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On the Stability of Pseudo-Linear Systems

by

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Abstract

A stability method for nonlinear systems of the form $\dot{x} = A(x)x$ with the origin as the only equilibrium point has been obtained. Examples are given to demonstrate the method.

Keywords: Nonlinear Systems, Lie Algebras.

1 Introduction

The Lyapunov approach to the stability of nonlinear systems has been widely studied by researchers. Due to the difficulty of finding a Lyapunov function which characterizes the nonlinear system in general, a new method has been developed to test the stability of a class of nonlinear systems. In this paper the nonlinear variation of parameters formula is used to derive a stability criterion for a nonlinear ordinary differential equation. In section 4 we prove a stability theorem for a class of nonlinear systems. Examples are given to illustrate the theorem. In the last section we consider the applications of this theorem to nonlinear systems which generate a semisimple Lie algebra.

2 Inequalities

The following inequality [1] will be used in the next section to derive a stability criterion for nonlinear systems.

Theorem 2.1: Assume that $v_i(t)$ $(i \in I)$ are nonnegative continuous functions defined on $[t_o,\infty)$, which satisfy the inequality

$$\dot{v}_i(t) \leq \lambda_i(t)v_i(t) + \sum_{j\in I} a_j^i(t)v_j(t) \tag{1}$$

Where $\lambda_i(t)$ is a nonpositive continuous function and $a_j^i(t)$ are nonnegative continuous functions, I is a countable index set.

If there exist $h > 0, \delta \in (0,1)$, and $d_i > 0$ such that

$$-\lambda_i(t) \geq h > 0, \quad \frac{1}{d_i}\sum_{j\in I} d_j \frac{a_j^i(t)}{-\lambda_i(t)} \leq \delta < 1, \quad \sup_{i\in I}\frac{v_i(t_o)}{d_i} < \infty \tag{2}$$

for all $t \geq t_o$ and $i \in I$, then there exist a constants $\omega > 0$ and $m \geq 1$ such that

$$\frac{v_i(t)}{d_i} \leq m \sup_{j\in I}\frac{v_j(t_o)}{d_j}\exp^{-\omega(t-t_o)} \tag{3}$$

for all $t \geq t_o$, and $i \in I$. For the proof of the above theorem see [1] □.

3 Nonlinear Systems

We shall study the general nonlinear system

$$\dot{x}(t) = f(x), x \in R^n \tag{4}$$

Where $f(x)$ is analytic function, and we shall assume that $x = 0$ is an isolated equilibrium point of (4), *i.e.*

$$f(0) = 0. \tag{5}$$

We can put the system (4) in the form

$$\dot{x}(t) = A(x)x, x \in R^n \tag{6}$$

where $A: R^n \rightarrow R^{n^2}$ is a matrix valued function and merely continuous.

Let us consider the case when $A(x)$ is a diagonal matrix

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdot & 0 \\ 0 & \lambda_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix} \tag{7}$$

and $\lambda_i: R \rightarrow R$, $i = 1, 2, ..., n$, are analytic functions. $x = 0$ is the only equlibrum point *i.e*: $\dot{x} = 0$ for $x = 0$.

Since $\lambda_i(x)$, $i = 1, ..., n$ are analytic functions, $\dot{x}_i(t)$, $i = 1, ..., n$ can be put in the following form:

$$\dot{x}_i = \Gamma_i(x)x_i + \sum_{j=1}^{n} a_j^i(x)x_j \quad (8)$$

for $i = 1, ..., n$, where $\Gamma_i(x) < 0$ for $x_i \neq 0$.

Definition: The zero solution of (4) is said to be globaly exponentially stable if there exist $\omega > 0$,$\pi \geq 1$ such that

$$\| x \| \leq \pi e^{-\omega(t-t_o)} \| x_o \| \quad (9)$$

for all $t \geq t_o$,$x(t_o) \in R^n$.

4 Stability Theorem

Consider the nonlinear system (8)

$$\dot{x}_i = \Gamma_i(x)x_i + \sum_{j=1}^{n} a_j^i(x)x_j \quad (10)$$

for $i = 1, ..., n$, where $\Gamma_i(x) < 0$ for $x_i \neq 0$.

Theorem 4.1: The zero solution of (10) is globally exponentially asymptotically stable if it satisfies the following conditions:

1. $x = 0$ is the only equilibrium point.

2. There exist constants $h > 0$, $\delta > 0$ and $d_i > 0$ such that

$$\Gamma_i(x) \leq -h \leq 0, \qquad \frac{1}{d_i}\sum_{j=1}^{n} \frac{d_j \mid a_j^i(x) \mid}{\mid \Gamma_i(x) \mid} \leq \delta < 1 \tag{11}$$

for $i = 1, ..., n$, $x_i \neq 0$ and for all $t \geq t_o$.

Proof: From (10) it follows that

$$x_i(t) = x_i(t_o) \exp^{(\int_{t_o}^{t} \Gamma_i(x(\tau))d\tau)} + \sum_{j=1}^{n} \int_{t_o}^{t} a_j^i(x(s)) x_j(s) \exp^{(\int_{t_o}^{t} \Gamma_i(x(\tau))d\tau)} ds \tag{12}$$

hence

$$\mid x_i(t) \mid \leq \mid x_i(t_o) \mid \exp^{(\int_{t_o}^{t} \Gamma_i(x(\tau))d\tau)} + \sum_{j=1}^{n} \int_{t_o}^{t} \mid a_j^i(x(s)) \mid\mid x_j(s) \mid \exp^{(\int_{t_o}^{t} \Gamma_i(x(\tau))d\tau)} ds. \tag{13}$$

Let $\mid x_i \mid = v_i$, then we have

$$v_i \leq v_i(t_o) \exp^{(\int_{t_o}^{t} \Gamma_i(x(\tau))d\tau)} + \sum_{j=1}^{n} \int_{t_o}^{t} \mid a_j^i(x(s)) \mid\mid v_j(s) \mid \exp^{(\int_{t_o}^{t} \Gamma_i(x(\tau))d\tau)} ds \tag{14}$$

and moreover, we have

$$\dot{v_i}(t) \leq \Gamma_i(x(t)) v_i(x(t)) + \sum_{j=1}^{n} \mid a_j^i(x(t)) \mid v_j(t). \tag{15}$$

Let

$$\| x \| = \sum_{i=1}^{n} \mid x_i \mid . \tag{16}$$

Then, we have

$$\frac{v_i(t_o)}{d_i} = \frac{| x_i(t_o) |}{d_i} \leq \frac{\| x(t_o) \|}{d_i} \tag{17}$$

i.e.

$$\frac{v_i(t_o)}{d_i} \leq \frac{\| x(t_o) \|}{d_i} < \infty. \tag{18}$$

Hence

$$\sup_{j \in I} \frac{v_j(t_o)}{d_j} \leq \sup_{j \in I} \frac{\| x(t_o) \|}{d_j} < \infty. \tag{19}$$

By *Theorem 2.1*, there exist $\omega > 0$ and $m \geq 1$ such that

$$\frac{v_i(t)}{d_i} \leq m \sup_{j \in I} \frac{v_j(t_o)}{d_j} \exp^{-\omega(t-t_o)} \tag{20}$$

for all $t \geq t_o$ and $i = 1, ..., n$. Then

$$\frac{| x_i(t) |}{d_i} \leq m \sup_{j \in I} \frac{\| x(t_o) \|}{d_j} \exp^{-\omega(t-t_o)} \; | x_i(t) | \leq m d_i \sup_{j \in I} \frac{\| x(t_o) \|}{d_j} \exp^{-\omega(t-t_o)}$$

$$| x_1(t) | + | x_2(t) | + ... + | x_n(t) | \leq m \sup_{j \in I} \frac{\| x(t_o) \|}{d_j} \exp^{-\omega(t-t_o)} \sum_{k=1}^{n} d_k \tag{21}$$

$$\| x(t) \| \leq m \sum_{k=1}^{n} d_k \sup_{j \in I} \frac{\| x(t_o) \|}{d_j} \exp^{-\omega(t-t_o)}. \tag{22}$$

If

$$m \sum_{k=1}^{n} d_k \sup_{j \in I} \frac{\| x(t) \|}{d_j} = M \tag{23}$$

then

$$\| x(t) \| = M \exp^{-\omega(t-t_o)}. \tag{24}$$

This shows that the zero solution of (10) is globally exponentially asymptotically stable $\square$.

Example 4.1:

$$\begin{aligned} \dot{x_1}(t) &= -x_1^3 + 0.5x_1^2 x_2 \\ \dot{x_2}(t) &= -\left(1 + x_2^2\right) x_2 + x_1 x_2 \end{aligned}$$

This system can be put in the form

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} (-x_1^2 + 0.5x_2x_1) & 0 \\ 0 & -(1 + x_2^2) + x_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The origin is the only equlibrum point, $f(0) = 0$. Let $d_1 = d_2 = 1$.

$$\begin{aligned} \dot{x_1}(t) &= (-x_1^2)x_1 + 0.5(x_1^2)x_2 \\ &= \Gamma_1(x)x_1 + a_2^1(x)x_2 \\ \dot{x_2}(t) &= -(1 + x_2^2)x_2 + x_2x_1 \end{aligned}$$

$$\begin{aligned}
&= \Gamma_2(x)x_2 + a_1^2(x)x_1 \\
\Gamma_1(x) &= -x_1^2 < 0 \qquad for \quad x_1 \neq 0 \\
\Gamma_2(x) &= -(1+x_2^2) < 0 \qquad for \quad x_2 \neq 0 \\
\frac{\mid a_2^1(x) \mid}{\mid \Gamma_1(x) \mid} &= \frac{\mid 0.5x_1^2 \mid}{\mid -x_1^2 \mid} = 0.5 \leq \delta < 1 \\
\frac{\mid a_1^2(x) \mid}{\mid \Gamma_2(x) \mid} &= \frac{0.5x_2}{\mid -(1+x_2^2) \mid} \leq \delta < 1.
\end{aligned}$$

Hence, the two conditions are satisfied, and the zero solution of the above system is globally exponentially asymptotically stable. From this example we can see that for a nonlinear system to be stable it is not necessary that the eigenvalues be negative definite.

Example 4.2:

$$\begin{aligned}
\dot{x_1}(t) &= (-2x_1 + x_2)x_1^2 \\
\dot{x_2}(t) &= -(1 + x_2^2 - x_1 + x_2^2x_3^2)x_2 \\
\dot{x_3}(t) &= -(1 + x_3^2 + x_2^2 + x_1)x_3
\end{aligned}$$

This system can be put in the form

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} (-2x_1^2 + x_2)x_1 & 0 & 0 \\ 0 & -(1 + x_2^2 - x_1 + x_2^2x_3^2) & 0 \\ 0 & 0 & -(1 + x_3^2 + x_2^2 + x_1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The origin is the only equilibirum point, $f(0) = 0$. Let $d_1 = d_2 = d_3 = 1$.

$$\begin{aligned}
\dot{x_1}(t) &= -2x_1^2x_1 + x_1^2x_2 \\
&= \Gamma_1(x)x_1 + a_2^1(x)x_2 \\
\dot{x_2}(t) &= -(1 + x_2^2 + x_2^2x_3^2)x_2 + x_1x_2 \\
&= \Gamma_2(x)x_2 + a_1^2(x)x_1 \\
\dot{x_3}(t) &= -(1 + x_3^2 + x_2^2)x_3 - x_1x_3 \\
&= \Gamma_3(x)x_3 + a_1^3(x)x_1 \\
\Gamma_1(x) &= -2x_1^2 < 0 \qquad for \quad x_1 \neq 0 \\
\Gamma_2(x) &= -(1 + x_2^2 + x_2^2x_3^2) < 0 \qquad for \quad x_2 \neq 0 \\
\Gamma_3(x) &= -(1 + x_2^2 + x_3^2) < 0 \qquad for \quad x_3 \neq 0 \\
\frac{| a_2^1(x) |}{| \Gamma_1(x) |} &= \frac{| x_1^2 |}{| -2x_1^2 |} \leq \delta < 1 \\
\frac{| a_1^2(x) |}{| \Gamma_2(x) |} &= \frac{| x_2 |}{| -(1 + x_2^2 + x_2^2x_3^2) |} \leq \delta < 1 \\
\frac{| a_1^3(x) |}{| \Gamma_3(x) |} &= \frac{| -x_3 |}{| -(1 + x_2^2 + x_3^2) |} \leq \delta < 1.
\end{aligned}$$

Therefore the zero solution is globally exponentially asymptotically stable.

Now we will consider the following nonlinear system

$$\dot{x}(t) = f(x(t)) \tag{25}$$

where $f(x):R^n \rightarrow R^n$ is analytic function. Then

$$\dot{x}(t) = f(x(t)) = B(x)x + \sum_{j=1}^{n} a_j(x)x_j \tag{26}$$

where $B(x):R^n \rightarrow R^{n^2}$ is a diagonal matrix and $a_j(x):R^n \rightarrow R^n$ are column vectors. We can write the above system in the form

$$\dot{x}_i(t) = b_{ii}(x)x_i + \sum_{j=1}^{n} a_j^i(x)x_j. \tag{27}$$

Theorem 4.2: The zero solution of the above nonlinear system is globally exponentially asymptotically stable if the following conditions are met:

1. $x = 0$ is the only equilibrium point.

2. There exist constants $h < 0$, $\delta \in (0,1)$ and $d_i > 0$ for $i = 1, ..., n$ such that

$$\begin{aligned} &b_{ii}(x) \leq h < 0, x \neq 0 \\ &\sum_{j=1}^{n} \frac{d_j \mid a_j^i(x) \mid}{d_i \mid b_{ii}(x) \mid} \leq \delta < 1. \end{aligned} \tag{28}$$

Proof. The proof of the theorem is as in *Theorem 4.1* □.

Example 4.3:

$$\begin{aligned}\dot{x_1}(t) &= -(2+x_2^2)x_1 + x_2x_2 \\ \dot{x_2}(t) &= -(1+\exp^{(x_1)})x_2 + 0.5x_1\end{aligned}$$

$x=0$ is an isolated equilibrium point, $f(0)=(0)$. let $d_1=d_2=1$.

$$\begin{aligned}\dot{x_1}(t) &= -(2+x_2^2)x_1 + x_2x_2 \\ &= b_{11}(x)x_1 + a_2^1(x)x_2 \\ \dot{x_2}(t) &= -(1+\exp^{(x_1)})x_2 + 0.5x_1 \\ &= b_{22}(x)x_2 + a_1^2(x)x_1 \\ b_{11}(x) &< 0 \qquad for \quad x \neq 0, \\ b_{22}(x) &< 0 \qquad for \quad x \neq 0. \\ \frac{\mid a_2^1(x) \mid}{\mid b_{11}(x) \mid} &= \frac{\mid x_2 \mid}{\mid -(2+x_2^2) \mid} \leq \delta < 1 \\ \frac{\mid a_1^2(x) \mid}{\mid b_{22}(x) \mid} &= \frac{0.5}{\mid -(\exp^{(x_1)}+1) \mid} \leq \delta < 1.\end{aligned}$$

Hence the zero solution is globally exponentially asymptotically stable.

Example 4.4:

$$\dot{x_1}(t) = -(1+x_2^2)x_1 + x_2^2x_3$$

$$\dot{x_2}(t) = -(x_1^2 + x_2^2 + x_3^2)x_2$$

$$\dot{x_3}(t) = \frac{x_2^2 x_1}{2} + \frac{x_1^2 x_2}{2} - (2 + x_1^2 + x_2^2)x_3$$

$x = 0$ is an isolated equilibrium point, $f(0) = (0)$. let $d_1 = d_2 = d_3 = 1$.

$$\begin{aligned}
\dot{x_1}(t) &= -(1 + x_2^2)x_1 + x_2^2 x_3 \\
&= b_{11}(x)x_1 + a_3^1(x)x_3 \\
\dot{x_2}(t) &= -(x_1^2 + x_2^2 + x_3^2)x_2 \\
&= b_{22}(x)x_2 \\
\dot{x_3}(t) &= -(2 + x_1^2 + x_2^2)x_3 + \frac{x_2^2}{2}x_1 + \frac{x_1^2}{2}x_2 \\
&= b_{33}(x)x_3 + a_1^3(x)x_1 + a_2^3(x)x_2 \\
b_{11}(x) &< 0 \qquad for \quad x \neq 0 \\
b_{22}(x) &< 0 \qquad for \quad x \neq 0 \\
b_{33}(x) &< 0 \qquad for \quad x \neq 0 \\
\frac{\mid a_3^1(x) \mid}{\mid b_{11}(x) \mid} &= \frac{\mid x_2^2 \mid}{\mid -(1 + x_2^2) \mid} \leq \delta < 1 \\
\frac{0}{\mid b_{22}(x) \mid} &= \frac{0}{\mid (x_1^2 + x_2^2 + x_3^2) \mid} \leq \delta < 1 \\
\frac{\mid a_1^3(x) \mid + \mid a_2^3(x) \mid}{\mid b_{33}(x) \mid} &= \frac{\mid (x_1^2 + x_2^2) \mid}{2 \mid (2 + x_1^2 + x_2^2) \mid} \leq \delta < 1.
\end{aligned}$$

Hence the zero solution is globally exponentially asymptotically stable.

5 The Application of Semisimple Lie Algebras

We shall consider in this section the nonlinear system

$$\dot{x}(t) = A(x)x \tag{29}$$

where $A(x): R^n \rightarrow R^{n^2}$, $x \in R^n$.

If the Lie algebra generated by $A(x)$, (see [3]), is a semisimple Lie algebra, then the nonlinear system (29) can be put in the form

$$\dot{x}(t) = H(x)x + \sum_{\alpha\in\Delta} e_\alpha(x)F_\alpha x \tag{30}$$

where $H(x)$ is the Cartan subalgebra and F_α are the roots. $H(x)$ can be diagonalized by a linear transformation, say P.

$$\begin{aligned} y &= P^{-1}x \\ \dot{y} &= P^{-1}H(y)Py + \sum_{\alpha\in\Delta} e_\alpha(y)P^{-1}F_\alpha Py \\ \dot{y} &= \Lambda(y)y + \sum_{\alpha\in\Delta} e_\alpha(y)E_\alpha y \end{aligned} \tag{31}$$

In general, by this transformation, the original system is transformed to a simpler one. Then we can use the result obtained in the previous sections to test for the stability of the system. We can arrange the equations of the above system to be in the form

$$\dot{y_i} = \beta_i(y)y_i + \sum_{j=1}^{n} a_j^i(y)y_j \tag{32}$$

by splitting the coefficient of y_i in the equation $\dot{y_i}(t)$ into a negative definite part ($\beta_i(y)$) and the rest can be accommodated in the a_j^i functions.

Theorem 5.1: The zero solution of (32) is globally exponentially asymptotically stable if it satisfies the following conditions:

1. $x = 0$ is the only equilibrium point.

2. There exist constants $h > 0$, $\delta > 0$ and $d_i > 0$ such that

$$\beta_i(x) \leq -h \leq 0, \qquad \frac{1}{d_i}\sum_{j=1}^{n} \frac{d_j \mid a_j^i(x) \mid}{\mid \beta_i(x) \mid} \leq \delta < 1 \tag{33}$$

for $i = 1, ..., n$, $x \neq 0$ and for all $t \geq t_o$.

Proof: as in *Theorem 4.1* □.

Example 5.1: Let us consider the semisimple Lie algebra A_2, (other types of semisimple Lie algebras B_n, D_n,... etc can be applied similarly). Consider the following two dimensional nonlinear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_1(x) & f_{12}(x) \\ f_{21}(x) & -f_1(x) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where

$$\begin{aligned} f_1(x) &= -x_1^4 + x_2^2 \\ f_{12}(x) &= -2x_1x_2 \\ f_{21}(x) &= -2x_1^3x_2 + 0.5x_1^4 \end{aligned}$$

The functions $f_1(x)$, $f_{12}(x)$ and $f_{21}(x)$ are functionally independent. The Lie algebra generated by $A(x)$ is semisimple of the type A_n and spanned by the following matrices:

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where h_1 spans the Cartan subalgebra. E_{12} and E_{21} span the root spaces of the semisimple Lie algebra. By splitting the coefficient of x_i in the equation $\dot{x}_i(t)$ into a negative definite part (call it $\beta_i(x)$) and the rest into the functions

a_j^i, the system equations can be put in the following form:

$$\dot{x_1} = \beta_1(x)x_1 + a_1^1(x)x_1 + a_2^1(x)x_2,$$

where $a_j^1(x) = 0$, $j = 1, 2$ and $\beta_1(x) = -(x_1^4 + x_2^2)$, and

$$\dot{x_2} = \beta_2(x)x_2 + a_1^2(x)x_1 + a_2^2(x)x_2$$

where $\beta_2(x) = -(x_1^4 + x_2^2)$, $a_1^2(x) = 0.5x_1^4$ and $a_2^2(x) = 0$. The origin is an isolated equilibrium point. Let $d_1 = d_2 = 1$. From the above equations we have

$$\begin{aligned}
\beta_1(x) &= -(x_1^4 + x_2^2) < 0 \\
\frac{0}{\mid \beta_1(x) \mid} &< \delta < 1 \\
\beta_2(x) &= -(x_1^4 + x_2^2) < 0 \\
\frac{\mid a_1^2(x) \mid}{\mid \beta_2(x) \mid} &= \frac{0.5x_1^4}{x_1^4 + x_2^2} < \delta < 1.
\end{aligned}$$

Therefore the zero solution of the system is globally exponentially asymptotically stable.

Example 5.2: Consider the following nonlinear system

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} f_{11}(x) - f_{21}(x) + f_{31}(x) & a_{12}(x) & a_{13}(x) \\ f_{21}(x) + f_{31}(x) & a_{22}(x) & a_{23}(x) \\ f_{31}(x) & f_{31}(x) + f_{32}(x) & f_{22}(x) - 3f_{31}(x) - f_{32}(x) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where

$$
\begin{aligned}
a_{12}(x) &= 2f_{11}(x)+f_{22}(x)+f_{12}(x)-f_{21}(x)+2f_{31}(x)+2f_{32}(x) \\
a_{13}(x) &= -4f_{11}(x)+f_{22}(x)-f_{12}(x)+3f_{21}(x)+f_{13}(x)-6f_{31}(x) \\
&\quad -2f_{32}(x)-f_{23}(x) \\
a_{22}(x) &= -f_{11}(x)-f_{22}(x)+f_{21}(x)+f_{31}(x)+f_{32}(x) \\
a_{23}(x) &= f_{11}(x)+2f_{22}(x)-3f_{21}(x)-3f_{31}(x)-f_{32}(x)+f_{23}(x).
\end{aligned}
$$

Let $f_{11}(x), f_{22}(x), f_{12}(x), f_{21}(x), f_{13}(x), f_{31}(x), f_{32}(x), f_{23}(x)$ be functionally independent. Then the Lie algebra generated by $A(x)$ is semisimple with the following matrices as its basis:

$$
h_1 = \begin{pmatrix} 1 & 2 & -4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad F_{12} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
F_{21} = \begin{pmatrix} -1 & -1 & 3 \\ 1 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_{31} = \begin{pmatrix} 2 & 2 & -6 \\ 1 & 1 & -3 \\ 1 & 1 & -3 \end{pmatrix}
$$

$$F_{32} = \begin{pmatrix} 0 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad F_{23} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

where h_1 and h_2 span the Cartan subalgebra. There exists a linear transformation P, which diagonalizes the Cartan subalgebra:

$$P = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let

$$\begin{aligned}
f_{11} &= -(1-(x_2-x_3)^2) \\
f_{22} &= -(1+x_3^2) \\
f_{12} &= -3(x_1+x_2-3x_3)(x_2-x_3) \\
f_{21} &= \frac{-(2+2x_3^2+(x_1+x_2-3x_3)^2)}{(x_1+x_2-2x_3)}(x_2-x_3) \\
f_{31} &= x_3^2 \\
f_{13} &= -0.5(1+2(x_2-x_3)^2) \\
f_{32} &= -x_3^3(x_2-x_3) \\
f_{23} &= -x_3(x_2-x_3).
\end{aligned}$$

The above system can be put in the following form:

$$\begin{aligned}
\dot{x} &= A(x)x \\
\dot{x} &= f_{11}(x)h_1 + f_{22}(x)h_2 + f_{12}(x)F_{12} + f_{21}(x)F_{21} + f_{13}F_{13} + f_{31}(x)F_{31} \\
&+ f_{32}(x)F_{32} + f_{23}(x)F_{23} \\
x &= Py \\
\dot{y} &= P^{-1}\dot{x} = (P^{-1}H(x)P + \sum_{\alpha} f_{\alpha}(x)P^{-1}F_{\alpha}P)y \\
\dot{y} &= (\Lambda(y) + \sum_{\alpha} f'_{\alpha}(y)E_{\alpha})y \\
\dot{y} &= f'_{11}(y)h'_{11} + f'_{22}(y)h'_{22} + f'_{12}(y)E_{12} + f'_{21}(y)E_{21} + f'_{13}(y)E_{13} + f'_{31}(y)E_{31} \\
&+ f'_{32}(y)E_{32} + f'_{23}(y)E_{23}
\end{aligned}$$

where

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

E_{ij}, where $i \neq j$, are the matrices with 1 on the ij'th component and 0 elsewhere and

$$\begin{aligned}
f'_{11}(y) &= -(1 - y_2^2) \\
f'_{22}(y) &= -(1 + y_3^2)
\end{aligned}$$

$$
\begin{aligned}
f'_{12}(y) &= -3y_1y_2 \\
f'_{21}(y) &= \frac{-(2+2y_3^2+y_1^2)}{y_1}y_2 \\
f'_{31}(y) &= y_3^2 \\
f'_{13}(y) &= -0.5(1+2y_2^2) \\
f'_{32}(y) &= -y_3^3y_2 \\
f'_{23}(y) &= -y_3y_2.
\end{aligned}
$$

Now we can test for the stability of this system using the previous theorem. $x = 0$ is an isolated equlibrum point. Let $d_1 = d_2 = d_3 = 1$.

$$
\begin{aligned}
\dot{y_1} &= -(1-y_2^2)y_1 + (-3y_1y_2)y_2 - 0.5(1+2y_2^2)y_3 \\
\dot{y_1} &= -(1+3y_2^2)y_1 + y_2^2y_1 - 0.5(1+2y_2^2)y_3 \\
&= \beta_1(y)y_1 + a_1^1(y)y_1 + a_3^1(y)y_3 \\
\dot{y_2} &= (2+y_3^2-y_2^2)y_2 - (2+2y_3^2+y_1^2)y_2 + (-y_2y_3)y_3 \\
\dot{y_2} &= -(y_1^2+y_2^2+2y_3^2)y_2 \\
&= \beta_2(y)y_2 \\
\dot{y_3} &= -(1+y_3^2)y_3 + y_3^2y_1 - (y_3^3y_2)y_2 \\
\dot{y_3} &= -(1+y_3^2+y_3^2y_2^2) + y_3^2y_1 \\
\dot{y_3} &= \beta_3(y)y_3 + a_1^3(y)y_1
\end{aligned}
$$

$$
\begin{aligned}
\beta_1(y) &< 0 \\
\frac{|\, a_1^1(y) \,| + |\, a_3^1(y) \,|}{|\, \beta_1(y) \,|} &= \frac{y_2^2 + 0.5(1 + 2y_2^2)}{(1 + 3y_2^2)} = \frac{(0.5 + 2y_2^2)}{(1 + 3y_2^2)} < \delta < 1 \\
\beta_2(y) &< 0 \\
\frac{0}{|\, \beta_2(y) \,|} &< \delta < 1 \\
\beta_3(y) &< 0 \\
\frac{|\, a_1^3(y) \,|}{|\, \beta_3(y) \,|} &= \frac{y_3^2}{(1 + y_3^2 + y_3^2 y_2^2)} < \delta < 1.
\end{aligned}
$$

Hence the zero solution of the system is globally exponentially asymptotically stable.

References

[1] Banks, S. P. and Zhang Yi, On the Stability Analysis of Nonlinear Systems. IMA Journal of Mathematical Control & Information, (1991) 8, 275-285.

[2] Brauer,F., (1966) Perturbation of nonlinear systems of differential equations. J. Math. Anal. App. 14, 198-206.

[3] Banks, S. P. and Al-Jurani, S. K., Lie Algebras and the Stability of nonlinear Systems. To appear in Int. J. Control.

[4] Banks, S. P. and Mhana, K. J., Optimal Control and Stablization for Nonlinear Systems. IMA J. Math. Contr. Inf., 1992.

[5] Banks, S. P., 1988 Mathematical Theories of Nonlinear Systems. London:Prentice-Hall.