\$1 220*
	dead	\$975 610 *\$74 390*	\$1 024 390 *\$25 610*

words, there is a zero expected gain from pooling mortality over the month. Thus, at the end of the month, neither Alice nor Bob have any further actuarial obligation to each other and thus can take their money and go their separate ways.

The same approach can be used to pool mortality risk among a large group of people. Indeed, we can think of Alice as a proxy for an aggregate group of individuals. For example, she could represent a group of 100 individuals each with wealth \$10 000.

The toy example made the unrealistic assumption that the return on wealth is zero. We show in the sequel that the fund can be made actuarially fair at all instants in time, and not just on a monthly basis, while allowing for investment returns.

3 Theoretical operation of the fund

Here we show how the annuity overlay fund operates theoretically, which is on an instantaneous basis. We prove that the fund is actuarially fair, in the sense that the expected instantaneous actuarial gains of each participant is zero at all times. Consumption is ignored because it does not affect the results.

3.1 Setup

Suppose that there are $M \in \mathbb{N}$ groups of individuals who participate in the annuity overlay fund. We call the collection of M groups the *portfolio*. Within the mth group there are $L_0^m \geq 1$ individuals age x_m alive at time 0 (for example, we could have only one individual in each group so that $L_0^m = 1$ for each m). Individuals within a group are homogeneous in the sense that they have the same mortality characteristics, risk preferences and initial wealth.

We model the survival of the ith individual in group m by the Poisson process $N^{m,i} := \{N_t^{m,i}, t \geq 0\}$. We assume $N_0^{m,i} = 0$ for all m and i. If the ith individual in group m is alive at time t, then $N_t^{m,i} = 0$, and otherwise $N_t^{m,i} = 1$. The rate parameter, called the *force of mortality* or *instantaneous rate of mortality*, of the Poisson process $N^{m,i}$ is λ_t^m at time t. Deaths are assumed to occur independently of each other, so that the Poisson processes are independent processes.

Denoting by N_t^m the number of deaths which have occurred up to and including at time t in the mth group, we have the relationship

$$N_t^m := \sum_{i=1}^{L_0^m} N_t^{m,i}. \tag{1}$$

Define the number of people alive at time t in the mth group as $L_t^m = L_0^m - N_t^m$. Then $N^m := \{N_t^m, t \geq 0\}$ is a Poisson process with rate $\lambda_t^m L_{t-}^m$ at time t. As deaths occur independently, the processes $N^1, \ldots, N^M$ are independent.

The financial market consists of two traded assets: a risky asset and a risk-free asset. The risk-free asset has price B_t at time t with dynamics

$$dB_t = rB_t dt, \tag{2}$$

with constant risk-free rate of return $r > 0$. The price process S of the risky asset is driven by a 1-dimensional standard Brownian motion $\mathcal{Z}$, so that at time t it has dynamics

$$dS_t = S_t \left(\mu dt + \sigma d\mathcal{Z}_t\right), \qquad S_0 > 0 \text{ constant}, \tag{3}$$

with $\mu > r$ constant and $\sigma > 0$ constant.

The Brownian motion and Poisson processes are defined on the same complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and are independent processes. With $\mathcal{N}(\mathbb{P})$ denoting the $\mathbb{P}$-null sets in the probability space, the information at time $t \geq 0$ is represented by the filtration

$$\mathcal{F}_t = \sigma\{(N_s^{1,1}, \ldots, N_s^{1,L_0^1}, \ldots, N_s^{M,1}, \ldots, N_s^{M,L_0^M}, \mathcal{Z}_s), s \in [0,t]\} \vee \mathcal{N}(\mathbb{P}). \tag{4}$$

In other words, at each time t, it is known which individuals have died in each group and the price of the risky asset at all times up to and including at time t.

We assume that individuals have provided for any desired bequests in advance of committing any assets to the annuity overlay fund, for example by buying a life insurance policy or committing less than 100% of their assets to the fund.

3.2 Theoretical operation on an instantaneous basis

The pool of M groups of individuals participate in the annuity overlay fund. In addition to joining the fund, participants invest in the financial market. For simplicity, we assume here that participants only exit the fund due to their own death, although this assumption can be relaxed without changing the results.

Denote the wealth at time t of each participant in the mth group who is alive at time t by W_t^m, for any $t \geq 0$ and for each $m = 1, \ldots, M$. If an individual in the mth group dies during the short time interval (t_-, t) then her wealth W_{t-}^m is put in the notional mortality account.

Let U_t represent the amount of money which has passed through the notional mortality account up to time t. The amount of money which is put in the notional mortality account during the short time interval (t_-, t) is written mathematically as

$$dU_t = \sum_{m=1}^{M} W_{t-}^m dN_t^m. \tag{5}$$

The amount dU_t is then shared out at time t among all the participants who were alive at time t_-. The amount allocated to each participant is proportional to their individual wealth and force of mortality. Thus each participant in the kth group who was alive at time t_- receives a payment at time t of amount

$$\frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^{M} W_{t-}^m \lambda_t^m L_{t-}^m} dU_t \tag{6}$$

from the notional mortality account. The payment, which we call a *mortality credit* is made irrespective of whether or not the participant is alive at time t.

Formally we calculate the actuarial gains of each individual due to their participation in the fund over the time interval (t_-, t). This allows us to separate the gains due to investment in the financial market from the actuarial gains due to sharing mortality risk. We denote by $G_t^{k,i}$ the total actuarial gains up to time t of a fixed individual i in the kth group. Allowing for the wealth of those dying being transferred into the notional mortality account, the change in the

actuarial gains at time t of individual i in the kth group is given as

$$dG_t^{k,i} = \begin{cases} \frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^M W_{t-}^m \lambda_t^m L_{t-}^m} dU_t - W_{t-}^k, & \text{if individual } i \text{ dies during } (t_-, t), \\ \frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^M W_{t-}^m \lambda_t^m L_{t-}^m} dU_t, & \text{if individual } i \text{ is alive at time } t, \\ 0, & \text{if individual } i \text{ is dead at time } t_-. \end{cases} \tag{7}$$

As the change in the actuarial gains $dG_t^{k,i}$ is due to participation in the fund over the short time interval (t_-, t), we refer to the gains as the *instantaneous actuarial gains*. Since individuals must be alive at time t_- in order to participate in the fund over the time interval (t_-, t), they can not have any actuarial gain at time t if they are dead at time t_-.

At time t, any individual who is still alive can continue to participate in the fund for another instant in time, if they choose to do so.

Proposition 3.1. *The expected instantaneous actuarial gains for a participant in the annuity overlay fund are zero at all times, i.e. for individual i in the kth group,*

$$\mathbb{E}\left(dG_t^{k,i} \,\middle|\, \mathcal{F}_{t-} \right) = 0,$$

for all $t \geq 0$ and for each $i = 1, \ldots, L_0^k$ and $k = 1, \ldots, M$.

Proof. See Appendix A. □

Proposition 3.1 is a consequence of participants pooling their mortality risk only instantaneously. This stands in contrast to products like life annuities for which annuitants pool their mortality risk over their lifetime, and thus cannot exit either before death or without being charged an onerous financial penalty. In the annuity overlay fund, a participant can exit without financial penalty, leaving with the full value of their wealth.

However, even though the expected actuarial gains are zero, the incentive to join the annuity overlay fund is that the actuarial gains for a participant who survives are always nonnegative.

Proposition 3.2. *Conditional upon survival, the expected instantaneous actuarial gains for*

individual i in the kth group are

$$\mathbb{E}\left(dG_t^{k,i} \,\middle|\, \mathcal{F}_{t-},\, N_t^{k,i} = 0\right) = \lambda_t^k W_{t-}^k \left(1 - \frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^{M} W_{t-}^m \lambda_t^m L_{t-}^m}\right) dt, \tag{8}$$

for all $t \geq 0$ and for each $k = 1, \ldots, M$.

Proof. See Appendix A. □

Corollary 3.3. *Conditional upon survival, the expected instantaneous actuarial gains for a participant in the annuity overlay fund are nonnegative at all times.*

Corollary 3.3 shows that, as long as a participant survives, they do not lose financially from participating in the fund. This is an important point and it is a key difference between the annuity overlay fund and a life annuity. It means that the natural frame for the annuity overlay fund is an investment frame, which considers its risk and return features.

In contrast, the natural frame for evaluating the life annuity is a consumption frame, which focuses on what can be consumed over time. However, many individuals may prefer to evaluate the life annuity in an investment frame (Brown et al., 2008). Having paid a known single premium at the start of the contract, the individual may ask if they can live long enough to make back their original "investment" (Hu and Scott, 2007).

For example, consider an individual who pays a single premium of \$100 000 to buy a life annuity income of \$5 000 per annum, paid at the end of each year until the individual dies. If the individual dies in the sixth year after purchase, then they have received 5 payments of \$5 000. From the individual's perspective, the annuity's "internal rate of return" is -33.5% per annum[3]. The individual has to live at least 20 years in order for the annuity to break even, and live more than 26 years to have a return of 2% per annum or higher.

If living long enough to benefit financially is a criterion for buying an annuity, then it may not look like an attractive investment to people who under-estimate their future lifetime. That may be true for a large number of people. For example, in a survey of people age 45 years

[3]Of course, a guarantee can be purchased in conjunction with the life annuity so that the annuity income is guaranteed for, say, 10 years. However, as a guarantee can also be purchased in conjunction with the annuity overlay fund, it is not useful to consider guarantees in the analysis here.

to 80 years, Greenwald and EBRI (2012, Figure 15) report that 41% of the surveyed group guessed a personal future life expectancy that was 5 years or more below their expected future life expectancy, based on a population mortality table suitable for their age and sex.

This issue does not occur with the annuity overlay fund. Its structure means that the individual gains an explicit financial payment while alive due to the pooling of mortality. They do not lose any of their money from pooling mortality risk until they die, unlike in the life annuity where the "loss" occurs at the start of the contract. The annuity overlay fund may be more attractive to individuals simply because of the investment framing of the mortality gains.

Additionally, observe that the annuity overlay fund is closer in spirit to the *actuarial notes*[4] introduced and analyzed in Yaari (1965), than the life annuity. Although a group of people benefit from mortality gains in a life annuity contract, the gains to each individual can only be appreciated by using a lifetime approach, which involves assigning probabilities to each future possible lifetime. It requires a sophisticated and abstract calculation. With the annuity overlay fund, surviving individuals have an annual return that is at least as big as the return from investment in the financial market. They are not required to use a lifetime probability model to appreciate the financial benefits of pooling mortality[5].

3.3 Practical considerations

We have presented the annuity overlay fund in its most general form, allowing people to leave whenever they choose. Indeed, it is highly unlikely that the fund could realistically be operated without some restrictions. However, the main point is that, as the proposed fund is actuarially fair at every instant in time, it is very flexible and can be adapted to any required restrictions. For example, if the fund has a particular purpose, such as to pool mortality risk from any cause of death, then allowing individuals to exit at any time, or without paying a financial penalty, would not be advisable; individuals have more information on their own health than the other participants in the fund.

These points aside, one may wonder how to implement the mortality risk-sharing mechanism

[4] An actuarial note pays out at a fixed time upon survival to that time. A similar contract is the *Arrow annuity* defined in Davidoff et al. (2005).

[5] However, to switch to a consumption frame they do need such a model.

in practice. For example, we might know someone's date of death but not their exact time of death. This would imply that the distribution of money from the notional mortality account should be done at most daily. We can imagine that broadly the implementation steps could be:

- An age- and time-dependent force of mortality function is assigned to each participant upon joining the annuity overlay fund. This may incur an initial charge to each participant.
- The wealth of participants at the start of each day is recorded.
- Upon the notification of a death among the participants,
  - the wealth of the dead participant is liquidated and distributed among the participants, using a discretized version of equation (6). The calculation is done as at the date of death, using the wealth and the force of mortality appropriate to each participant at start of the date of death. However, the amount of money to be distributed from the notional mortality account must clearly be the current (liquidated) wealth of the dead participant.
  - The mortality credits are paid to the surviving participants, either as cash or invested in line with a participant's chosen investment strategy.
  - The mortality credit due to the dead participant is paid to their estate.
- Each year, participants receive an investment statement detailing their current individual wealth, how much they gained from their investments over the year, the amount of any mortality credit paid to them, and costs such as investment management fees, administration costs, and so on.
- Additionally, each participant could receive annual information on how much mortality credit they can reasonably anticipate from the annuity overlay fund over the next year, based on the composition of the annuity overlay fund and the participant's wealth and investment strategy to date. Thus we do not suggest that participants are supplied with details of each other's wealth and force of mortality, but that they are given an indication of the future mortality credit that they may receive from the annuity overlay fund.

- The mortality functions are updated periodically to allow for unanticipated changes in mortality.

We have shown actuarial fairness holds instantaneously in a theoretical model. In practice, performing the calculations daily, as suggested above, should give a reasonable approximation to continuous time and hence actuarial fairness. A critical question is when could actuarial fairness break down in a non-trivial way in the real world. Potential pitfalls include:

- Incorrect choice of mortality model for the participants, for reasons that may be due to moral hazard, adverse selection or incorrect assessment by the fund administrators.
- Large changes in the wealth of the participants over the course of a day. This could be allowed for by a suitable adjustment to the calculation of the mortality credits, such as using average wealth value of the participants over the day, if the data is available, or by having a fund in which all participants have the same investment strategy.

In general the choice of the forces of mortality will depend on the conditions placed on entering and exiting the fund. We do not consider in this paper what restrictions should be placed on a fund to meet a particular purpose. Neither do we explore the additional issue of adverse selection, which is a problem also faced by annuity providers. However, observe equation (6), which shows the share of the notional mortality account paid to each participant in the kth group. We see that the relative values of the forces of mortality are more important than the absolute values. Thus we need a mortality model which accurately captures the relative differences in mortality among participants, rather than their absolute differences, so that the notional mortality account is shared out equitably.

Furthermore the mortality model can be updated frequently to reflect current mortality, since the money in the notional mortality account is shared out immediately. Thus we can allow for longevity improvements and other variations in mortality through time, something which is not possible for many conventional life annuities.

4 The infinite annuity overlay fund and its wider connections

We have already observed that, compared to investment in the financial market alone, it is rational for an investor with no bequest motive and who prefers more money to less, to join the annuity overlay fund; this is the practical implication of Corollary 3.3. Here we describe an idealized version of the annuity overlay fund, called the *infinite annuity overlay fund*, in which there are infinitely-many participants in each group. The infinite annuity overlay fund is strongly connected to both the classical life annuity contract and a particular type of pooled annuity fund, as we show in Section 4.2.

Whether the infinite annuity overlay fund can be used as a satisfactory approximation to a specific finite annuity overlay fund depends on the number of participants and their wealth-mortality profile. Our results in the sequel suggest that, for a suitably diversified fund, the numbers of participants may be in the hundreds rather than the thousands for this approximation to be reasonable. However, we emphasize that actuarial fairness continues to hold in the annuity overlay fund regardless of the number of participants and the heterogeneity of the group. This is a very important point which should not be disregarded as mere actuarial nitpicking, particularly for the relevance of the proposed fund to a real-world application.

4.1 Description of the infinite annuity overlay fund

Here we determine the actuarial gains in the infinite annuity overlay fund. Consider an individual who has no bequest motive. Suppose the individual i joins the annuity overlay fund and is assigned to the kth group.

Proposition 4.1. *Conditional upon survival, the variance of the instantaneous actuarial gains for individual i in the kth group is*

$$Var\left(dG_t^{k,i} \,\middle|\, \mathcal{F}_{t-},\, N_t^{k,i} = 0\right) = \left(\frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^M W_{t-}^m \lambda_t^m L_{t-}^m}\right)^2 \left(\sum_{m=1}^M \left(W_{t-}^m\right)^2 \lambda_t^m L_{t-}^m - \left(W_{t-}^k\right)^2 \lambda_t^k\right) dt, \tag{9}$$

for all $t \geq 0$ and for each $k = 1, \ldots, M$.

Proof. See Appendix A. □

Further assume that at time $t > 0$, each group in the annuity overlay fund has exactly the same number of members, so that $L_{t-} := L^1_{t-} = L^2_{t-} = \cdots = L^M_{t-} > 0$. In that case, the instantaneous actuarial gains of the chosen individual, assuming they are alive at time t, are from (7),

$$dG^{k,i}_t = \frac{\lambda^k_t W^k_{t-}}{L_{t-} \sum_{m=1}^M W^m_{t-} \lambda^m_t} dU_t. \tag{10}$$

Now let the number of members in each group tend to infinity. From Proposition 3.2 we get

$$\mathbb{E}\left(dG^{k,i}_t \,\middle|\, \mathcal{F}_{t-},\, N^{k,i}_t = 0 \right) \to \lambda^k_t W^k_{t-} dt \quad \text{as } L_{t-} \to \infty. \tag{11}$$

From Proposition 4.1,

$$\mathrm{Var}\left(dG^{k,i}_t \,\middle|\, \mathcal{F}_{t-},\, N^{k,i}_t = 0 \right) \to 0 \quad \text{as } L_{t-} \to \infty. \tag{12}$$

Thus there is no volatility in the instantaneous actuarial gains as the number of participants in each group tends to infinity. In an infinite annuity overlay fund, deaths occur continuously, which releases a continuous flow of money into the notational mortality account. As this is shared among infinitely-many participants, their individual wealth increases at a continuous rate equal to their own force of mortality, with zero volatility. In this perfect pool, the volatility of return on wealth arises solely from investment in the financial market.

To see how the actuarial gains in the infinite annuity overlay fund affect the wealth dynamics of the participants, assume the financial market detailed in Section 3.1. Consider an individual i who is a member of the kth group in the annuity overlay fund. Denote by π_t the amount of the individual i's wealth invested in the risky stock at time t. Then ignoring consumption, the dynamics of their wealth process are

$$dW^k_t = \left(rW^k_t + (\mu - r)\pi_t + \lambda^k_t W^k_t \right) dt + \sigma \pi_t d\mathcal{Z}_t. \tag{13}$$

The benefit of joining the infinite annuity overlay fund is seen in the additional term $\lambda^k_t W^k_t dt$,

an increase in the wealth due to the pooling of mortality risk with infinitely-many other people.

4.2 Connection of the infinite annuity overlay fund to other annuities

The actuarial gains in the (heterogeneous) infinite annuity overlay fund are identical to those in the (homogeneous) infinite pooled annuity fund, analyzed by Stamos (2008). This can be seen by comparing equation (13) with Stamos (2008, equation (17))[6]. In the latter fund, there are an infinite number of participants who are independent and identical copies of each other. The wealth of the deceased are shared equally among all the survivors. Both Donnelly et al. (2013) and Stamos (2008) analyze this type of pooled annuity fund.

Consequently, the welfare analysis of Stamos (2008) can be applied directly to the infinite annuity overlay fund. His analysis shows significant utility gains for individuals participating in an infinite annuity overlay fund compared to a pure withdrawal plan. The welfare gains of the annuity overlay fund compared to a fixed-payout annuity depend on the individual's level of risk aversion: an individual with low to moderate levels of risk aversion would derive greater utility from joining the annuity overlay fund compared to buying a fixed-payout annuity, whereas the situation is the reverse for an individuals with a high level of risk aversion. We refer the interested reader to Stamos (2008) for the precise details.

We can also connect the infinite annuity overlay fund with a life annuity. Suppose that at time 0, the individual i invests her wealth of $\$w$ entirely in the risk-free asset and joins the kth group of the infinite annuity overlay fund. She consumes her wealth continuously at the constant rate $\$C$ per annum. Then the dynamics of her wealth process W_t as long as she is alive, are

$$dW_t^k = \left(rW_t^k + \lambda_t^k W_t^k - C\right) dt, \tag{14}$$

subject to $W_0^k = w$. Those familiar with life insurance reserving may recognize (14) as the dynamics of the reserve held by the insurance company for a single life annuity with annual payment $\$C$ paid continuously, when mortality risk is fully diversified. It is a version of the celebrated Thiele's differential equation (Dickson et al., 2009, Section 7.5.1). Thus equation (14)

[6]Note that Stamos (2008) uses π to denote the proportion of wealth invested in the risky asset, whereas we use it here to denote the amount of wealth.

tells us that the wealth of a surviving participant in the infinite annuity overlay fund matches the reserve held by an insurance company against each of its annuity policies, when they both use the same assumptions and the annuity income paid by the insurance company matches the participant's consumption rate C.

5 Analysis of the finite annuity overlay fund

Here we consider an annuity overlay fund in which there are only a finite number of members in each group. It is important to consider how the heterogeneity among the participants can affect their actuarial gains.

For a member of the annuity overlay fund there are two sources of wealth volatility: the investment market and the membership of the fund. We assume that a member is indifferent to the source of volatility. For example, they do not care whether their wealth has increased due to a share dividend payment or due to another member dying. We want to analyze the impact of a heterogeneous fund (in terms of the mortality-wealth profile of the fund) on the wealth volatility of a participant in the fund, while allowing for the participants to invest their wealth in a financial market. It may be that the volatility due to deaths occurring in the fund is not significant compared to volatility from the financial market.

We assume that members' mortality distribution is known. While the expected return on wealth due to sharing mortality risk in the annuity overlay fund is consistent with the distribution, the actual return may differ due to volatility in the deaths in the fund.

We compare participation in the annuity overlay fund to membership of a benchmark fund called the *mortality–linked fund* (a more extensive discussion of the mortality–linked fund is provided by Donnelly et al. 2013). In the mortality–linked fund, wealth volatility arises from the investment market only. The random mortality credit of the annuity overlay fund is replaced by a deterministic mortality-linked interest rate paid by an insurer. In this context, the insurer is analogous to an annuity provider: they are indirectly pooling the mortality of the members of the mortality–linked fund.

The deterministic mortality-linked interest rate that the insurer pays on a member's wealth

is equal to the member's force of mortality but with a reduction to allow for *costs*. Note that, in this section, we use the word *costs* in a different sense to earlier. The costs are what the insurer of the mortality–linked fund charges to the individual to remove the latter's mortality risk. We emphasize that the mortality-linked interest rate is a deterministic interest rate. Exactly as in the annuity overlay fund, members of the mortality–linked fund are free to invest their wealth in the financial market as they choose. The costs are the tool that we use to analyze the differences between the annuity overlay fund and the mortality–linked fund.

Definition 5.1. The *instantaneous breakeven costs* applying at time t are the costs such that, for equal instantaneous volatilities of return on the wealth, a surviving individual has the same instantaneous expected return on wealth from the annuity overlay fund as from the mortality–linked fund at time t.

The idea is that we calculate first the volatility of return on wealth for a participant in the annuity overlay fund, given that some proportion of their wealth is invested in a risky financial asset. Next we calculate the proportion of wealth that an identical member of the mortality–linked fund would have to invest in the risky asset in order to have the same volatility of return on wealth. The proportion should be higher for the member of the mortality–linked fund since they have volatility from the financial market only.

Finally, we calculate the costs such that the expected return for the two individuals is the same, allowing for the different proportions of wealth invested in the risky asset. These are the instantaneous breakeven costs. If the actual costs charged by the mortality–linked fund are higher than the instantaneous breakeven costs, then an individual can obtain a higher expected return from the annuity overlay fund for the same amount of volatility of return on wealth.

In Sections 5.1 and 5.2 we write down the expected returns and volatility of a chosen individual in the annuity overlay fund and mortality–linked fund. This allows us to write down a mathematical expression for the instantaneous breakeven costs in Section 5.3.

5.1 Finite annuity overlay fund

As before, we assume that there are $M \in \mathbb{N}$ groups of individuals in the annuity overlay fund. Each surviving participant in the kth group has wealth W_t^k, force of mortality λ_t^k and invests a proportion p_t^k of their wealth in the risky asset at time t. The remaining proportion of wealth $1 - p_t^k$ is invested in the risk-free asset. Thus the wealth W_t^k of an individual i in the kth group in the finite annuity overlay fund has the dynamics

$$dW_t^k = \Big(r + p_t^k(\mu - r)\Big) W_{t-}^k dt + \sigma p_t^k W_{t-}^k d\mathcal{Z}_t + dG_t^{k,i}, \tag{15}$$

subject to $W_0^k = w_0^k > 0$. The first two terms on the right-hand side are due to the investment in the financial market. The third term, $dG_t^{k,i}$, represents the instantaneous actuarial gains from participation in the fund.

Conditional on individual i surviving to time t, her instantaneous expected return on wealth is calculated from the dynamics given by equation (15) and Proposition 3.2 to be

$$\mathbb{E}\left(\frac{dW_t^k}{W_{t-}^k}\,\middle|\,\mathcal{F}_{t-},\, N_t^{k,i} = 0\right) = \left(r + p_t^k(\mu - r) + \lambda_t^k\left(1 - \frac{W_{t-}^k \lambda_t^k}{\sum_{m=1}^M W_{t-}^m \lambda_t^m L_{t-}^m}\right)\right) dt, \tag{16}$$

in which we recall that L_{t-}^m represents the number of individuals in the mth group who are alive at time t_-.

Similarly, the instantaneous variance of the return on wealth conditional on individual i surviving to time t is

$$\mathrm{Var}\left(\frac{dW_t^k}{W_{t-}^k}\,\middle|\,\mathcal{F}_{t-},\, N_t^{k,i} = 0\right) = \left(\left(\sigma p_t^k\right)^2 + \left(\lambda_t^k\right)^2 \frac{\sum_{m=1}^M \left(W_{t-}^m\right)^2 \lambda_t^m L_{t-}^m - \left(W_{t-}^k\right)^2 \lambda_t^k}{\left(\sum_{m=1}^M W_{t-}^m \lambda_t^m L_{t-}^m\right)^2}\right) dt. \tag{17}$$

The same decomposition is seen for the instantaneous expected return on wealth and the instantaneous variance of return on wealth: there is a component due to individual i's investment in the financial market, and a component from her actuarial gains.

5.2 Mortality–linked fund with costs

Suppose instead that the individual i decides to join the mortality–linked fund, which is operated by an insurer. As long as she survives, a mortality-linked interest rate is paid by the insurer on her wealth, less the costs which are specified below.

In the mortality–linked fund, let $\tilde{p}_t^k$ be the proportion of wealth invested in the risky asset at time t by individual i. The remaining proportion of wealth $1-\tilde{p}_t^k$ is invested in the risk-free asset. The costs that the insurer charges to individual i are represented by a_t^k. As long as individual i survives, her wealth $\tilde{W}_t^k$ has the dynamics

$$d\tilde{W}_t^k = \left(r + \tilde{p}_t^k(\mu - r)\right)\tilde{W}_{t-}^k dt + \sigma\tilde{p}_t^k\tilde{W}_{t-}^k d\mathcal{Z}_t + (1-a_t^k)\lambda_t^k\tilde{W}_{t-}^k dt,$$

subject to $\tilde{W}_0^k = w_0^k > 0$. The term $(1-a_t^k)\lambda_t^k\tilde{W}_{t-}^k dt$ represents the amount of mortality credit paid by the insurer to individual i at time t. Note that for $a_t^k = 0$, the wealth dynamics for the surviving members of the mortality–linked fund match those of an infinite annuity overlay fund, in which there are an infinite number of members in each group of the annuity overlay fund[7].

The instantaneous expected return on wealth conditional on individual i being alive at time t is

$$\mathbb{E}\left(\frac{d\tilde{W}_t^k}{\tilde{W}_{t-}^k}\,\middle|\,\mathcal{F}_{t-},\, N_t^{k,i}=0\right) = \left(r + \tilde{p}_t^k(\mu-r) + (1-a_t^k)\lambda_t^k\right)dt. \tag{18}$$

The instantaneous variance of the return of the wealth, conditional on individual i being alive at time t, is

$$\mathrm{Var}\left(\frac{d\tilde{W}_t^k}{\tilde{W}_{t-}^k}\,\middle|\,\mathcal{F}_{t-},\, N_t^{k,i}=0\right) = \left(\sigma\tilde{p}_t^k\right)^2 dt. \tag{19}$$

Unlike in the annuity overlay fund, there is no uncertainty about the mortality credit for the participant[8]; the insurer pays it to individual i as long as individual i is alive. The only source of volatility for a survivor in the mortality–linked fund is the financial market; compare

[7] It can be shown that the infinite annuity overlay fund coincides with the pooled annuity fund analyzed in Donnelly et al. 2013 and Stamos 2008.

[8] Instead, it is borne by the insurer. Additionally, the insurer is exposed to *model risk* if the mortality index is not representative of the participant's actual mortality.

(17) and (19).

5.3 Instantaneous breakeven costs

Here we calculate the instantaneous breakeven costs. For ease of notation, we use bold notation to denote a vector of length M. For example, $\boldsymbol{p}_t = (p_t^1, \ldots, p_t^M)^\top$, $\mathbf{W}_{t-} = (W_{t-}^1, \ldots, W_{t-}^M)^\top$ and so on, where we use to denote $\mathbf{X}^\top$ the transpose of the vector $\mathbf{X}$. We also define the useful short-hand notation

$$S^k(\mathbf{w}, \boldsymbol{\ell}, \boldsymbol{\lambda}) := \frac{\sum_{m=1}^M (w^m)^2 \lambda^m \ell^m - \left(w^k\right)^2 \lambda^k}{\left(\sum_{m=1}^M w^m \lambda^m \ell^m\right)^2}, \tag{20}$$

for all $(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\ell}) \in \mathbb{R}_+^M \times \mathbb{R}_+^M \times \mathbb{N}^M$ and for each $k = 1, \ldots, M$.

Lemma 5.2 (Instantaneous breakeven costs)**.** *Suppose an individual i, who is in the kth group of the annuity overlay fund, invests the proportion p_t^k of her wealth in the risky asset. To have the same instantaneous volatility of wealth in the mortality–linked fund, she must invest the proportion $\tilde{p}^k(\boldsymbol{p}_t, \mathbf{W}_{t-}, \boldsymbol{\lambda}_t, \mathbf{L}_{t-})$ of her wealth in the risky asset, with*

$$\tilde{p}^k(\boldsymbol{p}, \mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\ell}) := \left(\left(p^k\right)^2 + \left(\frac{\lambda^k}{\sigma}\right)^2 S^k(\mathbf{w}, \boldsymbol{\ell}, \boldsymbol{\lambda})\right)^{1/2}, \tag{21}$$

for all $(\boldsymbol{p}, \mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\ell}) \in \mathbb{R}_+^M \times \mathbb{R}_+^M \times \mathbb{R}_+^M \times \mathbb{N}^M$. Then the instantaneous breakeven costs are $a_t^{k\star} = a^{k\star}(\boldsymbol{p}_t, \mathbf{W}_{t-}, \boldsymbol{\lambda}_t, \mathbf{L}_{t-})$, with

$$a^{k\star}(\boldsymbol{p}, \mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\ell}) := \frac{\mu - r}{\lambda_t^k}\left[\tilde{p}^k(\boldsymbol{p}, \mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\ell}) - p^k\right] + \frac{w^k \lambda^k}{\sum_{m=1}^M w^m \lambda^m \ell^m}, \tag{22}$$

for all $(\boldsymbol{p}, \mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\ell}) \in \mathbb{R}_+^M \times \mathbb{R}_+^M \times \mathbb{R}_+^M \times \mathbb{N}^M$.

Proof. To show (21), equate the instantaneous volatilities, given by equations (17) and (19), and rearrange. To show (22), equate the instantaneous expected returns, given by equation (16) and equation (18), and rearrange to find $a_t^{k\star}$. □

Thus the breakeven costs at which the expected returns from the funds are equal decomposes

into two components, one due to the financial market and the other due to the pooling of mortality. The first term on the right-hand side of equation (22) represents the extra expected return from higher investment in the risky asset in the mortality–linked fund. The second is the fraction of the money in the notional mortality account received by the participant at time t.

However, it is difficult to understand from equation (22) the main factors affecting the breakeven costs since the proportion of wealth invested in the risky asset in the mortality–linked fund also depends on the wealth-mortality profile of the annuity overlay fund. To understand these, we apply a Taylor series expansion to (22) to get the first-order approximation to the breakeven costs:

$$a^{k\star}(\boldsymbol{p}_t, \mathbf{W}_{t-}, \boldsymbol{\lambda}_t, \mathbf{L}_{t-}) \approx \frac{\mu - r}{2\sigma^2} \frac{\lambda_t^k}{p_t^k} S^k(\mathbf{W}_{t-}, \boldsymbol{\lambda}_t, \mathbf{L}_{t-}) + \frac{W_{t-}^k \lambda_t^k}{\sum_{m=1}^M W_{t-}^m \lambda_t^m L_{t-}^m}. \tag{23}$$

The first-order approximation suggests that the spread of the wealth weighted by the expected number of deaths in each group, as approximated by $S^k(\mathbf{W}_{t-}, \boldsymbol{\lambda}_t, \mathbf{L}_{t-})$, is a critical factor in the determination of the breakeven costs. The reason is that a high value of $S^k(\mathbf{W}_{t-}, \boldsymbol{\lambda}_t, \mathbf{L}_{t-})$ indicates a higher volatility in the amount and timing of money that is credited to the notional mortality account. We explore the impact of heterogeneity in the numerical illustrations next.

5.4 Numerical illustrations

We explore the impact of heterogeneity in the annuity overlay fund by comparing it with the mortality–linked fund (the benchmark fund), for three different heterogeneous portfolios. As the analysis is done over an instant in time, we do not need to consider consumption. The results suggest that

(a) there only has to be a few hundred participants in the portfolio for the breakeven costs to be very low, allowing for moderate heterogeneity among the participants, but

(b) severe heterogeneity in the portfolio may invalidate the above conclusion. Therefore, heterogeneity needs to be studied further in the context of annuity overlay funds.

The low breakeven costs are very interesting. They suggest that a group of a few hundred individuals, who are willing to accept volatility in the return on wealth from deaths, can obtain a higher expected return from forming an annuity overlay fund together than from the mortality–linked fund, for the same volatility of return on wealth.

5.4.1 Description of the portfolios and calculations

The three portfolios that we study are detailed in Table 5. The intention is that they reflect, in a simple way, the wealth that we might expect individuals to have at certain ages. Each portfolio consists of a fixed number of groups, with the members of each group k having the same age x_k, force of mortality λ^k and wealth w^k. There is the same number of members L in each group.

The "Old Spenders" portfolio is formed by people over 60 years. These people have been spending the money that they accumulated over their working life, and so the older people in this portfolio have less money than the younger people in the same portfolio. The "Young Spenders" portfolio is formed by participants aged less than 60 years. These people are saving for their retirement, and in this portfolio, the older people have more money than the younger people in the same portfolio. Finally, we combine these two portfolios into the "Combined Portfolio".

For the numerical calculation of the breakeven costs for a representative group member in each portfolio, we assume the financial market parameters $\mu = 0.06$, $\sigma = 0.18$ and $r = 0.02$ for the market behaviour. For simplicity, all participants in the annuity overlay fund invest the proportion $p = \frac{(\mu - r)}{5\sigma^2} \approx 25\%$ of their wealth in the risky asset. The force of mortality of each member of group k is $\lambda^k = \frac{1}{b} e^{(x_k - m)/b}$, with $m = 86.85$ and $b = 9.98$.[9]

Using the above values in addition to the age and wealth values in Table 5, we calculate the breakeven costs $a^{k\star}$ for a representative member of each group k by substituting the values into equation (22). This is done for each of the portfolios in turn assuming that there is only one member in each group, i.e. $L = 1$ (which corresponds to having 30 members in the "Old

[9]The parameter m is the modal age at death and b is the dispersion coefficient. The mortality law, which is a standard Gompertz law, was fitted by Stamos (2008) to US female population mortality data.

Table 5: Description of the portfolios. The number of members L in each group is varied in the numerical simulations.

Portfolio:	**Old Spenders**	**Young Spenders**	**Combined Portfolio**
Number of groups in the portfolio	30	30	60
Number of members within a group	L	L	L
Total number of members in portfolio	$30L$	$30L$	$60L$
Characteristics of the groups in the portfolio, expressed as (group number k, age x_k of members in group, wealth w^k of each member in the group)	(1, 60 years, \$30) (2, 61 years, \$29) ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ (29, 88 years, \$2) (30, 89 years, \$1)	(1, 30 years, \$1) (2, 31 years, \$2) ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ (29, 58 years, \$29) (30, 59 years, \$30)	(1, 30 years, \$1) (2, 31 years, \$2) ⋮ (29, 58 years, \$29) (30, 59 years, \$30) (31, 60 years, \$30) (32, 61 years, \$29) ⋮ (59, 88 years, \$2) (60, 89 years, \$1)

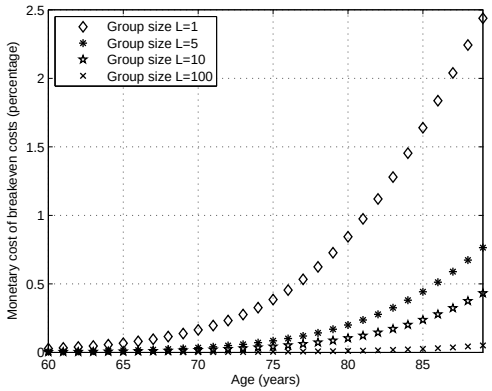
Spenders" portfolio, 30 members in the "Young Spenders" portfolio and 60 members in the "Combined Portfolio" portfolio). We repeat the calculations assuming that there is $L = 5$ people in each group within a portfolio, then again assuming a group size of $L = 10$ and finally we assume a group size of $L = 100$.

Figure 1 shows the results of the calculations. In it, the breakeven costs are expressed as a monetary rate per unit of wealth, namely $1 - e^{-\lambda^k a^{k\star}}$. Note that, to further ease interpretation, we plot the costs for a group against the age of the members in that group, rather than using the group number.

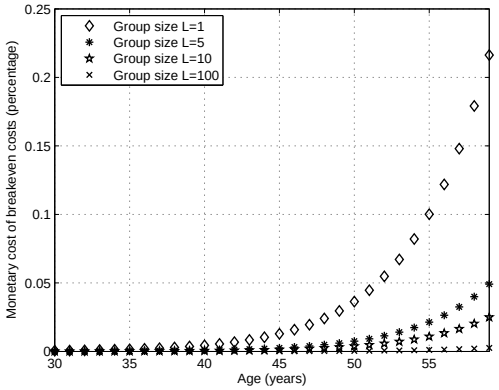
5.4.2 Discussion of the numerical results

The first observation is that all the results show an approximately inverse relationship to the total number of participants, regardless of whether we are considering the Old Spenders, the Young Spenders or the Combined Portfolio.

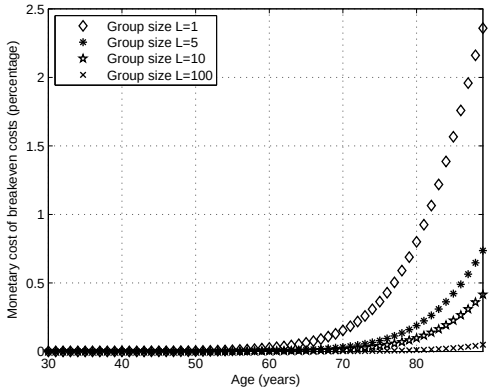
The second, more interesting, observation is that the breakeven costs are low across all portfolios, even when the total number of participants is small. From Figure 1(a), for the Old Spenders with groups of size $L = 10$ (meaning the total number of participants is 300) the



(a) Old Spenders.



(b) Young Spenders.



(c) Combined Portfolio.

Figure 1: Monetary cost of breakeven costs $1 - e^{-\lambda^k a^{k\star}}$.

breakeven costs are less than 0.5% per annum of wealth across all groups. This means that if the costs in the mortality–linked fund equate to more than 0.5% per annum of wealth, then the members of this particular Old Spenders portfolio can obtain a higher expected return on wealth

from participating in the annuity overlay fund than in the mortality–linked fund, for the same volatility of return on wealth. For example, if the costs in the mortality–linked fund were 1% per annum of wealth and assuming individuals are indifferent to the sources of volatility, then the 300 participants obtain a higher expected return on wealth from forming an Old Spenders portfolio together, rather than joining the mortality–linked fund.

For the Young Spenders, the breakeven costs are much lower. From Figure 1(b) for groups of size $L = 10$, so that the total number of participants is 300, they are less than 0.05% per annum of the members' wealth. The low breakeven costs in the Young Spenders are a reflection of the low mortality rate of the participants, who are less than 60 years old, meaning that deaths occur rarely.

In the Combined Portfolio, the breakeven costs are also low when the total number of participants is 300. As there are 60 groups in the Combined Portfolio, 300 total participants corresponds to groups of size $L = 5$. From Figure 1(c), we see that the monetary cost of the breakeven costs are less than 0.75% per annum of wealth across all groups in the portfolio.

Thus, even though the three portfolios have different wealth-mortality profiles, for a fixed total number of participants the monetary cost of the breakeven costs is low. A mortality–linked fund has to charge less than 0.75% per annum of wealth in order to be attractive to the three considered portfolios of 300 participants. These would be very low costs indeed for the insurer of the mortality–linked fund to charge for removing the volatility caused by deaths, when we consider their additional costs for writing such business: the need for reserves, reinsurance, hedging costs, regulatory costs and profit.

However, while in absolute terms the breakeven costs are low, heterogeneity in the wealth-mortality profile of each portfolio certainly does have an impact. We observe that there are quite large relative differences in the breakeven costs between the three portfolios. For example, consider an individual who is age over 60 years old. The monetary cost rate for the individual calculated assuming a Combined Portfolio with 300 total participants (corresponding to groups of size $L = 5$ in the Combined Portfolio) is approximately 2 times higher than that calculated assuming the Old Spenders with 300 total participants (corresponding to groups of size $L = 10$ in the Old Spenders); compare Figure 1(c) for $L = 5$ with Figure 1(a) for $L = 10$.

The relative differences between the breakeven costs calculated for the Young Spenders and the Combined Portfolio are even more extreme. Consider an individual who is age less than 60 years old. The monetary cost rate for the individual calculated assuming a Combined Portfolio with 300 total participants (corresponding to groups of size $L = 5$ in the Combined Portfolio) is approximately 7 times lower than that calculated assuming the Young Spenders with 300 total participants (corresponding to groups of size $L = 10$ in the Young Spenders); compare Figure 1(c) for $L = 5$ with Figure 1(b) for $L = 10$.

Thus the relative attractiveness of the annuity overlay fund to the mortality–linked fund, as measured by the breakeven costs, depends on the heterogeneity of the portfolio that the individual can join. One way to measure the heterogeneity is by the statistic

$$\frac{\sum_{k=1}^{M} \left(w^k\right)^2 \lambda^k L}{\left(\sum_{k=1}^{M} w^k \lambda^k L\right)^2}, \tag{24}$$

which should be a rough approximation to the function S^k defined by (20). It is a measure of how "spread out" is the wealth-mortality profile of the portfolio. Moreover, the first-order approximation to the breakeven costs given by (23) suggests that as S^k increases, and hence as the above statistic increases, the breakeven costs increase. For $L = 10$, the above statistic is 0.132 for the Old Spenders and, for $L = 5$, it is 0.252 for the Combined Portfolio, i.e. the Combined Portfolio is about 2 times more heterogeneous that the Old Spenders, for identical total numbers of participants. This is consistent with our earlier observation that the breakeven costs in the Combined Portfolio are about twice those in the Old Spenders. Similarly, the above statistic is 1.799 for the Young Spenders, suggesting that the Combined Portfolio is about 7 times less heterogeneous than the Young Spenders, for the same total number of participants. Again, this is consistent with our earlier observation about the corresponding breakeven costs.

The three portfolios that we have studied are of moderate heterogeneity and so, although there are differences in the breakeven costs for each portfolio, in absolute terms these differences are not significant. But consider the impact on one of these portfolios of the addition of a small group of high-wealth individuals. The above statistic (24) suggests that they would increase the heterogeneity of the portfolio considerably, and hence increase the breakeven costs.

The consequence may be a decline in the attractiveness of the resulting annuity overlay fund compared to the mortality–linked fund.

6 Summary

We have described the theoretical operation of the proposed annuity overlay fund on an instantaneous basis. The actuarial fairness of the fund at all instants in time makes it a highly flexible and adaptable product.

The features of the proposed annuity overlay fund can be summarized as follows.

- Costs can be categorized and each charged separately to the participants.
- Survivors in the fund benefit from participation in the fund, by gaining a nonnegative return on their wealth.
- Participants are free to invest their wealth in a financial market as they individually choose, with no restrictions on their investment strategy.
- As the longevity risk is borne by the participants of the fund, the annuity overlay fund should have lower costs than products in which the insurer is responsible for longevity risk. The participants do not pay the insurer, and indirectly the reinsurer and other financial institutions, to hedge and manage their longevity risk.
- The annuity overlay fund works for any group of participants regardless of their individual wealth, mortality and investment strategy.
- The number of participants and the wealth-mortality profile of those participants is critical in determining the expected value and volatility of the payments from the notional mortality account.
- Participants can exit the fund whenever they choose without paying a financial penalty, unlike conventional life annuities[10].

[10]In practice, it is likely that conditions would be placed on exiting the fund. For example, a participant must demonstrate that they are in good health before they are allowed to leave.

- The mortality law used to allocate the money in the notional mortality account among the participants can reflect the mortality of each participant individually. This means that we can allow for participants of different socio-economic status, professions and demographics, all within the same portfolio.
- The mortality law used to allocate the money in the notional mortality account among the participants can be updated regularly. Thus we can allow for longevity improvements and other variations in mortality.

We have highlighted the impact of the wealth-mortality profile on the expected value and volatility of the return on wealth. The low breakeven costs in the numerical simulations suggest that individuals may be willing to accept the volatility in the actuarial gains.

7 Conclusion

We have contributed to the management of retirement wealth with a new product that reconsiders annuities and their classical longevity protection scheme. We have introduced an unusual annuity fund that works for any heterogeneous group of participants and we have provided new results on the way these funds can be compared to some annuity schemes.

In particular, the metholodogy and the new type of fund that we have presented are innovative in several aspects:

(i) Participants can join an annuity fund and do not need to purchase a share. However their wealth plays a role in reallocating mortality credit. This is crucial, because the principle of accumulating a number of shares by the same person implies that shares are not independent: when that person dies, all their shares are released simultaneously.

(ii) The underlying assumptions on the force of mortality can be updated and personalized to match each individual participant's demographic and socio-economic status, as well as their lifestyle habits.

(iii) We have studied the role played by the number of participants in the proposed fund. Our initial intuition was that, in order to be competitive versus a fund with no mortality volatility,

a pooled annuity fund would need thousands of participants. On the contrary, we found that the pooled fund needs only a moderate number of participants. As the pooled fund can avoid the regulatory, prudential and administration costs of a mortality-linked fund, in which the insurer guarantees a credit proportional to a mortality reference index, it should be more cost-efficient to the customers than a mortality-linked fund.

We have assumed in the paper that there is no longevity risk. However, it is clear that the fund shares longevity risk among the participants, in contrast to life annuities which transfer it to the insurer. Although this should result in lower costs and thus higher expected returns for the participants, further work is required to analyze the trade-off between the guaranteed life income stream from an annuity and the income stream from the annuity overlay fund, in the more realistic scenario of stochastic financial market parameters and stochastic mortality models.

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A Proofs

Proof of Proposition 3.1:

Fix an arbitrary individual i in the kth group. The Poisson process $N_t^{k,i}$ indicates whether the chosen individual i is alive or dead at time t. By the properties of the Poisson process, this means

$$\mathbb{E}\left(dN_t^{k,i}\middle|\mathcal{F}_{t-}\right) = \lambda_t^k \mathbb{1}\left[N_{t-}^{k,i}=0\right] dt, \tag{25}$$

in which $\mathbb{1}[A]$ denotes the zero-one indicator function of the set $A \subset \Omega$.

Next, conditional on the information available at time t_-, the expected amount of money in the notional mortality account over the time interval (t_-, t) is

$$\mathbb{E}\left(dU_t\middle|\mathcal{F}_{t-}\right) = \sum_{m=1}^{M} W_{t-}^m \lambda_t^m L_{t-}^m dt. \tag{26}$$

Writing the instantaneous actuarial gains $dG_t^{k,i}$ given by (7) in the compact form

$$dG_t^{k,i} = \left(\frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^M W_{t-}^m \lambda_t^m L_{t-}^m} dU_t - W_{t-}^k dN_t^{k,i}\right) \mathbb{1}\left[N_{t-}^{k,i}=0\right], \tag{27}$$

we use equation (25) and equation (26) to show that the expected instantaneous actuarial gains are zero:

$$\begin{aligned}
&\mathbb{E}\left(dG_t^{k,i}\,\middle|\,\mathcal{F}_{t-}\right) \\
&= \mathbb{E}\left(\frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^M W_{t-}^m \lambda_t^m L_{t-}^m} dU_t - W_{t-}^k dN_t^{k,i}\,\middle|\,\mathcal{F}_{t-}\right) \mathbb{1}\left[N_{t-}^{k,i}=0\right] \\
&= \left(\frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^M W_{t-}^m \lambda_t^m L_{t-}^m} \mathbb{E}\left(dU_t|\mathcal{F}_{t-}\right) - W_{t-}^k \mathbb{E}\left(dN_t^{k,i}\,|\,\mathcal{F}_{t-}\right)\right) \mathbb{1}\left[N_{t-}^{k,i}=0\right] \\
&= \left(\lambda_t^k W_{t-}^k dt - \lambda_t^k W_{t-}^k dt\right) \mathbb{1}\left[N_{t-}^{k,i}=0\right] \\
&= 0.
\end{aligned} \tag{28}$$

□

Proof of Proposition 3.2:

As

$$\mathbb{E}\left(dN_t^{k,i}\middle|\mathcal{F}_{t-},\, N_t^{k,i}=0\right)=0 \tag{29}$$

and, excluding the individual i from possible deaths in the kth group,

$$\mathbb{E}\left(dU_t\middle|\mathcal{F}_{t-},\, N_t^{k,i}=0\right)=\left(\sum_{m=1}^{M}W_{t-}^m\lambda_t^m L_{t-}^m-W_{t-}^k\lambda_t^k\right)dt, \tag{30}$$

it follows that

$$\begin{aligned}
&\mathbb{E}\left(dG_t^{k,i}\,\middle|\,\mathcal{F}_{t-},\, N_t^{k,i}=0\right)\\
=&\mathbb{E}\left(\frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^{M}W_{t-}^m\lambda_t^m L_{t-}^m}dU_t-W_{t-}^k dN_t^{k,i}\,\middle|\,\mathcal{F}_{t-},\, N_t^{k,i}=0\right)\\
=&\frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^{M}W_{t-}^m\lambda_t^m L_{t-}^m}\mathbb{E}\left(dU_t\middle|\mathcal{F}_{t-},\, N_t^{k,i}=0\right)-W_{t-}^k\mathbb{E}\left(dN_t^{k,i}\,\middle|\,\mathcal{F}_{t-},\, N_t^{k,i}=0\right)\\
=&\frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^{M}W_{t-}^m\lambda_t^m L_{t-}^m}\left(\sum_{m=1}^{M}W_{t-}^m\lambda_t^m L_{t-}^m-W_{t-}^k\lambda_t^k\right)dt\\
=&\lambda_t^k W_{t-}^k\left(1-\frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^{M}W_{t-}^m\lambda_t^m L_{t-}^m}\right)dt.
\end{aligned} \tag{31}$$

□

Proof of Proposition 4.1:

As

$$\mathrm{Var}\left(dN_t^{k,i}\middle|\mathcal{F}_{t-},\, N_t^{k,i}=0\right)=0, \tag{32}$$

$$\mathrm{Cov}\left(dU_t, dN_t^{k,i}\middle|\mathcal{F}_{t-},\, N_t^{k,i}=0\right)=0, \tag{33}$$

and, excluding individual i from deaths in the kth group,

$$\mathrm{Var}\left(dU_t\middle|\mathcal{F}_{t-},\, N_t^{k,i}=0\right)=\left(\sum_{m=1}^{M}\left(W_{t-}^m\right)^2\lambda_t^m L_{t-}^m-\left(W_{t-}^k\right)^2\lambda_t^k\right)dt, \tag{34}$$

it follows that

$$\begin{aligned}
&\mathrm{Var}\left(dG_t^{k,i}\,\middle|\,\mathcal{F}_{t-},\, N_t^{k,i}=0\right)\\
=&\mathrm{Var}\left(\frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^{M}W_{t-}^m\lambda_t^m L_{t-}^m}dU_t-W_{t-}^k dN_t^{k,i}\,\middle|\,\mathcal{F}_{t-},\, N_t^{k,i}=0\right)\\
=&\left(\frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^{M}W_{t-}^m\lambda_t^m L_{t-}^m}\right)^2\mathrm{Var}\left(dU_t\middle|\mathcal{F}_{t-},\, N_t^{k,i}=0\right)\\
=&\left(\frac{\lambda_t^k W_{t-}^k}{\sum_{m=1}^{M}W_{t-}^m\lambda_t^m L_{t-}^m}\right)^2\left(\sum_{m=1}^{M}\left(W_{t-}^m\right)^2\lambda_t^m L_{t-}^m-\left(W_{t-}^k\right)^2\lambda_t^k\right)dt.
\end{aligned} \tag{35}$$

□