Book Section:

Rathjen, M (2014) Omega-models and well-ordering proofs. In: Tennant, N, (ed.) Foundational Adventures. College Publications , 179 - 212. ISBN 978-1-84890-117-9

ω-Models and Well-ordering Principles

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ABSTRACT. The purpose of the paper is to present a general methodology which in many cases allows one to establish an equivalence between two types of statements. The first type is concerned with the existence of ω models of a theory whereas the second type asserts that a certain (usually well-known) elementary operation on orderings preserves the property of being well-ordered. These results have their roots in a theorem of Harvey Friedman (see [11]) which characterizes the theory $\mathbf{ATR}_0$ by means of a Π^1_2 sentence of the form *"if X is well ordered then $f(X)$ is well ordered"*, where f is a standard proof theoretic function from ordinals to ordinals. The approach taken here, however, is rather different. The proofs are entirely different from the ones in Friedman's work. The methods used are purely proof-theoretic and crucially involve cut elimination theorems in infinitary logic with ordinal bounds.

The main result presented in this paper is that the following two statements are equivalent over $\mathbf{RCA}_0$:

(i) Every set is contained in an ω-model of $\mathbf{ATR}$;

(ii) If $\mathfrak{X}$ is a well-ordering, then so is $\Gamma_{\mathfrak{X}}$.

Albeit this result is just an example, it may stand for many others as the methodology exemplified in its proof lends itself to a wide range of applications. One could say that every cut elimination theorem in ordinal-theoretic proof theory encapsulates a theorem of this type. Moreover, the technique has the potential for generalization in that it can be lifted up to β-models and functors acting on ordinal functions.
MSC 03B30 03F05 03F15 03F35 03F35

1 Introduction

The present paper can be viewed as a continuation of [2, 25]. Its aim is to present a general proof-theoretic machinery for investigating statements about well-orderings from a reverse mathematics point of view. These statements are of the form

$$\mathbf{WOP}(f) \qquad \text{“}\textit{if } X \textit{ is well ordered then } f(X) \textit{ is well ordered}\text{”} \quad (1)$$

where f is a standard proof theoretic function from ordinals to ordinals. There are by now several examples of functions f where the statement $\mathbf{WOP}(f)$ has turned out to be equivalent to one of the theories of reverse mathematics over a weak base theory (usually $\mathbf{RCA}_0$). The first example is due to Girard [13].

Theorem 1.1. *(Girard 1987)* Let $\mathbf{WO}(\mathfrak{X})$ express that $\mathfrak{X}$ is a well ordering. Over $\mathbf{RCA}_0$ the following are equivalent:

(i) Arithmetic Comprehension

(ii) $\forall \mathfrak{X}\, [\mathbf{WO}(\mathfrak{X}) \to \mathbf{WO}(2^{\mathfrak{X}})]$.

More recently two new theorems appeared in preprints [18, 11] and were finally published in [19]. These results give characterizations of the form (1) for the theories $\mathbf{ACA}_0^+$ and $\mathbf{ATR}_0$, respectively, in terms of familiar proof-theoretic functions. $\mathbf{ACA}_0^+$ denotes the theory $\mathbf{ACA}_0$ augmented by an axiom asserting that for any set X the ω-th jump in X exists while $\mathbf{ATR}_0$ asserts the existence of sets constructed by transfinite iterations of arithmetical comprehension. $\alpha \mapsto \varepsilon_\alpha$ denotes the usual ε function while φ stands for the two-place Veblen function familiar from predicative proof theory (cf. [27]). More detailed descriptions of $\mathbf{ATR}_0$ and the function $\mathfrak{X} \mapsto \varphi\mathfrak{X}0$ will be given shortly. Definitions of the familiar subsystems of reverse mathematics can be found in [34].

Theorem 1.2. *(Montalban, Marcone)* Over $\mathbf{RCA}_0$ the following are equivalent:

(i) $\mathbf{ACA}_0^+$

(ii) $\forall \mathfrak{X}\, [\mathbf{WO}(\mathfrak{X}) \to \mathbf{WO}(\varepsilon_{\mathfrak{X}})]$.

A proof-theoretic proof for Theorem 1.2 was given by Afshari and Rathjen [2].

Theorem 1.3. *(Friedman, unpublished)* Over $\mathbf{RCA}_0$ the following are equivalent:

(i) $\mathbf{ATR}_0$

(ii) $\forall \mathfrak{X}\, [\mathbf{WO}(\mathfrak{X}) \to \mathbf{WO}(\varphi\mathfrak{X}0)]$.

There is a proof of this result in [18] and again there is a proof using proof theory which is due to Rathjen and Weiermann [26]. The original proofs of Theorem 1.3 and 1.2 used recursion-theoretic and combinatorial results about linear orderings. They build on a result from [9] to the effect that there is no arithmetic sequence of degrees descending by ω-jumps. The latter result was then improved by Steel [35] to descent by Turing jumps: If $Q \subseteq \mathrm{Pow}(\omega) \times \mathrm{Pow}(\omega)$ is arithmetic, then there is no sequence $\{A_n \mid n \in \omega\}$ such that (a) for every n, A_{n+1} is the unique set such that $Q(A_n, A_{n+1})$, (b) for every n, $A'_{n+1} \leq_T A_n$.

For a proof theorist, theorems 1.2 and 1.3 bear a striking resemblance to cut elimination theorems for infinitary logics. Hearing the statements, but not the proofs, the author was prompted to look for proof-theoretic ways of obtaining these results. The hope was that this would also unearth a common pattern behind them and possibly lead to generalizations. The project commenced in [2] in collaboration with Bahareh Afshari, where a purely proof-theoretic proof of Theorem 1.2 was presented. Joint work with Andreas Weiermann led to [25], giving a new (and again proof-theoretic) proof of 1.3. The main result I want to prove in this paper is the following:

Theorem 1.4. Over $\mathbf{RCA}_0$ the following are equivalent:

(i) $\forall \mathfrak{X}\, [\mathbf{WO}(\mathfrak{X}) \rightarrow \mathbf{WO}(\Gamma_{\mathfrak{X}})]$

(ii) Every set is contained in a countable ω-model of **ATR**.

At this point it might be useful to state precisely what a countable coded ω-model is.

Definition 1.5. *Let T be a theory in the language of second order arithmetic, L_2. A* countable coded ω-model of T *is a set $W \subseteq \mathbb{N}$, viewed as encoding the L_2-model*

$$\mathbb{M} = (\mathbb{N}, \mathcal{S}, +, \cdot, 0, 1, <)$$

with $\mathcal{S} = \{(W)_n \mid n \in \mathbb{N}\}$ such that $\mathbb{M} \models T$ (where $(W)_n = \{m \mid \langle n, m\rangle \in W\}$; $\langle\, ,\rangle$ some coding function).

This definition can be made in $\mathbf{RCA}_0$ (see [34], Definition VII.2).

We write $X \in W$ if $\exists n\, X = (W)_n$.

Another result in the same vein as Theorem 1.4 is from impredicative proof theory. Here we turn to the ordinal representation system used for the ordinal analysis of the theory $\mathbf{ID}_1$ of non-iterated inductive definitions, which can be expressed in terms of the θ-function (cf. [6]). $\mathbf{ID}_1$ has the same strength as the subsystem of second order arithmetic based on bar induction, **BI** (cf. [6, 7, 26]). In Simpson's book the acronym used for **BI** is Π^1_∞-$\mathbf{TI}_0$ (cf. [34, §VII.2]). In place of the function θ we prefer to work with simpler ordinal representations based on the ψ-function introduced in [5] or the ϑ-function of [26]. For definiteness we refer to [26]. Given a well-ordering $\mathfrak{X}$, the relativized versions $\vartheta_{\mathfrak{X}}$ and $\psi_{\mathfrak{X}}$ of the ϑ-function and the ψ-function, respectively, are obtained by adding all the ordinals from $\mathfrak{X}$ to the sets $C_n(\alpha, \beta)$ of [26, §1] and $C_n(\alpha)$ of [26, Definition 3.1] as initial segments, respectively. The resulting well-orderings $\vartheta_{\mathfrak{X}}(\varepsilon_{\Omega+1})$ and $\psi_{\mathfrak{X}}(\varepsilon_{\Omega+1})$ are equivalent owing to [26, Corollary 3.2].

The next Theorem is obtained by the same methodology as 1.4, but it will not be proved in this paper as its proof is too long to be incorporated.

Theorem 1.6. Over $\mathbf{RCA}_0$ the following are equivalent:

(i) *Every set is contained in a countable coded ω-model of* $\mathbf{BI}$.

(ii) $\forall \mathfrak{X}\ [\mathbf{WO}(\mathfrak{X}) \to \mathbf{WO}(\psi_{\mathfrak{X}}(\varepsilon_{\Omega+1}))]$.

At first glance, Theorems 1.4 and 1.6 appear to be of a different type than Theorems 1.2 and 1.3. But the similarity becomes more apparent owing to the next result.

Theorem 1.7. ($\mathbf{RCA}_0$)

(i) $\mathbf{ACA}_0^+$ is equivalent to the statement that every set is contained in a countable coded ω-model of $\mathbf{ACA}$.

(ii) $\mathbf{ATR}_0$ is equivalent to the statement that every set is contained in a countable coded ω-model of Δ^1_1-$\mathbf{CA}$ (or Σ^1_1-$\mathbf{DC}$).

Proof: (i) follows from [2, Lemma 3.4]. (ii) follows from [34, VIII.4.19]. $\square$

As a consequence of Theorems 1.2, 1.3, and 1.7 we have:

Corollary 1.8. ($\mathbf{RCA}_0$)

(i) $\forall \mathfrak{X}\ [\mathbf{WO}(\mathfrak{X}) \to \mathbf{WO}(\varepsilon_{\mathfrak{X}})]$ is equivalent to the statement that every set is contained in a countable coded ω-model of $\mathbf{ACA}$.

(ii) $\forall \mathfrak{X}\ [\mathbf{WO}(\mathfrak{X}) \to \mathbf{WO}(\varphi\mathfrak{X}0)]$ is equivalent to the statement that every set is contained in a countable coded ω-model of Δ^1_1-$\mathbf{CA}$ (or Σ^1_1-$\mathbf{DC}$).

Taking any of the Theorems 1.2, 1.3, 1.7, 1.6 the proof of the direction $(i) \Rightarrow (ii)$ can be directly inferred or gleaned from a result or proof in the proof-theoretic literature. The harder part is always the implication $(ii) \Rightarrow (i)$. For a theory T let $\mathrm{Mod}_\omega(T)$ be the statement that every set is contained in a countable coded ω-model of T. There exists, however, an Ansatz which given a theory T (a subsystem of second order arithmetic) can help one to find a function f on orderings such that the appertaining statements $\mathrm{Mod}_\omega(T)$ and $\mathbf{WOP}(f)$ are equivalent. The first step consists in an attempt to find an ω-model of T via the method of search trees in

ω-logic. This gives rise to a tree $\mathcal{D}$. In case one finds an infinite path $\mathbb{P}$ on $\mathcal{D}$ it can be used to define an ω-model of T. The less desirable outcome would be that all paths in $\mathcal{D}$ are finite, ending in a simple axiom. To get out of this predicament one can scour the proof-theoretic literature to find an infinitary proof system T_∞ which enjoys cut elimination such that $\mathcal{D}$ can be viewed as a skeleton of an infinitary proof in T_∞ of the empty sequent. If T_∞ is chosen optimal, the desired function f is the one which measures the cost of cut elimination in T_∞. As there is no cut free proof of the empty sequent, $\mathbf{WOP}(f)$ implies that $\mathcal{D}$ must possess an infinite path after all, and hence there is an ω-model of T as desired.

Acknowledgements

This research was supported by a Royal Society International Joint Projects award (2006/R3) and the *Swedish Collegium for Advanced Study*. I would like to thank the *Swedish Collegium for Advanced Study* in Uppsala for providing an excellent research environment. I am grateful for very inspiring discussions with Antonio Montalbán during his visits to Leeds in November 2010.

2 The ordering $\Gamma_{\mathfrak{X}}$

In this paper we use ordinal functions stemming from the early days of ordinal representation systems. Before we give a formal definition of $\Gamma_{\mathfrak{X}}$ it might be useful to recall some of the historical background.

2.1 A brief history of early ordinal representation systems

In 1904, Hardy [14] wanted to "construct" a subset of $\mathbb{R}$ of size $\aleph_1$. His method was to represent countable ordinals via increasing sequence of natural numbers and then to correlate a decimal expansion with each such sequence. Hardy used two processes on sequences which led to explicit representations for all ordinals $< \omega^2$. Veblen [37] in 1908 extended the initial segment of the countable for which fundamental sequences can be given effectively. The new tools he devised were the operations of **derivation** and **transfinite iteration** applied to **continuous increasing functions** on ordinals.

Definition 2.1. *Let* **ON** *be the class of ordinals. A (class) function f :* **ON** $\to$ **ON** *is said to be* **increasing** *if $\alpha < \beta$ implies $f(\alpha) < f(\beta)$ and* **continuous** *(in the order topology on* **ON***) if*

$$f(\lim_{\xi<\lambda} \alpha_\xi) = \lim_{\xi<\lambda} f(\alpha_\xi)$$

holds for every limit ordinal λ and increasing sequence $(\alpha_\xi)_{\xi<\lambda}$. f is called **normal** *if it is increasing and continuous.*

The function $\beta \mapsto \omega + \beta$ is normal while $\beta \mapsto \beta + \omega$ is not continuous at ω since $\lim_{\xi<\omega}(\xi + \omega) = \omega$ but $(\lim_{\xi<\omega} \xi) + \omega = \omega + \omega$.

Definition 2.2. *The* **derivative** f' *of a function* $f : \mathbf{ON} \to \mathbf{ON}$ *is the function which enumerates in increasing order the solutions of the equation* $f(\alpha) = \alpha$*, also called the* **fixed points** *of* f*.*

If f is a normal function, $\{\alpha : f(\alpha) = \alpha\}$ is a proper class and f' will be a normal function, too.

Definition 2.3. Now, given a normal function $f : \mathbf{ON} \to \mathbf{ON}$, define a hierarchy of normal functions as follows:

$$f_0 = f \qquad f_{\alpha+1} = f'_\alpha$$
$$f_\lambda(\xi) \;=\; \xi^{th} \text{ element of } \bigcap_{\alpha<\lambda} (\text{Range of } f_\alpha) \qquad \text{for } \lambda \text{ a limit ordinal.}$$

In this way, from the normal function f we get a two-place function, $\varphi_f(\alpha, \beta) := f_\alpha(\beta)$. Veblen then discusses the hierarchy when $f(\alpha) = 1 + \alpha$. We shall use the starting function $\ell(\alpha) = \omega^\alpha$. Instead of $\varphi_\ell(\alpha, \beta)$ it is customary to simply write $\varphi\alpha\beta$.

The least ordinal $\gamma > 0$ closed under $\varphi = \varphi_\ell$, i.e. the least ordinal > 0 satisfying $(\forall\alpha, \beta < \gamma)\ \varphi\alpha\beta < \gamma$ is the famous ordinal Γ_0 which Feferman [8] and Schütte [28, 29] determined to be the least ordinal 'unreachable' by *predicative means*.

In general, Γ_α denotes the α^{th} ordinal closed under φ.

2.2 Definition of $\Gamma_{\mathfrak{X}}$

Via simple coding procedures, countable well-orderings and functions on them can be expressed in the language of second order arithmetic, L_2. Variables $X, Y, Z, \ldots$ are supposed to range over subsets of $\mathbb{N}$. Using an elementary injective pairing function $\langle, \rangle$ (e.g. $\langle n, m\rangle := (n+m)^2+n+1$), every set X encodes a sequence of sets $(X)_i$, where $(X)_i := \{m \mid \langle i, m\rangle \in X\}$. We also adopt from [34], II.2 the method of encoding a finite sequence $(n_0, \ldots, n_{k-1})$ of natural numbers as a single number $\langle n_0, \ldots, n_{k-1}\rangle$.

Definition 2.4. *Every set of natural numbers Q can be viewed as encoding a binary relation $<_Q$ on $\mathbb{N}$ via $n <_Q m$ iff $\langle n, m\rangle \in Q$. The* **field** *of Q, fld(Q) is the set $\{n \mid \exists m\, [n <_Q m \vee m <_Q n]\}$.*

We say that Q is a **well-ordering** *if $<_Q$ is a well-ordering, that is $<_Q$ is a linear ordering of its field and every non-empty subset U of fld(Q) has a $<_Q$-least element.*

Definition 2.5. *Let Q be a linear ordering. Let $\Gamma_u := \langle 0, u\rangle$, $\varphi ua := \langle 1, \langle u, a\rangle\rangle$ and $\alpha_1 + \ldots + \alpha_n := \langle 2, \langle \alpha_1, \ldots, \alpha_n\rangle\rangle$ if $n > 1$.* **SC** *is the set*

$\{\Gamma_u \mid u \in \mathit{fld}(Q)\}$.

We introduce the ordering Γ_Q *by inductively defining its field* $\mathit{fld}(\Gamma_Q)$*, the ordering* $<_{\Gamma_Q}$*, the set* $\mathbf{H}$ *of additive principal members of* $\mathit{fld}(\Gamma_Q)$ *and the critical level function* $\mathbf{h}(\alpha)$ *for* $\alpha \in \mathit{fld}(\Gamma_Q)$*:*

1. $0 \in \mathit{fld}(\Gamma_Q)$ *and* $\mathbf{h}(0) = 0$.

2. $0 <_{\Gamma_Q} \alpha$ *if* $\alpha \in \mathit{fld}(\Gamma_Q)$ *and* $\alpha \neq 0$.

3. *If* $u \in \mathit{fld}(Q)$ *then* $\Gamma_u \in \mathit{fld}(\Gamma_Q)$, $\Gamma_u \in \mathbf{H}$ *and* $\mathbf{h}(\Gamma_u) = \Gamma_u$.

4. $\Gamma_u <_{\Gamma_Q} \Gamma_v$ *iff* $u <_Q v$.

5. *If* $\alpha, \beta \in \mathit{fld}(\Gamma_Q)$, $\alpha \notin \mathbf{SC}$ *and* $\mathbf{h}(\beta) \leq_{\Gamma_Q} \alpha$ *then* $\varphi\alpha\beta \in \mathit{fld}(\Gamma_Q)$, $\varphi\alpha\beta \in \mathbf{H}$, *and* $\mathbf{h}(\varphi\alpha\beta) = \alpha$.

6. *If* $\alpha, \beta \in \mathit{fld}(\Gamma_Q)$, $\alpha \in \mathbf{SC}$, $\mathbf{h}(\beta) \leq_{\Gamma_Q} \alpha$ *and* $\beta \neq 0$ *then* $\varphi\alpha\beta \in \mathit{fld}(\Gamma_Q)$, $\varphi\alpha\beta \in \mathbf{H}$, *and* $\mathbf{h}(\varphi\alpha\beta) = \alpha$.

7. *If* $\varphi\alpha\beta, \Gamma_u \in \mathit{fld}(\Gamma_Q)$ *then*
$$\begin{array}{lll} \varphi\alpha\beta <_{\Gamma_Q} \Gamma_u & \textit{iff} & \alpha, \beta < \Gamma_u, \\ \Gamma_u <_{\Gamma_Q} \varphi\alpha\beta & \textit{iff} & \Gamma_u \leq_{\Gamma_Q} \alpha \ \vee\ \Gamma_u \leq_{\Gamma_Q} \beta. \end{array}$$

8. *If* $\alpha_1, \ldots, \alpha_n \in \mathit{fld}(\Gamma_Q)$, $n > 1$, $\alpha_1, \ldots, \alpha_n \in \mathbf{H}$, *and* $\alpha_n \leq_{\Gamma_Q} \ldots \leq_{\Gamma_Q} \alpha_1$, *then*
$$\alpha_1 + \ldots + \alpha_n \in \mathit{fld}(\Gamma_Q)$$
and $\mathbf{h}(\alpha_1 + \ldots + \alpha_n) = 0$.

9. *If* $\alpha_1 + \ldots + \alpha_n,\ \beta_1 + \ldots + \beta_m \in \mathit{fld}(\Gamma_Q)$, *then*
$$\begin{array}{l} \alpha_1 + \ldots + \alpha_n <_{\Gamma_Q} \beta_1 + \ldots + \beta_m \ \textit{iff} \\ n < m \wedge \forall i \leq n\ \alpha_i = \beta_i \quad \textit{or} \\ \exists i \leq \min(n, m)[\alpha_i <_{\Gamma_Q} \beta_i \wedge \forall j < i\, \alpha_j = \beta_j]. \end{array}$$

10. *If* $\alpha_1 + \ldots + \alpha_n \in \mathit{fld}(\Gamma_Q)$ *and* $\beta \in \mathbf{H}$ *then*
$$\begin{array}{lll} \beta <_{\Gamma_Q} \alpha_1 + \ldots + \alpha_n & \textit{iff} & \beta \leq_{\Gamma_Q} \alpha_1 \\ \alpha_1 + \ldots + \alpha_n <_{\Gamma_Q} \beta & \textit{iff} & \alpha_1 <_{\Gamma_Q} \beta. \end{array}$$

11. If $\varphi\xi\alpha, \varphi\zeta\beta \in fld(\Gamma_Q)$, then

$$\varphi\xi\alpha <_{\Gamma_Q} \varphi\zeta\beta \quad \textit{iff} \quad \begin{array}{l} \xi <_{\Gamma_Q} \zeta \wedge \alpha <_{\Gamma_Q} \varphi\zeta\beta \quad \textit{or} \\ \xi = \zeta \wedge \alpha <_{\Gamma_Q} \beta \quad \textit{or} \\ \zeta <_{\Gamma_Q} \xi \wedge \varphi\xi\alpha <_{\Gamma_Q} \beta. \end{array}$$

Lemma 2.6. ($\mathbf{RCA}_0$)

(i) If Q is a linear ordering then so is Γ_Q.

(ii) Γ_Q is elementary recursive in Q.

3 Proof of the Main Theorem: The easy direction

The implication $(ii) \Rightarrow (i)$ of Theorem 1.4 even holds on the basis of intuitionistic logic. Assume (ii) and suppose $\mathbf{WO}(\mathfrak{X})$. Let U be an arbitrary set of natural numbers. By (2) we can pick an ω-model $\mathbb{A}$ of $\mathbf{ATR}$ which contains $\mathfrak{X}$ and U. Inside $\mathbb{A}$ we have transfinite induction on $\mathfrak{X}$ for arbitrary formulae with parameters from $\mathbb{A}$. It therefore follows from [23, Lemma 4.13,4.16] that $\mathbb{A} \models \mathbf{WO}(\Gamma_{\mathfrak{X}})$. Since $U \in \mathbb{A}$ it follows that U has a $\Gamma_{\mathfrak{X}}$-least element unless $U = \emptyset$. Consequently $\mathbf{WO}(\Gamma_{\mathfrak{X}})$ holds as U was an arbitrary set of naturals. $\square$

4 Proof of the Main Theorem: The hard direction part 1

Given a set $Q \subseteq \mathbb{N}$ we are to find an ω-model $\mathbb{M}$ of $\mathbf{ATR}$ containing Q. To find $\mathbb{M}$ we follow Schütte's method of proof search (deduction chains) from [27, II§4] which he used to prove the completeness theorem for first order logic (cf. [27, Theorem 5.7]). The method has to be extended to ω-logic, though. Rather than working in the Schütte calculus of positive and negative forms we work in a Gentzen sequent calculus with finite sets of formulas, called sequents. Before we embark on the technical details let's recall the history of this method.

The method of search trees in ω-logic

An extremely elegant and efficient proof procedure for first order logic consists in producing the search or decomposition tree (in German "Stammbaum") of a given formula. It proceeds by decomposing the formula according to its logical structure and amounts to applying logical rules backwards. This decomposition method has been employed by Schütte [31, 30] to prove the completeness theorem. It is closely related to the method of "semantic tableaux" of Beth [4] and the tableaux of Hintikka [17]. Ultimately, the whole idea derives from Gentzen [12].

The decomposition tree method can also be extended to prove the ω-completeness theorem due to Henkin [15] and Orey [22]. Schütte [32] used it to prove ω-completeness in the arithmetical case.

ω-logic is obtained from first-order logic by adding the rule

$$(\omega) \qquad \frac{F(0), F(1), \ldots, F(m), \ldots}{\forall x\, F(x)}$$

with infinitely many premisses. The ω-rule is usual attributed to Hilbert [16], though Tarski [36] says that he introduced the rule in 1927 in an unpublished talk to the Polish Philosophical Society at Warsaw. The restriction of the rule, with the premisses being enumerated by a recursive function, is sometimes referred to as *Novikov's rule* who in [21] introduced calculi with "constructive" infinite conjunctions and disjunctions.

4.1 Deduction chains in ω-logic

For what follows it is convenient (but by no means essential) that $\mathbf{ATR}_0$ can be axiomatized via a single sentence.

Lemma 4.1. $\mathbf{ATR}_0$ can be axiomatized via a single Π^1_2 sentence $\forall X\, C(X)$.

Proof: $\mathbf{ATR}_0$ is equivalent over $\mathbf{ACA}_0$ to the statement that every two well-orderings are comparable (see [34, Theorem V.6.8]). This statement can be expressed via a Π^1_2 sentence. Moreover, $\mathbf{ACA}_0$ can be axiomatized via a single Π^1_2 sentence (see [34, Lemma VIII.1.5]). $\Box$

Our formalization of the language of second order arithmetic, L_2, will slightly deviate from standard procedures in that it will not have any function symbols. Instead it has a constant $\bar{n}$ for each natural number n and symbols for primitive recursive relations (though we usually omit the bar on top of 0).

Definition 4.2.

(i) Let $U_0, U_1, U_2, \ldots$ be an enumeration of the free set variables of L_2. We shall assume that all predicate symbols of the language L_2 are symbols for primitive recursive relations. L_2 contains predicate symbols for the primitive recursive relations of equality and inequality and possibly more (or all) primitive recursive relations. If R is a symbol in L_2 for a primitive recursive relation we denote by $R^{\mathbb{N}}$ the primitive recursive relation it stands for. The formula $R(\bar{k}_1, \ldots, \bar{k}_r)$ ($\neg R(\bar{k}_1, \ldots, \bar{k}_r)$) is said to be true *if $R^{\mathbb{N}}(k_1, \ldots, k_r)$ is true (is false).*

(ii) *Henceforth a* **sequent** *will be a finite set of* L_2*-formulas without free number variables.*

(iii) *A sequent* Γ *is* **axiomatic** *if it satisfies at least one of the following conditions:*

1. Γ *contains a true* **literal**, *i.e. a true formula of either form* $R(\bar{k}_1, \ldots, \bar{k}_r)$ *or* $\neg R(\bar{k}_1, \ldots, \bar{k}_r)$, *where* R *is a predicate symbol in* L_2 *for a primitive recursive relation.*
2. Γ *contains formulae* $\bar{k} \in U$ *and* $\bar{k} \notin U$ *for some set variable* U *and number* k.

(iv) *A sequent is* **reducible** *or a* **redex** *if it is not axiomatic and contains a formula which is not a literal.*

Definition 4.3. *For* $Q \subseteq \mathbb{N}$ *define*

$$\bar{Q}(n) = \begin{cases} \bar{n} \in U_0 & \textit{if } n \in Q \\ \bar{n} \notin U_0 & \textit{otherwise} \end{cases}$$

Definition 4.4. *Fix* $Q \subseteq \mathbb{N}$. *Let* $\forall X\, C(X)$ *be the sentence of Lemma 4.1 which axiomatizes* $\mathbf{ATR}_0$.

A Q**-deduction chain** *is a finite string*

$$\Gamma_0, \Gamma_1, \ldots, \Gamma_k$$

of sequents Γ_i *constructed according to the following rules:*

(i) $\Gamma_0 \;=\; \neg\bar{Q}(0), \neg C(U_0)$.

(ii) Γ_i *is not axiomatic for* $i < k$.

(iii) *If* $i < k$ *and* Γ_i *is not reducible then*

$$\Gamma_{i+1} \;=\; \Gamma_i, \neg\bar{Q}(i+1), \neg C(U_{i+1}).$$

(iv) *Every reducible* Γ_i *with* $i < k$ *is of the form*

$$\Gamma_i', E, \Gamma_i''$$

where E *is not a literal and* Γ_i' *contains only literals.*
E *is said to be the* **redex** *of* Γ_i.

Let $i < k$ *and* Γ_i *be reducible.* Γ_{i+1} *is obtained from* $\Gamma_i = \Gamma_i', E, \Gamma_i''$ *as follows:*

1. *If* $E \equiv E_0 \vee E_1$ *then*

$$\Gamma_{i+1} \;=\; \Gamma_i', E_0, E_1, \Gamma_i'', \neg\bar{Q}(i+1), \neg C(U_{i+1}).$$

2. *If* $E \equiv E_0 \wedge E_1$ *then*

$$\Gamma_{i+1} \;=\; \Gamma_i', E_j, \Gamma_i'', \neg\bar{Q}(i+1), \neg C(U_{i+1})$$

where $j = 0$ *or* $j = 1$.

3. *If* $E \equiv \exists x\, F(x)$ *then*

$$\Gamma_{i+1} \;=\; \Gamma_i', F(\bar{m}), \Gamma_i'', \neg\bar{Q}(i+1), \neg C(U_{i+1}), E$$

where m *is the first number such that* $F(\bar{m})$ *does not occur in* $\Gamma_0, \ldots, \Gamma_i$, *providing* x *occurs free in* $F(x)$, *and* $m = 0$ *if* x *does not occur free in* $F(x)$.

4. *If* $E \equiv \forall x\, F(x)$ *then*

$$\Gamma_{i+1} \;=\; \Gamma_i', F(\bar{m}), \Gamma_i'', \neg\bar{Q}(i+1), \neg C(U_{i+1})$$

for some m.

5. *If* $E \equiv \exists X\, F(X)$ *then*

$$\Gamma_{i+1} \;=\; \Gamma_i', F(U_m), \Gamma_i'', \neg\bar{Q}(i+1), \neg C(U_{i+1}), E$$

where m *is the first number such that* $F(U_m)$ *does not occur in* $\Gamma_0, \ldots, \Gamma_i$, *providing* X *occurs free in* $F(X)$, *and* $m = 0$ *if* X *does not occur free in* $F(X)$.

6. *If* $E \equiv \forall X\, F(X)$ *then*

$$\Gamma_{i+1} \;=\; \Gamma_i', F(U_m), \Gamma_i'', \neg\bar{Q}(i+1), \neg C(U_{i+1})$$

where m *is the first number such that* $m \neq i+1$ *and* U_m *does not occur in* Γ_i.

The set of Q-deduction chains forms a tree $\mathcal{D}_Q$ labeled with strings of sequents. We will now consider two cases.

Case I: $\mathcal{D}_Q$ is not well-founded. Then $\mathcal{D}_Q$ contains an infinite path $\mathbb{P}$. Now define a set M via

$$(M)_i \quad = \quad \{k \mid \bar{k} \notin U_i \text{ occurs in } \mathbb{P}\}.$$

Set $\mathbb{M} = (\mathbb{N}; \{(M)_i \mid i \in \mathbb{N}\}, +, \cdot, 0, 1, <)$.

For a formula F, let $F \in \mathbb{P}$ mean that F occurs in $\mathbb{P}$, i.e. $F \in \Gamma$ for some $\Gamma \in \mathbb{P}$.

Claim: Under the assignment $U_i \mapsto (M)_i$ we have

$$F \in \mathbb{P} \qquad \Rightarrow \qquad \mathbb{M} \models \neg F. \tag{2}$$

The Claim will imply that $\mathbb{M}$ is an ω-model of **ATR**. Also note that $(M)_0 = Q$, thus Q is in $\mathbb{M}$. The proof of (2) follows by induction on F using Lemma 4.5 below. The upshot of the foregoing is that we can prove Theorem 1.4 under the assumption that $\mathcal{D}_Q$ is ill-founded for all sets $Q \subseteq \mathbb{N}$.

Lemma 4.5. Let Q be an arbitrary subset of $\mathbb{N}$ and $\mathcal{D}_Q$ be the corresponding deduction tree. Moreover, suppose $\mathcal{D}_Q$ is not well-founded. Then $\mathcal{D}_Q$ has an infinite path $\mathbb{P}$. $\mathbb{P}$ has the following properties:

1. $\mathbb{P}$ does not contain literals which are true in $\mathbb{N}$.
2. $\mathbb{P}$ does not contain formulas $s \in U_i$ and $t \notin U_i$ for constant terms s and t such that $s^{\mathbb{N}} = t^{\mathbb{N}}$.
3. If $\mathbb{P}$ contains $E_0 \vee E_1$ then $\mathbb{P}$ contains E_0 and E_1.
4. If $\mathbb{P}$ contains $E_0 \wedge E_1$ then $\mathbb{P}$ contains E_0 or E_1.
5. If $\mathbb{P}$ contains $\exists x F(x)$ then $\mathbb{P}$ contains $F(\bar{n})$ for all n.
6. If $\mathbb{P}$ contains $\forall x F(x)$ then $\mathbb{P}$ contains $F(\bar{n})$ for some n.
7. If $\mathbb{P}$ contains $\exists X F(X)$ then $\mathbb{P}$ contains $F(U_m)$ for all m.
8. If $\mathbb{P}$ contains $\forall X F(X)$ then $\mathbb{P}$ contains $F(U_m)$ for some m.
9. $\mathbb{P}$ contains $\neg C(U_m)$ for all m.
10. $\mathbb{P}$ contains $\neg\bar{Q}(m)$ for all m.

Proof: Standard. $\square$

Corollary 4.6. If $\mathcal{D}_Q$ is ill-founded then there exists a countable coded ω-model of **ATR** which contains Q.

5 Proof of the Main Theorem: The hard direction part 2

The remainder of the paper will be devoted to ruling out the possibility that for some Q, $\mathcal{D}_Q$ could be a well-founded tree. This is the place where the principle $\forall\mathfrak{X}\ [\mathbf{WO}(\mathfrak{X}) \rightarrow \mathbf{WO}(\Gamma_{\mathfrak{X}})]$ in the guise of cut elimination for an infinitary proof system enters the stage. Aiming at a contradiction, suppose that $\mathcal{D}_Q$ is a well-founded tree. Let $\mathfrak{X}_0$ be the Kleene-Brouwer ordering on $\mathcal{D}_Q$ (see [34, Definition V.1.2]). Then $\mathfrak{X}_0$ is a well-ordering. In a nutshell, the idea is that a well-founded $\mathcal{D}_Q$ gives rise to a derivation of the empty sequent (contradiction) in the infinitary proof systems $\mathcal{T}_Q^\infty$ from [24, Section 3]. To make this step more transparent we introduce two intermediate systems $\mathbf{KPl}_0$ and $\mathbf{KPl}_Q^\infty$. $\mathbf{KPl}_0$ is a formal set theory with the natural numbers as urelements. It has a constant $\mathbb{N}$ for the set of natural numbers and a unary predicate symbol $\mathbf{Ad}$ to convey that a set is an admissible set. The axioms of $\mathbf{KPl}_0$ are the usual axioms of Peano arithmetic for the urelements plus the schema of induction on the naturals for arbitrary formulae, extensionality for sets, an axiom saying that the natural numbers (urelements) form a set, an axiom saying that every set is contained in an admissible set and axioms saying that every admissible set is transitive and satisfies the axioms of Kripke-Platek set theory (see [3]), $\mathbf{KP}$, but with the axiom of foundation omitted. It is easy to show that $\mathbf{ATR}$ can be viewed as a subtheory of $\mathbf{KPl}_0$ wherein the second order quantifiers of $\mathbf{ATR}$ are interpreted as set quantifiers ranging over subsets of $\mathbb{N}$.

Lemma 5.1. $\mathbf{ATR}$ is a subtheory of $\mathbf{KPl}_0$.

Proof: We argue informally in $\mathbf{KPl}_0$. Suppose $\prec$ is a well-ordering on a subset of $\mathbb{N}$, u, v are the free variables of a bounded formula $B(u, v)$, i.e. all quantifiers in $B(u, v)$ are restricted. According to the axioms of $\mathbf{KPl}_0$ we can find an admissible set $\mathbb{A}$ such that $\mathbb{N}, \prec \in \mathbb{A}$ and such that all parameters occurring in $B(u, v)$ are also elements of $\mathbb{A}$. We use induction on $\prec$ to show that for every $n \in \mathbb{N}$ the following statement $C(n)$ holds: there exists a function $f_n \in \mathbb{A}$ with domain $\{i \in \mathbb{N} \mid i \preceq n\}$ such that

$$(\forall i \preceq n)\ \ f_n(i) \quad = \quad \{\langle j, m\rangle \in \mathbb{N} \times \mathbb{N} \mid j \prec i \ \wedge\ B(m, \bigcup_{l \prec j} f_n(l))\}. \qquad (3)$$

Note that each function f_n is uniquely determined by (3). Inductively assume that we have a function f_n of this form for all $n \prec k$. By Σ collection, bounded separation and union in $\mathbb{A}$, $g_k := \bigcup_{n \prec k} f_n$ is a set in $\mathbb{A}$. Thus the function f_k with domain $\{i \mid i \preceq k\}$ defined by

$$f_k(i) = \begin{cases} g_k(i) & \text{if } i \prec k \\ \{\langle k, m\rangle \mid m \in \mathbb{N} \wedge B(m, \bigcup_{n \prec k} f_n(n))\} & \text{if } i = k \end{cases}$$

is also an element of $\mathbb{A}$. Moreover, f_k satisfies (3) (when we replace f_n by f_k), whence $C(k)$ holds.

In view of the foregoing, in order to verify that transfinite arithmetical recursion is provable in $\mathbf{KPl}_0$ we only need to ensure that the above employment of transfinite induction is legitimate. To this end pick an admissible set $\mathbb{B}$ such that $\mathbb{A} \in \mathbb{B}$. Then $\{n \in \mathbb{N} \mid C(n)\}$ is a set by bounded separation in $\mathbb{B}$. □

5.1 A sequent calculus for $\mathbf{KPl}_0$

The *language* of $\mathbf{KPl}_0$, $\mathcal{L}$, consists of: *free variables* $a_1, a_2, a_3, \ldots$, *bound variables* $x_1, x_2, x_3, \ldots$, *constants*, *predicate symbols* , the *logical symbols* $\neg, \wedge, \vee, \forall, \exists$; and parentheses.

The *constants* are $\mathbb{N}$ for the set of natural numbers and for each natural number n a constant $\bar{n}$. The *terms* are the constants and free variables and will be denoted by letters $s, t, s_0, t_0, \ldots$.

The *predicate symbols* are $\in$ for elementhood, $\mathbf{Ad}$ for the class of admissible sets, a unary predicate $\mathbf{Set}$ to signify that an object is a set, a unary predicate P to denote an arbitrary set of natural numbers, two binary predicates $\mathbf{SUC}, =_{\mathbb{N}}$ for the successor relation and the identity on natural numbers, respectively. Further, there are two ternary relations $\mathbf{ADD}, \mathbf{MULT}$ for the graphs of addition and multiplication on natural numbers, respectively.

Formulae are built from atomic and *negated* atomic formulae by means of the connectives $\wedge, \vee$ and the following construction steps: If t is a term, a is a free variable and $F(a)$ is a formula in which the bound variable x does not occur, then $(\forall x{\in}t)F(x), (\exists x{\in}t)F(x), \forall x F(x), \exists x F(x)$ are formulae.

A formula which contains only bounded quantifiers, i.e. quantifiers of the form $(\forall x{\in}t), (\exists x{\in}s)$, is said to be a Δ_0*–formula*.

The *negation*, $\neg A$, of a non–atomic formula A is defined to be the formula obtained from A by (i) putting $\neg$ in front any atomic subformula, (ii) replacing $\wedge, \vee, (\forall x{\in}t), (\exists x{\in}t), \forall x, \exists x$ by $\vee, \wedge, (\exists x{\in}t), (\forall x{\in}t), \exists x, \forall x$, respectively, and (iii) dropping double negations.

Equality is defined by

$$\begin{aligned} a = b \quad :\Leftrightarrow \quad & \big(\neg\mathbf{Set}(a) \wedge \neg\mathbf{Set}(b) \wedge a =_{\mathbb{N}} b\big) \vee \\ & \big(\mathbf{Set}(a) \wedge \mathbf{Set}(b) \wedge (\forall x \in a)(x \in b) \wedge (\forall x \in b)(x \in a)\big). \end{aligned}$$

As a result of this, we will have to state the Axiom of Extensionality in a different way than usually.

We use $A, B, C, ..., F(a), G(a), ..$ as meta–variables for formulae. Upper case Greek letters $\Delta, \Gamma, \Lambda, ...$ range over finite sets of formulae. The meaning of $\{A_1, \ldots, A_n\}$ is the disjunction $A_1 \vee \cdots \vee A_n$. Γ, A stands for $\Gamma \cup \{A\}$. As usual, $A \to B$ abbreviates $\neg A \vee B$. We shall write $s = \{y \in t : F(y)\}$ for $(\forall y \in s)[y \in t \wedge F(y)] \wedge (\forall y \in t)[F(y) \to y \in t]$. We use $(\forall x_1, \ldots, x_n \in \mathbb{N})$ as an abbreviation for $(\forall x_1 \in \mathbb{N}) \ldots (\forall x_n \in \mathbb{N})$.

The axioms of $\mathbf{KPl}_0$ fall into several groups.

Logical axioms

1. $\Gamma, A, \neg A$ for each atomic formula A.

Ontological axioms.

1. $\Gamma,\ \mathbf{Set}(s) \leftrightarrow s \notin \mathbb{N}$.
2. $\Gamma,\ \bar{n} \in \mathbb{N}$
   for every number constant $\bar{n}$.
3. $\Gamma,\ t \in s \to \mathbf{Set}(s)$.
4. $\Gamma,\ J(s_1, \ldots, s_n) \to s_1 \in \mathbb{N} \wedge \ldots \wedge s_n \in \mathbb{N}$
   when J is one of the symbols $\mathrm{P}, \mathbf{SUC}, =_{\mathbb{N}}, \mathbf{ADD}, \mathbf{MULT}$.

Number-theoretic axioms.

1. $(\forall x \in \mathbb{N})\neg\mathbf{SUC}(x, \bar{0})$.
2. $(\forall x \in \mathbb{N})[x \neq_{\mathbb{N}} \bar{0} \to (\exists y \in \mathbb{N})\,\mathbf{SUC}(y, x)]$.
3. $(\forall x \in \mathbb{N})\,(\exists y \in \mathbb{N})\,\mathbf{SUC}(x, y)$.
4. $\Gamma,\ \mathbf{SUC}(\bar{n}, \overline{n+1})$ for all numbers n.
5. $(\forall x, y, z \in \mathbb{N})\,[\mathbf{SUC}(x, y) \wedge \mathbf{SUC}(x, z) \to y =_{\mathbb{N}} z]$.
6. $(\forall x, y, z \in \mathbb{N})\,[\mathbf{SUC}(y, x) \wedge \mathbf{SUC}(z, x) \to y =_{\mathbb{N}} z]$.
7. $(\forall x, y, z \in \mathbb{N})\,(\forall v \in \mathbb{N})\,[\mathbf{ADD}(y, x, z) \wedge \mathbf{ADD}(y, x, v) \to z =_{\mathbb{N}} v]$.
8. $(\forall x, y \in \mathbb{N})\,(\exists z \in \mathbb{N})\,\mathbf{ADD}(x, y, z)$.
9. $(\forall x \in \mathbb{N})\,\mathbf{ADD}(x, \bar{0}, x)$.
10. $(\forall x, y, z, v, w \in \mathbb{N})\,[\mathbf{ADD}(x, y, z) \wedge \mathbf{SUC}(y, v) \wedge \mathbf{SUC}(z, w) \to \mathbf{ADD}(x, v, w)]$.

11. $(\forall x, y, z, v \in \mathbb{N})\,[\mathbf{MULT}(y, x, z) \wedge \mathbf{MULT}(y, x, v) \rightarrow z =_{\mathbb{N}} v]$.
12. $(\forall x, y \in \mathbb{N})\,(\exists z \in \mathbb{N})\,\mathbf{MULT}(x, y, z)$.
13. $(\forall x \in \mathbb{N})\,\mathbf{MULT}(x, \bar{0}, \bar{0})$.
14. $(\forall x, y, z, v, w \in \mathbb{N})\,[\mathbf{MULT}(x, y, z) \wedge \mathbf{SUC}(y, v) \wedge \mathbf{ADD}(z, x, w) \rightarrow \mathbf{MULT}(x, v, w)]$.

Equality and Extensionality axioms.

1. $\Gamma,\ s = s$.
2. $\Gamma,\ s = t \wedge A(s) \rightarrow A(t)$ for all atomic formulae A.

$\mathbb{N}$-induction.

$\bar{0} \in s \wedge (\forall x, y \in \mathbb{N})\,[x \in s \wedge \mathbf{SUC}(x, y) \rightarrow y \in s] \rightarrow (\forall x \in \mathbb{N})\,(x \in s)$

Set-theoretic axioms

Ad-*Limit:*

$$\Gamma,\ \exists y\,(t \in y \wedge \mathbf{Ad}(y)).$$

Ad-*Linearity:*

$$\Gamma,\ \mathbf{Ad}(s) \wedge \mathbf{Ad}(t) \rightarrow s \in t \vee s = t \vee t \in s.$$

Ad-Axioms

($\mathbf{Ad}1$)*:* $\Gamma,\ \mathbf{Ad}(s) \rightarrow \mathbb{N} \in s \wedge (\forall x \in s)\,(\forall z \in x)\,z \in s$

($\mathbf{Ad}2$)*:* $\Gamma,\ \mathbf{Ad}(s) \rightarrow A^s$

where A^s is the relativization of A to s and A is a universal closure of one of the following axioms:

Pairing: $\exists x\ (x = \{a, b\})$

Union: $\exists x\ (x = \bigcup a)$

Δ_0*–Separation:*

$$\exists x\ \big(x = \{y \in a : F(y)\}\big)$$

for all Δ_0–formulae $F(b)$

Δ_0*–Collection:*

$$(\forall x \in a)\exists y G(x, y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)G(x, y)$$

for all Δ_0–formulae $G(b, c)$.

The *logical rules of inference* are:

$$(\wedge)\ \frac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \wedge A'} \qquad (\vee)\ \frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1}\ \text{ if } i \in \{0,1\}$$

$$(b\forall)\ \frac{\Gamma, a \in s \rightarrow F(a)}{\Gamma, (\forall x \in s)F(x)} \qquad (\forall)\ \frac{\Gamma, F(a)}{\Gamma, \forall x F(x)}$$

$$(b\exists)\ \frac{\Gamma, t \in s \wedge F(t)}{\Gamma, (\exists x \in s)F(x)} \qquad (\exists)\ \frac{\Gamma, F(t)}{\Gamma, \exists x F(x)}$$

$$(Cut)\ \frac{\Gamma, A \qquad \Gamma, \neg\, A}{\Gamma}$$

where in $(\forall)$ and $(b\forall)$ the free variable a is not to occur in the conclusion of the inference.

5.2 The infinitary calculus $\mathbf{KPl}_Q^\infty$

In what follows we fix $Q \subseteq \mathbb{N}$. In the main, the infinitary version of $\mathbf{KPl}_0$, denoted $\mathbf{KPl}_Q^\infty$, is obtained from $\mathbf{KPl}_0$ by adding the ω-rule and the basic diagram of Q. $\mathbf{KPl}_0$ and $\mathbf{KPl}_Q^\infty$ have the same language. Again, we will be working in a Tait-style formalization of set theory with formulae in negation normal form, i.e. negations only in front of atomic formulas.

$\mathbf{KPl}_Q^\infty$, has the following axioms and rules.

Basic Axioms

1. **Logical axioms:**

   $\Gamma, A, \neg A$ for each atomic formula A.

2. **Ontological** and **number-theoretic axioms**:

   (A1) $\Gamma, s \notin t,\ s \in t$.

   (A2) $\Gamma, \mathrm{P}(\bar{n})$ if $n \in Q$.

   $\Gamma, \neg\mathrm{P}(\bar{n})$ if $n \notin Q$.

   (A3) $\Gamma, R(\bar{n}_1, \ldots, \bar{n}_k)$ if $R(n_1, \ldots, n_k)$ is true, where R is one of the symbols $\mathbf{SUC}, \mathbf{ADD}, \mathbf{MULT}, =_{\mathbb{N}}$.

   (A4) $\Gamma, \neg R(\bar{n}_1, \ldots, \bar{n}_k)$ if $R(n_1, \ldots, n_k)$ is false, where R is one of the symbols $\mathbf{SUC}, \mathbf{ADD}, \mathbf{MULT}, =_{\mathbb{N}}$.

   (A5) $\Gamma, \bar{n} \in \mathbb{N}$.

   (A6) $\Gamma, s \in \mathbb{N}, \mathbf{Set}(s)$.

   (A7) $\Gamma, \neg\mathbf{Set}(s), s \notin \mathbb{N}$.

   (A8) $\Gamma, \neg\mathbf{Set}(\bar{n})$.

(A9) $\Gamma, s \in \mathbb{N}, \neg \mathrm{P}(s)$.

(A10) $\Gamma, s_i \in \mathbb{N}, \neg J(s_1, \ldots, s_k)$ if $1 \leq i \leq k$ and J is one of the symbols $\mathbf{SUC}, \mathbf{ADD}, \mathbf{MULT}, =_{\mathbb{N}}$.

(A11) $\Gamma,\ s \notin \bar{n}$.

3. **Equality** and (at the same time) **extensionality axioms**:
   $\Gamma,\ s = t\ \wedge\ A(s) \rightarrow A(t)$ for all atomic formulae A.

4. **Set-theoretic axioms**:

Ad-*Limit:*	$\Gamma,\ \exists y\,(s \in y\ \wedge\ \mathbf{Ad}(y))$.
Ad-*Linearity:*	$\Gamma,\ \mathbf{Ad}(s)\ \wedge\ \mathbf{Ad}(t)\ \rightarrow\ s \in t\ \vee\ s = t\ \vee\ t \in s$.
(**Ad**1)*:*	$\Gamma,\ \mathbf{Ad}(s)\ \rightarrow\ \mathbb{N} \in s\ \wedge\ (\forall x \in s)\,(\forall z \in x)\,z \in s$.
(**Ad**2)*:*	$\Gamma,\ \mathbf{Ad}(s)\ \rightarrow\ A^s$, where A is a universal closure of one of the following axioms:
Pairing:	$\exists x\ (x = \{a, b\})$.
Union:	$\exists x\ (x = \bigcup a)$.
Δ_0–*Separation:*	$\exists x\ \big(x = \{y \in a : F(y)\}\big)$ for all Δ_0–formulae $F(b)$
Δ_0–*Collection:*	$(\forall x \in a)\exists y G(x, y) \rightarrow \exists z (\forall x \in a)(\exists y \in z) G(x, y)$ for all Δ_0–formulae $G(b, c)$.

Below a, b always denote free variables. The rules of $\mathbf{KPl}_Q^\infty$ are:

$$(\wedge) \quad \frac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \wedge A'}$$

$$(\vee) \quad \frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \quad \text{if } i = 0 \text{ or } i = 1$$

$$(\forall_{\mathbb{N}}) \quad \frac{\Gamma, F(\bar{n}) \text{ for all number constants } \bar{n}}{\Gamma, (\forall x{\in}\mathbb{N})F(x)}$$

$$(\exists_{\mathbb{N}}) \quad \frac{\Gamma, F(\bar{n}) \text{ for some number constant } \bar{n}}{\Gamma, (\exists x{\in}\mathbb{N})F(x)}$$

$$(b\forall^{\infty}) \quad \frac{\Gamma, b{\in}s \rightarrow F(b) \text{ for all } b}{\Gamma, (\forall x \in s)F(x)}$$

$$(b\exists^{\infty}) \quad \frac{\Gamma, t{\in}s \wedge F(t) \text{ for some } t}{\Gamma, (\exists x \in s)F(x)}$$

$$(\forall^{\infty}) \quad \frac{\Gamma, F(b) \text{ for all } b}{\Gamma, \forall x\, F(x)}$$

$$(\exists^{\infty}) \quad \frac{\Gamma, F(t) \text{ for some } t}{\Gamma, \exists x\, F(x)}$$

$$(\text{Cut}) \quad \frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

The *degree* of a formula A of $\mathcal{L}(\mathbf{KPl}_Q^{\infty})$, $\deg(A)$, is defined as follows:

1. $\deg(A) = 0$ if A is Δ_0.
2. $\deg((\exists x{\in}t)F(x)) := \deg((\forall x{\in}t)F(x)) := \deg(F(\bar{0})) + 2$ if $F(\bar{0})$ is not Δ_0.
3. $\deg(\exists x\, F(x)) := \deg(\forall x\, F(x)) := \deg(F(\bar{0})) + 1$.
4. $\deg(A \wedge B) := \deg(A \vee B) := \max\{\deg(A), \deg(B)\} + 1$ if $A \wedge B$ is not Δ_0.

The relation $\mathbf{KPl}_Q^{\infty} \overset{\beta}{\underset{k}{\vdash}} \Gamma$ is inductively defined as follows:

1. If Γ is an axiom of $\mathbf{KPl}_Q^{\infty}$, then $\mathbf{KPl}_Q^{\infty} \overset{\beta}{\underset{k}{\vdash}} \Gamma$ for all β and k.

2. If $\mathbf{KPl}_Q^\infty \mid\!\frac{\beta_i}{k}\, \Gamma_i$ and $\beta_i < \beta$ hold for every premise Γ_i of a rule other than (Cut), then $\mathbf{KPl}_Q^\infty \mid\!\frac{\beta}{k}\, \Gamma$ if Γ is the conclusion of that rule.

3. If $\mathbf{KPl}_Q^\infty \mid\!\frac{\beta_0}{k}\, \Gamma, A$, $\mathbf{KPl}_Q^\infty \mid\!\frac{\beta_1}{k}\, \Gamma, \neg A$, $\beta_0, \beta_1 < \beta$ and $\deg(A) < k$, then $\mathbf{KPl}_Q^\infty \mid\!\frac{\beta}{k}\, \Gamma$.

If Γ is a set of formulae we use the notation $\Gamma[a_1, \ldots, a_r]$ to convey that all free variables of formulae in Γ are contained in the set $\{a_1, \ldots, a_r\}$. We use $F[a_1, \ldots, a_r]$ to convey the analogous thing for a formula F.

Theorem 5.2. If $\mathbf{KPl}_0$ proves a sequent $\Gamma[a_1, \ldots, a_r]$, then there exist $k < \omega$ and $\alpha < \omega + \omega$ such that for all terms $s_1, \ldots, s_n$,

$$\mathbf{KPl}_Q^\infty \mid\!\frac{\alpha}{k}\, \Gamma[s_1, \ldots, s_r]\,.$$

Proof: This is routine. For induction one has to use the ω-rule $(\forall_{\mathbb{N}})$. $\square$

Theorem 5.3. Let $k < \omega$ and Θ be a finite set of arithmetical sentences. Then we have:

$$\mathbf{KPl}_Q^\infty \mid\!\frac{\alpha}{k}\, \Theta \quad \Rightarrow \quad \mathbf{KPl}_Q^\infty \mid\!\frac{\Gamma_{\varepsilon_{\omega^\alpha+2}}}{0}\, \Theta\,.$$

Proof: This follows from Corollary 7.6 at the very end of the paper. $\square$

There are different ways of formalizing infinite deductions in theories like **PA**. We just mention [33] and [10].

5.3 Finishing the proof of the main Theorem

Recall that in order to finish the proof of Theorem 1.4 we want to show that $\mathcal{D}_Q$ is not well-founded. Aiming at a contradiction, suppose that $\mathcal{D}_Q$ is a well-founded tree, i.e. all paths in $\mathcal{D}_Q$ are finite, and thus every maximal path ends in a sequent which contains a basic axiom. It is then possible to conceive of $\mathcal{D}_Q$ as a skeleton of a proof in $\mathbf{KPl}_Q^\infty$. Each formula A of L_2 can be identified with a formula of $\mathbf{KPl}_Q^\infty$ arising by the following two steps:

1. Replace all second order quantifiers $\forall X \ldots$ and $\exists X \ldots$ by $\forall X(X \subseteq \mathbb{N} \to \ldots)$ and $\exists X(X \subseteq \mathbb{N} \wedge \ldots)$, respectively. (Here we adopt the convention that variables of L_2 other than U_0 are also variables of the language of $\mathbf{KPl}_Q^\infty$ and $X \subseteq \mathbb{N}$ is an abbreviation for $(\forall v \in X)(v \in \mathbb{N})$.)

2. Replace each subformula of the form $t \in U_0$ by $\mathrm{P}(t)$.

As for (2), note that the variable U_0 is axiomatically linked in deduction chains to the set Q. In $\mathbf{KPl}_Q^\infty$ this role is taken over by the predicate symbol P.

Now $\mathcal{D}_Q$ can be conceived of as a derivation of the empty sequent $\emptyset$ with **hidden cuts** involving cut formulae of the shape $\neg\bar{Q}(i)$ and $\neg C(U_i)$. Note that by Theorem 5.2 and Lemma 5.1, we have the following result:

Lemma 5.4. There exist fixed $k_0, k_1 < \omega$ such that for all $i < \omega$:

(i) $\mathbf{KPl}_Q^\infty \mid\!\frac{\omega+k_0}{k_1}\, \neg U_i \subseteq \mathbb{N},\, C(U_i)\,.$

(ii) $\mathbf{KPl}_Q^\infty \mid\!\frac{0}{0}\, \bar{Q}(i)\,.$

Thus if Λ is the sequent attached to a node τ of $\mathcal{D}_Q$ and $(\Lambda_i)_{i\in I}$ is an enumeration of the sequents attached to the immediate successor nodes of τ in $\mathcal{D}_Q$ then the transition

$$\frac{(\Lambda_i)_{i\in I}}{\Lambda}$$

can be viewed as a combination of three inferences in $\mathbf{KPl}_Q^\infty$, the first one being logical inferences and the other two being cuts. To make this formally precise, let $\mathfrak{X}_0$ be the Kleene-Brouwer ordering of this tree (see [34, Definition V.1.2]). Note that $\mathfrak{X}_0$ has a maximal element which is the bottom node $\langle\rangle$ of the tree. Next let $\mathfrak{X}_1$ be the well-ordering $\mathfrak{X}_0 \cdot \omega$ defined in [25, Definition 2.2]. At each node τ of $\mathcal{D}_Q$ the pertaining sequent is of the form $\Gamma_\tau, \neg\bar{Q}(j), \neg C(U_j)$, where j is the highest number such that the pair $\neg\bar{Q}(j), \neg C(U_j)$ belongs to the sequent. We shall convey this by writing:

$$\mathcal{D}_Q \models^{\tau} \Gamma_\tau, \neg\bar{Q}(j), \neg C(U_j)\,.$$

We then have the following result.

Lemma 5.5. Let $\tau \in \mathfrak{X}_0$ and suppose the free set variables of Γ_τ with indices > 0 are among $U_{i_1}, \ldots, U_{i_r}$. Then there exists a fixed k such that

$$\mathcal{D}_Q \models^{\tau} \Gamma_\tau, \neg\bar{Q}(j), \neg C(U_j)$$

implies

$$\mathbf{KPl}_Q^\infty \mid\!\frac{\omega+\tau\cdot\omega}{k}\, \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \Gamma_\tau\,. \tag{4}$$

Here the ordinal $\omega + \tau \cdot \omega$ is a member of $\mathfrak{X}_1$.

Proof: Below we use the notation $\tau * 0$ to denote the node of the tree obtained by appending 0 to the string τ, i.e. what is usually denoted by $\tau^\frown\langle 0\rangle$.

Note that $\deg(C(U_i)) = 3$. So we may put $k := \max(k_1, 4)$ with k_1 taken from Lemma 5.4.

The proof proceeds by induction on τ with respect to the well-ordering $\mathfrak{X}_1$. If $\Gamma_\tau, \neg\bar{Q}(j), \neg C(U_j)$ is axiomatic (in the sense of Definition 4.2 (iii)), then $\Gamma_\tau, \neg\bar{Q}(j)$ an axiom of $\mathbf{KPl}^\infty_Q$ according to (A1), (A3) or (A4). Thence, by Lemma 5.4, (Cut) and weakening:

$$\mathbf{KPl}^\infty_Q \mid\!\frac{\omega+\tau\cdot\omega}{k}\, \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \Gamma_\tau\,.$$

Now suppose Γ_τ has a redex E of the form $E \equiv \exists X\, F(X)$. Then $\Gamma_\tau = \Gamma'_\tau, E, \Gamma''_\tau$ for some $\Gamma'_\tau, \Gamma''_\tau$, and, moreover,

$$\mathcal{D}_Q \mid\!\frac{\tau*0}{}\, \Gamma'_\tau, F(U_m), \Gamma''_\tau, E, \neg\bar{Q}(j), \neg C(U_j), \neg\bar{Q}(i+1), \neg C(U_{i+1})$$

for some i, m. Since $\tau * 0$ is smaller than τ in the Kleene-Brouwer ordering, by the induction hypothesis (and a little help from weakening):

$$\begin{aligned}\mathbf{KPl}^\infty_Q \mid\!\frac{\omega+(\tau*0)\cdot\omega}{k}\, & \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \neg U_m \subseteq \mathbb{N}, \neg U_j \subseteq \mathbb{N},\\ & \neg U_{i+1} \subseteq \mathbb{N}\Gamma'_\tau, F(U_m), \Gamma''_\tau, E, \neg\bar{Q}(j), \neg C(U_j),\\ & \neg\bar{Q}(i+1), \neg C(U_{i+1}).\end{aligned}$$

Owing to Lemma 5.4 we get for some k_0, using several cuts:

$$\begin{aligned}\mathbf{KPl}^\infty_Q \mid\!\frac{\omega+(\tau*0)\cdot\omega+k_0}{k}\, & \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \neg U_m \subseteq \mathbb{N}, \neg U_j \subseteq \mathbb{N},\\ & \neg U_{i+1} \subseteq \mathbb{N}, \Gamma'_\tau, F(U_m), \Gamma''_\tau, E.\end{aligned}$$

Since $\mathbf{KPl}^\infty_Q \mid\!\frac{k'}{0}\, U_m \subseteq \mathbb{N}, \neg U_m \subseteq \mathbb{N}$ for some k' we get

$$\begin{aligned}\mathbf{KPl}^\infty_Q \mid\!\frac{\omega+(\tau*0)\cdot\omega+k_0+1}{k}\, & \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \neg U_m \subseteq \mathbb{N}, \neg U_j \subseteq \mathbb{N},\\ & \neg U_{i+1} \subseteq \mathbb{N}, \Gamma'_\tau, U_m \subseteq \mathbb{N} \wedge F(U_m), \Gamma''_\tau, E\end{aligned}$$

via $(\wedge)$ and thus

$$\begin{aligned}\mathbf{KPl}^\infty_Q \mid\!\frac{\omega+(\tau*0)\cdot\omega+k_0+2}{k}\, & \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \neg U_m \subseteq \mathbb{N}, \neg U_j \subseteq \mathbb{N},\\ & \neg U_{i+1} \subseteq \mathbb{N}, \Gamma'_\tau, \Gamma''_\tau, E\end{aligned}$$

since $E \equiv \exists X(X \subseteq \mathbb{N} \wedge F(X))$. If $i+1, j, m \in \{i_1, \ldots, i_r\}$ we are done. If this is not the case we can substitute the set constant $\mathbb{N}$ for any of the variables whose index does not belong to $\{i_1, \ldots, i_r\}$ everywhere in the derivation. This does not change the length of the derivation. As a result we have:

$$\mathbf{KPl}_Q^\infty \stackrel{\omega+(\tau*0)\cdot\omega+k_0+2}{\vert\overline{}}_k \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \neg\mathbb{N} \subseteq \mathbb{N}, \Gamma'_\tau, \Gamma''_\tau, E\,.$$

Since $\mathbf{KPl}_Q^\infty \stackrel{k''}{\vert\overline{}}_0 \mathbb{N} \subseteq \mathbb{N}$, a cut yields

$$\mathbf{KPl}_Q^\infty \stackrel{\omega+(\tau*0)\cdot\omega+k_0+3}{\vert\overline{}}_k \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \Gamma'_\tau, \Gamma''_\tau, E$$

and hence

$$\mathbf{KPl}_Q^\infty \stackrel{\omega+\tau\cdot\omega}{\vert\overline{}}_k \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \Gamma_\tau\,.$$

Next suppose Γ_τ has a redex E of the form $E \equiv \forall X\, F(X)$. Then $\Gamma_\tau = \Gamma'_\tau, E, \Gamma''_\tau$ for some $\Gamma'_\tau, \Gamma''_\tau$, and, moreover,

$$\mathcal{D}_Q \stackrel{\tau*0}{\vert\overline{}} \Gamma'_\tau, F(U_m), \Gamma''_\tau, E, \neg\bar{Q}(j), \neg C(U_j), \neg\bar{Q}(i+1), \neg C(U_{i+1})$$

for some i, m with the proviso that U_m occurs only in $F(U_m)$, $i+1 \neq m$ and $j \neq m$. By the induction hypothesis and weakening we have:

$$\begin{aligned}\mathbf{KPl}_Q^\infty \stackrel{\omega+(\tau*0)\cdot\omega}{\vert\overline{}}_k\ & \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \neg U_m \subseteq \mathbb{N}, \neg U_j \subseteq \mathbb{N}, \\ & \neg U_{i+1} \subseteq \mathbb{N}, \Gamma'_\tau, F(U_m), \Gamma''_\tau, E, \neg\bar{Q}(j), \neg C(U_j), \\ & \neg\bar{Q}(i+1), \neg C(U_{i+1}).\end{aligned}$$

Owing to Lemma 5.4 we get

$$\begin{aligned}\mathbf{KPl}_Q^\infty \stackrel{\omega+(\tau*0)\cdot\omega+k_0}{\vert\overline{}}_k\ & \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \neg U_j \subseteq \mathbb{N}, \neg U_{i+1} \subseteq \mathbb{N} \\ & \Gamma'_\tau, U_m \subseteq \mathbb{N} \to F(U_m), \Gamma''_\tau, E,\end{aligned}$$

for some k_0, using several cuts and $(\vee)$ twice. As U_m is an eigenvariable we infer (via $(\forall^\infty)$) that

$$\begin{aligned}\mathbf{KPl}_Q^\infty \stackrel{\omega+(\tau*0)\cdot\omega+k_0+1}{\vert\overline{}}_k\ & \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \neg U_j \subseteq \mathbb{N}, \neg U_{i+1} \subseteq \mathbb{N}, \\ & \Gamma'_\tau, \Gamma''_\tau, E\end{aligned}$$

since $E \equiv \forall X(X \subseteq \mathbb{N} \to F(X))$. If $i+1, j \in \{i_1, \ldots, i_r\}$ we are done. If this is not the case we can substitute everywhere in the derivation the set constant $\mathbb{N}$ for any of the variables whose index does not belong to $\{i_1, \ldots, i_r\}$. This does not change the length of the derivation. As a result we have:

$$\mathbf{KPl}_Q^\infty \stackrel{\omega+(\tau*0)\cdot\omega+k_0+1}{\vert\overline{}}_k \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \neg\mathbb{N} \subseteq \mathbb{N}, \Gamma'_\tau, \Gamma''_\tau, E\,.$$

Since $\mathbf{KPl}_Q^\infty \mathrel{\frac{k''}{0}} \mathbb{N} \subseteq \mathbb{N}$, a cut yields

$$\mathbf{KPl}_Q^\infty \mathrel{\Big|\frac{\omega+(\tau*0)\cdot\omega+k_0+2}{k}} \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \Gamma'_\tau, \Gamma''_\tau, E$$

and hence

$$\mathbf{KPl}_Q^\infty \mathrel{\Big|\frac{\omega+\tau\cdot\omega}{k}} \neg U_{i_1} \subseteq \mathbb{N}, \ldots, \neg U_{i_r} \subseteq \mathbb{N}, \Gamma_\tau\,.$$

Finally, if the redex E is of the form $E_0 \vee E_1$ or $E_0 \wedge E_1$ the desired assertion follows by similar (simpler) considerations. □

We can now finish the proof of Theorem 1.4. Let τ_0 be the bottom node of the tree $\mathcal{D}_Q$. By Lemma 5.5 we have

$$\mathbf{KPl}_Q^\infty \mathrel{\Big|\frac{\omega+\tau_0\cdot\omega}{k}} \emptyset\,.$$

Going to the well-ordering $\Gamma_{\mathfrak{X}_1}$ we can employ Theorem 5.3, arriving at

$$\mathbf{KPl}_Q^\infty \mathrel{\Big|\frac{\Gamma_{\varepsilon_{\omega^{\tau_0}+2}}}{0}} \emptyset\,.$$

However, this is impossible since a cut free derivation in $\mathbf{KPl}_Q^\infty$ cannot produce the empty sequent as any derivation starts from axioms and formulae can only disappear via cuts.

6 Prospectus

A statement of the form $\mathbf{WOP}(f)$ is Π^1_2 and therefore cannot be equivalent to a theory whose axioms have a higher complexity, like for instance Π^1_1-comprehension. After ω-models come β-models and the theory Π^1_1-$\mathbf{CA}$ has a characterization in terms of countable coded β-models (see [34, VII]), namely via the statement "every set belongs to a countably coded β-model". An ω-model $\mathfrak{A}$ is a β-model if the concept of well ordering is absolute with respect to $\mathfrak{A}$.

The question arises whether the methodology of this paper can be extended to more complex axiom systems, in particular to those characterizable via β-models? The answer will be couched as a conjecture. First of all, to get equivalences one has to climb up in the type structure. Given a functor

$$F : (\mathbb{LO} \to \mathbb{LO}) \to (\mathbb{LO} \to \mathbb{LO}),$$

where $\mathbb{LO}$ is the class of linear orderings, we consider the statement:

$$\mathbf{WOPP}(F): \qquad \forall f \in (\mathbb{LO} \to \mathbb{LO})\, [\mathbf{WOP}(f) \to \mathbf{WOP}(F(f))].$$

There is also a variant of $\mathbf{WOPP}(F)$ which should basically encapsulate the same "power". Given a functor

$$G : (\mathbb{LO} \to \mathbb{LO}) \to \mathbb{LO}$$

consider the statement:

$$\mathbf{WOPP}_1(G): \qquad \forall f \in (\mathbb{LO} \to \mathbb{LO})\ [\mathbf{WOP}(f) \to \mathbf{WOP}(G(f))].$$

Conjecture 6.1. Statements of the form $\mathbf{WOPP}(F)$ (or $\mathbf{WOPP}_1(F)$), where F comes from some ordinal ordinal representation system used for an ordinal analysis of a theory T_F, are equivalent to statements of the form "every set belongs to a countable coded β-model of T_F".

The conjecture may be a bit vague, but it has been corroborated in some cases (around Π^1_1-$\mathbf{CA}$), and, what is perhaps more important, the proof technology exhibited in this paper seems to be sufficiently malleable as to be applicable to the extended scenario of β-models, too.

At this point I'd like to point out that Antonio Montálban has made several (precise) conjectures about statements of the type 6.1 being equivalent to Π^1_1-$\mathbf{CA}$ in [20].

7 Appendix: The infinitary calculus $\mathcal{T}_Q^\infty$

We still do not have a complete proof of Theorem 1.4 because we haven't proved the cut elimination Theorem 5.3 for $\mathbf{KPl}_Q^\infty$. This appendix is devoted to this task. It requires the introduction of yet another proof calculus, the system $\mathcal{T}_Q^\infty$ from [24].

In addition to the constants and relation symbols of $\mathbf{KPl}_Q^\infty$, the language of $\mathcal{T}_Q^\infty$, $\mathcal{L}(\mathcal{T}_Q^\infty)$, has the following symbols:

- constants $\mathbf{M}_\alpha$ for all $1 \leq \alpha \leq \omega$.
- free variables $a^i, b^i, c^i, \ldots$ for all $i < \omega$.

Letting $\mathbf{M}_0 := \mathbb{N}$, the intended meaning of $\mathbf{M}_{n+1}$ is the least admissible set above $\mathbf{M}_n$ while $\mathbf{M}_\omega = \bigcup_{n<\omega} \mathbf{M}_n$.

Variables a^i are supposed to range over elements of $\mathbf{M}_{i+1}$ which are not numbers, i.e. sets.

The *terms of* $\mathcal{L}(\mathcal{T}_Q^\infty)$ consist of the constants and variables. Each term t possesses a *level*, $|\,t\,|$, which is defined as follows:

$$\begin{array}{rcl} |\,\bar{n}\,| & := & 0 \\ |\,\mathbb{N}\,| & := & 0 \\ |\,\mathbf{M}_\alpha\,| & := & \alpha \\ |\,a^i\,| & := & i. \end{array}$$

The *atomic formulae* of $\mathcal{L}(\mathcal{T}_Q^\infty)$ are obtained from atomic formulae of $\mathcal{L}(\mathbf{KPl}_Q^\infty)$ by replacing all its free variables with terms of $\mathcal{L}(\mathcal{T}_Q^\infty)$ having levels $< \omega$ (hence $\mathbf{M}_\omega$ does not appear in atomic formulae of $\mathcal{L}(\mathcal{T}_Q^\infty)$).

Formulae are built from atomic and *negated* atomic formulae by means of the connectives $\wedge, \vee$ and the following construction step: If s is a term and a^α is a free variable of $\mathcal{L}(\mathcal{T}_Q^\infty)$ and $\mathcal{F}(a^\alpha)$ is a formula in which the bound variable x does not occur, then $(\forall x{\in}s)\mathcal{F}(x)$ and $(\exists x \in s)\mathcal{F}(x)$ are formulae.

Notice that formally $\mathcal{L}(\mathcal{T}_Q^\infty)$ formulae do not have unbounded quantifiers, albeit the quantifiers $(\forall x \in \mathbf{M}_\omega)$ and $(\exists x \in \mathbf{M}_\omega)$ can be viewed as unbounded as they range over the entire universe of discourse of $\mathcal{T}_Q^\infty$.

Below we use the relation $\equiv$ to mean syntactical identity. For terms s, t we set

$$s \lhd t \quad :\Leftrightarrow \quad \begin{cases} & [\, s \text{ is a numeral and } t \equiv \mathbb{N}\,] \\ \text{or} & [\, |\,s\,| < |\,t\,| \text{ and } t \equiv \mathbf{M}_\alpha \text{ for some } \alpha > 0\,] \\ \text{or} & [\, |\,s\,| \leq |\,t\,| \text{ and } t \equiv a^i \text{ for some } i\,]. \end{cases}$$

For terms s, t with $s \lhd t$ we set

$$s \mathring{\in} t \quad :\equiv \quad \begin{cases} \bar{0} =_{\mathbb{N}} \bar{0} & \text{if } t \equiv \mathbb{N} \\ \bar{0} =_{\mathbb{N}} \bar{0} & \text{if } t \equiv \mathbf{M}_\alpha \text{ for some } \alpha > 0 \\ s \in t & \text{if } t \equiv a^i \text{ for some } i. \end{cases}$$

The *rank* of formulae and terms is determined as follows.

1. $rk(t) := \omega \cdot |\,t\,|$.

2. $rk(s{\in}t) := rk(s{\notin}t) := \max(rk(s) + 6, rk(t) + 1)$.

3. $rk(\mathbf{Ad}(s)) := rk(\neg\mathbf{Ad}(s)) := rk(s) + 9$.

4. $rk(J(s_1, \ldots, s_n)) = rk(\neg J(s_1, \ldots, s_n)) = \max(rk(s_1), \ldots, rk(s_n)) + 1$ if J is a predicate symbol other than $\in$ and $\mathbf{Ad}$.

5. $rk((\exists x{\in}t)F(x)) := rk((\forall x{\in}t)F(x)) := max(rk(t), rk(F(\bar{0})) + 2)$ provided that t is not a variable.

6. $rk((\exists x{\in}a^i)F(x)) := rk((\forall x{\in}a^i)F(x)) := \max(rk(a^i){+}6, rk(F(a^i))){+}2$.

7. $rk(A \wedge B) := rk(A \vee B) := \max(rk(A), rk(B)) + 1$.

Let $0 < k < \omega$. A formula of $\mathcal{T}_Q^\infty$ is $\Delta_0(k)$ if all the terms occurring in it have levels $< k$.

A formula is $\Sigma(k)$ if it is in the smallest class of formulae containing the $\Delta_0(k)$ formulae which is closed under $\wedge, \vee$ and bounded quantifiers $(\exists x \in t)$, $(\forall x \in s)$, $(\exists x \in a^i)$, $(\forall x \in a^i)$, providing $|s|, i < k$ and $|t| \leq k$, where s, t are closed terms.

A formula is $\Sigma_\infty(k)$ if it is in the smallest class of formulae containing the $\Delta_0(k)$ formulae which is closed under $\wedge, \vee$ and bounded quantifiers $(\exists x \in t)$, $(\forall x \in s)$, $(\exists x \in a^i)$, $(\forall x \in a^i)$, providing $i < k$ and $|s|, |t| \leq k$, where s, t are closed terms.

Observe that if A is $\Delta_0(k)$, then $rk(A) < \omega \cdot k$. If B is $\Sigma_\infty(k)$, then $rk(B) < \omega \cdot k + \omega$. If C is of the form $(\exists x \in t)F(x)$, where t is a closed term with $|t| = k$ and $F(\bar{0})$ is $\Delta_0(k)$, then $rk(C) = \omega \cdot k$.

The *logical, ontological and arithmetic axioms* of $\mathcal{T}_Q^\infty$ are:

(A1) $\Gamma, s \notin t,\ s \in t$.

(A2) $\Gamma, \mathrm{P}(\bar{n})$ if $n \in Q$.
$\Gamma, \neg\mathrm{P}(\bar{n})$ if $n \notin Q$.

(A3) $\Gamma, R(\bar{n}_1, \ldots, \bar{n}_k)$ if $R(n_1, \ldots, n_k)$ is true, where R is one of the symbols $\mathbf{SUC}, \mathbf{ADD}, \mathbf{MULT}, =_{\mathbb{N}}$.

(A4) $\Gamma, \neg R(\bar{n}_1, \ldots, \bar{n}_k)$ if $R(n_1, \ldots, n_k)$ is false, where R is one of the symbols $\mathbf{SUC}, \mathbf{ADD}, \mathbf{MULT}, =_{\mathbb{N}}$.

(A5) $\Gamma, \bar{n} \in \mathbb{N}$.

(A6) $\Gamma, s \notin \mathbb{N}$ if s is not a numeral.

(A7) $\Gamma, \mathbf{Set}(s)$ if s is not a numeral.

(A8) $\Gamma, \neg\mathbf{Set}(\bar{n})$.

(A9) $\Gamma, \neg\mathrm{P}(s)$ if s is not a numeral.

(A10) $\Gamma, \neg J(s_1, \ldots, s_k)$ if, for some $1 \leq i \leq k$, s_i is not a numeral and J is one of the symbols $\mathbf{SUC}, \mathbf{ADD}, \mathbf{MULT}, =_{\mathbb{N}}$.

(A11) $\Gamma,\ a^i \notin a^i$.

Let $0 < \kappa < \omega$. The *set-theoretical axioms* of $\mathcal{T}_Q^\infty$ are:

(Extens.)	$\Gamma, r \neq s, t \neq t', s \notin t, r \in t'$	t not a numeral.
(Pair)	$\Gamma, (\exists x \in \mathbf{M}_\kappa)(s \in x \wedge t \in x)$	if $s, t \lhd \mathbf{M}_\kappa$.
(Union)	$\Gamma, (\exists x \in \mathbf{M}_\kappa)(\forall y \in s)(\forall z \in y)(z \in x)$	if $s \lhd \mathbf{M}_\kappa$.
(Δ_0-Sep)	$\Gamma, (\exists x \in \mathbf{M}_\kappa)\ \big(x = \{y \in s : \mathcal{F}[y, \vec{t}]\}\big)$	if $\mathcal{F}$ is Δ_0 and $s, \vec{t} \lhd \mathbf{M}_\kappa$.

The *rules* of $\mathcal{T}_Q^\infty$ are:

$$(\wedge) \qquad \frac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \wedge A'}$$

$$(\vee) \qquad \frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \quad \text{if } i = 0 \text{ or } i = 1$$

$$(\forall) \qquad \frac{\cdots \Gamma, s \mathring{\in} t \to F(s) \cdots \text{(for all } s \lhd t)}{\Gamma, (\forall x \in t) F(x)}$$

$$(\exists) \qquad \frac{\Gamma, s \mathring{\in} t \wedge F(s)}{\Gamma, (\exists x \in t) F(x)} \quad \text{if } s \lhd t$$

$$(\notin) \qquad \frac{\cdots \Gamma, s \mathring{\in} t \to r \neq s \cdots\cdots \text{(for all } s \lhd t)}{\Gamma, r \notin t}$$

$$(\in) \qquad \frac{\Gamma, s \mathring{\in} t \ \wedge \ r = s}{\Gamma, r \in t} \quad \text{if } s \lhd t.$$

$$(\neg \mathbf{Ad}) \qquad \frac{\cdots \Gamma, \mathbf{M}_\kappa \neq t \cdots (\kappa \leq |\, t\,|)}{\Gamma, \neg \mathbf{Ad}(t)}$$

$$(\mathbf{Ad}) \qquad \frac{\Gamma, \mathbf{M}_\kappa = t}{\Gamma, \mathbf{Ad}(t)} \quad \text{if } \kappa \leq |\, t\,|$$

$$(\text{Cut}) \qquad \frac{\Gamma, A \qquad \Gamma, \neg\, A}{\Gamma}$$

$$(\Delta_0(\kappa)\text{-Col}) \qquad \frac{\Gamma,\ (\forall x \in s)(\exists y \in \mathbf{M}_\kappa)\mathcal{F}(x,y)}{\Gamma,\ (\exists z \in \mathbf{M}_\kappa)(\forall x \in s)(\exists y \in z)\mathcal{F}(x,y)} \quad \mathcal{F}(\bar{0},\bar{0})\ \Delta_0(\kappa).$$

The relation $\mathcal{T}_Q^\infty \overset{\beta}{\underset{\rho}{\vdash}} \Gamma$ is inductively defined as follows:

1. If Γ is an axiom of $\mathcal{T}_Q^\infty$, then $\mathcal{T}_Q^\infty \overset{\beta}{\underset{\rho}{\vdash}} \Gamma$ for all β and ρ.

2. If $\mathcal{T}_Q^\infty \overset{\beta_i}{\underset{\rho}{\vdash}} \Gamma_i$ and $\beta_i < \beta$ hold for every premise Γ_i of a rule other than (Cut), then $\mathcal{T}_Q^\infty \overset{\beta}{\underset{\rho}{\vdash}} \Gamma$ if Γ is the conclusion of that rule.

3. If $\mathcal{T}_Q^\infty \overset{\beta_0}{\underset{\rho}{\vdash}} \Gamma, A$, $\mathcal{T}_Q^\infty \overset{\beta_1}{\underset{\rho}{\vdash}} \Gamma, \neg A$, $\beta_0, \beta_1 < \beta$ and $rk(A) < \rho$, then $\mathcal{T}_Q^\infty \overset{\beta}{\underset{\rho}{\vdash}} \Gamma$.

Theorem 7.1. Let $k, m < \omega$. For every finite set $\Gamma[a_1, \ldots, a_r]$ of $\mathbf{KPl}_Q^\infty$ formulae and $\mathcal{T}_Q^\infty$ terms $s_1, \ldots, s_r$ of levels $< \omega$,

$$\mathbf{KPl}_Q^\infty \vdash^{\alpha}_{k} \Gamma[a_1, \ldots, a_r] \quad \Rightarrow \quad \mathcal{T}_Q^\infty \vdash^{\omega+\omega^\alpha}_{\omega^2+k} \Gamma[s_1, \ldots, s_r]^{\mathbf{M}_\omega} ,$$

where $\Gamma[s_1, \ldots, s_r]^{\mathbf{M}_\omega}$ arises from $\Gamma[a_1, \ldots, a_r]$ by substituting s_i for a_i and replacing unbounded quantifiers $\forall x$ and $\exists x$ by $(\forall x \in \mathbf{M}_\omega)$ and $(\exists x \in \mathbf{M}_\omega)$, respectively.

Proof: Note that the highest rank a term of $\mathcal{T}_Q^\infty$ can have is ω^2. One easily computes that whenever a formula $F[\vec{a}]$ of $\mathbf{KPl}_Q^\infty$ has degree m, then $rk(F[\vec{s}]) \leq \omega^2 + m$ holds for all $\mathcal{T}_Q^\infty$ terms $\vec{s}$ of levels $< \omega$.

The proof of this theorem, which proceeds by induction on α, is a simplification of the proof of [24, Theorem 3.17] for $\mathcal{T}^\infty$ as the rule (Δ_0-Coll) does not exist in $\mathcal{T}_Q^\infty$ and in the case of (Cut) one can just apply the induction hypothesis and the above observation about the rank of $F[\vec{s}]$. $\square$

Below we use the function ε which is defined by $\varepsilon_\alpha := \varphi 1 \alpha$.

Theorem 7.2. *(Cut elimination I)* Let Γ be a set of formulas of rank $< \rho + \omega$, where $\rho := \omega \cdot \alpha$. Furthermore, we will assume that all the derivations considered below neither contain variables of level α nor Extensionality axioms with terms of level α.

(i) If $\mathcal{T}_Q^\infty \vdash^{\delta}_{\rho+m+2} \Gamma$, then $\mathcal{T}_Q^\infty \vdash^{\omega^\delta}_{\rho+m+1} \Gamma$.

(ii) If α is a limit or 0 and $\mathcal{T}_Q^\infty \vdash^{\delta}_{\rho+m+1} \Gamma$, then $\mathcal{T}_Q^\infty \vdash^{\omega^\delta}_{\rho+m} \Gamma$.

(iii) If $\mathcal{T}_Q^\infty \vdash^{\beta}_{\rho+n+1} \Gamma$, then $\mathcal{T}_Q^\infty \vdash^{\omega_n(\beta)}_{\rho+1} \Gamma$.

(iv) If α is a limit or 0 and $\mathcal{T}_Q^\infty \vdash^{\beta}_{\rho+n} \Gamma$, then $\mathcal{T}_Q^\infty \vdash^{\omega_n(\beta)}_{\rho} \Gamma$.

(v) If $\mathcal{T}_Q^\infty \vdash^{\beta}_{\rho+\omega} \Gamma$, then $\mathcal{T}_Q^\infty \vdash^{\varepsilon_\beta}_{\rho+1} \Gamma$.

(vi) If α is a limit or 0 and $\mathcal{T}_Q^\infty \vdash^{\beta}_{\rho+\omega} \Gamma$, then $\mathcal{T}_Q^\infty \vdash^{\varepsilon_\beta}_{\rho} \Gamma$.

Proof: [24, Theorem 3.21]. $\square$

Theorem 7.3. *(Cut elimination II)* Let $n = k + 1$ and Γ be a set of $\Sigma_\infty(k)$ formulae. Let $\rho := \omega \cdot n$.

If $\mathcal{T}_Q^\infty \vdash^{\beta}_{\rho+1} \Gamma$, then $\mathcal{T}_Q^\infty \vdash^{\varphi\varepsilon_{\beta+n+1}0}_{\rho} \Gamma$.

Proof: [24, Theorem 3.22]. $\square$

Define $\omega_0(\beta) := \beta$ and $\omega_{n+1}(\beta) := \omega^{\omega_n(\beta)}$.

Theorem 7.4. *(Cut elimination III)* Let $k < \omega$ and Λ be a set of $\Sigma_\infty(k)$ formulae. Then:

$$\mathcal{T}_Q^\infty \,\Big|\!\frac{\alpha}{\omega^2}\, \Lambda \quad \Rightarrow \quad \mathcal{T}_Q^\infty \,\Big|\!\frac{\Gamma_\alpha}{\omega \cdot k+1}\, \Lambda\,.$$

Proof: The proof proceeds by induction on α.

If the last inference was not (Cut), then the desired assertion follows easily from the induction hypotheses applied to the premises, using the same inference rule. Now suppose that the last inference was (Cut). Then

$$\mathcal{T}_Q^\infty \,\Big|\!\frac{\alpha_0}{\omega^2}\, \Lambda, A \quad \text{and} \quad \mathcal{T}_Q^\infty \,\Big|\!\frac{\alpha_0}{\omega^2}\, \Lambda, \neg A \tag{5}$$

for some $\alpha_0 < \alpha$. Let m_0 be minimal such that $A, \neg A$ are $\Sigma_\infty(m_0)$ formulae.

Case 1: $m_0 < k$. The induction hypothesis then yields

$$\mathcal{T}_Q^\infty \,\Big|\!\frac{\Gamma_{\alpha_0}}{\omega \cdot k+1}\, \Lambda, A \quad \text{and} \quad \mathcal{T}_Q^\infty \,\Big|\!\frac{\Gamma_{\alpha_0}}{\omega \cdot k+1}\, \Lambda, \neg A\,.$$

Thus since $rk(A) < \omega \cdot k$, via (Cut) we get

$$\mathcal{T}_Q^\infty \,\Big|\!\frac{\Gamma_\alpha}{\omega \cdot k+1}\, \Lambda\,.$$

Case 2: $k \leq m_0$. The induction hypothesis then yields

$$\mathcal{T}_Q^\infty \,\Big|\!\frac{\Gamma_{\alpha_0}}{\omega \cdot m_0+1}\, \Lambda, A \quad \text{and} \quad \mathcal{T}_Q^\infty \,\Big|\!\frac{\Gamma_{\alpha_0}}{\omega \cdot m_0+1}\, \Lambda, \neg A\,.$$

Thus, by (Cut),

$$\mathcal{T}_Q^\infty \,\Big|\!\frac{\Gamma_{\alpha_0}+1}{\omega \cdot k+l}\, \Lambda \tag{6}$$

for some $l < \omega$.

Let $\mu := \omega^{rk(\mathbf{M}_{m_0} \in \mathbf{M}_{m_0})}$. By substituting $\mathbf{M}_{m_0}$ for variables of level m_0 occurring in derivation (6) and subsequently replacing Extensionality axioms with occurrences of $\mathbf{M}_{m_0}$ by derivations according to [24, Lemma 3.12], we arrive at $\mathcal{T}_Q^\infty \,\Big|\!\frac{\mu+\mu+\Gamma_{\alpha_0}+1}{\omega \cdot m_0+l}\, \Lambda$. To the latter we may apply cut elimination I (Theorem 7.2, (iii)) to obtain

$$\mathcal{T}_Q^\infty \,\Big|\!\frac{\omega_l(\mu+\mu+\Gamma_{\alpha_0}+1)}{\omega \cdot m_0+1}\, \Lambda\,. \tag{7}$$

As $\mu, \Gamma_{\alpha_0} < \Gamma_\alpha$, we have $\omega_l(\mu+\mu+\Gamma_{\alpha_0}+1) < \Gamma_\alpha$; thus $\mathcal{T}_Q^\infty \mid\frac{\Gamma_\alpha}{\omega\cdot m_0+1} \Lambda$. So we are done if $m_0 = k$.

Suppose $m_0 > k$. Let $\beta_0 := \omega_l(\mu+\mu+\Gamma_{\alpha_0}+1)$. By applying cut elimination II (Theorem 7.3) to (7), we obtain

$$\mathcal{T}_Q^\infty \mid\frac{\varphi(\varepsilon_{\beta_0+m_0+1})0}{\omega\cdot(m_0-1)+\omega} \Lambda\,. \tag{8}$$

Let $\eta := \omega^{rk(\mathbf{M}_{m_0-1}\in\mathbf{M}_{m_0-1})}$. By substituting $\mathbf{M}_{m_0-1}$ for variables of level m_0-1 occurring in the derivation (8) and subsequently replacing Extensionality axioms with occurrences of $\mathbf{M}_{m_0-1}$ by derivations according to [24, Lemma 3.12], we arrive at

$$\mathcal{T}_Q^\infty \mid\frac{\eta+\eta+\varphi(\varepsilon_{\beta_0+m_0+1})0}{\omega\cdot(m_0-1)+\omega} \Lambda\,.$$

Hence, letting $\beta_1 := \eta+\eta+\varphi(\varepsilon_{\beta_0+m_0+1})0$, cut elimination I (Theorem 7.2,(v)) yields

$$\mathcal{T}_Q^\infty \mid\frac{\varepsilon_{\beta_1}}{\omega\cdot(m_0-1)+1} \Lambda\,. \tag{9}$$

If $k = m_0-1$ we are done. If $k < m_0-1$, we have to repeat the above procedure again and again until we arrive after finitely many steps at an ordinal β_r such that

$$\mathcal{T}_Q^\infty \mid\frac{\varepsilon_{\beta_r}}{\omega\cdot k+1} \Lambda\,. \tag{10}$$

Since we started out with ordinals $< \Gamma_\alpha$ and applied the functions $+$, $\cdot$ and φ finitely many times to these ordinals we arrive at an ordinal ε_{β_r} which is still smaller than Γ_α. Thus from (10) we conclude that

$$\mathcal{T}_Q^\infty \mid\frac{\Gamma_\alpha}{\omega\cdot k+1} \Lambda\,.$$

$\square$

Corollary 7.5. If $k < \omega$ and Θ is a finite set of arithmetical sentences, then:

$$\mathcal{T}_Q^\infty \mid\frac{\beta}{\omega^2+k} \Theta \quad\Rightarrow\quad \mathcal{T}_Q^\infty \mid\frac{\Gamma_{\varepsilon_{\beta+1}}}{0} \Theta\,.$$

Proof: First we use cut elimination I (Theorem 7.2(ii)) to get:

$$\mathcal{T}_Q^\infty \mid\frac{\omega_k(\beta)}{\omega^2} \Theta\,.$$

By cut elimination III (Theorem 7.4) we obtain

$$\mathcal{T}_Q^\infty \;\Big|\!\frac{\Gamma_{\omega_k(\beta)}}{\omega+1}\; \Theta\,. \tag{11}$$

Let $\beta_0 := \Gamma_{\omega_k(\beta)}$. Applying cut elimination II (Theorem 7.3) to (11) we have

$$\mathcal{T}_Q^\infty \;\Big|\!\frac{\varphi\varepsilon_{\beta_0+2}0}{\omega}\; \Theta\,. \tag{12}$$

The derivation of (12) may contain free variables of level 0. We can get rid of them by substituting $\mathbb{N} = \mathbf{M}_0$ for these variables and subsequently we can replace extensionality axioms with occurrences of $\mathbf{M}_0$ in this derivation using [24, Lemma 3.12]. As a result, since $\omega^{rk(\mathbf{M}_0 \in \mathbf{M}_0)} = \omega^6$ we also have

$$\mathcal{T}_Q^\infty \;\Big|\!\frac{\varphi\varepsilon_{\beta_0+2}0}{\omega}\; \Theta\,, \tag{13}$$

where the derivation no longer contains free variables nor extensionality axioms. Let $\beta_1 := \varphi\varepsilon_{\beta_0+2}0$. Via a final application cut elimination I (Theorem 7.2(v)) we therefore get:

$$\mathcal{T}_Q^\infty \;\Big|\!\frac{\varepsilon_{\beta_1}}{0}\; \Theta\,.$$

One easily computes that $\varepsilon_{\beta_1} < \Gamma_{\varepsilon_{\beta+1}}$; whence $\mathcal{T}_Q^\infty \;\Big|\!\frac{\Gamma_{\varepsilon_{\beta+1}}}{0}\; \Theta\,.$ □

Corollary 7.6. Let $k < \omega$ and Θ be a finite set of arithmetical sentences.

(i) $\mathbf{KPl}_Q^\infty \;\Big|\!\frac{\alpha}{k}\; \Theta \quad \Rightarrow \quad \mathcal{T}_Q^\infty \;\Big|\!\frac{\Gamma_{\varepsilon_{\omega^\alpha+2}}}{0}\; \Theta\,.$

(ii) $\mathbf{KPl}_Q^\infty \;\Big|\!\frac{\alpha}{k}\; \Theta \quad \Rightarrow \quad \mathbf{KPl}_Q^\infty \;\Big|\!\frac{\Gamma_{\varepsilon_{\omega^\alpha+2}}}{0}\; \Theta\,.$

Proof: (i) follows from Theorem 7.1 and Corollary 7.5.

(ii) follows from (i) and the observation that a cut free derivation of Θ in $\mathcal{T}_Q^\infty$ consists entirely of arithmetical sentences and all the inferences and axioms used therein are inferences and axioms of $\mathbf{KPl}_Q^\infty$ too. Thus such a derivation is a derivation of $\mathbf{KPl}_Q^\infty$ as well. □

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