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Dynamical paths and universality in continuous-variable open systems

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We address the dynamics of quantum correlations in continuous-variable open systems and analyze the evolution of bipartite Gaussian states in independent noisy channels. In particular, we introduce the notion of *dynamical path* through a suitable parametrization for symmetric states and focus attention on phenomena that are common to Markovian and non-Markovian Gaussian maps under the assumptions of weak coupling and the secular approximation. We find that the dynamical paths in the parameter space are universal, that is, they depend only on the initial state and on the effective temperature of the environment, with non-Markovianity that manifests itself in the velocity of running over a given path. This phenomenon allows one to map non-Markovian processes onto Markovian ones and may reduce the number of parameters needed to study a dynamical process, e.g., it may be exploited to build constants of motions valid for both Markovian and non-Markovian maps. Universality is also observed in the value of Gaussian discord at the separability threshold, which itself is function of the initial conditions only, in the limit of high temperature. We also prove the existence of excluded regions in the parameter space, i.e., sets of states that cannot be linked by any Gaussian dynamical map.

I. INTRODUCTION

As soon as quantum correlations [1,2] had been recognized as a resource for quantum information processing, it was realized that decoherence is the main obstacle to overcome in order to effectively implement quantum technologies. Decoherence appears whenever a system interacts with its environment so that its dynamics is no longer unitary but rather described by a nonunitary completely positive quantum map, irreversibly driving the system towards relaxation and the loss of quantum coherence [3,4]. The main effect of the interaction with the environment is to set up a time scale τ_M over which the dynamics of the system is effectively described by a coarse-grained Markovian process towards equilibrium. Conversely, for times shorter than τ_M, the dynamics is more involved and the correlations with and within the environment play a major role [4–9]. In this regime, decoherence may be less detrimental and the dynamics may even induce recoherence: This is why a great deal of attention has been devoted to the study of the corresponding non-Markovian maps, e.g., in different continuous-variable systems ranging from quantum optics to mechanical oscillators and harmonic lattices [10–14]. In addition, there is evidence that non-Markovian open quantum systems [15–18] can be useful for quantum technologies [19,20]. As a consequence, much attention is currently devoted to the analysis of system-environment coupling in order to characterize, control, and possibly reduce decoherence in the

most effective way [21,22], e.g., by taking advantage of the backflow of information from the environment.

As a matter of fact, non-Markovian models are usually more involved than the corresponding Markovian ones and only a few cases can be solved analytically [4,23,24]. However, these cases are also of great interest since they display a rich phenomenology that is relevant for practical implementations. This is especially true for continuous-variable systems [25], where considering a set of quantum oscillators excited in a Gaussian state and then linearly interacting with their thermal environment provides an excellent model for a large class of physical systems in order to study non-Markovianity and the decoherence of quantum correlations. Motivated by these considerations, we address in details the dynamics of quantum correlations between two quantum oscillators, each interacting with a local thermal environment. We assume a weak coupling between the oscillators and the corresponding environment, as well as the validity of the secular approximation. These are the minimal assumptions to have a model that displays remarkable differences between Markovian and non-Markovian dynamics and at the same time allows the use of analytic tools to describe results. We also assume that the oscillators are initially prepared in a symmetric Gaussian state and study in details the evolution of their quantum correlations as described by their dynamical paths, i.e., the time evolution in a suitably chosen parameter space.

We start by noting that the set of Gaussian states, i.e., states with a Gaussian Wigner function [26], does not constitute a manifold and is not convex and thus geometrical approaches to its dynamics are not considered particularly appealing. At variance with this belief, we address the

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study of decoherence by representing dynamical paths in a suitable, overcomplete, parameter space, involving Gaussian entanglement (negativity) [27], Gaussian discord [28,29], and the overall purity of the state. The use of these variables offers a suitable framework to compare non-Markovian maps and their Markovian counterparts and to show which properties do and notably *do not* distinguish Markovian and non-Markovian processes. In particular, upon describing the dynamics as a path in the three-dimensional space individuated by the above variables, we observe universality: the dynamical paths do not depend on the specific features of the environment's spectrum and are determined only by the initial state and the effective temperature of the environment. The non-Markovianity of the system changes only the velocity of running over a given path. This behavior allows one to map non-Markovian processes onto Markovian ones and it may reduce the number of parameters needed to study a dynamical process, e.g., it may be exploited to build constants of motion valid for both Markovian and non-Markovian maps. Universality is also observed in the value of discord at the separability threshold, which moreover is a function of the initial conditions only, in the limit of high temperature. Finally, we find that the geometrical constraints provided by the structure of the parameter space imply the existence of excluded regions, i.e., sets of states that cannot be linked by any Gaussian dynamical map.

The paper is structured as follows. In Sec. II we introduce the interaction model and briefly review its solution for Gaussian states. We also introduce symmetric Gaussian states and the quantities used to quantify their quantum correlations, i.e., Gaussian entanglement and Gaussian discord. The dynamics of the system is then described in details in both the Markovian and the non Markovian regimes, illustrating universality of the dynamical paths. Section III closes the paper with a concluding discussion and some remarks.

II. KINEMATICS AND DYNAMICS

Here we consider the dynamical decoherence of two oscillators of frequency ω_0, each coupled to its own bosonic environment made of modes at frequencies ω_k. The baths are separated and of equal structure (see [11,13] for the interaction with a common bath). The system-bath interaction Hamiltonian is given by (we use natural units)

$$H_I = \alpha \sum_k j_k (X_1 q_{1k} + X_2 q_{2k}),$$

where α is the coupling, the complex $\{j_k\}_{k=1,2,\dots}$ modulate the dispersion over the bath's modes, and $X_s = (a_s + a_s^\dagger)/\sqrt{2}$ ($s = 1,2$) and $q_{sk} = (b_{sk} + b_{sk}^\dagger)/\sqrt{2}$ denote the canonical operators of the systems' and baths' modes, respectively, i.e., $[a_s, a_{s'}^\dagger] = \delta_{ss'}$ ($s = 1,2$) and $[b_k, b_{k'}^\dagger] = \delta_{kk'}$. In the weak-coupling limit $\alpha \ll \omega_0$ and performing the secular approximation we can write a non-Markovian time-local master equation for the dynamical evolution of the density operator ϱ describing the quantum state of the two oscillators in the interaction picture [23]

$$\dot{\varrho}(t) = -\sum_k (\Delta(t)[X_k,[X_k,\varrho(t)]]$$
$$- i\gamma(t)[X_k,\{P_k,\varrho(t)\}]), \qquad (1)$$

where $[A,B]$ and $\{A,B\}$ denote the commutator and anticommutator between the operators A and B. Upon defining the spectrum of the environment as

$$j(\omega) = \sum_k |j_k|^2 \delta(\omega - \omega_k),$$

the diffusion (heating) and dissipation (damping) coefficients are given by [30]

$$\Delta(t) = \alpha^2 \int_0^t ds \int_0^\infty d\omega \, j(\omega) \coth(\omega\beta/2) \cos(\omega s) \cos(\omega_0 s),$$

$$\gamma(t) = \alpha^2 \int_0^t ds \int_0^\infty d\omega \, j(\omega) \sin(\omega s) \sin(\omega_0 s), \qquad (2)$$

respectively, with $\beta = 1/T$. At high temperatures the damping coefficient $\gamma(t)$ is negligible and the diffusion $\Delta(t)$ is dominant, while at lower temperatures they have the same order of magnitude.

It is worth noting that the assumptions of weak coupling and a secular approximation are the minimal ones to have a model that displays differences between Markovian and non-Markovian dynamics. At the same time the dynamical equations remain simple enough to allow the use of analytic tools to describe results.

In fact, the master equation (1) may be transformed into a differential equation for the two-mode symmetrically ordered characteristic function associated with the density operator ϱ [10,31]

$$\chi(\mathbf{\Lambda}) = \text{Tr}[\varrho D(\lambda_1) \otimes D(\lambda_2)],$$

where $D(\lambda) = \exp\{\lambda a^\dagger - \lambda^* a\}$, $\lambda \in \mathbb{C}$, is the displacement operator and $\mathbf{\Lambda} = (x_1, y_1, x_2, y_2)$, $\lambda_k = (x_k + iy_k)/\sqrt{2}$. The solution of this equation may be written as

$$\chi_t(\mathbf{\Lambda}) = \exp\{-\mathbf{\Lambda}^T (\widetilde{W}_t \oplus \widetilde{W}_t)\mathbf{\Lambda}\}$$
$$\times \chi_0(e^{-\Gamma(t)/2}(O_t^{-1} \oplus O_t^{-1})\mathbf{\Lambda}), \qquad (3)$$

where $\chi_t(\mathbf{\Lambda})$ is the characteristic function at time t and $\chi_0(\mathbf{\Lambda})$ the corresponding quantity at $t = 0$. The quantity $\Gamma(t)$ represents an effective time-dependent damping factor, given by

$$\Gamma(t) = \int_0^t ds \, \gamma(s).$$

The 2×2 matrices $\widetilde{W}_t$ and O_t are given by

$$\widetilde{W}_t = e^{-\Gamma(t)} O_t W_t O_t^T, \qquad (4)$$

$$O_t = \begin{pmatrix} \cos \omega_0 t & \sin \omega_0 t \\ -\sin \omega_0 t & \cos \omega_0 t \end{pmatrix}. \qquad (5)$$

Finally,

$$W_t = \int_0^t ds \, e^{\Gamma(s)} \widetilde{M}_s$$

with $\widetilde{M}_s = O_s^T M_s O_s$ and

$$M_s = \begin{pmatrix} \Delta(s) & 0 \\ 0 & 0 \end{pmatrix}. \qquad (6)$$

Gaussian states have a Gaussian characteristic function and are fully characterized by the mean values of the canonical

operators X_k and $P_k = i(a_k^\dagger - a_k)/\sqrt{2}$ and by the covariance matrix (CM) σ, which is written as

$$\sigma_{jk} = \tfrac{1}{2}\text{Tr}[\varrho(R_j R_k + R_k R_j)],$$

with $\mathbf{R} = (X_1, P_1, X_2, P_2)$. Since the Gaussian character of an input state is preserved by the master equation (1) and we are considering Gaussian states, we need to address the evolution of the first two moments. In addition, we can focus attention on the evolution of the CM only, since the quantum correlations are independent of the first moments.

In particular, we assume that the initial state is a two-mode Gaussian state ϱ_0 with zero amplitude, i.e., $\text{Tr}[\varrho_0 X_k] = \text{Tr}[\varrho_0 P_k] = 0$, $k = 1,2$, and with covariance matrix σ_0. According to Eq. (3), the evolved state at time t is still a Gaussian state with zero mean value and the covariance matrix given by [12,31,32]

$$\sigma_t = e^{-\Gamma(t)}(O_t \oplus O_t)\sigma_0(O_t \oplus O_t)^T + \widetilde{W}_t \oplus \widetilde{W}_t. \quad (7)$$

Upon retaining only the terms consistent with the secular approximation we arrive at the expression

$$\sigma_t = e^{-\Gamma(t)}\sigma_0 + \tfrac{1}{2}\Delta_\Gamma(t)\mathbb{I}_4, \quad (8)$$

where

$$\Delta_\Gamma(t) = e^{-\Gamma(t)} \int_0^t ds\, e^{\Gamma(s)}\Delta(s)$$

is a time-dependent effective diffusion factor.

The non-Markovian features are embodied in the time dependence of the coefficients $\Delta_\Gamma(t)$ and $\Gamma(t)$, which describe diffusion and damping, respectively. For times $t \lesssim \tau_M$ both coefficients are strongly influenced by the whole spectrum of the environment [12]. In contrast, for times $t \gg \tau_M$ the coefficients achieve their Markovian limiting values. In particular we have

$$\lim_{t\to+\infty} \gamma(t) = \alpha^2 |j(\omega_0)|^2 \equiv \gamma_M,$$

such that

$$\Gamma(t) = \gamma_M t, \quad \Delta_\Gamma(t) = (1 - e^{-\gamma_M t})(2n_T + 1),$$

and the solution (8) is rewritten as

$$\sigma(t) = e^{-\gamma_M t}\sigma_0 + (1 - e^{-\gamma_M t})\sigma_T,$$

where $\sigma_T = (n_T + \tfrac{1}{2})\mathbb{I}_4$ is the CM of the stationary state, i.e., a thermal equilibrium state at temperature $1/\beta$ and in turn a population of $n_T = (e^{\beta\omega} - 1)^{-1}$ thermal photons.

A. Symmetric Gaussian states

A bipartite Gaussian state is symmetric if its CM can be recast (via local operations) in a form depending on two real parameters a and c, that is,

$$\sigma = a\mathbb{I}_4 + c\sigma_1 \otimes \sigma_3, \quad (9)$$

the σ_j's being Pauli matrices. Note that uncertainty relations impose a constraint that reads [33] $|a - c| \geqslant \tfrac{1}{2}$. The evolution under the master equation (1) preserves the symmetry [see Eq. (8)], therefore, the evolved CM at time t may still be

written as $\sigma(t) = a(t)\mathbb{I}_4 + c(t)\sigma_1 \otimes \sigma_3$, where

$$a(t) = a_0 e^{-\Gamma(t)} + \Delta_\Gamma(t), \quad (10)$$

$$c(t) = c_0 e^{-\Gamma(t)}, \quad (11)$$

with $a_0 = a(0)$ and $c_0 = c(0)$.

A symmetric CM of the form (9) corresponds to the preparation of the two oscillators in a squeezed thermal state (STS), i.e., a state with a density operator of the form

$$\varrho(r,\nu_T) = S(r)\nu \otimes \nu S^\dagger(r),$$

where ν denotes a single-mode thermal state with ν_T photons and $S(r) = e^{r(a_1 a_2 - a_1^\dagger a_2^\dagger)}$ is the two-mode squeezing operator. For $\nu_T = 0$ the state $\varrho(r,0)$ reduces to the so-called two-mode squeezed vacuum state or twin-beam state, i.e., the maximally entangled state of two oscillators at fixed energy.

The parameters of the CM are connected to the physical parameters as

$$a = (\nu_T + \tfrac{1}{2})\cosh(2r), \quad c = (\nu_T + \tfrac{1}{2})\sinh(2r).$$

Furthermore, by introducing the (equal) population (mean photon number) of the two subsystems

$$\bar{n} = \sinh^2 r(2\nu_T + 1) + \nu_T,$$

the diagonal elements may be written as $a = \tfrac{1}{2} + \bar{n}$, while the c coefficients describe the correlations among them. It is worth noting that any two-mode entangled Gaussian state can be converted into a symmetric one by local operations and classical communication [34,35]. Therefore, our results about the dynamics of quantum correlations actually hold for more general initial states than the symmetric ones, including any initially entangled state.

Indeed, the representation in terms of the coefficients a and c does not fully illustrate the correlation properties of a state. In particular, it does not allow one to analyze the relations between different kinds of quantum correlations, such as entanglement or discord, in a dynamical context and to compare their robustness against dissipation and noise. To this aim we introduce a different (overcomplete) parametrization involving the overall purity of the state

$$\mu = \text{Tr}[\varrho(r,\nu_T)^2] = \frac{1}{4\sqrt{\det\sigma}} = \frac{1}{(2\nu_T + 1)^2}, \quad (12)$$

its Gaussian entanglement expressed in terms of the minimum symplectic eigenvalue

$$\lambda = a - c = (\nu_T + \tfrac{1}{2})e^{-2r}$$

(the state is separable if and only if $\lambda \geqslant \tfrac{1}{2}$), and the Gaussian quantum discord, which for symmetric Gaussian states may be written as [28]

$$D(a,c) = h(a) - 2h(\sqrt{a^2 - c^2}) + h\left(a - \frac{2c^2}{1 + 2a}\right)$$

$$\equiv D(\mu,\lambda), \quad (13)$$

where

$$h(x) = (x + \tfrac{1}{2})\ln(x + \tfrac{1}{2}) - (x - \tfrac{1}{2})\ln(x - \tfrac{1}{2}).$$

The parameter space individuated by μ, λ, and D is overcomplete and the third parameter is a function of the other two [36]

at any time. In the following, we will describe the dynamics of the system by paths in the three-dimensional space (μ, λ, D) according to the following definition.

Definition. A dynamical path for a symmetric Gaussian state is a line in the three-dimensional space (μ, λ, D) individuated by the overall purity of the state μ, its least symplectic eigenvalue λ, and its Gaussian discord D.

Dynamical paths lay on the surface individuated by the constraint (13) and in the region satisfying the uncertainty relations. In terms of the parameters (μ, λ, D) these constraints correspond to

$$D = D(\mu, \lambda), \quad \mu < \frac{1}{4\lambda^2}. \qquad (14)$$

A dynamical path describes the evolution of a symmetric Gaussian state in a noisy Gaussian channel with no explicit dependence on time. This allows one to compare non-Markovian maps and their Markovian counterparts and to show which properties do and do not distinguish Markovian and non-Markovian processes. At the same time, it allows us to reveal the relationships among the different kinds of quantum correlations in a dynamical context. In other words, each dynamical path actually describes an equivalence class of dynamical time-dependent trajectories (including both Markovian and non-Markovian ones), characterized by a specific dependence of the Gaussian discord on the other two parameters.

B. Markovian dynamics

The Markovian master equation depends on the (effective) environment's temperature and the damping γ_M; nonetheless, the Markovian dynamical paths depend exclusively on the (effective) temperature of the environment. The damping affects only the speed of running over a dynamical path, but not its shape, and the rate $c(t)/c_0 = e^{-\gamma_M t}$ determines in a unique way the rate $a(t)/a_0$. In the left panel of Fig. 1 we show Markovian paths for different values of the temperature, assuming that the two oscillators are initially prepared in a twin-beam state $\varrho(r_0, 0)$, i.e., a pure maximally entangled state of the two oscillators. As it is apparent from the plot, two limiting paths emerge at low and high temperatures. The transition from one regime to the other occurs continuously by raising the temperature and we see that the high-temperature limit is already achieved for temperatures corresponding to

$$n_T / \sinh^2(r_0) \gtrsim 3.$$

Two other phenomena are revealed by this representation: (i) The value of the discord at the separability threshold ($\lambda = \frac{1}{2}$) depends only on the initial squeezing r_0 and approaches a universal curve in the high-temperature limit and (ii) for a given initial state $\varrho(r_0, 0)$ there are STSs that cannot be reached during any Markovian decoherence process, despite the fact that they have reduced entanglement and purity compared to the initial state.

C. Non-Markovian dynamics

As mentioned in the Introduction, non-Markovian dynamics may display remarkable differences from their Markovian counterpart during the initial transient when $t \lesssim \tau_M$. Entanglement oscillations may occur and the separability threshold
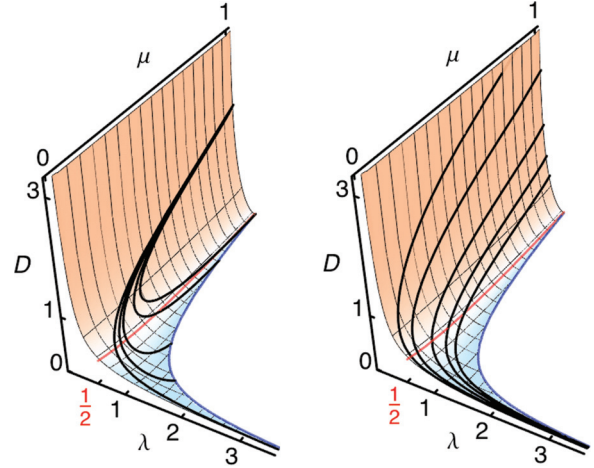

FIG. 1. (Color online) On the left are Markovian thermalization paths at different temperatures. The initial state of the two oscillators is a two-mode squeezed vacuum with $r_0 = 1.2$; the different paths (black lines) correspond to different numbers of thermal photons in the environment. From top to bottom we have paths for $n_T = 0, 0.1, 0.5, 1.0, 10$. The solid blue line corresponds to thermal product states $\nu \otimes \nu$ with zero discord. The solid red line denotes the separability threshold. The high-temperature limit is already achieved for $n_T \gtrsim 3$. On the right are dynamical paths in the high-temperature limit for the two oscillators initially prepared in a two-mode squeezed vacuum state with different values of initial entanglement. The curves correspond to (from bottom to top) $r_0 = 0.5, 0.7, 1.0, 1.5, 2.0$.

may be delayed or accelerated depending on the spectrum of the environment. A question thus arises as to whether these differences also affect significantly the dynamical path in the space of parameters. As we will see, this is not the case and universality occurs. The results about the dynamics that we are going to discuss are independent of the particular choice of environment spectrum, which is a crucial point of our analysis. However, in order to show some numerical solutions, we employ a few examples corresponding to white noise and both Ohmic and super-Ohmic spectral densities with a cutoff ω_c. More specifically, we are going to consider the Ohmic spectrum

$$j(\omega) \propto \frac{\omega \omega_c^2}{\omega^2 + \omega_c^2},$$

which leads to non-Markovian features when out of resonance, i.e., when $\omega_0 \gg \omega_c$, the super-Ohmic spectrum

$$j(\omega) \propto \frac{\omega^2 \omega_c}{\omega^2 + \omega_c^2}$$

and white noise spectrum $j(\omega) \propto \omega_c$.

We start by analyzing the high-temperature regime, where over a time scale $\tau \sim \tau_M$ we can neglect the damping $\Gamma(t)$ (it becomes relevant over times $\tau \sim \gamma_M^{-1} \gg \tau_M$, which is definitely in the Markovian regime). Short-time non-Markovian dynamics is thus due to the behavior of the heating function $\Delta_\Gamma(t)$ and in turn is very sensitive to the details of the environment spectrum $j(\omega)$. In this limit non-Markovian effects can be seen during the whole decoherence process, with entanglement oscillation across the separability threshold [10]. The dynamics is driven by the approximate dynamical

equation

$$\sigma_t \simeq \sigma_0 + \int_0^t ds\, \Delta(s) \frac{\mathbb{I}_4}{2} \qquad (15)$$

corresponding to

$$a(t) = a_0 + \frac{1}{2}\int_0^t d\tau\, \Delta(\tau) \qquad (16)$$

$$c(t) = c_0. \qquad (17)$$

The minimum symplectic eigenvalue is thus given by

$$\lambda(t) = \lambda_0 + \frac{1}{2}\int_0^t d\tau\, \Delta(\tau). \qquad (18)$$

The condition $c(t) = c_0$ imposes a constraint on the dynamical paths, which is the same independently of whether the dynamic of $a(t)$ is Markovian or displays oscillations, as long as $a(t) \geqslant a_0\ \forall t$ and $a(t) \to a_T$. In other words, the paths are the same as in the Markovian case and the possible oscillations of $a(t)$ influence only the speed of running over the dynamical path. In the right panel of Fig. 1 we show the dynamical paths for different values of the initial squeezing r_0.

D. Discord at the separability threshold

The condition $c(t) = c_0$ also implies that the Gaussian discord may be written as

$$D(a,c) \simeq D(\lambda + c_0, c_0), \quad T \gg 1,$$

i.e., it depends on the temperature and the initial squeezing [37] only through the minimum symplectic eigenvalues. At the separability threshold, i.e., for $t = t_{\text{sep}}$ such that $\lambda(t_{\text{sep}})$, we have

$$\begin{aligned}D_{\text{sep}} &\equiv D_{\text{sep}}(r_0) \\ &= D\big(\tfrac{1}{2}[1 + \sinh(2r_0)], \sinh(2r_0)\big),\end{aligned} \qquad (19)$$

i.e., the discord at separability is a universal function of the initial squeezing. In Fig. 2 we show the Gaussian discord at separability as a function the initial squeezing. The solid black line corresponds to the above high-temperature approximation $D_{\text{sep}}(r_0)$ and the colored symbols correspond to the full non-Markovian solutions for $n_T = 10$, obtained taking into account the damping and different environment spectra. As it is apparent from the plot, there is excellent agreement between the two solutions, independently of the environment's spectrum. We also notice that D_{sep} saturates to a limiting value

$$d_* = \lim_{r_0 \to \infty} D_{\text{sep}}(r_0) = -1 + 2\ln 2 \simeq 0.3863$$

as the initial squeezing increases. The initial squeezing needed to achieve the saturation regime increases with temperature. As it may be seen from the plot, for high temperatures, i.e., for $n_T \gtrsim 1$, it is about $r_0 \simeq 2$.

For lower temperature the approximation $c(t) \simeq c_0$ is no longer valid and the Gaussian discord at separability is given by (Markovian expression)

$$D_{\text{sep}} \equiv D_{\text{sep}}(r_0, n_T) = D\big(\tfrac{1}{2} + c(t_{\text{sep}}), c(t_{\text{sep}})\big).$$

In Fig. 2 we show $D_{\text{sep}}(r_0, n_T)$ as a function of r_0 for different values of n_T (dashed gray lines). We also report
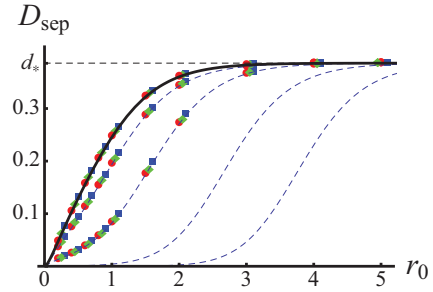

FIG. 2. (Color online) Discord at the separability threshold as a function of the initial squeezing. The solid black line denotes the universal function $D_{\text{sep}}(r_0)$ of Eq. (19) obtained in the high-temperature limit, whereas the colored symbols are the solutions of the full non-Markovian dynamics in Eq. (8) for $n_T = 10$ and for three different environment spectra corresponding to Ohmic, super-Ohmic, and white-noise spectral density, respectively (red circles, green diamonds, and blue squares). The horizontal dashed line is the high-temperature high-squeezing limiting value $d_* \simeq 0.3863$. The dashed gray lines denote the low-temperature Markovian curves $D(\tfrac{1}{2} + c(t_{\text{sep}}), c(t_{\text{sep}}))$ for $n_T = 0.5, 0.1, 10^{-2}, 10^{-3}$ respectively. We also report the solutions of the full non-Markovian dynamics for $n_T = 0.5, 0.1$ and the same three spectra, whereas for $n_T = 10^{-2}, 10^{-3}$ the separability threshold is definitely in the Markovian regime.

the values obtained from the full non-Markovian solutions for different environment spectra and not too low temperature, i.e., $n_T = 0.5, 0.1$. As it is apparent from the plot, the two solutions are in excellent agreement and this may be understood as follows. At low temperatures the damping $\gamma(t)$ and the heating function $\Delta(t)$ become of the same order of magnitude and thus the separability threshold t_{sep} depends on the environment's spectrum. In contrast, separability is always achieved in the Markovian regime and thus D_{sep} is a universal quantity. The plot confirms that this argument holds also if the temperature is not too low, i.e., for $n_T = 0.5, 0.1$. For times $t \lesssim \tau_M$, there is a competition between $\gamma(t)$ and $\Delta(t)$ and, in principle, one would not expect a universal behavior. However, low temperature and weak coupling make the effect of damping and heating very weak, with appreciable perturbation of the initial state only after a long time. In other words, any dynamical effect of the interaction is taking place in the Markovian regime, thus regaining universality and independence of the environment's spectrum. This also means that the dynamical paths in the left panel of Fig. 1 legitimately describe non-Markovian dynamical trajectories at low temperatures.

E. Universality of constants of motion

Any path-dependent property may be checked analytically using the set of Markovian equations and then extended to the non-Markovian regime, where an analytic approach would be unfeasible. In particular, we introduce the rescaled time $\tau = \Gamma t$ and recall that in the Markovian regime we have

$$\partial_\tau \lambda = e^{-\tau}(\lambda_T - \lambda_0),$$
$$\partial_\tau(\lambda\mu)^{-1} = e^{-\tau}[(\lambda_0\mu_0)^{-1} + 4\lambda_T],$$

where the subscript 0 (T) refers to the initial (stationary) state. Then any constant of motion, e.g., $C = \lambda + y/4\lambda\mu$, with $y = (\lambda_T - \lambda_0)/\lambda_T + (4\mu_0\lambda_0)^{-1}$, built using the Markovian

dynamical equation is a constant of motion also in the non-Markovian regime, independently of the environment's spectrum, and with potential application for the development of general channel engineering strategies. The temperature dependence disappears in the high-temperatures limit.

III. DISCUSSION AND CONCLUSIONS

We have addressed the dynamics of quantum correlations in continuous-variable open systems and analyzed the evolution of bipartite Gaussian states in independent noisy channels. We have assumed weak coupling between the system and the environment as well as the secular approximation. These are the minimal assumptions to have a model that displays remarkable differences between Markovian and non-Markovian dynamics and at the same time allows the use of analytic tools to describe results.

In describing the noisy evolution of two-mode symmetric Gaussian states we introduced the concept of dynamical paths, i.e., lines in the three-dimensional space individuated involving Gaussian entanglement, Gaussian discord, and the overall purity of the state. Dynamical paths describe the evolution of symmetric Gaussian states with no explicit dependence on time. This has been proven suitable to address the decoherence effects of both Markovian and non-Markovian Gaussian maps and to reveal which properties do and do not distinguish Markovian and non-Markovian processes. At the same time, dynamical paths allow us to reveal the relationships among the different kinds of quantum correlations in a dynamical context. Each dynamical path actually describes an equivalence class of dynamical time-dependent trajectories (including both Markovian and non-Markovian ones), characterized by a specific dependence of the Gaussian discord on the other two parameters.

Upon describing the dynamics as a path in the three-dimensional space individuated by the above variables, we have observed universality: The dynamical paths do not depend on the specific features of the environment's spectrum and are determined only by the initial state and the effective temperature of the environment. Non-Markovianity manifests itself in the velocity of running over a given path. This phenomenon allows one to map non-Markovian processes onto Markovian ones and may reduce the number of parameters needed to study a dynamical process, e.g., it may be exploited to build constants of motion valid for both Markovian and non-Markovian maps.

Universality is also observed for the value of discord at the separability threshold, which moreover depends on the initial squeezing in the high-temperature limit. We also found that the geometrical constraints provided by the structure of the parameter space imply the existence of excluded regions, i.e., sets of Gaussian states that cannot be linked by any Gaussian dynamical map, despite the fact that they have reduced entanglement and purity compared to the initial one.

Our results have been obtained for Gaussian states and are not directly transferable to the non-Gaussian sector of the Hilbert space. Indeed, there are no necessary and sufficient criteria to individuate and quantify non-Gaussian entanglement and there are no analytic formulas to evaluate non-Gaussian quantum discord. The interplay between Gaussian and non-Gaussian quantum correlations has been discussed in recent years [38–41], but a complete understanding has not yet been achieved.

Finally, we emphasize once again that the universality of dynamical paths does not depend on the environment spectrum, i.e., it is a consequence of the assumptions of weak coupling and the linear interaction between the system and environment. It may therefore be conjectured that universality represents a more general feature, characterizing any open quantum system admitting a Markovian limit.

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