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Lagrangian manifolds and asymptotically optimal stabilizing feedback control

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Abstract. In this paper, under some local controllability hypotheses, we present an algorithm for finding an approximation to the optimal control for an infinite horizon linear analytic optimal regulator problem. This approximation is asymptotically optimal, in that it converges to the optimal control close to the origin, and is extremely simple to implement as it involves only the solution of algebraic Riccati equations. We demonstrate its effectiveness on a simulated inverted pendulum and present a test for determining in advance a stability region.

1 Introduction

Consider the following infinite time optimal regulator problem:

$$V(\xi) = \inf_{u(.)\in L_2(0,\infty)} \int_0^\infty \frac{1}{2}\left(x(t)^T q(x(t))x(t) + u(t)^T r(x(t))u(t)\right) dt \qquad (1)$$

subject to $\dot{x} = f(x) + g(x)u$, $x(0) = \xi$, $\lim_{t\to\infty} x(t) = 0$ where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and f, g, q and r are analytic functions of the appropriate dimensions. We assume there is an equilibrium at the origin, i.e. $f(0) = 0$, and that $q(x)$ is positive semi-definite and $r(x)$ is positive definite for all x. In this paper, under some local controllability hypotheses, we present an algorithm for finding an approximation to the optimal control for the above problem. This approximation is asymptotically optimal, in that it converges to the optimal control as $x \to 0$, and is extremely simple to implement as it involves only the solution of algebraic Riccati equations. We demonstrate its effectiveness on a simulated inverted pendulum and present a test for determining in advance a stability region.

To begin with we make some comments about the existence of the dynamic programming solution to the above problem in order to justify our later hypotheses. Ideally one would like to say that the value function $V(x)$ is a stationary solution to the Cauchy problem for the associated Hamilton-Jacobi-Bellman

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(HJB) equation

$$\max_u \left\{ -\frac{\partial V}{\partial x}(f(x) + g(x)u) - \frac{1}{2}x^T q(x)x - \frac{1}{2}u^T r(x)u - \frac{\partial V}{\partial t} \right\} = 0. \qquad (2)$$

One approach is to consider the family of finite time problems

$$V_T(\xi, \tau) = \inf_{u(.)\in L_2(\tau,T)} \int_\tau^T \frac{1}{2}\left(x(t)^T q(x(t))x(t) + u(t)^T r(x(t))u(t)\right) dt$$

subject to $\dot{x} = f(x) + g(x)u$, $x(\tau) = \xi$ and then attempt to obtain $V(x)$ as $\sup_{T>0} V_T(x, 0) = \lim_{T\to\infty} V_T(x, 0)$. This is the standard approach to solving the infinite horizon linear-quadratic regulator (see (Russell, 1979) for instance) and requires stabilizability and detectability. This is extended to nonlinear systems of the type considered in (1) by Lukes (1969) for optimal control, and by van der Schaft (1991, 1992) for H_∞ control. The key to their analysis is the link between stationary solutions to (2) and stable Lagrangian manifolds for the corresponding Hamiltonian dynamics, i.e. the dynamics of x and the adjoint variable y arising from the maximum principle. The existence of a solution to the linearised problem at the origin implies the existence of a stable Lagrangian manifold L through the origin. Furthermore, it implies that locally L projects onto state space and the corresponding stationary solution $V(x)$ is smooth. $V(x)$ is in fact the generating function for L, i.e. L is the set of points $\{x, y = -\partial V/\partial x\}$ in phase space and $V(x) = -\int_x^0 y dx$ along trajectories of the Hamiltonian flow lying on L (see (Arnold, 1989) or (van der Schaft, 1991) for details). It also follows that the optimal control is the feedback $u = -r^{-1}(x)g^T(x)\partial V/\partial x$. This is the nonlinear extension of the feedback which solves the linear problem.

Smoothness breaks down when the optimal trajectories start to cross (going backwards in time). At such points singularities develop in the projection of L onto state space and $-\int y dx$ no longer gives a well defined function of x. However, the value function $V(x)$ for the optimal control problem is still well defined beyond such points and is in fact a stationary viscosity solution to (2), provided it is locally bounded. See (Soravia, 1996) for a proof covering general H_∞ control or (Fleming and Soner, 1993). For an introduction to viscosity solutions see the previous reference or (Crandall *et al.*, 1984).

For the particular case considered here where local assumptions imply the existence of a stable manifold L, it is shown by Day (1998) how to construct from L a stationary viscosity solution $V(x)$ to (2) under the assumption that $V(x)$ is locally Lipschitz. The construction uses the fact that arbitrarily large regions of L can be obtained as deformation retracts, along the dynamics, of open balls containing the origin and so are simply connected. Thus, within such regions, $-\int y dx$ still gives a well defined function $S(x, y)$ on L beyond points at which it fails to be well defined over x. In any neighbourhood on L which project onto state space, $S(x, y)$, as a locally defined function of x, is the generating function for L. $V(x)$ is obtained by taking the minimum value of $S(x, y)$ over all points $(x, y) \in L$ which project onto x.

The problem of finding the optimal feedback when $V(x)$ is a viscosity solution is discussed in Section 6 of (Soravia, 1996) and is, in general, open. For the special case considered here we would guess from Day's construction that it is $u = r^{-1}(x)g^T(x)y$ where (x, y) is the point on L at which $S(x, y)$ takes its minimum value over x. We know of no reference for this in the literature yet, but it can probably be proved using the type of convexity arguments employed in Section 3 of this paper. In any case, we don't use this fact.

So, to summarise, we shall assume that the linearisation of (1) at the origin is stabilizable and detectable to ensure that there exists a locally smooth optimal solution $V(x)$. We shall also assume there exists a larger region of the origin on which $V(x)$ is locally Lipschitz and can be obtained as $\min\{S(x,y) : y \text{ such that } (x,y) \in L\}$ where $S(x,y)$ satisfies $dS = -ydx$ on L. We will show in a future paper that the Lipschitz condition follows from the topological properties of L. For the present, however, we simply assume this and note that, at the very worst, it will hold on the region of the origin on which $V(x)$ is smooth and, in general, on a larger region. In the worst case, the stability arguments of Section 3 reduce to the standard smooth Lyapunov type arguments.

To end this introduction, we note that another approach to generalised solutions to (2) is via idempotent analysis where the standard arithmetic operations of $(+, \times)$ are replaced with $(\min, +)$. In this setting, (2) becomes linear and solutions can be defined in the weak sense of distributions (Kolokoltsov and Maslov, 1989). Again, given local assumptions implying the existence of a stable manifold, it is shown in (Dobrokhotov *et al.*, 1992) that stationary solutions to (2) in this sense can also be identified with those branches of the stable manifold on which the generating function achieves its minimum.

2 An asymptotically optimal solution

As described above, we assume that the linearised system $(\partial f/\partial x(0),\ g(0),\ q^{1/2}(0))$ is stabilizable and detectable. We factor $f(x)$ in the form $A(x)x$ for some matrix valued function $A(x)$ such that $A(x) \to \partial f/\partial x(0)$ as $x \to 0$. In the region where $V(x)$ is a smooth stationary solution to (2) the maximum is achieved by $u = -r^{-1}(x)g^T(x)\partial V/\partial x$, and V is thus the solution to

$$-\frac{\partial V}{\partial x}f(x) + \frac{1}{2}\frac{\partial V}{\partial x}^T g(x)r^{-1}(x)g^T(x)\frac{\partial V}{\partial x} - \frac{1}{2}x^T q(x)x = 0. \tag{3}$$

It is shown in (van der Schaft, 1991) that $\frac{\partial V}{\partial x}(0) = 0$ and so we can write $\frac{\partial V}{\partial x} = P(x)x$ for some matrix valued function $P(x)$ and (3) becomes

$$x^T\left(-P^T(x)A(x) - A^T(x)P(x) + P^T(x)g(x)r^{-1}(x)g^T(x)P(x) - q(x)\right)x = 0.$$

To apply the algorithm we ignore the requirement that $P(x)x$ be the gradient of some function and assume that $P(x)$ is symmetric. Then, at any given x, the algorithm consists of finding the positive semi-definite solution $P(x)$ to the algebraic Riccati equation

$$P(x)A(x) + A^T(x)P(x) - P(x)g(x)r^{-1}(x)g^T(x)P(x) + q(x) = 0 \tag{4}$$

and applying, at that x, the control $u = -r^{-1}(x)g^T(x)P(x)x$. There are several points to be made about this algorithm.

1. Since $A(x) \to \partial f/\partial x(0)$ as $x \to 0$, $P(x)$ tends to the solution to the algebraic Riccati equation for the linearised problem at the origin. Hence in a sufficently small neighbourhood of the origin the feedback from the algorithm is arbitrarily close to the optimal feedback.

2. Although the heuristic derivation of the algorithm took place in the region where $V(x)$ is smooth, it can clearly be applied independently of this assumption and, in fact, the stability analysis of the next section assumes only that $V(x)$ is Lipschitz.

3. The clearest benefit of this algorithm is its simplicity and its apparent effectiveness. At any x it involves only the solution of an algebraic Riccati equation. There is no attempt to solve the HJB equation (3) as outlined, for example, in (Lukes, 1969). In order that (4) have a positive semi-definite solution for all x, it is sufficient that the 'frozen' linear systems $(A(x), g(x), q^{1/2}(x))$ be stabilizable and detectable for all x. The algorithm then gives a smooth feedback. Even if this is not true for all x, the algorithm still works as the following example shows, although the feedback then becomes discontinuous and there are values of x at which the algorithm blows up.

4. Implementing the algorithm, at least for simulations, is easy in, for instance, MATLAB. The step length between successive solutions of (4) is set by the Runge-Kutta routine. It is a question for the future how small the step length should be for real-time control of a physical system and how much computing power is required to solve (4) within this time step.

5. A finite time version of this algorithm was presented in (Banks and Mhana, 1992) and shown to be stable under certain bounds on the derivatives of $A(x)$ and $g(x)$. This paper also contained a discussion of the infinite time case but, as pointed out in (Gong and Thompson, 1995), the arguments given applied only in the scalar case.

6. A similar algorithm can be applied to find an approximate solution to the state feedback H_∞ control problem as posed by van der Schaft (van der Schaft, 1991, 1992).

As an illustration of the effectiveness of the algorithm, we applied it to the simulated stabilization around the vertical upright position of a pendulum mounted on a cart on a linear track subject to a control force acting along the track. Taking the pendulum to be 1m long with a 1kg bob and 1kg cart and gravitational acceleration to be $10\mathrm{ms}^{-2}$, the equations of motion can be written in the factored form $A(x)x$ as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{20 \sin x_1}{(1+\sin^2 x_1)x_1} & \frac{-(\sin x_1 \cos x_1)x_2}{1+\sin^2 x_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-10 \sin x_1 \cos x_1}{(1+\sin^2 x_1)x_1} & \frac{(\sin x_1)x_2}{1+\sin^2 x_1} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{-\cos x_1}{1+\sin^2 x_1} \\ 0 \\ \frac{1}{1+\sin^2 x_1} \end{pmatrix} u$$

where x_1, x_2 are the angular position and velocity respectively of the pendulum measured from the upright vertical position and x_3, x_4 are the linear position and velocity respectively of the cart on the track. The integrand of the cost function is taken to be

$$\frac{1}{2}x_4^2 + \frac{1}{2}(x_4 + x_2 \cos x_1)^2 + \frac{1}{2}x_2^2 \sin^2 x_1 + 10(1 - \cos x_1) + \frac{1}{2}u^2.$$

This is the mechanical Lagrangian of the unforced system, i.e. the kinetic minus the potential energy with the potential taken to be zero at the vertical upright position, plus a control cost term. The factorisation $1/2x^T q(x)x$ is obtained in the obvious way by writing

$$1 - \cos x_1 = \frac{(1 - \cos x_1)}{x_1^2} x_1^2.$$

One can check that this system is stabilizable and detectable at the origin, i.e. the corresponding Hamiltonian dynamics arising from the maximum principle have a hyperbolic equilibrium at the origin. It can also be clearly seen that the matrix $A(x)$ tends to the linearisation of the system at the origin. The expression for the feedback $u = -r^{-1}(x)g^T(x)P(x)x$ reduces, in this case, to

$$u = \frac{\eta_2 \cos x_1 - \eta_4}{1 + \sin^2 x_1}$$

where $\eta = P(x)x$. Implementation involves writing u as a function of x_1, x_2, x_3, x_4 using a Riccati routine, in MATLAB for instance, and the simulation from a given starting condition is then a straightforward Runge-Kutta. The resulting controlled trajectories are shown in Fig. 1 for a number of initial angles starting from rest. The algorithm provides stabilization from all initial angles except $\pm\pi/2$. This is because the frozen linear system $(A(x), g(x))$ is unstabilizable at $x_1 = \pm\pi/2$. At these points the feedback algorithm blows up. This is dealt with by imposing bounds in the function for u in a neighbourhood of these points. Starting from rest from initial angles close to $\pm\pi/2$, the size of the initial control impulse produces a large overshoot which then requires a number of oscillations or complete revolutions to dissipate. However, even in these extreme cases the algorithm recovers and eventually stabilizes. Note, there is a trade-off in the size of the bound imposed on u between the speed of response and the amount of overshoot. The other consequence of unstabilizability is a discontinuity in the feedback produced by the algorithm at $\pm\pi/2$. The is because the positive definite root of (4) switches as the pendulum passes through $\pm\pi/2$ and the feedback goes from pushing to pulling or vice versa. For systems in which the 'frozen' linear system is stabilizable and detectable for all x, these effects will not be observed.

3 A stablility test

We now assume, in addition to the local stabilizability and detectability assumed in Section 2, that the value function $V(x)$ for problem (1) is locally Lipschitz in a region Ω of the origin and is equal to $\min\{S(x,y) : y \text{ such that } (x,y) \in L\}$ where $S(x,y)$ satisfies $dS = -ydx$ on the stable manifold L. Recall that L exists by the stabilizability and detectability hypotheses and note that Ω has to be covered by L, i.e. for all $x \in \Omega$ there exists y such that $(x,y) \in L$. We also assume now that $q(x)$ is positive definite for all $x \neq 0$ from which it follows that $V(x) > 0$ for $x \neq 0$, $V(0) = 0$. Note that these are basically sufficient conditions for asymptotic stability of the exact solution to (1). Our aim is to give a condition under which $V(x)$ is also a Lyapunov function for the approximate solution given by the feedback algorithm. The following is

Figure 1: Controlled trajectories (pendulum angle x_1 in radians versus time in seconds) starting from rest for different initial angles.

therefore a modification of the proof that a stationary viscosity solution to (2) gives a Lyapunov function (see (Soravia, 1996) for instance) with the additional element of using the stable manifold and convexity to reduce the condition to one which is more easily tested. The use of convexity prevents the following analysis being extended to prove stability of the feedback algorithm when applied to H_∞ control.

We start by noting that the positive semi-definite Riccati matrix solving the linearised problem at the origin is $\partial^2 V/\partial x^2(0)$, as shown by van der Schaft (1991). Further, since our factorisation $f(x) = A(x)x$ satisfies $A(0) = \partial f/\partial x(0)$, the solution $P(0)$ to (4) at $x = 0$ satisfies $P(0) = \partial^2 V/\partial x^2(0)$. We also note that points $(x, y) \in L$ can be generated by following trajectories of the Hamiltonian dynamics corresponding to (2) backwards in time from final conditions $(x_f, y_f) \in L$ lying close to the origin. The dynamics are given by

$$\dot{x} = \frac{\partial H}{\partial y} \qquad\qquad \dot{y} = -\frac{\partial H}{\partial x} \tag{5}$$

where

$$\begin{aligned} H &= \max_u \{ y^T f(x) + y^T g(x) u - \frac{1}{2} x^T q(x) x - \frac{1}{2} u^T r(x) u \} \\ &= y^T f(x) + \frac{1}{2} y^T g(x) r^{-1}(x) g^T(x) y - \frac{1}{2} x^T q(x) x. \end{aligned}$$

For points lying in a small ball $B_\epsilon \backslash \{0\}$, the final condition $(x_f, y_f) \in L$ can be approximated by taking (x_f, y_f) to lie on the tangent plane to L at the origin. This is given by

$$y_f = -\frac{\partial^2 V}{\partial x^2}(0) x_f = -P(0) x_f.$$

The proof of the local stable manifold theorem (see for instance (Lukes, 1969) or any standard text on differential equations) shows how to obtain higher order approximations to L, should greater accuracy be required. In the following let Ω_t, for $t > 0$, be the set of all $x \in \mathbf{R}^n$ which are projections of points $(x, y) \in L$ which can be reached in time t along reverse trajectories of (5) starting from some $(x_f, y_f) \in L$, $x_f \in B_\epsilon \backslash \{0\}$. We can now state the stability test, which essentially involves checking that the error between the true feedback and the approximation is small in some sense.

Proposition 3.1. *For any $t > 0$ such that $\Omega_t \subset \Omega$, $V(x)$ is strictly decreasing along trajectories of the feedback algorithm $\dot{x} = f(x) - g(x) r^{-1}(x) g^T(x) P(x) x$ for all $x \in \Omega_t \backslash \{0\}$ provided*

$$\begin{aligned} &\frac{1}{2}(y + P(x)x)^T g(x) r^{-1}(x) g^T(x)(y + P(x)x) \\ &\qquad - \frac{1}{2} x^T P(x) g(x) r^{-1}(x) g^T(x) P(x) x \le 0 \end{aligned} \tag{6}$$

for all $(x, y) \in L$ such that $x \in \Omega_t \backslash \{0\}$.

Proof. $V(x)$ is a viscosity solution of

$$\max_u \left\{ -\frac{\partial V}{\partial x}(f(x) + g(x)u) - \frac{1}{2} x^T q(x) x - \frac{1}{2} u^T r(x) u \right\} = 0. \tag{7}$$

So, in particular, it is a supersolution, i.e. for all $p \in D^-V$, the subdifferential of V, we have

$$-p^T f + \frac{1}{2} p^T g r^{-1} g^T p - \frac{1}{2} x^T q x \geq 0$$

where, for a given $p \in D^-V$, the maximum in (7) is achieved by $u = -r^{-1} g^T p$. Then, denoting $P(x)x$ by $\hat{p}$, the 'subderivative' of V along trajectories of the feedback algorithm is

$$-p(f - gr^{-1}g^T\hat{p}) \geq -\frac{1}{2} pgr^{-1}g^T p + \frac{1}{2} x^T q x + pgr^{-1}g^T\hat{p}. \tag{8}$$

By Theorem I.14 of (Crandall and Lions, 1983), V is strictly decreasing along trajectories of $\dot{x} = f - gr^{-1}g^T\hat{p}$ provided (8) is strictly positive for all $p \in D^-V$. Since $q(x)$ is positive definite for $x \neq 0$, this will follow provided

$$\frac{1}{2} pgr^{-1}g^T p - pgr^{-1}g^T\hat{p} = \frac{1}{2}(p - \hat{p})^T gr^{-1}g^T (p - \hat{p}) - \frac{1}{2}\hat{p}gr^{-1}g^T\hat{p} \leq 0 \tag{9}$$

for all $p \in D^-V$. Since V is Lipschitz, $D^-V \subset \partial V$, the generalised gradient (Frankowska, 1989). Further, it is shown in the proof of Theorem 3 of (Day, 1998) that for $V(x)$ of the form $\min\{S(x,y) : y \text{ such that } (x,y) \in L\}$, $\partial V \subset \text{co}\{-y : (x,y) \in L\}$ where co denotes the convex hull. Since the expression for p in (9) is convex, (9) will follow if

$$\begin{aligned} &\frac{1}{2}(-y - \hat{p})^T gr^{-1}g^T(-y - \hat{p}) - \frac{1}{2}\hat{p}gr^{-1}g^T\hat{p} \\ =&\frac{1}{2}(y + \hat{p})^T gr^{-1}g^T(y + \hat{p}) - \frac{1}{2}\hat{p}gr^{-1}g^T\hat{p} \leq 0 \end{aligned}$$

as required. □

Note that since $P(x)x \to -y$ as $x \to 0$, (6) will hold in a sufficiently small ball B_ϵ centred on the origin. The stability test then involves following trajectories of (5) backwards in time from points $x_f \in \partial B_\epsilon$, $y_f = -\partial^2 V/\partial x^2(0)x_f$ and estimating the largest t for which (6) holds in Ω_t. The feedback algorithm will then be asymptotically stable in the sublevel set $\{x \in \mathbf{R}^n : V(x) \leq c\}$ where $c = \min\{V(x) : x \in \partial\Omega_t\}$. The above analysis could be done with weaker conditions than $q(x)$ positive definite for $x \neq 0$; for example some form of zero-state detectability (see (van der Schaft, 1992) and the references therein). However, such weaker conditions are naturally more difficult to verify.

We end by noting that for the pendulum example, (6) holds out to about $x_1 = 1.5$ radians on a sample of trajectories, which we have seen is where a discontinuity occurs in the feedback algorithm. For this example $q(x)$ is positive definite if one ignores x_3, the position on the track. This is justified as x_3 does not, effectively, enter the dynamics and we have not imposed any cost on x_3. Consequently, there is an equilibrium for $x_1 = x_2 = x_4 = 0$ for all x_3.

It is also worth remarking that there are important numerical issues involved in generating a sufficiently wide spread of points $(x,y) \in L$. If α and ω are the largest and smallest (in absolute magnitude) eigenvalues for the linearised system $\dot{x} = \partial f/\partial x(0)x - g(0)r^{-1}(0)g^T(0)\partial^2 V/\partial x^2(0)x$, then trajectories on L, when projected onto state space, tend to approach the origin along the eigenvector corresponding to ω and leave (in reverse time) along the eigenvector corresponding to α. It is thus necessary to concentrate the final conditions $x_f \in \partial B_\epsilon$ along the eigendirection corresponding to ω in order to generate a spread of points on L in reverse time. Another issue is the accuracy of the Runge-Kutta routine used to estimate trajectories of (5). The calculation may be improved by the use of a symplectic integration routine (Sanz-Serna, 1992). These issues are left for the future.

4 Conclusion

We have presented an algorithm for finding an asymptotically optimal stabilizing feedback control for infinite horizon linear analytic optimal regulator problems which are locally stabilizable and detectable. We have also presented a test for estimating off-line the extent of the region on which the feedback will be stabilizing. The main advantage of the algorithm is its computational simplicity and its effectiveness, as demonstrated on a simulated inverted pendulum. There remains further work to be done on testing the algorithm on a physical plant and on numerical issues associated with estimating the region of stability.

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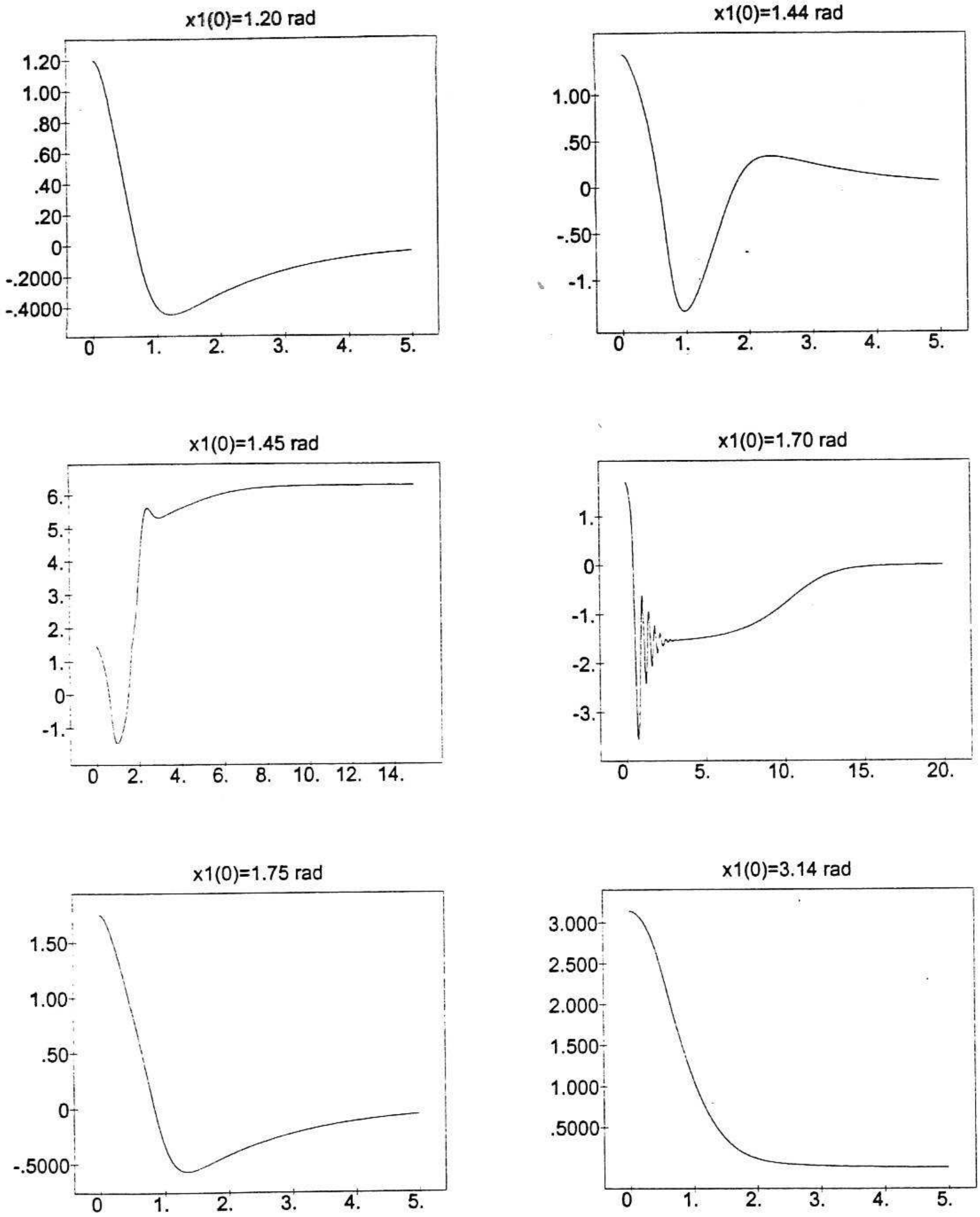
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1.20 1.00 .80 .60 .40 .20 0 -.2000 -.4000
0 1. 2. 3. 4. 5.
x1(0)=1.44 rad
1.00 .50 0 -.50 -1.
0 1. 2. 3. 4. 5.
x1(0)=1.45 rad
6. 5. 4. 3. 2. 1. 0 -1.
0 2. 4. 6. 8. 10. 12. 14.
x1(0)=1.70 rad
1. 0 -1. -2. -3.
0 5. 10. 15. 20.
x1(0)=1.75 rad
1.50 1.00 .50 0 -.5000
0 1. 2. 3. 4. 5.
x1(0)=3.14 rad
3.000 2.500 2.000 1.500 1.000 .5000
0 1. 2. 3. 4. 5.

FIG. 1