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Symmetric Heyting relation algebras with applications to hypergraphs

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Abstract

A relation on a hypergraph is a binary relation on the set consisting of all the nodes and the edges, and which satisfies a constraint involving the incidence structure of the hypergraph. These relations correspond to join preserving mappings on the lattice of sub-hypergraphs. This paper introduces a generalization of a relation algebra in which the Boolean algebra part is replaced by a Heyting algebra that supports an order-reversing involution. A general construction for these symmetric Heyting relation algebras is given which includes as a special case the algebra of relations on a hypergraph. A particular feature of symmetric Heyting relation algebras is that instead of an involutory converse operation they possess both a left converse and a right converse which form an adjoint pair of operations. Properties of the converses are established and used to derive a generalization of the well-known connection between converse, complement, erosion and dilation in mathematical morphology. This provides part of the foundation necessary to develop mathematical morphology on hypergraphs based on relations on hypergraphs.

Keywords: relation algebra, symmetric Heyting algebra, hypergraph, mathematical morphology, left converse, right converse

1. Introduction

The notion of a relation on a set U is well-known and the study of properties of these subsets of $U \times U$ has led to the abstract notion of a relation algebra [1]. Relations on U can be identified with the join preserving functions on the Boolean algebra of subsets of U . Practical applications of this identification include mathematical morphology [2, 3] where taking U to be a set of pixels, the subsets correspond to monochrome images. In this context the

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structuring elements used in image processing correspond to certain relations on U , and some of the basic properties of morphological operations appear as special cases of facts about relations on the set U .

A number of authors [4, 5, 6, 7] have proposed approaches to mathematical morphology on subgraphs of a graph, or of a hypergraph, as an extension to operations on subsets of a set. While the lattice theoretic basis of set-based mathematical morphology has long been understood [8, 3], this is not the case when we move to graphs and hypergraphs.

Previous work [9, 10] has shown how to characterize join preserving functions on the lattice of subgraphs as relations of a particular kind. Further algebraic properties of these relations were investigated in [11], which showed that the familiar involutory converse operation for relations on a set becomes two weaker operations forming an adjunction or a Galois connection. The algebraic structure of a bi-Heyting algebra which is used in [9] also appears in a different application to mathematical morphology in the work of Perret [12].

In the present paper, the constructions in [11], are extended to a more general setting by moving from the concrete case of the full relation algebra on U to a relation algebra. This clarifies both the features which are essential in these constructions, and also the relationship between the two converse operations and the two complement-like operations, the pseudocomplement and its dual. A relation algebra consists of a Boolean algebra equipped with additional operations which include converse and composition. When dealing with relations on a hypergraph we no longer have a Boolean algebra, but we have a symmetric Heyting algebra. A notion of ‘symmetric Heyting relation algebra’ (SHRA) is proposed, and constructions which generate such algebras are identified. These constructions include the relations on a hypergraph.

The structure of the paper is as follows. Section 2 reviews some well-known basic concepts that we need later. Section 3 introduces hypergraphs and some basic properties of relations on a hypergraph. These relations are not closed under the usual operations of converse or complement, which raises the question of whether there is a suitable generalization of relation algebra which fits the case of relations on a hypergraph. To answer this question, Section 4 recalls the existing notion of a symmetric Heyting algebra, that is a Heyting algebra with an order-reversing involution. We then see that besides the pseudocomplement and its dual found in any bi-Heyting algebra, a symmetric Heyting algebra possesses two other unary operations which we call the left and right converses.

In Section 5 we extend symmetric Heyting algebras with an additional binary operation to model the composition of relations. This leads to the definition of a symmetric Heyting relation algebra, or SHRA for short. It is then justified that these structures generalize relation algebras, and we establish some basic properties that are used in Section 6 to provide a means of constructing SHRAs. This construction takes an SHRA containing a pre-order element and uses this element to find a sub-SHRA. Since a relation algebra is a special case of an SHRA we can use this construction to obtain an SHRA inside a relation algebra. In the special case of the full relation algebra on a set U this construction yields the algebra of relations on a hypergraph as demonstrated in Section 7. An

application to mathematical morphology on hypergraphs is presented in Section 8 and we end with conclusions in Section 9.

2. Preliminaries

This section provides preliminaries, to establish notation and terminology.

2.1. Lattices

Definition 1. A **semilattice** $(A, *, 1)$ is a commutative monoid where $x*x = x$ for all $x \in A$.

Definition 2. A **lattice** is an algebra $(A, \vee, \wedge, \perp, \top)$ such that

1. both (A, \vee, \perp) and (A, \wedge, \top) are semilattices, and
2. for all $x, y \in A$, $a \vee b = b$ iff $b \wedge a = a$.

In a lattice we write \leq for the partial order defined by $x \leq y$ if $a \vee b = b$. Given any $x, y \in A$, the operation $x \mapsto x \wedge y$ will be denoted $_ \wedge y$. This underscore notation will be used in a similar way for other operations.

Definition 3. A **distributive lattice** is a lattice satisfying the identity

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Definition 4. A **Boolean algebra** is an algebra $(A, \vee, \wedge, \bar{}, \perp, \top)$ where

1. $(A, \vee, \wedge, \perp, \top)$ is a distributive lattice, and
2. for all $x \in A$, $x \wedge \bar{x} = \perp$ and $x \vee \bar{x} = \top$.

2.2. Adjoints between pre-orders

We will use the category-theoretic terminology of adjunctions for situations that are also known as Galois connections.

Definition 5. Let X and Y be sets each equipped with a pre-order (i.e. a reflexive and transitive relation), and let $f : X \rightarrow Y$ and $g : Y \rightarrow X$ be functions. If the condition $fx \leq y \Leftrightarrow x \leq gy$ holds for all $x \in X$ and all $y \in Y$, we say f is **left adjoint** to g .

Note that the condition $fx \leq y \Leftrightarrow x \leq gy$ implies that both f and g are order preserving. For if $x_1 \leq x_2$ we have $x_2 \leq gfx_2$ since $fx_2 \leq fx_2$, and hence $x_1 \leq gfx_2$ so that $fx_1 \leq fx_2$.

The notation $f \dashv g$ is used to mean that f is left adjoint to g , or equivalently that g is **right adjoint** to f . The idea [13, p152], of viewing \dashv as an arrow (with the horizontal dash as the shaft of the arrow, and the vertical dash as the head of the arrow) proceeding from the left adjoint to the right adjoint is also adopted in diagrams where \dashv may appear rotated as an arrow between two other arrows as follows.

$$X \begin{array}{c} \xrightarrow{f} \\ \dashv \\ \xleftarrow{g} \end{array} Y$$

2.3. Heyting algebras

Definition 6. A **Heyting algebra** is an algebra $(A, \vee, \wedge, \rightarrow, \perp, \top)$ such that

1. $(A, \vee, \wedge, \perp, \top)$ is a lattice, and
2. for all $x, y, z \in A$, $x \leq y \rightarrow z$ iff $x \wedge y \leq z$.

The basic results in Proposition 1 and Proposition 2 are proved in [14].

Proposition 1. *An algebra $(A, \vee, \wedge, \rightarrow, \perp, \top)$ is a Heyting algebra iff*

1. $(A, \vee, \wedge, \perp, \top)$ is a lattice, and
2. the following identities hold for all $x, y, z \in A$.

$$\begin{array}{ll} \text{(i)} & x \rightarrow x = \top \\ \text{(ii)} & x \wedge (x \rightarrow y) = x \wedge y \\ \text{(iii)} & y \wedge (x \rightarrow y) = y \\ \text{(iv)} & x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z) \end{array}$$

□

Definition 7. A **co-Heyting algebra** is an algebra $\mathbf{A} = (A, \vee, \wedge, \succ, \perp, \top)$ such that

1. $(A, \vee, \wedge, \perp, \top)$ is a lattice, and
2. for all $x, y, z \in A$, $x \succ y \leq z$ iff $x \leq y \vee z$.

Note that a co-Heyting algebra with the ordering reversed will yield a Heyting algebra, but the implication operation in this algebra will be $x \rightarrow y = y \succ x$ and not $x \succ y$. The operation \succ , with a different notation, is called ‘subtraction’ by Lawvere [15, p9].

Definition 8. A **bi-Heyting algebra** is an algebra $(A, \vee, \wedge, \succ, \rightarrow, \perp, \top)$ such that

1. $(A, \vee, \wedge, \rightarrow, \perp, \top)$ is a Heyting algebra, and
2. $(A, \vee, \wedge, \succ, \perp, \top)$ is a co-Heyting algebra.

The properties of the Heyting implication and the co-Heyting subtraction can be written as adjunctions:

$$_ \wedge y \dashv y \rightarrow _, \quad \text{and} \quad _ \succ y \dashv y \vee _.$$

Every Boolean algebra is a bi-Heyting algebra by defining

$$x \rightarrow y = \bar{x} \vee y \quad \text{and} \quad x \succ y = x \wedge \bar{y},$$

and \bar{x} is equal both to $x \rightarrow \perp$ and to $\top \succ x$. In an arbitrary bi-Heyting algebra however, these two expressions provide distinct weaker versions of the complement which play an important role later in the paper. They are known as the pseudocomplement: $\neg x = x \rightarrow \perp$ and the dual pseudocomplement: $\neg x = \top \succ x$.

Proposition 2. *A Heyting algebra is a Boolean algebra iff $\neg\neg x = x$ holds for all x .* □

Proposition 3. *Let A be a bi-Heyting algebra and $x \in A$. Then $\neg x \leq \lrcorner x$. The equality $\neg x = \lrcorner x$ holds for all x iff A is a Boolean algebra.*

PROOF. From $\neg x \wedge x = \perp$ we get $(\neg x \wedge x) \vee \lrcorner x = \lrcorner x$ so that $(\neg x \vee \lrcorner x) \wedge (x \vee \lrcorner x) = \lrcorner x$ and thus $\neg x \vee \lrcorner x = \lrcorner x$.

If $\neg x = \lrcorner x$ holds then the operation satisfies both the requirements for a complement. Conversely, in a Boolean algebra $w \wedge x \leq y$ iff $w \leq \bar{x} \vee y$, so by uniqueness of adjoints $x \rightarrow y = \bar{x} \vee y$ and hence $\neg x = \bar{x}$. Similarly $x \multimap y = x \wedge \bar{y}$ and $\lrcorner x = \bar{x}$. \square

2.4. Relation algebras

The notion of a relation algebra can be given in many equivalent ways. We recall here the definition due to Chin and Tarski as presented in [16, p3] although we have numbered the equations differently.

Definition 9. A **relation algebra** is an algebra $\mathbf{A} = (A, \vee, \wedge, \bar{}, \top, \perp, ;, \cdot, 1, \smile)$ such that:

1. $(A, \vee, \wedge, \bar{}, \top, \perp)$ is a Boolean algebra,
2. $(A, ;, 1)$ is a monoid, and
3. the following hold for all $x, y, z \in A$:

$$\begin{aligned} \text{(RA-i)} \quad (x; y)^\smile &= \check{y}; \check{x}, & \text{(RA-ii)} \quad \check{\check{x}} &= x, & \text{(RA-iii)} \quad (x \vee y)^\smile &= \check{x} \vee \check{y}, \\ \text{(RA-iv)} \quad x; (y \vee z) &= (x; y) \vee (x; z), & \text{(RA-v)} \quad \check{x}; \overline{x; y} &\leq \bar{y}. \end{aligned}$$

Note that we use the notation 1 rather than $1'$ for the identity element for the composition operation $;$ and that 0 is used for the complement of 1. The top and bottom elements will always be denoted \top and \perp .

Every $x \in A$ gives rise to an operation $_ / x$ on A which has a right adjoint, namely the operation $_ / x$ defined by $y / x = \bar{y}; \check{x}$. Composition on the left also has a right adjoint. The operation $x; _$ has the operation $x \setminus _$ as its right adjoint where $x \setminus y = \check{\check{x}}; \bar{y}$. The two adjunctions $_ ; y \dashv _ / y$ and $x; _ \dashv x \setminus _$ mean that

$$x; y \leq z \text{ iff } x \leq z / y \quad \text{and} \quad x; y \leq z \text{ iff } y \leq x \setminus z.$$

3. Hypergraphs

3.1. Basic concepts

A hypergraph can be defined as consisting of a set N of nodes and a set E of edges (or hyperedges) together with an incidence function $i : E \rightarrow \mathcal{P}N$ from E to the powerset of N . This approach allows several edges to be incident with the same set of nodes, and also allows edges incident with the empty set of nodes. An example is shown on the left in Figure 1 (adapted from [10]) where the edges are drawn as curves enclosing the nodes with which they are incident.

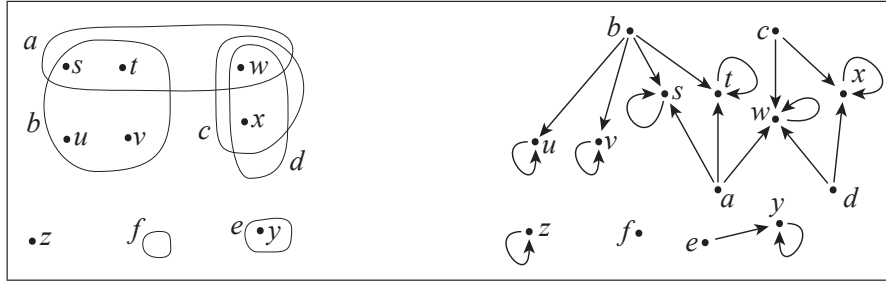


Figure 1: Left hand side: hypergraph with edge set $E = \{a, b, c, d, e, f\}$ and node set $N = \{s, t, u, v, w, x, y, z\}$. On the right: the corresponding relation φ on $N \cup E$.

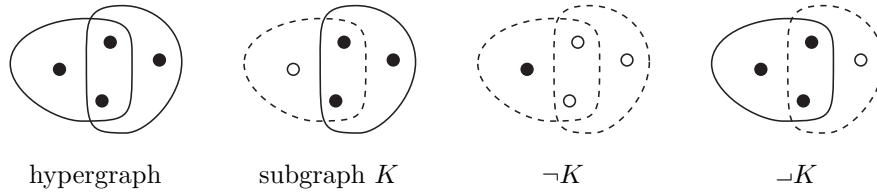


Figure 2: Example showing the pseudocomplement and dual pseudocomplement of a subgraph

When studying relations on these structures it is more convenient to use an equivalent definition, in which there is a single set comprising both the edges and nodes together. This has been used in [9, 10] and is based on using a similar approach to graphs in [17].

Definition 10. A **hypergraph** consists of a set U and a relation $\varphi \subseteq U \times U$ such that for all $u, v, w \in U$,

1. if $(u, v) \in \varphi$ then $(v, v) \in \varphi$, and
2. if $(u, v) \in \varphi$ and $(v, w) \in \varphi$ then $v = w$.

From a hypergraph described in this way we can obtain the edges as those $u \in U$ for which $(u, u) \notin \varphi$, whereas the nodes satisfy $(u, u) \in \varphi$. Figure 1 shows an example. A **subgraph** of (U, φ) is defined as a subset $K \subseteq U$ for which $k \in K$ and $(k, u) \in \varphi$ imply $u \in K$. The **dual hypergraph** of (U, φ) is (U, φ^*) , where $\varphi^* = \bar{\varphi} \wedge (1 \vee \bar{\varphi})$. In the dual the edges and nodes are interchanged.

The subgraphs of a given hypergraph (U, φ) form a complete lattice which is a sublattice of the Boolean algebra $\mathcal{P}U$. This sublattice is not a Boolean algebra as it is not closed under complementation. It is a bi-Heyting algebra for which the special case of graphs rather than hypergraphs has been discussed by Lawvere [18] and Reyes and Zolfaghari [19]. Figure 2 shows a subgraph K , indicated by the unbroken edges and the nodes as solid discs, together with its pseudocomplement and dual pseudocomplement.

3.2. Relations on a hypergraph

The relations on a set U correspond to those functions, f , on the lattice of subsets of U which preserve arbitrary joins. The preservation of arbitrary joins means that $f(\bigcup_{i \in \mathcal{I}} A_i) = \bigcup_{i \in \mathcal{I}} (f A_i)$ for all \mathcal{I} indexed families of subsets of U for any \mathcal{I} . In particular this includes the case $\mathcal{I} = \emptyset$, so that join preserving functions satisfy $f(\emptyset) = \emptyset$.

To identify the ‘right’ notion of relation on a hypergraph, one approach is to take these relations to be equivalent to the join preserving functions on the lattice of subgraphs of (U, φ) . This can be done, and it results in identifying the relations on (U, φ) as relations on U which meet the constraints in the following definition.

Definition 11. A relation x on U is a **graphical relation** on the hypergraph (U, φ) when $\varphi ; x \leq x$ and $x ; \varphi \leq x$.

It is, however, more convenient to work with an alternative characterization of these relations, using what we will call h -relations, where h is a relation on U of the following kind.

Definition 12. A relation h on U is an **incidence relation** if $1 \leq h$ and $(h \wedge 0) ; (h \wedge 0) = \perp$. Given a set U with incidence relation h , a relation x on U is an **h -relation** if $x = h ; x ; h$.

Writing $h = (h \wedge 1) \vee (h \wedge 0) = 1 \vee (h \wedge 0)$ we can see that $h ; h \leq h$, so that h is transitive as well as reflexive. In fact, h will be a poset, but antisymmetry is not essential in what follows and we shall see later that it only needs to be a pre-order.

The h -relations are actually a special case of the category-theoretic notion of a distributor or a pro-functor [20, 21]. Although we do not need the full generality of this notion here, it is worth noting that Bénabou [21] describes distributors as “relations between (small) categories” and introduces the special case of the categories being posets. Taking the poset h in this sense, the distributors from h to h are exactly the h -relations.

We now consider the precise sense in which graphical relations correspond to h -relations. It is straightforward to check that when (U, φ) is a hypergraph the reflexive closure, φ° , of φ is an incidence relation on U . Further, any incidence relation h arises in this way since defining $\varphi = (1 \wedge (h ; (h \wedge 0)) \vee (h \wedge 0))$, gives a hypergraph (U, φ) with $\varphi^\circ = h$.

Distinct hypergraphs (U, φ_1) and (U, φ_2) can have $(U, \varphi_1^\circ) = (U, \varphi_2^\circ)$ when, for example, an element $u \in U$ appears as an isolated node in (U, φ_1) but as an empty edge in (U, φ_2) . However, the following easily established result shows that if $(U, \varphi_1^\circ) = (U, \varphi_2^\circ)$ then the graphical relations on (U, φ_1) and on (U, φ_2) coincide. This justifies the use of h -relations rather than graphical relations.

Proposition 4. *Let (U, φ) be a hypergraph and x a relation on U . Then x is a graphical relation on (U, φ) iff x is an h -relation where $h = \varphi^\circ$. \square*

For a hypergraph (U, φ) , the dual hypergraph (U, φ^*) is related to the converse of the corresponding incidence relation. Clearly $(\varphi^*)^\circ = (\varphi^\circ)^\circ$ and h is an incidence relation iff \check{h} is an incidence relation.

The subgraphs of a hypergraph given by an incidence relation h on a set U are readily seen to be the subsets $K \subseteq U$ such that $k \in K$ and $(k, u) \in h$ imply $u \in K$.

3.3. Basic properties of h -relations

Although the following basic facts about h -relations follow from a more general setting later in the paper, they provide a useful concrete motivation at this point.

Lemma 5. *Let h be a pre-order on U , let x and y be h -relations, and let z be any relation on U .*

1. h is an h -relation, and $x; h = x = h; x$.
2. $z \leq h; z; h$.
3. The relations $U \times U$ and \emptyset are both h -relations.
4. $x \vee y$ is an h -relation.
5. $x \wedge y$ is an h -relation.
6. $x; y$ is an h -relation.

PROOF. We use basic properties of relations and the fact that $h; h = h$, which immediately establishes part 1.

2. Since $1 \leq h$, we have $1; z; 1 \leq h; z; h$.
3. Since $1 \leq h$, we have $U \times U = h; (U \times U); h$. Clearly $\emptyset = h; \emptyset; h$.
4. $h; (x \vee y); h = (h; x; h) \vee (h; y; h) = x \vee y$.
5. $h; (x \wedge y); h \leq (h; x; h) \wedge (h; y; h) = x \wedge y$. The reverse inclusion comes from part 2.
6. Using $h; h = h$ and the fact that x and y are h -relations we have $h; (x; y); h = h; (h; x; h); (h; y; h); h = x; y$. □

We use the notation $h\text{-Rel}$ for the set of all h -relations. The special case of $h = 1$ includes all relations on U as h -relations. The set of all relations on U will be denoted Rel . The above results show that $h\text{-Rel}$ is closed under the operations of union, intersection and composition in Rel , that h is the identity element for composition in $h\text{-Rel}$, and that the greatest and least elements of $h\text{-Rel}$ are those in Rel . However, $h\text{-Rel}$ is neither closed under complementation nor under taking converses, as can be seen from the example in Figure 3. Note that in this figure the relation h modelling the incidence structure is the reflexive closure of the relation denoted by φ in the earlier discussion. This relation φ would not include (a, a) .

Let \mathcal{L} denote the lattice of all subgraphs for a given (U, h) . Given any h -relation x , and any subgraph $K \in \mathcal{L}$, we can check that defining $x(K) = \{u \in U : \exists k \in K ((k, u) \in x)\}$ makes the assignment $K \mapsto x(K)$ a join preserving operation on \mathcal{L} . The appropriate algebraic setting for these operations is a quantale [22] and it is straightforward to check that we have the following result [10].

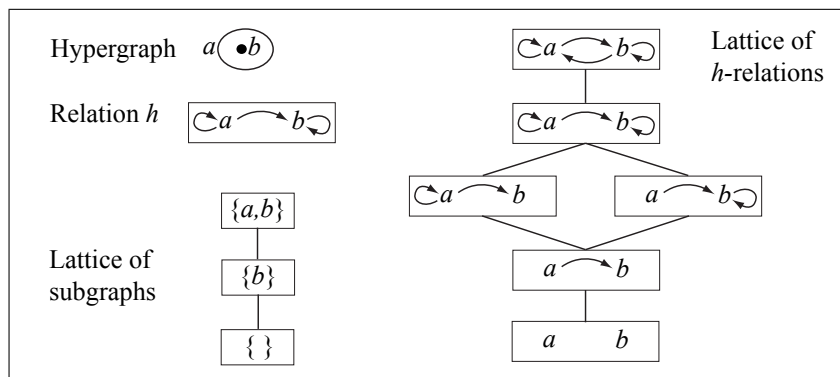


Figure 3: Hypergraph with edge a and node b showing the lattice of h -relations

Theorem 6. *The h -relations in $h\text{-Rel}$ form a quantale under composition of relations, with unit h , which is isomorphic to the quantale of join preserving functions from the lattice \mathcal{L} to itself.* \square

What this result does not tell us is whether relations on a hypergraph have analogues of the operations we find in a relation algebra, in particular converse and complementation. We shall see that the analogue of the complement consists, hardly surprisingly in view of the bi-Heyting structure of the lattice of subgraphs, of a pseudocomplement and its dual. In addition there are two converse operations which we will now explore together with their connection to the weaker complements.

3.4. The converse-complement operation

Although h -relations are closed neither under complement nor under converse, they are closed under their combination, that is whenever x is an h -relation then so is $\check{\check{x}} = \bar{\bar{x}}$.

Lemma 7. *Let x be any relation. Then $h ; x ; h = x$ iff $\check{\check{h}} ; \bar{x} ; \check{\check{h}} = \bar{x}$.*

PROOF. The operation $h ; _ ; h : \text{Rel} \rightarrow \text{Rel}$ is left adjoint to $h \setminus _ / h$. Hence $x \leq h \setminus x / h$ if and only if $h ; x ; h \leq x$. Writing $h \setminus x / h$ as $(\check{\check{h}} ; \bar{x} ; \check{\check{h}})$ gives $h ; x ; h \leq x$ if and only if $\check{\check{h}} ; \bar{x} ; \check{\check{h}} \leq \bar{x}$. The result follows since $x \leq h ; x ; h$ and $\bar{x} \leq \check{\check{h}} ; \bar{x} ; \check{\check{h}}$ by Lemma 5. \square

An immediate corollary is the following which establishes the important fact that h -relations are closed under the converse-complement.

Proposition 8. *For $x \in \text{Rel}$ the following four statements are equivalent*

- (i) $x \in h\text{-Rel}$ (ii) $\bar{x} \in \check{\check{h}}\text{-Rel}$ (iii) $\check{\check{x}} \in \check{\check{h}}\text{-Rel}$ (iv) $\bar{\bar{x}} \in h\text{-Rel}$.

PROOF. The equivalence of (i) and (ii) is immediate from Lemma 7. For the other parts, $h ; x ; h = x$ iff $\check{h} ; \check{x} ; \check{h} = \check{x}$ iff $h ; \bar{x} ; h = \bar{x}$ by Lemma 7. \square

It is this converse-complement operation that is key to understanding the algebra of relations on a hypergraph. However, the general picture is not restricted to hypergraphs and we now begin to explore the more general setting by considering Heyting algebras equipped with an order-reversing involution.

4. Symmetric Heyting algebras

The notion of symmetric Heyting algebra is due to A. Monteiro in the 1960s and the extensive development was brought together in the substantial paper [23]. Here we only need some basic concepts, but as far as we have been able to determine the significance of these structures in the algebra of relations on hypergraphs has not been noted before.

4.1. Basic properties

Definition 13. A **symmetric Heyting algebra** is an algebra $(A, \vee, \wedge, \rightarrow, \perp, \top, \frown)$ where

1. $(A, \vee, \wedge, \rightarrow, \perp, \top)$ is a Heyting algebra,
2. the operation $\frown : A \rightarrow A$ satisfies
 - (a) $\frown \frown x = x$,
 - (b) $x \leq y \Rightarrow \frown y \leq \frown x$ where \leq is the order in the Heyting algebra.

In a symmetric Heyting algebra, the operation \frown will be called the **symmetry**. Examples of symmetric Heyting algebras include the following.

1. Any Boolean algebra with complement as the symmetry.
2. Any relation algebra with the converse-complement as the symmetry. Thus symmetries need not be unique.
3. The algebra of relations on a hypergraph with the converse complement operation as the symmetry (justified by the results in Section 7).
4. If B is a Boolean algebra, consider the set

$$A = \{(x_1, x_2) \in B \times B \mid x_1 \leq x_2\}.$$

This is a Heyting algebra with joins and meets defined componentwise and where

$$(x_1, x_2) \rightarrow (y_1, y_2) = ((\bar{x}_1 \vee y_1) \wedge (\bar{x}_2 \vee y_2), \bar{x}_2 \vee y_2).$$

The symmetry $\frown(x_1, x_2) = (\bar{x}_2, \bar{x}_1)$ makes this a symmetric Heyting algebra.

We gather together some basic properties of symmetric Heyting algebras. The proofs of these are straightforward calculations.

Proposition 9. *Let $(A, \vee, \wedge, \rightarrow, \perp, \top, \frown)$ be a symmetric Heyting algebra. Then*

1. $\frown(x \vee y) = \frown x \wedge \frown y$ and $\frown(x \wedge y) = \frown x \vee \frown y$,
2. $x \wedge y = \frown(\frown x \vee \frown y)$ and $x \vee y = \frown(\frown x \wedge \frown y)$,
3. $\top = \frown \perp$ and $\perp = \frown \top$. □

The first part of Proposition 9 shows that the symmetry satisfies the de Morgan laws. Symmetric Heyting algebras have also been called de Morgan-Heyting algebras [24] because they can be defined as de Morgan algebras with additional structure. The second and third parts of Proposition 9 show that the definition of a symmetric Heyting algebra is somewhat redundant as some of the operations can always be defined in terms of others using the symmetry. An alternative but equivalent approach would be to require a structure $(A, \wedge, \rightarrow, \top, \frown)$ where

1. (A, \wedge, \top) is a semilattice,
2. the operation $\frown : A \rightarrow A$ satisfies
 - (a) $\frown \frown x = x$,
 - (b) $x \leq y \Rightarrow \frown y \leq \frown x$ where \leq is the order in the semilattice.
3. $x \wedge y \leq z$ iff $x \leq y \rightarrow z$

Proposition 10. *Let \succ be defined by $x \succ y = \frown(\frown y \rightarrow \frown x)$. Then $(A, \vee, \wedge, \rightarrow, \succ, \perp, \top)$ is a bi-Heyting algebra. □*

4.2. Left and right converses

In any bi-Heyting algebra the operations of pseudocomplement and its dual are well known. In the case of a symmetric Heyting algebra two other operations are available and play a key role in what follows.

Definition 14. In any symmetric Heyting algebra the following four unary operations can be defined.

$$\begin{array}{ll} \text{pseudocomplement:} & \neg x = x \rightarrow \perp & \text{right converse:} & \smile x = \frown x \rightarrow \perp \\ \text{dual pseudocomplement:} & \lrcorner x = \top \succ x & \text{left converse:} & \smile x = \top \succ \frown x \end{array}$$

Proposition 11. *In any symmetric Heyting algebra $y \succ \frown x \leq z$ iff $x \leq \frown z \rightarrow \frown y$.*

PROOF. $y \succ \frown x \leq z$ holds iff $\frown(x \rightarrow \frown y) \leq z$ by definition of \succ . But this is equivalent to $\frown z \leq x \rightarrow \frown y$ and thus to $x \wedge \frown z \leq \frown y$ by the adjunction between implication and meet. □

In particular, taking $y = \top$ we have the corollary:

Corollary 12. *Left converse is left adjoint to right converse: $\smile \dashv \smile$. □*

Proposition 13. *The following identities hold.*

$$\begin{array}{llll} \smile x = \neg \frown x & \smile x = \lrcorner \frown x & \neg x = \smile \frown x & \lrcorner x = \smile \frown x \\ \smile x = \frown \neg x & \smile x = \frown \lrcorner x & \lrcorner x = \frown \smile x & \neg x = \frown \smile x \\ \smile \smile x = \neg \neg x & \smile \smile x = \lrcorner \lrcorner x & \neg \lrcorner x = \smile \smile x & \lrcorner \neg x = \smile \smile x \end{array}$$

PROOF. The definitions of \cup and \smile immediately give the first row. The first two items in the second row follow from $x \rightarrow y = \wedge(\wedge y \neg \wedge x)$, and the remaining ones follow from the earlier ones using the fact that \wedge is an involution. \square

Proposition 14. *Let A be a symmetric Heyting algebra and $x \in A$. Then $\smile x \leq \cup x$. The equality $\smile x = \cup x$ holds for all x iff A is a Boolean algebra.*

PROOF. Using Propositions 3 and 13 we have $\smile x = \neg \wedge x \leq \neg \wedge x = \cup x$. Using Proposition 13 again, $\smile x = \cup x$ holds iff $\wedge \neg x = \wedge \neg x$ iff $\neg x = \neg x$. This is equivalent to A being Boolean by Proposition 3. \square

Proposition 15. *For all x, y in any symmetric Heyting algebra,*

$$\begin{aligned} \text{(i)} \quad \cup(x \vee y) &= \cup x \vee \cup y, & \text{(iii)} \quad \cup(x \wedge y) &= \neg \neg(\cup x \wedge \cup y), \\ \text{(ii)} \quad \smile(x \wedge y) &= \smile x \wedge \smile y, & \text{(iv)} \quad \smile(x \vee y) &= \neg \neg(\smile x \vee \smile y). \end{aligned}$$

PROOF. Parts (i) and (ii) are immediate since \cup is a left adjoint and \smile is a right adjoint. For part (iii) we have $\cup(x \wedge y) = \neg \wedge(x \wedge y) = \neg(\wedge x \vee \wedge y)$. But in any bi-Heyting algebra $\neg(a \vee b) = \neg \neg(\neg a \wedge \neg b)$, so the result follows using Proposition 13. The fourth part is similar. \square

5. Symmetric Heyting relation algebras

5.1. A generalization of relation algebra

The next definition provides a generalization of relation algebras in which instead of a Boolean algebra part there is a bi-Heyting algebra which arises from a symmetric Heyting algebra.

Definition 15. A **symmetric Heyting relation algebra**, \mathbf{A} , is an algebra

$$\mathbf{A} = (A, \vee, \wedge, \rightarrow, \top, \perp, ;, 1, \wedge)$$

subject to conditions 1–3 below. We use \leq to denote the partial order in the Heyting algebra, and use the derived operations \smile and \cup defined earlier.

1. $(A, \vee, \wedge, \rightarrow, \top, \perp, \wedge)$ is a symmetric Heyting algebra,
2. $(A, ;, 1)$ is a monoid,
3. the following hold for all $x, y, z \in A$.

$$\text{LCV:} \quad \cup(x ; y) \leq (\cup y) ; (\cup x)$$

$$\text{ACR:} \quad x ; y \leq z \text{ iff } x \leq \wedge(y ; \wedge z)$$

We will use the acronym ‘SHRA’ as a shorter form of ‘symmetric Heyting relation algebra’ from now on. The condition LCV (Left ConVerse) dictates how the left converse interacts with composition, and ACR (Adjointness of Composition on the Right) requires composition on the right to have a specified right adjoint. Composition on the left also has a right adjoint as follows.

Proposition 16. *In any SHRA, $x ; y \leq z \Leftrightarrow y \leq \wedge(\wedge z ; x)$.*

PROOF.

$$\begin{aligned}
x ; y \leq z &\Leftrightarrow x \leq \wedge(y ; \wedge z) && \text{by ACR} \\
&\Leftrightarrow y ; \wedge z \leq \wedge x && \wedge \text{ reverses order and is an involution} \\
&\Leftrightarrow y \leq \wedge(\wedge z ; \wedge x) && \text{by ACR} \\
&\Leftrightarrow y \leq \wedge(\wedge z ; x) && \square
\end{aligned}$$

It follows immediately from Proposition 16 and axiom ACR, using the comment after Definition 5, that the composition operation $;$ is order preserving in both arguments.

Proposition 17. *In any SHRA $(\smile y) ; (\smile x) \leq \smile(x ; y)$.*

PROOF. By Corollary 12, the right converse is order preserving and we have both $x \leq \smile\smile x$ and $\smile\smile x \leq x$. Using these facts together with LCV and the order preserving property of composition, we have:

$$(\smile y) ; (\smile x) \leq \smile\smile((\smile y) ; (\smile x)) \leq \smile((\smile\smile x) ; (\smile\smile y)) \leq \smile(x ; y). \quad \square$$

We now clarify the sense in which SHRAs are generalizations of relation algebras. Given any relation algebra $(A, \vee, \wedge, \bar{}, \top, \perp, ;, \smile, \smile)$ we obtain a symmetric Heyting relation algebra by defining $x \rightarrow y = \bar{x} \vee y$ and $\wedge x = \check{x}$. This means that $\smile x = \check{x} = \smile x$ and it is then straightforward to verify the conditions 1–3 in the definition using basic properties of relation algebras. Conversely, a symmetric Heyting relation algebra in which the Heyting part is Boolean is a relation algebra in the following sense.

Theorem 18. *Let \mathbf{A} be a symmetric Heyting relation algebra. Suppose that the Heyting part of \mathbf{A} is Boolean, and let \bar{x} denote the complement of x in this Boolean algebra. If \check{x} is defined by $\check{x} = \wedge \bar{x}$ then $(A, \vee, \wedge, \bar{}, \top, \perp, ;, \smile, \smile)$ is a relation algebra.*

PROOF. We use the formulation of relation algebra given in Definition 9. RA-i follows immediately from Proposition 14 together with LCV and Proposition 17. Note that from Proposition 13 we can also use the equality of $\bar{}$ with \perp and of \smile with \smile to obtain

$$\bar{\check{x}} = \wedge x = \check{\check{x}}. \quad (1)$$

Since $\check{\check{x}} = \smile\smile x$ we get RA-ii from Proposition 13. By Proposition 15 we have RA-iii. The axiom RA-iv holds because the operation $x ; _$ is a left adjoint.

RA-v holds as $\check{y} ; \check{x} \leq \check{y} ; \check{x}$ yields from ACR that $\check{y} \leq \wedge(\check{x} ; \wedge(\check{y} ; \check{x}))$ which implies $\check{x} ; \wedge(\check{y} ; \check{x}) \leq \wedge\check{y}$. Using equation (1) and the previously established RA-i, we get $\check{x} ; \bar{x} ; \bar{y} \leq \bar{y}$ as required. \square

There are non-Boolean examples of SHRAs, in particular the algebra of relations on a hypergraph. These examples arise from a general means of constructing SHRAs which is presented in Section 6 below.

Definition 19. An element h in an SHRA is

1. **reflexive** if $1 \leq h$,
2. **transitive** if $h ; h \leq h$,
3. a **pre-order** if it is both reflexive and transitive.

Note that when h is a pre-order, $h ; h = h$. From now on we will always use h to denote a pre-order in an SHRA.

Definition 20. An element $x \in A$ is **h -stable** if $x = h ; x ; h$. The set of all h -stable elements in A is denoted A_h .

Note that h is itself h -stable. Since h is reflexive and composition is order preserving in both arguments we have that $x \leq h ; x ; h$ for any x . This means that $x \in A_h$ iff $h ; x ; h \leq x$. Using equation (2) above we can also express h -stability by $x \in A_h$ iff $x \leq h \setminus x / h$.

Lemma 22. *The inclusion of A_h in A has both a left adjoint and a right adjoint. These are respectively $h ; _ ; h$ and $h \setminus _ / h$.*

PROOF. Let $x \in A$ and $y \in A_h$, then we have the following calculations.

For the left adjoint, $h ; x ; h \leq y$ implies $x \leq y$ as $x \leq h ; x ; h$, and conversely, $x \leq y$ implies $h ; x ; h \leq h ; y ; h = y$. For the right adjoint, $y \leq h \setminus x / h$ iff $h ; y ; h \leq x$ iff $y \leq x$. \square

From the adjoints identified in Lemma 22 we have two corollaries that follow from general properties of adjunctions. Note that the first of these means that x being h -stable is equivalent to $x = h \setminus x / h$.

Corollary 23. *For any $x \in A$, $h \setminus x / h \leq x$.* \square

Corollary 24. *A_h is closed under all joins and meets which exist in the Heyting part of \mathbf{A} . In particular, $\perp, \top \in A_h$ and if $x, y \in A_h$ then $x \vee y \in A_h$ and $x \wedge y \in A_h$.* \square

General properties of adjunctions also tell us that for any $x \in A$, the elements $h ; x ; h$ and $h \setminus x / h$ are respectively the least h -stable element greater than or equal to x and the greatest h -stable element less than or equal to x . Symbolically,

$$h ; x ; h = \bigwedge \{y \in A_h \mid x \leq y\}, \quad (3)$$

and

$$h \setminus x / h = \bigvee \{y \in A_h \mid y \leq x\}. \quad (4)$$

Next we see that the h -stable elements are closed under the symmetry \smile in \mathbf{A} as well as under composition and under the operations $\setminus, /$ and \dagger from Definitions 16 and 18.

Proposition 25. *Let $x, y \in A_h$, then:*

1. $\neg x \in A_h$
2. $x ; h = x = h ; x$
3. $x ; y \in A_h$
4. $x \setminus y \in A_h$
5. $x / y \in A_h$
6. $x \dagger y \in A_h$

PROOF.

1. $\neg x \in A_h$ iff $\neg x \leq h \setminus (\neg x) / h$ iff $\neg x \leq \neg(h ; x ; h)$, by Lemma 20, iff $h ; x ; h \leq x$.
2. $x ; h = h ; x ; h ; h = x = h ; h ; x ; h = h ; x$
3. $h ; (x ; y) ; h = (h ; x) ; (y ; h) = x ; y$
4. $x \setminus y = \neg(\neg y ; x)$ which lies in A_h by parts 1 and 3.

Parts 5 and 6 are similar to part 4 because $/$ and \dagger are expressible in terms of \neg and $;$. \square

The h -stable elements are not closed under the implication, \rightarrow , but they do support their own implication which is defined by means of the right adjoint to the inclusion of A_h in A .

Definition 21. The operation \xrightarrow{h} in A_h is defined by

$$x \xrightarrow{h} y = h \setminus (x \rightarrow y) / h.$$

Proposition 26. $\mathbf{A}_h = (A_h, \vee, \wedge, \xrightarrow{h}, \top, \perp, \neg)$ is a symmetric Heyting algebra.

PROOF. Since $(A, \vee, \wedge, \top, \perp)$ is a lattice, Corollary 24 implies that the h -stable elements form a lattice $(A_h, \vee, \wedge, \top, \perp)$. To show that we have a Heyting algebra, consider the following adjunctions.

$$\begin{array}{ccccc}
A_h & \xrightarrow{\quad \perp \quad} & A & \xrightarrow{\quad \neg \wedge y \quad} & A & \xrightarrow{\quad h ; _ ; h \quad} & A_h \\
& & \xleftarrow{\quad h \setminus _ / h \quad} & & \xleftarrow{\quad y \rightarrow _ \quad} & & \xleftarrow{\quad _ \quad}
\end{array}$$

Composing the adjunctions shows $h ; (_ \wedge y) ; h \dashv h \setminus (y \rightarrow _) / h$, but for any $x, y \in A_h$, we have $h ; x \wedge y ; h = x \wedge y$ so that $_ \wedge y \dashv h \setminus (y \rightarrow _) / h$. \square

Definition 22. In the symmetric Heyting algebra $(A_h, \vee, \wedge, \xrightarrow{h}, \top, \perp, \neg)$ the symbols $\overset{h}{\succ}, \overset{h}{\curvearrowright}, \overset{h}{\curvearrowleft}, \overset{h}{\dashv}, \overset{h}{\perp}$ will denote respectively the dual implication, the left converse, the right converse, the pseudocomplement and the dual pseudocomplement.

Proposition 27. *In the symmetric Heyting algebra $(A_h, \vee, \wedge, \overset{h}{\rightarrow}, \top, \perp, \frown)$, the operations $\succ, \overset{h}{\cup}, \overset{h}{\smile}, \overset{h}{\neg}, \overset{h}{\dashv}$ can be expressed in terms of the structure in \mathbf{A} :*

$$\begin{aligned} x \succ y &= h ; (x \rightarrow y) ; h, \\ \overset{h}{\cup} x &= h ; \cup x ; h, & \overset{h}{\smile} x &= h \setminus \smile x / h, \\ \overset{h}{\dashv} x &= h ; \dashv x ; h, & \overset{h}{\neg} x &= h \setminus \neg x / h. \end{aligned}$$

PROOF. For the dual implication, by Proposition 10 we have to show that $x \succ y = \frown(\frown y \overset{h}{\rightarrow} \frown x)$ which is a straightforward calculation using Lemma 20. The other four parts are similar calculations. \square

Theorem 28. *The structure $\mathbf{A}_h = (A_h, \vee, \wedge, \overset{h}{\rightarrow}, \top, \perp, ;, h, \frown)$ is an SHRA.*

PROOF. By Proposition 26, $(A_h, \vee, \wedge, \overset{h}{\rightarrow}, \top, \perp, \frown)$ is a symmetric Heyting algebra, and by Proposition 25, $(A_h, ;, h)$ is a monoid.

To verify LCV we need $\overset{h}{\cup}(x;y) \leq (\overset{h}{\cup}y);(\overset{h}{\cup}x)$. Using Proposition 27 we have $\overset{h}{\cup}(x;y) = h ; (\cup(x;y)); h \leq h ; (\cup y; \cup x); h \leq (h ; \cup y; h); (h ; \cup x; h) = \overset{h}{\cup}y ; \overset{h}{\cup}x$.

The condition ACR is immediate as it holds in \mathbf{A} . \square

7. The special case of relations on a hypergraph

7.1. Boolean-specific properties of the construction

In the special case that the SHRA, \mathbf{A} , in the last section is Boolean (equivalently, is a relation algebra in the sense of Definition 9) we find that the construction of \mathbf{A}_h has some additional properties. In this section we use \mathbf{B} to denote a relation algebra, and \mathbf{B}_h the SHRA constructed from a pre-order h in \mathbf{B} as described in the last section.

Proposition 29. *If h is a pre-order in \mathbf{B} then \check{h} is also a pre-order.* \square

We note in passing that when x is a pre-order in a general SHRA, neither $\cup x$ nor $\smile x$ need be a pre-order. It is straightforward to construct examples from the case of relations on a hypergraph where $\smile x = \perp$ holds (so is not reflexive) and where $\cup x$ need not be transitive.

Proposition 30. *If $x \in B_h$ then $\bar{x} \in B_{\check{h}}$ and $\check{x} \in B_{\check{h}}$.*

PROOF. As $x \in B_h$, we have $\bar{x} = \overline{h ; x ; h} = \check{h} \setminus \bar{x} / \check{h}$, so $\bar{x} \in B_{\check{h}}$. The fact that $\check{x} \in B_{\check{h}}$ is immediate from properties of the converse. \square

Corollary 31. *The following are equivalent:*

1. $x \in B_h$

2. $\bar{x} \in B_{\check{h}}$
3. $\check{x} \in B_{\bar{h}}$
4. $\bar{\check{x}} \in B_h$

□

To summarize, we have the following, which in part re-states Proposition 27 in convenient notation in the special case that \mathbf{A} is a relation algebra.

Theorem 32. *In a relation algebra $\mathbf{B} = (B, \vee, \wedge, \neg, \top, \perp, ;, 1, \smile)$, the set of all h -stable elements forms an SHRA $\mathbf{B}_h = (B_h, \vee, \wedge, \rightarrow, \top, \perp, ;, 1, \smile)$ where \rightarrow is defined in terms of the operations in \mathbf{B} by*

$$x \rightarrow y = h \setminus (\bar{x} \vee y) / h.$$

In addition, the derived operations are expressible as follows.

$$\begin{aligned} x \succ y &= h ; (x \wedge \bar{y}) ; h, \\ \smile x &= h ; \check{x} ; h, & \smile x &= h \setminus \check{x} / h, \\ \neg x &= h ; \bar{x} ; h, & \neg x &= h \setminus \bar{x} / h. \end{aligned}$$

□

7.2. Relations on a hypergraph

Suppose we have a hypergraph as in Section 3 consisting of a set U and a relation h on U which is an incidence relation in the sense of Definition 12. Taking the full relation algebra on U to be \mathbf{B} , then the elements of \mathbf{B}_h are precisely the h -relations from the same definition. Thus the relations on a hypergraph constitute an SHRA and in particular the two converses are given by the expressions in Theorem 32. The extent to which these converses are weaker than the classical converse can be seen in the following specific examples.

1. In the case shown in Figure 3 we can use the expressions in Theorem 32 to construct $\smile h$ and $\smile h$. We find that $\smile h = \perp$ and $\smile h = \top$.
2. Taking $U = \{u, v, w, x, y\}$ and defining h to be the reflexive closure of the relation $\{(x, u), (x, v), (y, v), (y, w)\}$, we find that $(u, v) \in \smile h$ and $(v, w) \in \smile h$ but $(u, w) \notin \smile h$ so that $\smile h$ is not transitive. We also find in the same example that $(x, w) \in \smile h$ but that $(w, x) \notin \smile h$ although we do have $(w, x) \in \smile \smile h$. Since h is the identity for composition we thus have $1 \neq \smile 1$ and $\smile \smile 1 \neq \smile 1$ and $\smile \smile 1 \neq 1$.
3. We can construct an example of an h -relation x where not only is the iterated left converse $\smile^2 x = \smile \smile x$ strictly greater than x , but repeatedly applying this operation produces an infinite sequence of successively strictly greater relations. Consider the graph with edges $\{2n + 1 : n \in \mathbb{Z}\}$ and nodes $\{2n : n \in \mathbb{Z}\}$ and where edge $2n + 1$ is incident only with nodes $2n$ and $2n + 2$. This means we have $U = \mathbb{Z}$ and, denoting the identity relation on U by 1_U ,

$$h = 1_U \cup \{(2n + 1, 2n) : n \in \mathbb{Z}\} \cup \{(2n + 1, 2n + 2) : n \in \mathbb{Z}\}.$$

Define x to be the relation $\{(1, 2)\}$. It can be checked that $(-1, 4) \in \cup^2 x$, that $(-3, 6) \in \cup^4 x$. In general $\cup^{n+2} x$ is strictly larger than $\cup^n x$ for all n .

Similarly, the right-converse can be repeated on a suitable infinite relation on the same graph to demonstrate that even powers of this operation can become successively strictly smaller. Of course neither $\cup\cup x$ nor $\cup\cup x$ is the appropriate analogue of \check{x} , for which we need to consider $\cup\cup x$ and $\cup\cup x$ in view of Corollary 12.

8. Application to mathematical morphology on hypergraphs

In mathematical morphology on sets certain relationships between complement, converse, and the operations of dilation and erosion are well-known and are used in developing the basic theory. In this section we see how the properties of relations on a hypergraph by virtue of their SHRA structure can be used to establish some basic properties of mathematical morphology on hypergraphs. We start by recalling some background in the set-based case.

8.1. Mathematical morphology on sets

Given an arbitrary relation x on a set U , two operations are defined on the powerset $\mathcal{P}U$.

Definition 23. For $K \subseteq U$ the **dilation**, $K \oplus x$, and **erosion**, $x \ominus K$, are:

$$\begin{aligned} K \oplus x &= \{u \in U : \exists v ((v, u) \in x \text{ and } v \in K)\}, \\ x \ominus K &= \{u \in U : \forall v ((u, v) \in x \text{ implies } v \in K)\}. \end{aligned}$$

The motivation for writing the subset on the left in dilation and on the right in erosion comes from the identities $K \oplus (x ; y) = (K \oplus x) \oplus y$ and $(x ; y) \ominus K = x \ominus (y \ominus K)$. These operations are, of course, well known and appear in many contexts other than mathematical morphology.

Erosion and dilation by a relation x are operations on subsets of U . However, subsets can be regarded as relations and erosion and dilation can easily be extended to operations on arbitrary relations. In this context they appear simply as composition with x and right residuation by x . The well-known property in mathematical morphology that $K \oplus \check{x} = -(x \ominus -K)$, where $-K$ is the complement of K in U , is then nothing more than $y ; \check{x} = (\overline{y} / x)$. However, this more general context is not the appropriate setting for considering erosion and dilation on subgraphs. In this case we can look for an analogue of $K \oplus \check{x} = -(x \ominus -K)$ in which the complement and the converse are each replaced by one of the weaker operations described above. This is achieved in Theorem 37 but it only holds when K is a subgraph and not an arbitrary relation. To do this we need to identify subgraphs as particular relations, namely the range elements.

8.2. Range elements

We recall the following notion from [1, p294].

Definition 24. A **range element** in a relation algebra \mathbf{B} is an element x for which $\top ; x = x$.

The significance of range elements is that when \mathbf{B} is the algebra of all relations on a set U , then the range elements are exactly the relations of the form $U \times K$ for $K \subseteq U$, and thus they provide one way in which subsets of U can be identified with particular relations on U .

Lemma 33. *Let y be a range element.*

1. \bar{y} is a range element.
2. If x is reflexive then $x ; y = y$.
3. If z is any element, then $y ; z$ is a range element.
4. If z is any element, then y / z is a range element.

PROOF. For (1) see [1, p316] Theorem 305. For (2) we have $1 \leq x \Rightarrow 1 ; \top \leq x ; \top$ so that $\top \leq x ; \top$ and thus $x ; \top = \top$. Then $x ; y = x ; \top ; y = \top ; y = y$. Part (3) is an immediate consequence of the definition, and part (4) follows by writing $y / z = \bar{y} ; \check{z}$ and using parts (1) and (3). \square

As noted in [1, p316], the range elements of \mathbf{B} form a subalgebra of the Boolean part of \mathbf{B} . This can be extended to \mathbf{B}_h as follows.

Theorem 34. *The set of all range elements in \mathbf{B} which are h -stable forms a subalgebra of the bi-Heyting part of \mathbf{B}_h .*

PROOF. Let R_h denote the set of all h -stable range elements in \mathbf{B} . Since \mathbf{B}_h is closed under unions and intersections, and as the range elements in \mathbf{B} are closed under these operations too, this is true for all elements of R_h .

To show that R_h is closed under relative pseudocomplementation, suppose that $x, y \in R_h$. We have that $x \rightarrow y = \check{h} ; (x \wedge \bar{y}) ; \check{h}$, but since $x \wedge \bar{y}$ is a range element and \check{h} is reflexive, $\check{h} ; (x \wedge \bar{y}) = x \wedge \bar{y}$. Also using the fact that $x \wedge \bar{y}$ is a range element, we get that $(x \wedge \bar{y}) ; \check{h}$ is a range element so that $x \rightarrow y \in R_h$. Closure under the dual relative pseudo-complement is shown in a similar way using the fact that $x \succ y = h ; (x \wedge \bar{y}) ; h$. \square

The elements of \mathbf{B} act on the set \mathbf{R} of all range elements in two ways

$$\begin{aligned} \oplus : \mathbf{R} \times \mathbf{B} &\rightarrow \mathbf{R} & \text{where} & & r \oplus x &= r ; x \\ \ominus : \mathbf{B} \times \mathbf{R} &\rightarrow \mathbf{R} & \text{where} & & x \ominus r &= r / x \end{aligned}$$

and these actions satisfy $r \oplus (x ; y) = (r \oplus x) \oplus y$ and $(x ; y) \ominus r = x \ominus (y \ominus r)$. In the special case that \mathbf{B} is the full relation algebra on a set U , the elements of \mathbf{R} correspond to subsets of U and elements of \mathbf{B} correspond exactly to the join preserving functions on the Boolean algebra \mathbf{R} .

Now, the two actions restrict to actions of \mathbf{B}_h on \mathbf{R}_h by Lemma 33. We know that for a hypergraph given in terms of h , the relations on the hypergraph are the elements of \mathbf{B}_h . We can now identify the subgraphs with the elements of \mathbf{R}_h in view of the fact that $K \subseteq U$ is a subgraph iff $(U \times K) ; h \subseteq (U \times K)$ and the following result.

Proposition 35. *Let $r \in \mathbf{B}$ be a range element. Then r is h -stable iff $r ; h \leq r$.*

PROOF. If $r ; h \leq r$ then $h ; r ; h \leq h ; r$ but h is reflexive so $h ; r = r$ by Lemma 33. Hence $h ; r ; h \leq r$. Conversely, suppose $h ; r ; h \leq r$. We then have $r ; h \leq h ; r ; h \leq r$. \square

Theorem 36. *Suppose \mathbf{B} is the full relation algebra on a set U . In this case the elements of \mathbf{B}_h can be identified with join preserving functions on the bi-Heyting algebra \mathbf{R}_h .*

PROOF. The known correspondence between elements of \mathbf{B} and join preserving functions on the set of range elements \mathbf{R} , identifies $x \in B$ with the function $_ ; x$. Now a join preserving function f on R_h extends to all of R by defining $f'(u) = f(u ; h)$ for any $u \in R$ as $u ; h \in R_h$ by the previous result. So every join preserving function on R_h can be represented in the form $_ ; x$ for some $x \in B$. But for any $r \in R_h$ we have $r ; h ; x ; h = r ; x$ since r is h -stable and $r ; x \in R_h$. Thus the functions $_ ; x$ and $_ ; (h ; x ; h)$ agree on all h -stable range elements and we have $h ; x ; h \in B_h$.

So every join preserving function on R_h can be expressed as $_ ; x$ where $x \in B_h$. But this expression is uniquely determined because if $x, y \in B_h$ and $_ ; x$ agrees with $_ ; y$ on all of R_h , then for any $r \in R$ we get $(r ; h) ; x = (r ; h) ; y$ as $r ; h \in R_h$, and hence $x = y$ as $h ; x = x$ and $h ; y = y$. \square

8.3. The morphology equations

In any relation algebra, the equation $y / x = \overline{\overline{y} ; \check{x}}$ is equivalent to $y / \check{x} = \overline{\overline{y} ; x}$ and to $y ; \check{x} = \overline{\overline{y} / x}$ using just the fact that complementation and conversion are involutions. When we consider the h -stable elements we no longer have these properties, but we can establish the following.

Theorem 37. *Let B be any relation algebra, and let h be any pre-order element of B . Let $x, y \in B_h$ and assume that y is a range element. Then*

1. $y / \cup x = \neg(\neg y ; x)$, and
2. $y ; \cup x = \neg(\neg y / x)$.

PROOF. For (1), we have $y / \cup x = \overline{\overline{y} ; (\cup x)^\vee} = \overline{\overline{y} ; \check{h} ; x ; \check{h}} = \overline{\overline{y} ; x ; \check{h}}$ as \check{h} is h -stable.

Expanding the RHS of (1) we get $h \setminus \overline{\overline{h} ; \check{y} ; h} / h = \overline{\overline{\check{h} ; h ; \check{y} ; x ; \check{h}}}$ using the h -stability of x . Now $\check{h} ; h$ is reflexive, so by Lemma 33 $\check{h} ; h ; \check{y} = \check{y}$ which establishes (1).

For (2) we have $\lrcorner(\neg y/x) = h; \overline{(h \setminus \bar{y} / h)}; x; h$. A straightforward calculation, making use of the \check{h} -stability of \check{x} , establishes that $\lrcorner(\neg y/x) = h; \check{h}; y; \check{x}; h$ so, by Lemma 33, $\lrcorner(\neg y/x) = y; \check{x}; h$. Now, $y; \cup x = y; h; \check{x}; h = y; \check{x}; h$ since y is h -stable. \square

We noted earlier that the identity $K \oplus \check{x} = -(x \ominus -K)$ connecting erosion, dilation, converse and complement for a subset K and a relation x could be deduced immediately from the more general fact that $y; \check{x} = \overline{(\bar{y} / x)}$ for arbitrary relations x and y on a set U . To see that morphology on subgraphs of a hypergraph is more subtle than this consider Theorem 37 part (2). If this generalizes to a statement about arbitrary h -relations x and y we can take $x = y = h$. This leads to $h; \cup h = \lrcorner(\neg h / h)$ which implies $\cup h = \lrcorner \neg h$. But $\lrcorner \neg h = \cup \cup h$ by Proposition 13 so we would have $\cup h = \cup \cup h$, which we have seen need not be the case in Section 7.2.

9. Conclusions

We have seen that relations on a hypergraph provide a concrete example of a generalization of the algebraic properties of relations in which, instead of a single converse operation that is an involution there is a pair of adjoint operations. The notion of a symmetric Heyting relation algebra (SHRA) has been introduced as an appropriate abstract setting for relations with this type of structure. A general construction has been given which includes as a special case the algebra of relations on a hypergraph.

The setting of symmetric Heyting relation algebras has been applied to mathematical morphology for hypergraphs by establishing a connection between the operations of erosion and dilation and the left converse and the pseudocomplement and its dual. This generalizes a very basic fact linking erosion and dilation with converse and complement for relations on sets, but in the case of hypergraphs we saw we cannot express this link just in terms of relations: it is necessary to consider the action of relations on range elements.

There are three directions for further work. We noted that relations on a hypergraph are a special case of the category-theoretic notion of a distributor or profunctor. However the weaker kinds of converse we have analysed here do not appear to have been studied in the context of distributors, and investigating to what extent this is possible is one direction for future work. In another direction, the foundations for mathematical morphology on hypergraphs will be developed by determining the extent to which more properties known in the set-based case can be extended to hypergraphs. Finally, the left converse operation as described in [11] has been used [25, 26] in providing semantics for a bi-intuitionistic modal logic. Connections between modal logic and the more general setting in the current paper remain to be investigated.

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