Reliability of Strongly Nonlinear Single Degree of Freedom Dynamic Systems by the Path Integration Method

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Reliability of Strongly Nonlinear Single Degree of Freedom Dynamic Systems by the Path Integration Method

This paper presents a first passage type reliability analysis of strongly nonlinear stochastic single-degree-of-freedom systems. Specifically, the systems considered are a dry friction system, a stiffness controlled system, an inertia controlled system, and a swing. These systems appear as a result of implementation of the quasioptimal bounded in magnitude control law. The path integration method is used to obtain the reliability function and the first passage time. [DOI: 10.1115/1.2967896]

Keywords: probability density function, path integration, strongly nonlinear systems, control, swings, random vibrations, outcrossing rate, reliability

1 Introduction

Reliability and safety are the major concerns in designing and developing modern mechanical systems. A system's reliability may be considered as the probability that no system failure occurs within a given time interval. Often the reliability problem is associated with finding the probability that a system's response stays within a prescribed domain, an outcrossing of which leads to immediate failure. A problem of this type is called *the first passage problem* [1–3], and it has been extensively studied by a number of authors. The first passage problem is directly related to a solution of the corresponding Pontryagin equation, written with respect to the first excursion time T. Unfortunately, an exact analytical solution to this problem, even for a linear system, is yet to be found. A few strategies have been proposed over the years to deal with this type of problems. One of them is based on an averaging procedure and further problem reformulation for the system's response amplitude or energy. The Markov property of the energy envelope has been used to evaluate the probability of the first passage time for a linear system [4], systems with nonlinear stiffness [5], or nonlinear damping [6].

A number of problems have been solved numerically and analytically since then. A numerical solution to the Pontryagin equation has been developed in Refs. [7–9], whereas a numerical solution to the backward Kolmogorov–Feller equation, for a system subjected to a Poisson driven train of impulses, has been proposed in Ref. [10].

New analytical and numerical approaches have been reported in Refs. [11–13]. The method proposed in Ref. [11] can only be used in practice for problems where the stochastic aspect can be represented by a very limited number of random variables. Hence, it would not seem to be applicable for problems with stochastic process inputs of the kind studied in this paper. In Ref. [12], a method is described for estimating the exceedance probability of time variant systems with random parameters by using an improved response surface technique. However, the accuracy of such a method for the problems considered in this paper is hard to assess. Recently, a new tail-equivalent linearization method has been developed in Ref. [14], which may be used for reliability estimates for single as well as multiple-degree-of-freedom (MDOF) systems for stationary inputs.

Special attention should be paid to the reliability of systems, which appears as a result of some design or optimization procedures. Indeed, the purpose of these procedures is to satisfy certain criteria, often not related to the system's reliability. In fact, their implementation may lead to a deterioration of the system's reliability. For instance, consider a stochastic optimal control problem, which aims to reduce the mean response energy of a single-degree-of-freedom (SDOF) undamped linear oscillator, subjected to a zero-mean external Gaussian white noise, by means of a bounded in magnitude control force. It has been demonstrated in Ref. [15] that an optimal control law for a steady-state response is represented by a dry friction law. On the other hand, it has been shown by asymptotic analysis in Ref. [16] that a stochastic system with dry friction is less reliable than that of a system with linear damping. Therefore, a reliability investigation of controlled stochastic systems may be of special importance.

This paper is devoted to a reliability investigation of four types of controlled systems by application of the numerical path integration (PI) method [17]. The PI code is validated by comparing some results to the results of the Monte Carlo simulations as well as results obtained for an equivalent linear system. The latter makes sense only for "weak" nonlinearities, i.e., for small values of the control parameter r ($r \ll 1$). First, a system with dry friction is studied; its asymptotic analysis has been made in Ref. [16] with respect to the system's energy. The other three SDOF systems under consideration are systems with parametric control of their parameters. They appear as a result of application of bounded in magnitude control forces, applied consequently to the system's stiffness, inertia, or by varying a pendulum's length (swings) [18–20]. Although the idea of controlling a system's response by changing its parameters is far from being new (see examples in Ref. [21]), the proposed strategy leads to control forces of the signum type. It makes these systems *strongly nonlinear* and their analysis highly complicated, especially for large "amplitude" of jumps at switching (values of r close to unity). Since the available asymptotic techniques provide reliable estimates for nonlinear systems only in the case of small nonlinearities, it was decided to conduct a numerical investigation, comparing some obtained results to the reliability results for an equivalent linear system. The latter is constructed using values of an equivalent viscous damping coefficient and effective frequency. The path integration

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method has been used earlier for these systems to estimate the stationary response probability density function (PDF) of the state space variables [22]. Here the PI method has been adapted to obtain the reliability characteristics of the considered systems. The approach is based on previous work reported in Ref. [23].

2 Problem Statement and Numerical Approach

2.1 Path Integration Approach to Reliability. The SDOF dynamic systems to be investigated in this paper can all be written in the following form:

$$\ddot{X} + g(X,\dot{X}) = \xi(t) \tag{1}$$

where $g(\cdot,\cdot)$ denotes a function to be specified in each particular case, while $\xi(t)$ throughout denotes a zero-mean stationary Gaussian white noise process satisfying $E[\xi(t)\xi(t+\tau)]=D\delta(\tau)$ for a positive intensity parameter D. The application of the external quasioptimal control policy leads to a dry friction law, whereas the application of the quasioptimal control force results in parametrically controlled systems with a jumpwise variation of either the system's stiffness, moment of inertia, or both. The latter happens through a variation of the pendulum's length and such a system is well known as a swing.

Equation (1) will be construed as an Itô stochastic differential equation (SDE), that is,

$$dZ(t) = h(Z(t))dt + bdB(t) \tag{2}$$

where the state space vector process $Z(t)=(X(t),Y(t))^T=(X(t),\dot{X}(t))^T$ has been introduced, $h=(h_1,h_2)^T$ with $h_1(Z)=Y$ and $h_2(Z)=-g(X,Y)$, $b=(0,\sqrt{D})^T$, and $B(t)$ denotes a standard Brownian motion process. From Eq. (2) it follows immediately that $Z(t)$ is a Markov process, and it is precisely the Markov property that will be used in the formulation of the PI procedure.

The reliability is defined in terms of the displacement response process $X(t)$ in the following manner, assuming that all events are well defined:

$$R(T|x_0,0,t_0) = \text{Prob}\{x_l < X(t) < x_c; t_0 < t \le T | X(t_0) = x_0, Y(t_0) = 0\} \tag{3}$$

where x_l, x_c are the lower and upper threshold levels defining the safe domain of operation. Hence, the reliability $R(T|x_0,0,t_0)$, as we have defined it here, is the probability that the system response $X(t)$ stays above the threshold x_l and below the threshold x_c throughout the time interval (t_0,T) given that it starts at time t_0 from x_0 with zero velocity $(x_l<x_0<x_c)$. In general, it is impossible to calculate the reliability exactly as it has been specified here since it is defined by its state in continuous time, and for most systems the reliability has to be calculated numerically, which inevitably will introduce a discretization of the time. Assuming that the realizations of the response process $X(t)$ are piecewise differentiable with bounded slope with probability one, the following approximation is introduced:

$$R(T|x_0,0,t_0) \approx \text{Prob}\{x_l < X(t_j) < x_c, j = 1, \ldots, n | X(t_0) = x_0, Y(t_0) = 0\} \tag{4}$$

where $t_j=t_0+j\Delta t$, $j=1,\ldots,n$, and $\Delta t=(T-t_0)/n$. With the assumptions made, the right hand side (rhs) of this equation can be made to approximate the reliability as closely as desired by appropriately choosing Δt, or equivalently n. Within the adopted approximation, it is realized that the reliability can now be expressed in terms of the joint conditional PDF

$$f_{X(t_1)\cdots X(t_n)|X(t_0),Y(t_0)}(\cdot, \ldots, \cdot | x_0, 0)$$

as follows, which is just a rephrasing of Eq. (4):

$$R(T|x_0,0,t_0) \approx \int_{x_l}^{x_c} \cdots \int_{x_l}^{x_c} f_{X(t_1)\cdots X(t_n)|X(t_0)Y(t_0)} \times (x_1, \ldots, x_n | x_0, 0) dx_1 \cdots dx_n \tag{5}$$

Due to the Markov property of the state space vector process $Z(t)=(X(t),Y(t))^T$, we may express the joint PDF of $Z(t_1),\ldots,Z(t_n)$ in terms of the transition probability density function

$$p(z,t|z',t') = f_{Z(t)|Z(t')}(z|z') = f_{Z(t)Z(t')}(z,z')/f_{Z(t')}(z'), \quad (f_{Z(t')}(z') \neq 0)$$

in the following way:

$$f_{Z(t_1)\ldots Z(t_n)|Z(t_0)}(z_1, \ldots, z_n|z_0) = \prod_{j=1}^{n} p(z_j,t_j|z_{j-1},t_{j-1}) \tag{6}$$

This leads to the expression $(z_0=(x_0,0)^T$, $dz_j=dx_jdy_j$, $j=1,\ldots,n)$.

$$R(T|x_0,0,t_0) \approx \int_{-\infty}^{\infty}\int_{x_l}^{x_c} \cdots \int_{-\infty}^{\infty}\int_{x_l}^{x_c} \prod_{j=1}^{n} p(z_j,t_j|z_{j-1},t_{j-1}) dz_1 \cdots dz_n \tag{7}$$

which is the path integration formulation of the reliability problem. The numerical calculation of the reliability is done iteratively in an entirely analogous way as in standard path integration. To show that, let us introduce a reliability density function (RDF) $q(z,t|z_0,t_0)$ as follows:

$$q(z_2,t_2|z_0,t_0) = \int_{-\infty}^{\infty}\int_{x_l}^{x_c} p(z_2,t_2|z_1,t_1)p(z_1,t_1|z_0,t_0)dz_1 \tag{8}$$

and $(n>2)$,

$$q(z_k,t_k|z_0,t_0) = \int_{-\infty}^{\infty}\int_{x_l}^{x_c} p(z_k,t_k|z_{k-1},t_{k-1})q(z_{k-1},t_{k-1}|z_0,t_0)dz_{k-1}, k = 3, \ldots, n \tag{9}$$

The reliability is then finally calculated approximately as $(T=t_n)$.

$$R(T|x_0,0,t_0) \approx \int_{-\infty}^{\infty}\int_{x_l}^{x_c} q(z_n,t_n|z_0,t_0)dz_n \tag{10}$$

The complementary probability distribution of the time to failure T_e, i.e., the first passage time, is given by the reliability function. The mean time to failure $\langle T_e \rangle$ can thus be calculated by the equation

$$\langle T_e \rangle = \int_0^{\infty} R(\tau|x_0,0,t_0)d\tau \tag{11}$$

To evaluate the reliability function, it is required to know the transition probability density function $p(z,t|z',t')$, which is unknown for the considered nonlinear systems. However, from Eq. (2), it is seen that for a small $t-t'$ it can be determined approximately, which is what is needed for the numerical calculation of the reliability. A detailed discussion of this and the iterative integrations of Eqs. (8) and (9) are given in Refs. [22,24]. Concerning the integrations, there is, however, one small difference between the present formulation and that described in these references. In Eqs. (8) and (9), the integration in the x-variable only extends over the interval (x_l,x_c). The infinite upper and lower limits on the y-variable are replaced by suitable constants determined by, e.g., an initial Monte Carlo simulation (MCS).

If the system response $Z(t)$ has a stationary response PDF $f_Z(z)$ as $t\to\infty$, it follows that the conditional response PDF

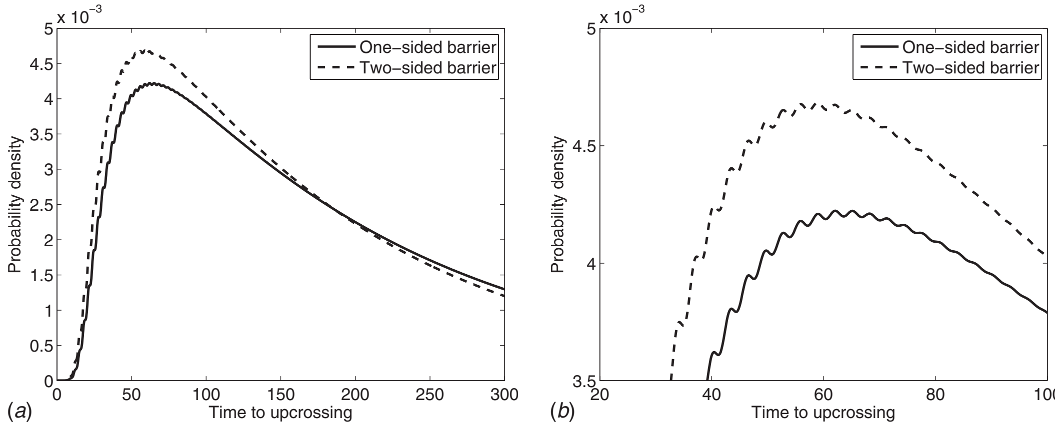

Fig. 1 Probability density of time to failure for the dry friction problem with reliability level 2.5 standard deviations and r=0.15 for the one- and two-sided cases

$f_{\{Z(t_n)|Z(0),x_l<X(t_j)<x_c;0\le j\le n-1\}}(z)$ will also reach a stationary density, say, $q^*(z)$, when $t_n \to \infty$, provided the system has finite memory. This means that the reliability process eventually becomes memoryless, and hence the RDF converges $q(z,t_n|z_0,t_0)\to q^*(z)Ke^{-\nu t_n}$ for some constants K and ν as $t_n\to\infty$. Also the numerical method should reach stationarity in the conditional density. This also implies that the numerically estimated reliability function must be exponential, since the same relative amount of probability mass leaves the system at every iteration. Hence in the end, the only thing one should need for a good reliability estimate is the behavior in the transient phase and the exponential decay thereafter.

2.2 General Comments About the Numerical Procedure. The numerical calculations were performed for a 256×256 mesh in the state space, with very high grid resolution around the axes for the inertia controlled system and swing system, because the PDFs have discontinuities along the axes and high spikes at the discontinuity that requires a well adapted spline representation [22]. More specifically, the grid resolution was determined by an exponentially decaying function away from each coordinate axis. Because of the discontinuities, there are no grid points on the axes themselves. However, the interpolant will be globally smooth and will assume finite values also on the axes. Hence, there is no true discontinuity in the interpolant even if the gradients of the interpolant may be very large at the axes. The time step was 0.01 for all simulations, and the noise intensity D was set to 1.0. The initial choice of time step is determined by the characteristic time constants of the dynamic system, which can be either seen from the system equations or from a short Monte Carlo simulation of the dynamic response of the system. As is typically done for verifying the convergence of numerical solutions, the accuracy of the calculated PI solution may be checked by changing repeatedly, if required, the size of the time step, for example, by a factor 2.

For all simulations, the reliability was computed using the barriers $x_c=2.5\sigma_x$, $x_c=3.0\sigma_x$, and $x_c=3.5\sigma_x$. The lower barrier is either $x_l=-\infty$, one-sided barrier case, or $x_l=-x_c$ for two-sided reliability. These bounds were far enough out in the tails that interpolation of the RDF from Eqs. (8) and (9) was no problem.

It should be mentioned that for all the systems studied in this paper, the calculated reliability function displayed a distinctive exponential behavior asymptotically, as one would expect. That is, after some transient time, the reliability function could not be distinguished from a straight line when plotted on a logarithmic scale. In addition, the PDF for the time to failure has a right tail that is exponential with the same exponent, which again is verified by plotting the PDF on a logarithmic scale. The oscillatory behavior of the PDFs of the time to failure, as seen on the close ups, largely reflects the transient dynamics of the systems due to initial conditions (see Fig. 1, for instance).

3 Monte Carlo Simulation

To check the numerical results, MCSs have been run for a few selected cases. The main problem is that the probability of crossing a high reliability level is small, so the simulation will have to run for a long time before this happens. Since a good approximation of the PDF for the first passage time needs a large number of Monte Carlo simulations, this easily becomes a very time consuming method. The verification of the numerical results by Monte Carlo simulations is therefore carried out on two levels. First, the expected first passage time is estimated directly from simulated response time histories for the lowest level $(=2.5\sigma)$, where σ equals the standard deviation of stationary response. For all the models investigated in this paper, the estimated expected first passage time obtained by MCS agreed with the corresponding one calculated by PI within the accuracy of the MCS estimate, that is, within a few percent.

The second method of verification was based on the observation that the reliability function and the PDF decays exponentially after a transient time. A focus on the estimation of the rate of decay reduces the number of required simulations considerably. That is, the main statistic to estimate from the stochastic upcrossing time T is ν, given the formula

$$P(T>t|T>t_{\mathrm{tr}})=e^{-\nu(t-t_{\mathrm{tr}})},\quad t>t_{\mathrm{tr}} \tag{12}$$

where t_{tr} stands for the *transient* time. Equation (12) is an approximation, since the transient never dies out completely. However, the equation is asymptotically correct as $t_{\mathrm{tr}}\to\infty$, and numerically valid for a transient time chosen sufficiently large. An adjusted maximum likelihood estimator (MLE) for ν, which is also unbiased for a fixed transient time t_{tr}, is

$$\hat{\nu}=\frac{n-1}{\sum_{i=1}^{n}(T_i-t_{\mathrm{tr}})} \tag{13}$$

for n independent upcrossing times T_i that are all larger than t_{tr}. This means that some Monte Carlo simulations with exit time shorter than the transient time will be discarded, but as the probability of exiting that early from a start at the origin is small, most results will be used.

It is important to note that estimating the full PDF, and here especially the transient behavior, is very time consuming with the Monte Carlo methods without a parametric model. Path integration, however, calculates this directly, and if only the transient behavior is needed, the PDF can be found with high accuracy with a fairly short simulation.

When comparing the results for the MC and the PI methods, remember that the strengths and weaknesses of the numerical

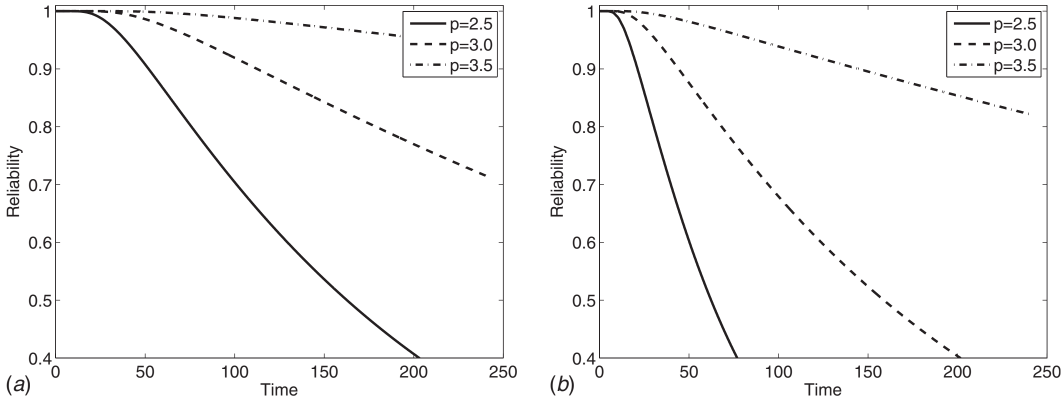

Fig. 2 One-sided reliability function of the dry friction system for different levels of *p*

methods are also very different. The main problem for the PI method is that the PDFs have sharp discontinuities or peaks that make the interpolation difficult.

The exponent can, however, be calculated between two time steps from long after the transient has died out.

The MC method relies on the parametric representation of the PDF after an estimated transient time, which is an approximation. The point estimates of parameter from MC are here calculated from 200 random samples and hence have a variance.

Since $T_i - t_{\mathrm{tr}}$ is assumed to be exponentially distributed, $\hat{\nu}/(n-1)$ has an inverse gamma distribution, and the variance is

$$\mathrm{Var}(\hat{\nu}) = \frac{\nu^2}{(n-2)} \tag{14}$$

Hence, instead of estimating the standard deviation of the test statistic $\hat{\nu}$ for the MC simulation, Eq. (14) gives that it is approximately $\hat{\nu}/\sqrt{198} = 0.07\hat{\nu}$.

The MC simulations were performed for all four systems, and the estimated values of ν from the PI calculations were checked against the 90% confidence intervals based on the MC results, and they were all accepted.

4 Results for a Stochastic Dry Friction System

In this section, some derivations made in Ref. [16] for a stochastic system with dry friction are recalled. It is worth mentioning that for the parametric systems the stochastic averaging procedure results in an exponential response PDF for response energy, whereas the dry friction system has an exponent in power of the square root of the response energy. Therefore, for the parametrically controlled systems, the case of small nonlinearity cannot be caught by the averaging procedure and needs to be investigated numerically. For the system with dry friction, it is possible to use an approximate analysis for a small value of the dry friction coefficient. Early results on the use of PI for an oscillator with dry friction are reported in Ref. [25].

Consider the following nonlinear system, subjected to the zero-mean, stationary Gaussian white noise $\xi(t)$ introduced above:

$$\ddot{X} + r\,\mathrm{sign}(\dot{X}) + \Omega^2 X = \xi(t), \quad 0 \le t \le t_f \tag{15}$$

Applying the stochastic averaging procedure and following the derivations made in Ref. [16] the first passage time may be found as

$$T(c) = \frac{[\mathrm{Ei}(2\lambda\sqrt{\bar{c}}) - \mathrm{Ei}(2\lambda\sqrt{c})]}{2\Omega\lambda^2} - \frac{\sqrt{\bar{c}} - \sqrt{c}}{\Omega\lambda} - \frac{\ln(\bar{c}/c)}{4\Omega\lambda^2}$$
$$c = \frac{E}{D/4\Omega}, \quad \bar{c} = \frac{\bar{E}}{D/4\Omega}, \quad \lambda = \frac{2\sqrt{2}\mu}{\pi}, \quad \mu = \frac{r}{\sqrt{D\Omega}} \tag{16}$$

where $\mathrm{Ei}(y)$ is the exponential integral function, D is the noise intensity, and $\bar{E}$ is the critical value of energy. Thus, an analytical expression (16) may be used for reliability estimates, keeping in mind that r should be small. This result may be compared with one, reported in Ref. [2], keeping in mind that the value of an equivalent viscous damping coefficient is equal to

$$\alpha_{\mathrm{eq}}^{df} = \frac{16r^2}{3\pi^2 D}$$

It can be seen from the comparison with the result for the linear system [2] that Eq. (16) has an additional second term, which is non-negative. Moreover, the exponential integral function (16) depends on the square root of the system's energy, whereas the formula for an equivalent linear system [2] predicts dependence on the system's energy itself. Both these facts indicate that the first passage time to failure for the dry friction system should be less than that for an equivalent linear system.

Numerical simulations, conducted using the PI method, have shown that the joint response PDF has a single peak, at small values of r, which splits into two peaks, moving away from each other, when the nonlinearity parameter r increases. A peak of the probability density of time to failure moves left when the value of r increases, which indicates deterioration of the system's reliability. Figure 1 demonstrates the results of a numerical simulation for one- and two-sided probability densities of time to failure. It can be seen from Fig. 1(a) that both densities have similar shapes and

Table 1 Expected time to upcrossing for the dry friction system. All numbers to be $\times 10^3$.

p / r	2.5σ One-sided	2.5σ Two-sided	3.0σ One-sided	3.0σ Two-sided	3.5σ One-sided	3.5σ Two-sided
0.15	0.2167	0.1947	0.6009	0.5344	2.6885	2.1427
0.20	0.1344	0.1170	0.3697	0.3166	1.7757	1.2995
0.25	0.0814	0.0689	0.2169	0.1791	1.0834	0.7344

Table 2 Linear system with a damping coefficient α: expected time to upcrossing. All numbers to be $\times 10^3$.

p α	2.5σ One-sided	2.5σ Two-sided	3.0σ One-sided	3.0σ Two-sided	3.5σ One-sided	3.5σ Two-sided
$16\cdot 0.15^2/(3\pi^2)$	0.5594	0.4884	1.7249	1.4670	4.7063	4.3196
$16\cdot 0.20^2/(3\pi^2)$	0.3808	0.3165	1.2338	0.9869	4.1552	3.5478
$0.10/\pi$	0.2859	0.2281	0.9370	0.7143	3.5829	2.8099
$16\cdot 0.25^2/(3\pi^2)$	0.2810	0.2225	0.9331	0.7046	3.6107	2.8128
$0.30/\pi$	0.1710	0.1149	0.6250	0.3927	2.9383	1.8043
$0.50/\pi$	0.1497	0.0909	0.5743	0.3265	2.8247	1.5878
$0.90/\pi$	0.1376	0.0749	0.5418	0.2831	2.6838	1.4232

are almost identical (except the peak value), which was expected, since the considered problem is symmetric. Similar behavior has been observed for other problems investigated here. Thus, figures for one- and two-sided probability densities of time to failure are not presented for parametric systems. Figure 2 presents the reliability function for $r=0.15$ (a) and $r=0.25$ (b) for different values of the crossing level $p=x_c/\sigma_x$. These results show strong dependence of the reliability function on r, i.e., an increase in r increases the slope of the reliability function, consequently decreasing the time to failure. At first glance, this may seem odd, but remember that an increase in r leads to a decrease in σ_x, and therefore in the critical level. On the other hand, an increase in the crossing level leads, as expected, to an increase in the first passage time value for a fixed value of r.

Table 1 presents the results of numerical simulations for the first passage time. The data in Table 1 may be compared with the data presented in the first, second, and fourth lines of Table 2. Direct comparison of these results, for the same level of energy dissipation in both systems, shows that the dry friction system has a significantly (at least twice) smaller value of failure time than that of the equivalent linear system, which indicates a relatively poor reliability of the dry friction system.

5 Results for Parametrically Controlled Systems

To verify the code and to qualitatively comprehend the new results, it is proposed to obtain numerical results for a linear system, subjected to external Gaussian white noise, in addition to the Monte Carlo method described in Sec, 3. The results, obtained by the PI method, very well agreed with the results of MC simulations. To compare these results with results for the considered nonlinear systems, for small values of nonlinearity parameter, one has to select a proper value of viscous damping coefficient. It should be reminded that the value of an equivalent viscous damping coefficient for stiffness and inertia controlled systems is $\alpha_{eq} = r\Omega/\pi$, and this value is tripled for a swing system [18,19]. Therefore, in order to compare the results it is decided to select $\alpha_{eq}=r/\pi$ for $r=0.1, 0.3, 0.5$ ($\Omega=1$). The results for the probability density function of failure time for the three largest mentioned values of α_{eq} are shown in Fig. 3. Table 2 presents the results for the mean time to failure for different values of the equivalent viscous damping coefficient, corresponding to different values of r, according to the above mentioned formulas.

5.1 System With Controlled Stiffness.

Consider a stochastic system with controlled stiffness, whose motion is governed by the following equation:

$$\ddot{X} + \Omega^2 X[1 + r\,\mathrm{sign}(X\dot{X})] = \xi(t), \quad 0 \le t \le t_f \tag{17}$$

where $0 < r < 1$. The probability densities are obtained numerically by extrapolating the probability of no upcrossing by an exponential function and differentiating this numerically. The sampling points were dense $\Delta t_{samp}=0.16$ s so that the differentiation proved to be very accurate. The transient seemed noisy, but a closer look would show that there is actually a smooth oscillation. The peak-to-peak period of this oscillation seems to coincide well with the effective period of the system [22]. An interpretation is that the system's variability has to reach a certain level before the probability of exceedance is substantially high. The first substantial removal of the high-displacement part of the probability density gradually starts to affect the exceedance probability approximately one period later. This behavior is consistent with previous observations, and the same kind of oscillations is also seen in the PI results for the equivalent linear system.

Figure 4 presents the results of the failure time PDF as a function of the nonlinearity parameter r. In Fig. 5 one can observe the reliability function for different values of the upcrossing level p,

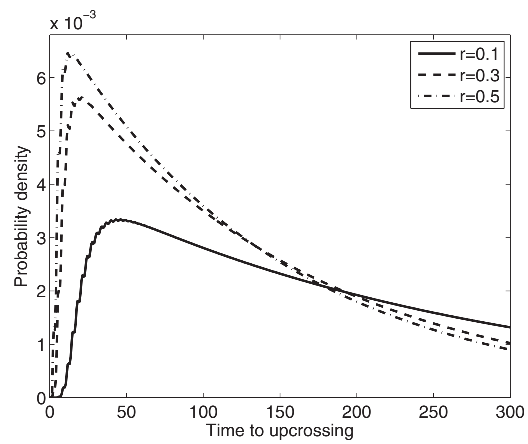

Fig. 3 Probability density of time to failure for the linear system with reliability level 2.5 standard deviations and $r=0.1$, 0.3, and 0.5

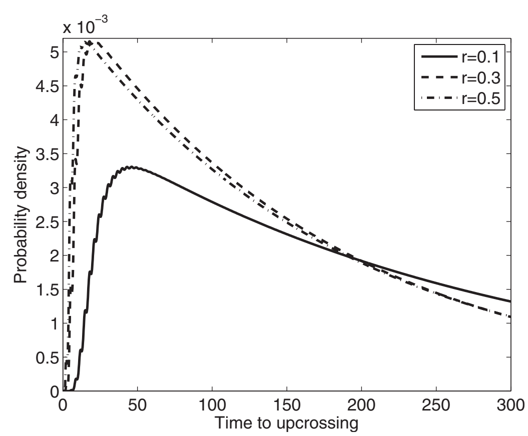

Fig. 4 Probability density of time to failure for the stiffness control problem with reliability level 2.5 standard deviations and $r=0.1, 0.3, 0.3$, and 0.5

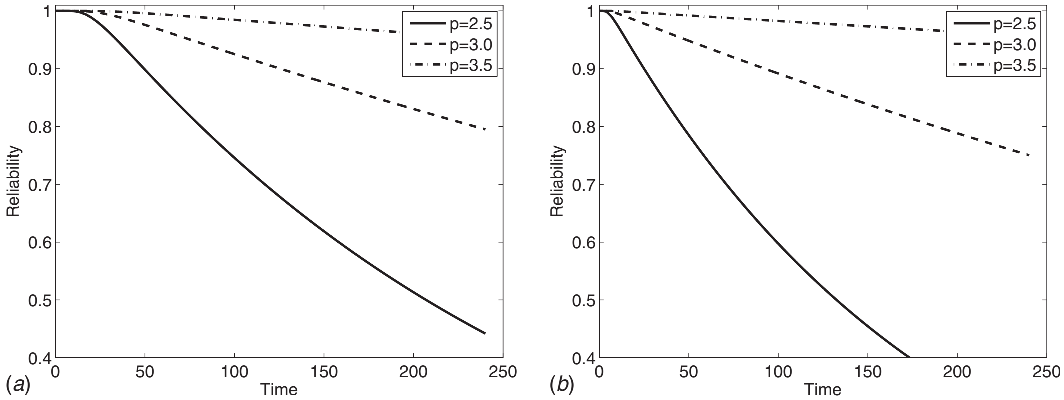

Fig. 5 Reliability function for the stiffness control problem with one-sided barrier for different levels of *p*

with $r=0.1$ (a) and $r=0.5$ (b) correspondingly. Comparing the results (a) and (b) indicates that an increase in nonlinearity influences the system's reliability much more when small values of the upcrossing level are selected. Results in Fig. 5 have been compared with results for an equivalent linear system. Direct comparison revealed that the behavior of the equivalent system, in terms of the reliability function, matches the behavior of the stiffness controlled system for small values of the nonlinearity parameter r. Meanwhile, detailed comparison for $r=0.5$ indicates that the stiffness controlled system's time to failure is bigger than that of the linear system with the same level of energy dissipation in both systems. This fact is reflected in Table 3, which shows the mean failure time for the stiffness controlled system. It should be reported that the observed behavior is different from the behavior of the two other parametric systems. Namely, it is seen, based on the numerical results, that there is no monotonic decrease in mean upcrossing time with a gradual increase in r. However, Monte Carlo simulations confirm these results for $2.5\sigma_x$.

5.2 System With Controlled Moment of Inertia. Consider a system with controlled moment of inertia:

$$\frac{d}{dt}\{[1 + r\,\mathrm{sign}(\phi\dot{\phi})]\dot{\phi}\} + \Omega^2\phi = \xi(t), \quad 0 \le t \le t_f \qquad (18)$$

where $0<r<1$. The exponential behavior for the absorbing probability density has been observed in this case. Figure 6 presents results for the probability density function of failure time for $p=2.5$ and different values of the nonlinearity parameter. One can clearly see a trend in the peak shift to the left with the increase of r. Figure 7 illustrates results for the reliability function for $r=0.1$ (a) and $r=0.5$ (b), respectively. Comparison with the results obtained for the stiffness controlled and linear system shows that the reliability function of the inertia controlled system has a less steep angle, which indicates that this system is "more" reliable. This fact is reflected in Table 4, where for small values of r one can find that $T_{\mathrm{in}} \ge T_{\mathrm{sc}} \approx T_{\mathrm{lin}}$, whereas for large values of r one obtains $T_{\mathrm{lin}} \le T_{\mathrm{in}} \le T_{\mathrm{sc}}$.

5.3 Swings. A governing equation of motion of a mathematical pendulum with controlled length or swings may be written as

$$\frac{d}{dt}(L^2\dot{\phi}) + \Omega^2 L\sin(\phi) = \xi(t), \quad 0 \le t \le t_f$$
$$L = [1 + r\,\mathrm{sign}(\phi\dot{\phi})], \quad \Omega^2 = g/L_0 \qquad (19)$$

For small values of ϕ the nonlinear term in Eq. (19) is changed to $\sin(\phi)\sim\phi$, thereby giving the linearized equation of a swing. The smooth oscillatory behavior of the PDF with a frequency close to its natural frequency has been observed. Figure 8 demonstrates the results of the numerical simulation for the PDF of failure time for different values of the nonlinearity parameter r. All peaks are shifted to the left compared with the peaks for the other systems investigated above. In Fig. 9 the numerically estimated reliability functions for $r=0.1$ (a) and $r=0.3$ (b) are presented. Since the equivalent damping coefficient for the linearized system (19) is three times bigger, the result (a) should be compared with the one obtained for the linear system with $r=0.3$. It should be reported that the reliability function of the swings has smaller decay rate, which results in a larger value of mean failure time. The latter can be observed from Table 5 for the corresponding values of the nonlinearity parameter.

6 Conclusions

In this paper, the authors have considered a first passage type reliability problem for strongly nonlinear stochastic systems, i.e., systems with signum type nonlinearity. The numerical results presented in this paper are obtained by the path integration method, which was adjusted to handle reliability problems. The results were verified by Monte Carlo simulations and the results obtained by the path integration method for an equivalent linear system. Generated results demonstrated that the reliability of all the considered systems strongly depends on the nonlinearity parameter r, especially for low values of the upcrossing level. It also has been shown that the systems with parametrically changing parameters have longer mean time to failure than those of equivalent linear systems. Thus, the parametrically controlled strongly nonlinear systems not only provide a way to dissipate the system's response energy, but also improve their first passage time reliability. On the other hand, the dry friction system or the system with an external

Table 3 Stiffness control: expected time to upcrossing. All numbers to be $\times 10^3$.

p r	2.5σ One-sided	2.5σ Two-sided	3.0σ One-sided	3.0σ Two-sided	3.5σ One-sided	3.5σ Two-sided
0.1	0.2889	0.2304	0.9513	0.7248	3.6376	2.8612
0.3	0.1865	0.1248	0.7151	0.4453	3.4306	2.1786
0.5	0.1886	0.1130	0.8181	0.4557	4.0836	2.6258

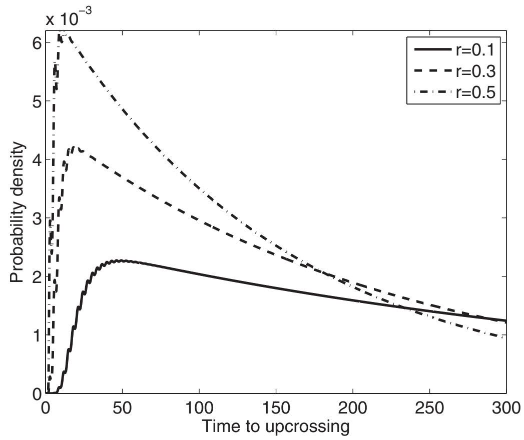

Fig. 6 Probability density of time to failure for the inertia control problem with reliability level 2.5 standard deviations and r =0.1, 0.3, and 0.5

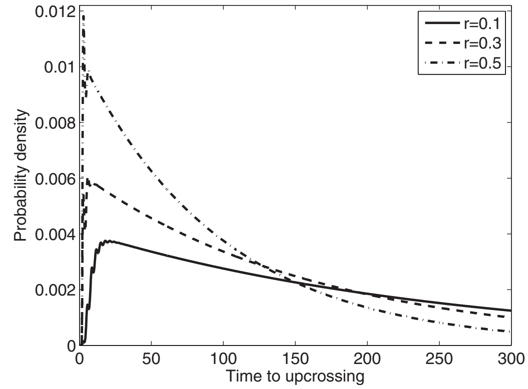

Fig. 8 Probability density of time to failure for the swing problem with reliability level 2.5 standard deviations and r=0.1, 0.3, and 0.5

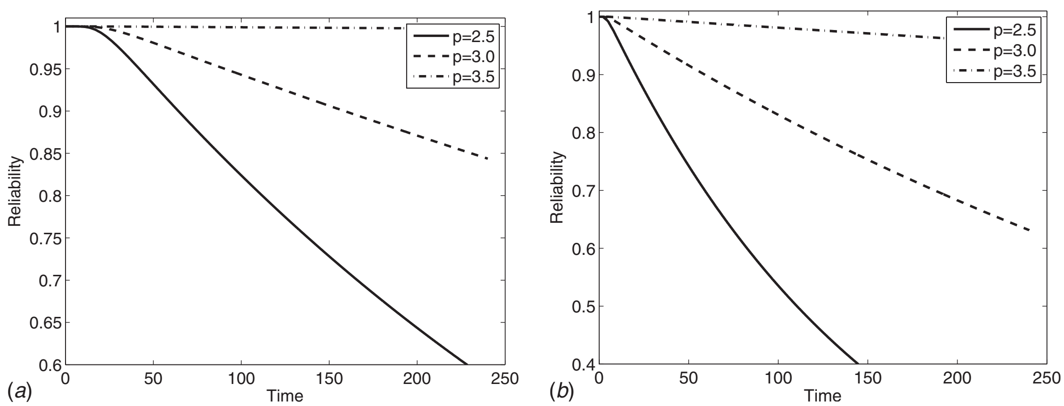

Fig. 7 Reliability function for the inertia control problem with one-sided barrier for different levels of p

Table 4 Inertia control: expected time to upcrossing. All numbers to be $\times 10^3$.

p r	2.5σ One-sided	2.5σ Two-sided	3.0σ One-sided	3.0σ Two-sided	3.5σ One-sided	3.5σ Two-sided
0.1	0.4261	0.3261	1.2842	0.8960	5.3885	5.2652
0.3	0.2323	0.1463	0.7717	0.4408	4.8281	4.2013
0.5	0.1574	0.0901	0.5158	0.2766	3.6161	2.3682

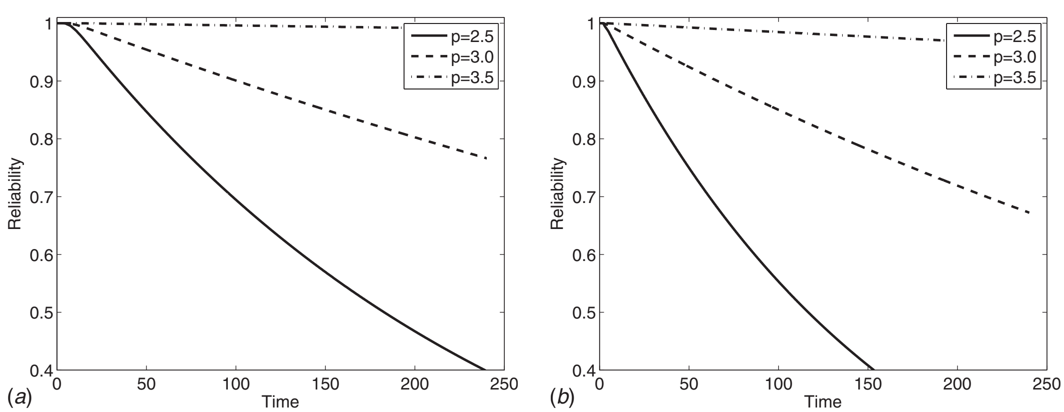

Fig. 9 Reliability function for the swing problem with one-sided barrier for different levels of p

Table 5 Linearized swing: expected time to upcrossing. All numbers to be $\times 10^3$.

p / r	2.5σ One-sided	2.5σ Two-sided	3.0σ One-sided	3.0σ Two-sided	3.5σ One-sided	3.5σ Two-sided
0.1	0.2601	0.1631	0.8759	0.4968	5.0722	4.6536
0.3	0.1671	0.0885	0.5999	0.3072	3.9651	2.7934
0.5	0.1004	0.0514	0.3256	0.1645	1.7827	0.9058

bounded in magnitude control law has poor reliability compared with its equivalent linear system, although it is capable of reducing the system's response energy.

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References

[1] Dimentberg, M. F., 1988, *Statistical Dynamics of Nonlinear and Time-Varying Systems*, Research Studies, Taunton, UK.
[2] Lin, Y. K., and Cai, G. Q., 1995, *Probabilistic Structural Dynamics*, McGraw-Hill, New York.
[3] Roberts, J. B., and Spanos, P. D., 1990, *Random Vibration and Statistical Linearization*, Wiley, New York.
[4] Roberts, J. B., 1976, "First Passage Time for the Envelope of a Randomly Excited Linear Oscillator," J. Sound Vib., **46**(1), pp. 1–14.
[5] Roberts, J. B., 1978, "First Passage Time for Oscillators With Non-Linear Restoring Forces," J. Sound Vib., **56**(1), pp. 71–86.
[6] Roberts, J. B., 1978, "First Passage Time for Oscillators With Nonlinear Damping," ASME J. Appl. Mech., **45**, pp. 175–180.
[7] Bergman, L. A., and Heinrich, J. C., 1982, "On the Reliability of the Linear Oscillator and Systems of Coupled Oscillators," Int. J. Numer. Methods Eng., **18**, pp. 1271–1295.
[8] Spencer, B. F., Jr., and Bergman, L. A., 1985, "On the Reliability of a Simple Hysteretic System," J. Eng. Mech., **111**(12), pp. 1502–1513.
[9] Bergman, L. A., and Spencer, B. F., Jr., 1987, "First Passage Time for Linear Systems With Stochastic Coefficients," Probab. Eng. Mech., **2**(1), pp. 46–53.
[10] Koyluoglu, H. U., Nielsen, S. R. K., and Iwankiewicz, R., 1994, "Reliability of Non-Linear Oscillators Subjected to Poisson Driven Impulses," J. Sound Vib., **176**(1), pp. 19–33.
[11] Chen, J. B., and Li, J., 2005, "Dynamic Response and Reliability Analysis of Non-Linear Stochastic Structures," Probab. Eng. Mech., **20**, pp. 33–44.
[12] Gupta, S., and Manohar, C. S., 2004, "Improved Response Surface Method for Time-Varying Reliability Analysis of Nonlinear Random Structures Under Non-Stationary Excitations," Nonlinear Dyn., **36**, pp. 267–280.
[13] Zhu, W. Q., Deng, M. L., and Huang, Z. L., 2002, "First-Passage Failure of Quasi-Integrable Hamiltonian Systems," ASME J. Appl. Mech., **69**, pp. 274–282.
[14] Fujimura, K., and Kiureghian, A., 2007, "Tail-Equivalent Linearization Method for Nonlinear Random Vibration," Probab. Eng. Mech., **22**, pp. 63–76.
[15] Iourtchenko, D. V., 2000, "Stochastic Optimal Bounded Control for a System With the Boltz Cost Function," J. Vib. Control, **6**, pp. 1195–1204.
[16] Dimentberg, M. F., Iourtchenko, D. V., and Bratus, A., 2000, "Optimal Bounded Control of Steady-State Random Vibrations," Probab. Eng. Mech., **15**, pp. 381–386.
[17] Risken, H., 1989, *The Fokker-Plank Equation: Methods of Solutions and Applications*, Springer-Verlag, New York.
[18] Iourtchenko D. V., and Dimentberg M. F., 2001, "Energy Balance for Random Vibrations of Piecewise-Conservative Systems," J. Sound Vib., **248**(5), pp. 913–923.
[19] Dimentberg, M. F., and Bratus, A., 2000, "Bounded Parametric Control of Random Vibrations," Proc. R. Soc. London, Ser. A, **456**, pp. 2351–2363.
[20] Iourtchenko, D. V., 2006, "Random Vibrations of Swings," J. Sound Vib., **295**, pp. 1011–1014.
[21] Panovko, Ia. G., and Gubanova, I. I., 1965, *Stability and Oscillations of Elastic Systems: Paradoxes, Fallacies, and New Concepts*, Consultants Bureau, New York.
[22] Iourtchenko, D. V., Mo, E., and Naess, A., 2006, "Response Probability Density Functions of Strongly Nonlinear Systems by the Path Integration Method," Int. J. Non-Linear Mech., **41**(5), pp. 693–705.
[23] Johnsen, J. M., 1992, "Response Statistics of Nonlinear Dynamic Systems," Ph.D. thesis, Department of Structural Engineering, Norwegian Institute of Technology, Trondheim.
[24] Naess, A., Kolnes, F. E., and Mo, E., 2007, "Stochastic Spur Gear Dynamics by Numerical Path Integration," J. Sound Vib., **302**, pp. 936–950.
[25] Naess, A., and Johnsen, J. M., 1993, "The Effect of Coulomb Damping on the Response Statistics of a SDOF Oscillator Subjected to White Noise Excitation," *Proceedings of the Second European Conference on Structural Dynamics (Eurodyn '93)*, Trondheim, Norway, Jun.