ON THE *SM* -OPERATORS

Soeparna Darmawijaya[1], Muslim Ansori[2], dan Supama[3]

[1,3]Mathematics Departement, FMIPA UGM , Yogyakarta
email: maspomo@yahoo.com

[2]Mathematics Departement, FMIPA Universitas Lampung
Jln. Soematri Brodjonegoro No 1 Bandar Lampung
email: ansomath@yahoo.com

ABSTRACT

This is a partial part of our results in studying generalization of Hilbert-Schmidt and Carleman operators in Banach spaces. This problem can be done if we preserve some intrinsic properties of Hilbert spaces involved; for examples, reflexivity and separability. The result of the generalization of Hilbert-Schmidt operator will be called *SM*-operator. Infact, almost all of properties of the *SM*-operator preserve almost all of properties of the Hilbert-Schmidt operators. The application on some classical Banach spaces will appear in the next publications.

Keywords: Orthonormal Schauder bases,separable and reflexive Banach spaces, Hilbert-Schmidt operator

Makalah diterima 17 September 2005

1. INTRODUCTION

One of the most important classes of bounded operators is the class of Hilbert-Schmidt operators. Let H_1 and H_2 be Hilbert spaces. A bounded operator $A: H_1 \to H_2$ is called a Hilbert-Schmidt operator if there exists an orthonormal bases $\{e_n\}$ of H_1 such that

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty.$$

This definition implies that $A: H_1 \to H_2$ is a Hilbert-Schmidt operator if and only if $A^*: H_2 \to H_1$ is a Hilbert-Schmidt operator ; in this case

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 = \sum_{n=1}^{\infty} \|A^*d_n\|^2,$$

for every orthonormal bases $\{e_n\}$ of H_1 and $\{d_n\}$ of H_2 . Now, question arises, whether such an operator can be developed in Banach spaces. The answer is positive whenever we preserve some instrinsic properties of the two Hilbert spaces,i.e. reflexivity and separability. The separability of Banach space X is to guarantee the existence of countable bases of a Banach space X and the reflexivity of a Banach space X is to guarantee that the bases of X is shrinking (Zippin,1968). Further, Johnson.,et al.,(1971) pointed out that the existence of bases in the dual X^* does imply that also X has a bases , see also (Dapa,2000;Morrisson,2001). More precisely, if a separable and reflexive Banach space X has a shrinking bases, so does the dual space X^*. For example, ℓ_p, $1 < p < \infty$, has a bases (Schauder bases) but ℓ_∞ has not.

2. PRELIMINARY

In what follows we shall always assume that the Banach spaces X, Y and Z are reflexive and separable normed space. Let X^* be the dual space of $(X, \| \|)$ that is the collection of all continuously linear functionals on X. We always write $\langle x, x^* \rangle$ to stand for $x^*(x)$ and vice versa, for every $x \in X$ and $x^* \in X^*$.

A sequence of linearly independent vectors $\{e_n\} \subset X$ is called a Schauder bases of X if for every vector $x \in X$ there is uniquely sequence of scalars $\{a_n\}$ such that

$$x = \sum_{k=1}^{\infty} a_k e_k .$$

Further, for simplicity and some reason we assume that $\|e_n\| = 1$ for every n. We define a sequence of vector $\{e_n^*\} \subset X^*$, which is called biorthonormal system of $\{e_n\}$, as follows:

$$\langle x, e_n^* \rangle = e_n^*(x) = e_n^*\left(\sum_{k=1}^{\infty} a_k e_k \right)$$

$$= \sum_{k=1}^{\infty} a_k e_n^*(e_k) = a_n ,$$

for every $n \in N$. It is true that $e_n^* \in X^*$, for e_n^* is linear and bounded, $\langle e_k, e_n^* \rangle = 0$ for every $k \neq n$ and $\langle e_k, e_n^* \rangle = 1$ for every $k = n$. The sequence $\{e_n^*\}$ forms a bases of the closed subspace $\overline{[\{e_n^*\}]} \subset X^*$. Especially, we have $\overline{[\{e_n^*\}]} = X^*$ if and only if $\{e_n\}$ is shrinking, i.e.,

$$\lim_{n \to \infty} \sum_{k=n}^{\infty} \langle e_k, x^* \rangle e_k^* = o^* ,$$

for every $x^* \in X^*$ (Lindenstrauss and Tzafriri, 1996, Proposition 1.b.1).

Again, in what follows we shall always assume that $\{e_n\}$ and $\{d_n\}$ are orthonormal Schauder bases, or in short, OSB of X and Y, respectively. If $A \in L_c(X, X)$, where $L_c(X, Y)$ is the collection of continuously linear operators from Banach space X into Banach space Y, the operator $A^* \in L_c(Y^*, X^*)$ is called the adjoint operator of A if for any $x \in X$ and $y^* \in Y^*$, we have

$$\langle Ax, y^* \rangle = \langle x, A^* y^* \rangle .$$

Then, we have

$$\langle Ae_n, d_k^* \rangle = \langle e_n, A^* d_k^* \rangle$$

where, for every $n, k = 1, 2, ...$

$$Ae_n = \sum_{k=1}^{\infty} d_k^*(Ae_n) d_k = \sum_{k=1}^{\infty} < Ae_n, d_k^* > d_k$$

$$= \sum_{k=1}^{\infty} < e_n, A^* d_k^* > d_k .$$

It implies

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle Ae_n, d_k^* \rangle = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} < e_n, A^* d_k^* >$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} < e_n, A^* d_k^* > .$$

The dual space X^* also has a dual. It is usually denoted by X^{**}, is called the second dual of $(X, \|\ \|)$ and consists of all continuously linear functionals on X^*. For each fixed $x \in X$ define $\hat{x}(f)$ to be $f(x)$ for all f in X^*. It is clear that $\hat{x}$ is a linear functional on X^*, and since

$$|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\| ,$$

we see that $\hat{x}$ in X^{**}. Hence we can define a map f from X into X^{**} by letting $f(x) = \hat{x}$ for each x in X. Since, for any nonzero element x_0 in $(X, \|\ \|)$, then there is an element $f^* \in X^*$ such that

$$\|f^*\| = 1 \text{ and } f^*(x_0) = \|x_0\| .$$

Thus, the map is linear and $\|f(x)\| = \|x\|$ for each x in X. As a consequence, we have

$$\|f(x)\| = \sup_{\|x^*\|=1} |\langle x^*, f(x) \rangle|$$

$$= \sup_{\|x^*\|=1} |\langle x, x^* \rangle| \qquad (1.1)$$

$$= \|x\| .$$

Thus f is also an isometry and sets up a congruence between X and X^{**}. The normed space is imbedded X into X^{**} by the canonical imbedding f in a isometrically isomorfic way and $f(X) = X^{**}$. Thus X can be considered as the normed space X^{**}.

3. MAIN RESULTS

Based on the results of the last discussion we start with the following definition.

Definition 1. *An operator* $A \in L_c(X,Y)$ *is called an SM-operator from* X *into* Y*, if*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\left\langle Ae_n, d_m^*\right\rangle\right| = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\left\langle e_n, A^* d_m^*\right\rangle\right| < \infty$$

for every OSB $\{e_n\}$ *of* X *and* $\{d_m\}$ *of* Y.

It is clear that if A is an SM-operator, then the number $|||A|||$:

$$|||A||| = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\left\langle Ae_n, d_m^*\right\rangle\right|$$

is nonnegative and it does not depend on the choice of an OSB $\{e_n\}$ of X and an OSB $\{d_n\}$ of Y. Let $SM(X,Y)$ be the collection of SM-operators from a Banach space X into a Banach space Y.

By Definition 1 and (1.1), we have the following theorem.

Theorem 2. *An operator* $A \in L_c(X,Y)$ *is an SM-operator if only if* A^* *is an SM-operator, that is,* $A \in SM(X,Y)$ *if and only if* $A^* \in SM(Y^*, X^*)$*, and*

$$\begin{aligned} |||A||| &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\left\langle Ae_n, d_m^*\right\rangle\right| \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\left\langle e_n, A^* d_m^*\right\rangle\right| \\ &= |||A^*|||, \end{aligned}$$

for every OSB $\{e_n\}$ *of* X *and* $\{d_k\}$ *of* Y.

Theorem 3. *Let* $\{e_n\}$ *and* $\{d_n\}$ *be an OSB of Banach space* X *and* Y*, respectively. Then,*

(i) $\|A\| \le |||A|||$ *, for every* $A \in SM(X,Y)$,

(ii) $SM(X,Y)$ *is a Banach space with respect to* $|||.|||$,

(iii) *If* $A \in SM(X,Y)$*, then* A *is compact.*

Proof: (i) For every $x \in X$, we have

$$Ax = \sum_{k=1}^{\infty}\left\langle x, e_n^*\right\rangle Ae_n$$

and

$$\begin{aligned} \|Ax\| &\le \sum_{n=1}^{\infty}\|x\|\|Ae_n\| \\ &= \|x\| \sum_{n=1}^{\infty}\|Ae_n\| \\ &= \|x\|\sum_{n=1}^{\infty}\left\|\sum_{k=1}^{\infty} < Ae_n, d_k^* > d_k\right\| \\ &\le \|x\|\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left|\left\langle Ae_n, d_k^*\right\rangle\right|\|d_k\| \\ &= \|x\|\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left|\left\langle Ae_n, d_k^*\right\rangle\right| \\ &= \|x\|\,|||A|||, \end{aligned}$$

which implies $\|A\| \le |||A|||$.

(ii). The space $SM(X,Y)$ is a normed space with respect to the norm $|||.|||$, for:

(ii.a). $\left|\,|||A|||\,\right| = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\left\langle Ae_n, d_m^*\right\rangle\right| \ge 0$, for every $A \in SM(X,Y)$.

$$\left|\,|||A|||\,\right| = 0 \Leftrightarrow \sum_{m=1}^{\infty}\left|\left\langle Ae_n, d_m^*\right\rangle\right| = 0 \Leftrightarrow A = O$$

(null operator),

(ii.b).For every scalar a and $A \in SM(X,Y)$, we have

$$\begin{aligned} \left|\,|||a A|||\,\right| &= \sum_{m=1}^{\infty}\left|\left\langle a\, Ae_n, d_m^*\right\rangle\right| \\ &= |a|\sum_{m=1}^{\infty}\left|\left\langle Ae_n, d_m^*\right\rangle\right| = |a|\,\left|\,|||A|||\,\right|, \end{aligned}$$

and

(ii.c). For every $A, B \in SM(X,Y)$, we have

$$\begin{aligned} |||A+B||| &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\left\langle (A+B)\, e_n, d_m^*\right\rangle\right| \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\left\langle Ae_n + Be_n, d_m^*\right\rangle\right| \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\left\langle Ae_n, d_m^*\right\rangle + \left\langle Be_n, d_m^*\right\rangle\right| \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\left\langle Ae_n, d_m^*\right\rangle\right| + \left|\left\langle Be_n, d_m^*\right\rangle\right| \end{aligned}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\left\langle Ae_n, d_m^*\right\rangle\right| + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\left\langle Be_n, d_m^*\right\rangle\right|$$
$$= |||A||| + |||B|||$$

or

$$\left|\, |||A+B||| \,\right| \le \left|\, |||A||| \,\right| + \left|\, |||B||| \,\right|.$$

The proof of the completeness of the space is as follows. Let $\{A_n\} \subset SM(X,Y)$ be an arbitrary Cauchy sequence. Then, for any number $e > 0$, there is a positive integer n_0 such that for every two positive integers $m,n \ge n_0$, we have

$$\left|\, |||A_m - A_n||| \,\right| < \frac{e}{2}.$$

We want to prove that there is $A \in SM(X,Y)$ such that

$$\lim_{n\to} |||A_n - A||| = 0 .$$

Since $L_c(X,Y)$ is complete and

$$\|A_m - A_n\| \le \left|\, |||A_m - A_n||| \,\right|,$$

there is $A \in L_c(X,Y)$ such that

$$\|A - A_n\| < \frac{e}{2},$$

for every $n \ge n_0$ or

$$\lim_{n\to\infty} \|A_n - A\| = 0 .$$

Thus, we have

$$\sum_{j=1}^{s}\left\|\sum_{k=1}^{t}\left\langle (A_n - A_m)e_j, d_k^*\right\rangle d_k\right\|$$
$$\le \sum_{j=1}^{s}\sum_{k=1}^{t}\left|\left\langle (A_n - A_m)e_j, d_k^*\right\rangle\right|$$
$$\le |||A_n - A_m|||$$
$$< \frac{e}{2}$$

for any integers s,t and $m,n \ge n_0$. By letting $m \to \infty$, we have

$$\sum_{j=1}^{s}\left\|\sum_{k=1}^{t}\left\langle (A_n - A)e_j, d_k^*\right\rangle d_k\right\|$$
$$\le \sum_{j=1}^{s}\sum_{k=1}^{t}\left|\left\langle (A_n - A)e_j, d_k^*\right\rangle\right|$$
$$\le \frac{e}{2}$$

for any integers s,t and $m,n \ge n_0$. Letting $s \to \infty$ and $t \to \infty$, we have

$$\sum_{j=1}^{\infty}\left\|\sum_{k=1}^{\infty}\left\langle (A_n - A)e_j, d_k^*\right\rangle d_k\right\|$$
$$\le \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left|\left\langle (A_n - A)e_j, d_k^*\right\rangle\right| < \frac{e}{2}$$

for every $n \ge n_0$. Therefore $A_n - A \in SM(X,Y)$ and hence $A = A_n + (A - A_n)$ in $SM(X,Y)$. Moreover,

$$|||A_n - A||| < e,$$

for every $n \ge n_0$. Hence,

$$\lim_{n\to} |||A_n - A||| = 0 .$$

(iii) If $A \in SM(X,Y)$ and $x \in X$, we have

$$Ax = \sum_{k=1}^{\infty}\left\langle Ax, d_k^*\right\rangle d_k$$

and for every positive integer n, we define an operator $B_n : X \to Y$:

$$B_n x = \sum_{k=1}^{n}\left\langle Ax, d_k^*\right\rangle d_k .$$

It is clear that $B_n \in L_c(X,Y)$, B_n is a finite rank operator, and

$$\lim_{n\to\infty} \|A - B_n\| = 0 .$$

Therefore, A is a compact operator.

Theorem 4. *Let X,Y and Z be Banach spaces. If $A \in SM(X,Y)$ and $B \in L_c(Y,Z)$, then $BA \in SM(X,Z)$ and $|||BA||| \le \|B\|\, |||A|||$.*

Proof: For every OSB $\{e_n\}$ of X, $\{d_m\} \subset Y$ and $\{f_j\} \subset Z$, we have

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\left|\left\langle BAe_n, f_j^*\right\rangle\right| \le \sum_{n=1}^{\infty}\|B\|\|Ae_n\|\|f_j^*\|$$
$$\le \|B\|\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\left\langle Ae_n, d_m^*\right\rangle\right|$$
$$= \|B\|\, |||A|||,$$

that is $BA \in SM(X,Z)$ and

$$|||BA||| \le \|B\|\, |||A|||.$$

Let $SM(X)$ and $L_c(X)$ stand for $SM(X,X)$ and $L_c(X,X)$, respectively. Combining the results of Theorem 3 and Theorem 4, we have proved that $SM(X)$ is $*-$algebra, where $*$ is an involution from $SM(X)$ into $SM(X)$ satisfying :

$$\left(A^*\right)^* = A\,;\,\left(AB\right)^* = B^*A^*$$

and

$$\left(a\,A+B\right)^* = a\,A^* + B^*$$

for every $A,B \in SM\left(X\right)$ and a real scalar a, as stated in the following theorem.

Theorem 5. *Let* X *be a Banach space having a shrinking* OSB. *Then,* $SM(X)$ *is a Banach* $*-$ *algebra and an ideal of* $L_c(X)$.

CONCLUSION

Generalization of Hilbert-Schmidt operators into Banach spaces can be done by preserving the instrinsic properties of Hilbert spaces, i.e., separable and reflexivity. The results, denoted by *SM*(*X*,*Y*) has in general the same properties of those of Hilbert-Schmidt operators.

The biorthonormal system

$$\left\{\left(\{e_n\},\{e_n^*\}\right):\{e_n\}\subset X,\{e_n^*\}\subset X^*\right\}$$

and

$$\left\{\left(\{d_m\},\{d_m^*\}\right):\{d_m\}\subset Y,\{d_m^*\}\subset Y^*\right\}$$

is the key to solve the condition of orthonormality in Hilbert-Schmidt operators, used later in $SM\left(X,Y\right)$. For further works, we have been using the operator in classical Banach spaces L_p and $\ell_p, 1 < p < \infty$.

ACKNOWLEDGEMENTS

The second author was partially supported by BPPS DIKTI. We also thank all the anonymous referees for reading the paper carefully.

REFERENCES

Conway, J.B., 1990. *A Course in Functional Analysis*, Springer Verlag, New York.

Dapa P.S., 2000. *On strong M-bases in Banach spaces with PRI*, Collect. Math. **51**, 3, 277-284.

Johnson W.B.,Roshental,H.P., and Zippin,M., 1971. *On bases, finite-dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. **9**, 488-506.

Lindenstrauss J and Tzafriri L, 1996. *Classical Banach Spaces I and II*, Springer Verlag, New York.

Morrison, T.J.,2001. *Functional Analysis*: *An Introduction to Banach Space Theory*, John Wiley & Sons. Inc. New York.

Weidmann, J., 1980. *Linear Operators in Hilbert Spaces*, Springer Verlag. New York.

Zippin, M., 1968. *A remark on bases and reflexivity in Banach spaces*, Israel J. Math. **6**, 74-79