TRUNCATION OF NONLINEAR SYSTEM EXPANSIONS IN THE FREQUENCY DOMAIN

S. A. Billings and Zi-Qiang Lang

Department of Automatic Control and Systems Engineering
The University of Sheffield, Mappin Street
Sheffield, S1 3JD, UK

Research Report No. 633

July 1996

TRUNCATION OF NONLINEAR SYSTEM EXPANSIONS IN THE FREQUENCY DOMAIN

S. A. Billings and Zi-Qiang Lang

Department of Automatic Control and Systems Engineering

The University of Sheffield, Mappin Street

Sheffield, S1 3JD, UK

Abstract: The truncation of Volterra series expansions is studied using the output frequency characteristics of nonlinear systems to develop a new algorithm for determining the terms to include in a Volterra series expansion. The results show the influence of both the generalised frequency response functions and properties of the input spectra on the significance of individual terms in the series. The effectiveness of the proposed method is demonstrated using simulation studies including the analysis of a single degree mechanical oscillator. Because nonlinear system analysis using Volterra series theory must always be based on a truncated Volterra series description the present study provides an effective strategy for determining which terms to include in the analysis of practical nonlinear systems based on Volterra series models.

1. INTRODUCTION

Many systems can be approximated by linear models when the input varies within a small neighbourhood of the operating point. This is a well-known concept that has been widely applied in engineering. A natural extension of this to practical systems which cannot be described by linear models is to allow the maximum order of the dominant nonlinearities of the system to be different for inputs with different amplitude levels. This extended concept justifies the application of the truncated Volterra series expansion for nonlinear system models in both the time and frequency domain.

The Volterra series was proposed in the 1940s and it has been proved that a wide class of nonlinear systems can be approximated by convergent Volterra series expansions (Boyd and Chua 1985). But there are only limited methods available (Rugh 1981, Thapar and Leon 1984) which can be applied to practically truncate the Volterra series expansion of nonlinear systems to yield a time domain description with a finite number of Volterra kernels. There appears to be no results concerning the truncation of the frequency domain Volterra series expansion, which involves the summation of the association of variables (Rugh 1981) up to an infinite order of the system nonlinearities. But frequency domain descriptions are often important for mechanical, electrical and electronic engineering systems because many physical phenomena associated with these systems are directly related to the system frequency domain properties. The truncation of Volterra series expansions in the frequency domain involves expressing the system output spectrum as a function of the input spectrum and a finite number of Generalised Frequency Response Functions (GFRFs). This is associated with the output frequency characteristics of nonlinear systems. In the linear case this concept is straightforward and leads to the classical relationship between the input spectrum, the system frequency response and the output spectrum. The frequency range of the output is therefore exactly the same as that of the input. But nonlinear system frequency domain outputs depend on the association of variables of each order of the system nonlinearities (Mitzel, et al. 1979, Peyton Jones and Billings 1990). This induces mixing and intermodulation effects of the input frequencies to produce outputs at new frequency modes which were regarded by Atherton (1982) as typical of nonlinear behaviour. These conclusions are supported by recent theoretical studies by Lang and Billings (1994,1995) who derived physically meaningful expressions for the nonlinear system output frequency responses and algorithms for computing the system output frequency ranges for both the multiple and general input cases. Billings and Lang (1995 (a) and (b)) also studied the bound on the output frequency responses of nonlinear systems to provide a simplified analysis of nonlinear systems.

In the present paper, the truncation of nonlinear system Volterra series expansions in the frequency domain is studied. A general method for determining the terms that need to be included in truncated frequency domain Volterra series expansions of nonlinear systems is derived. This provides valuable practical insight into the analysis of nonlinear systems based on frequency domain Volterra series models. In addition it has been shown that based on the frequency domain results, the truncation of Volterra series expansions in the time domain can be directly implemented. This is important since it is impossible to obtain the frequency domain results from the time domain because the Fourier transform of a time domain bound is zero except at zero frequency. Applications of the method are illustrated using examples including a single degree of freedom mechanical oscillator and are shown to provide significant insight into the behaviour of typical engineering systems.

2. OUTPUT FREQUENCY CHARACTERISTICS OF NONLINEAR SYSTEMS

In this section, results on the output frequency characteristics of nonlinear systems developed by Lang and Billings (1994, 1995) and Billings and Lang (1995 (a) and (b)) are briefly summarised to provide a basis for investigating the truncation of nonlinear system expansions in the frequency domain.

Consider a nonlinear system which is stable at the zero equilibrium point and which can be described in the neighbourhood of the equilibrium point by the Volterra series

$$y(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\ldots,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \tag{2.1}$$

where $y(t)$ and $u(t)$ are the system output and input, respectively, $h(\tau_1,\ldots,\tau_n)$ is the nth-order Volterra kernel, and N denotes the maximum order of the system nonlinearities.

When the system (2.1) is excited by the general input

$$u(t)=\tfrac{1}{2\pi}\int_{0}^{\infty}2|U(j\omega)|\cos[\omega t+\angle U(j\omega)]d\omega=\tfrac{1}{2\pi}\int_{-\infty}^{\infty}U(j\omega)e^{j\omega t}d\omega \tag{2.2}$$

It can be shown (Lang and Billings 1994) that the output frequency response can be expressed as

$$\mathbf{Y}(j\omega)=\sum_{n=1}^{N}\mathbf{Y}_n(j\omega)\qquad \text{for}\,\forall\omega \tag{2.3}$$

where

$$\mathbf{Y}_n(j\omega)=\frac{1/\sqrt{n}}{(2\pi)^{n-1}}\int_{\omega_1+,\ldots,+\omega_n=\omega}H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega \tag{2.4}$$

$\mathbf{Y}(j\omega)$ and $U(j\omega)$ represent the Fourier transforms of the system output and input, and

$$\int_{\omega_1+,\dots,+\omega_n=\omega} H_n(j\omega_1,\dots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega$$

denotes the integration of $H_n(j\omega_1,\dots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)$ over the n-dimensional hyperplane $\omega=\omega_1+\cdots+\omega_n$, $H_n(j\omega_1,\dots,j\omega_n)$ is called the nth-order Generalised Frequency Response Function or GFRF (George 1959).

Under a general but realistic assumption for the input spectrum defined by

$$U(j\omega)=\begin{cases}U(j\omega) & \text{when } |\omega|\in[a,b]\\ 0 & \text{otherwise}\end{cases} \tag{2.5}$$

the nonnegative frequency range produced by the nth-order nonlinear system output denoted by f_{Y_n} can be determined by using the following formula (Lang and Billings 1995)

$$f_{Y_n}=\begin{cases}\bigcup_{k=0}^{i^*-1}R_k & \text{when } \frac{nb}{(a+b)}-INT\left[\frac{na}{(a+b)}\right]<1\\ \bigcup_{k=0}^{i^*}R_k & \text{when } \frac{nb}{(a+b)}-INT\left[\frac{na}{(a+b)}\right]\geq 1\end{cases} \tag{2.6}$$

where $i^*=INT\left[\frac{na}{(a+b)}\right]+1$, $R_k=[na-k(a+b),nb-k(a+b)]$, for $k=0,\dots,i^*-1$, $R_{i^*}=[0,nb-i^*(a+b)]$ and $INT[.]$ denotes to take the integer part of $[.]$. It has also been shown that the relationship between f_{Y_n} and the nonnegative system output frequency range denoted by f_Y is given by

$$f_Y=f_{Y_N}\bigcup f_{Y_{N-(2p^*-1)}} \tag{2.7}$$

where p^* depends upon the orders of the existing system nonlinearities and can be determined such that if the system GFRFs satisfy

$$H_{N-(2i-1)}(.)=0,\quad \text{for } i=1,2,\dots,q-1$$

but $H_{N-(2q-1)}(.)\neq 0$ then $p^*=q$.

Equations (2.3), (2.4) and (2.7), (2.6) can be regarded as a natural extension of the concept of the output frequency characteristics of linear systems to the nonlinear case. Although the extension is introduced here based on continuous time nonlinear systems, the results are the same for discrete time systems.

The magnitude of $\mathbf{Y}(j\omega)$ provides valuable information and Billings and Lang (1995a) proposed the concept of a bound for $|\mathbf{Y}(j\omega)|$ to simplify the analysis of nonlinear systems and showed that the bound can be expressed as

$$\mathbf{Y}^B(\omega)=\sum_{n=1}^{N}\mathbf{Y}_n^B(\omega) \tag{2.8}$$

where

$$\mathbf{Y}_n^B(\omega) = \frac{1}{(2\pi)^{n-1}} \left| H_n(j\omega_{n1}^{\#},\ldots,j\omega_{nn}^{\#}) \right| \overbrace{\left|U(j\omega)\right| * \cdots * \left|U(j\omega)\right|}^{n} \tag{2.9}$$

$\{\omega_{n1}^{\#},\ldots,\omega_{nn}^{\#}\}$ represents the co-ordinate of a point on the n-dimensional hyperplane $\omega_1 +,\cdots,+\omega_n = \omega$, and $\overbrace{|U(j\omega)| * \cdots * |U(j\omega)|}^{n}$ denotes the n-dimensional convolution integration for $|U(j\omega)|$.

Billings and Lang (1995a) also suggested the following general procedures for the practical computation of this bound.

First compute $|H_n(j\omega_1,\ldots,j\omega_n)|_{\omega}^{B}$, a bound for $H_n(j\omega_1,\ldots,j\omega_n)$ with $\omega_1,\ldots,\omega_n$ satisfying the constraint $\omega_1 +,\cdots,+\omega_n = \omega$.

Then calculate $\overbrace{|U(j\omega)| * \cdots * |U(j\omega)|}^{n}$ using the algorithm

$$\begin{cases} \left|U(j\frac{2\pi}{MT}i)\right| * \cdots * \left|U(j\frac{2\pi}{MT}i)\right| = T\tilde{\bar{U}}\left[i + \left(\frac{M}{2}-1\right)n\right]\left(\frac{2\pi}{M}\right)^{n-1} \quad i = -n\left(\frac{M}{2}-1\right),\ldots,0,\ldots,n\left(\frac{M}{2}\right) \\ \left\{\tilde{\bar{U}}(0),\ldots,\tilde{\bar{U}}[n(M-1)]\right\} = Conv\left\{\overbrace{[\tilde{U}(0),\ldots,\tilde{U}(M-1)],\ldots,[\tilde{U}(0),\ldots,\tilde{U}(M-1)]}^{n}\right\} \\ \tilde{U}(i) = \left|U_d\left[j\frac{2\pi}{M}\left(i - \frac{M}{2}+1\right)\right]\right|, \quad i = 0,1,\ldots,M-1 \end{cases} \tag{2.10}$$

where $U_d(.)$ is the discrete time Fourier transform of a sampled sequence of $u(t)$ with sampling period T, M is a sufficiently large even number and is the length of the sampled sequence used to compute $\left|U_d\left(j\frac{2\pi}{M}l\right)\right|$ $l = -\left(\frac{M}{2}-1\right),\ldots,0,\ldots,\frac{M}{2}$. If the system is a discrete time system then $U_d(j\omega) = U(j\omega)$ and T in the algorithm is taken as 1.

Finally evaluate

$$\overline{\mathbf{Y}}^B(\omega) = \sum_{n=1}^{N} \overline{\mathbf{Y}}_n^B(\omega) \tag{2.11}$$

where

$$\overline{\mathbf{Y}}_n^B(\omega) = \frac{1}{(2\pi)^{n-1}} |H_n(j\omega_1,\ldots,j\omega_n)|_{\omega}^{B} \overbrace{|U(j\omega)| * \cdots * |U(j\omega)|}^{n} \tag{2.12}$$

to yield an approximation $\overline{\mathbf{Y}}^B(\omega)$ that possesses the same properties as $\mathbf{Y}^B(\omega)$.

3. FREQUENCY DOMAIN CRITERIA FOR THE TRUNCATION OF NONLINEAR SYSTEM EXPANSIONS

The output spectrum of a nonlinear system is given by

$$\mathbf{Y}(j\omega) = \sum_{n=1}^{\infty} \mathbf{Y}_n(j\omega) \tag{3.1}$$

and the truncation of nonlinear system expansions in the frequency domain involves determining a truncated expression

$$\mathbf{Y}(j\omega) = \sum_{n=1}^{N} \mathbf{Y}_n(j\omega)$$

such that the effects of the system nonlinearities above the Nth-order can be neglected. Obviously this truncated expression depends upon the contribution of each $\mathbf{Y}_n(j\omega)$, $1 \le n < \infty$, to the output spectrum.

If the contribution of $\mathbf{Y}_n(j\omega)$ to $\mathbf{Y}(j\omega)$ has been evaluated by obtaining a bound $B_1(n)$ on $\mathbf{Y}_n(j\omega)$ such that

$$|\mathbf{Y}_n(j\omega)| \le B_1(n) \quad for\ \omega \in f_{Y_n} \tag{3.2}$$

the effect of the nth-order time domain nonlinear output $y_n(t)$ on the output $y(t)$ can immediately be evaluated based on the following relationship between the nth-order outputs in the time and frequency domain

$$|y_n(t)| = \left| \int_{f_{Y_n}} 2|\mathbf{Y}_n(j\omega)| \cos[\omega t + \angle \mathbf{Y}_n(j\omega)] d\omega \right| \le \int_{f_{Y_n}} 2|\mathbf{Y}_n(j\omega)| d\omega \le 2B_1(n) \int_{f_{Y_n}} d\omega = B_2(n) \tag{3.3}$$

where $\int_{f_{Y_n}} (.) d\omega$ denotes the integration for (.) over the nth-order output frequency range f_{Y_n}.

Notice that it is impossible to obtain the frequency domain bound from the time domain result because the Fourier transform of the time domain bound is zero except at zero frequency. It is also difficult to determine the time domain bound from the frequency domain result using the inverse Fourier transform because no phase information is available.

Determination of the contribution of each $\mathbf{Y}_n(j\omega)$, $1 \le n < \infty$, to the output spectrum for the frequency domain truncation requires a criterion similar to $B_1(n)$ in (3.2) to evaluate the effect of $\mathbf{Y}_n(j\omega)$ on $\mathbf{Y}(j\omega)$. Such criteria will be analysed below using the results presented in Section 2 concerning the output frequency characteristics of nonlinear systems. Based on this analysis, a general method for the truncation of nonlinear system Volterra series expansions in the frequency domain will be proposed in the next section.

Beyond the frequency range produced by the nth-order system nonlinearity f_{Y_n},

$$|\mathbf{Y}_n(j\omega)| = 0$$

and $|\mathbf{Y}_n(j\omega)|$ with $\omega \in f_{Y_n}$ represents the contribution of the nth-order system nonlinearity to the output spectrum at the frequency ω. The contribution of the nth-order system nonlinearity to the output spectrum can therefore be evaluated using the criterion

$$J_1(n) = \max_{\omega \in f_{Y_n}} |\mathbf{Y}_n(j\omega)| \tag{3.4}$$

However, if the expression for $\mathbf{Y}_n(j\omega)$ given by (2.4) is substituted $|\mathbf{Y}_n(j\omega)|$ is too complex to calculate and therefore $J_1(n)$ is difficult to apply in practice.

To circumvent this difficulty, consider the relationship (Billings and Lang 1995(a))

$$|\mathbf{Y}_n(j\omega)| \le \mathbf{Y}_n^B(\omega) \le \overline{\mathbf{Y}}_n^B(\omega)$$

and consider the evaluation of $\overline{\mathbf{Y}}_n^B(\omega)$ using the procedures in Section 2 to define another criterion based on $\overline{\mathbf{Y}}_n^B(\omega)$ as

$$J_2(n) = \max_{\omega \in f_{Y_n}} \overline{\mathbf{Y}}_n^B(\omega) \tag{3.5}$$

$J_2(n)$ can be used in practice to determine the contribution of the nth-order system nonlinearity to the output spectrum.

Notice that evaluating $J_2(n)$ requires exact knowledge of the system input spectrum. This implies that $J_2(n)$ can only represent the effect on the output spectrum of the nth-order system nonlinearity for a particular input. However, from a practical viewpoint, it is more important to discuss the truncation of nonlinear system expressions for a class of inputs rather than for just one particular input. Consider therefore a class of inputs where the frequency spectra are defined by

$$U(j\omega) = \begin{cases} U(j\omega) & \text{with } |U(j\omega)| \le M_u \quad |\omega| \in [a,b] \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

and denote

$$U^{\#}(\omega) = \begin{cases} 1 & |\omega| \in [a,b] \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

to give from (3.6)

$$|U(j\omega)| \le M_u U^{\#}(\omega) \tag{3.8}$$

Substituting (3.8) and

$$\overbrace{|U(j\omega)| * \cdots * |U(j\omega)|}^{n} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |U(j\omega_1)| \cdots |U(j\omega_{n-1})| |U(j\omega - \omega_1 -, \cdots, -\omega_{n-1})| d\omega_1 \cdots d\omega_{n-1} \tag{3.9}$$

into the expression of $\overline{\mathbf{Y}}_n^B(\omega)$ given by (2.12) yields

$$\begin{aligned}
\overline{\mathbf{Y}}_n^B(\omega) &= \frac{1}{(2\pi)^{n-1}} |H_n(j\omega_1, \ldots, j\omega_n)|_\omega^B \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |U(j\omega_1)| \cdots |U[j(\omega - \omega_1 -, \cdots, -\omega_{n-1})]| d\omega_1 \cdots d\omega_{n-1} \\
&\le \frac{M_u^n}{(2\pi)^{n-1}} |H_n(j\omega_1, \ldots, j\omega_n)|_\omega^B \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} U^{\#}(\omega_1) \cdots U^{\#}(\omega - \omega_1 -, \cdots, -\omega_{n-1}) d\omega_1 \cdots d\omega_{n-1} \\
&= \frac{M_u^n}{(2\pi)^{n-1}} |H_n(j\omega_1, \ldots, j\omega_n)|_\omega^B \overbrace{U^{\#}(\omega) * \cdots * U^{\#}(\omega)}^{n} = \overline{\mathbf{Y}}_n^{B\#}(\omega)
\end{aligned} \tag{3.10}$$

Then define a criterion based on $\overline{\mathbf{Y}}_n^{B\#}(\omega)$ as

$$J_3(n) = \frac{M_u^n}{(2\pi)^{n-1}} \max_{\omega \in f_{Y_n}} |H_n(j\omega_1, \ldots, j\omega_n)|_\omega^B \max_{\omega \in f_{Y_n}} \overbrace{U^{\#}(\omega) * \cdots * U^{\#}(\omega)}^{n} \tag{3.11}$$

The significance of the nth-order system nonlinearity to the output spectrum under the class of inputs defined by (3.7) and (3.8) can then be determined based on the principle that if the value of $J_3(n)$ is negligible then the nth-order nonlinearity need not be considered in the expression of the output spectrum.

The above analysis indicates that $J_2(n)$ can be used in practice to decide on the orders of significant system nonlinearities in the system frequency domain description for a specific input. While $J_3(n)$ can be taken as a relatively general criterion for the truncation of nonlinear system frequency domain Volterra series expansions.

It is observed from (3.11) that $J_3(n)$ depends on both the system characteristics and properties of the input. The system characteristics are represented by

$$\max_{\omega\in f_{Y_n}} \left|H_n(j\omega_1,\ldots,j\omega_n)\right|_{\omega}^{B} \tag{3.12}$$

and the input properties are given by

$$\frac{M_u^n}{(2\pi)^{n-1}} \max_{\omega\in f_{Y_n}} \overbrace{U^{\#}(\omega)*\cdots*U^{\#}(\omega)}^{n} \tag{3.13}$$

where the latter mainly depend on M_u, a bound on the input spectrum, and the n-dimensional convolution integration for $U^{\#}(\omega)$.

Based on properties of the Fourier transform it can be shown that the n-dimensional convolution integration for $U^{\#}(\omega)$ can be written as

$$\overbrace{U^{\#}(\omega)*\cdots*U^{\#}(\omega)}^{n} = F^{-1}\left\{F\left[\overbrace{U^{\#}(\omega)*\cdots*U^{\#}(\omega)}^{n}\right]\right\} = F^{-1}\{F^n[U^{\#}(\omega)]\} \tag{3.14}$$

where $F(.)$ and $F^{-1}(.)$ denote the Fourier and inverse Fourier transform operator, respectively. According to the definition of the Fourier transform,

$$F\left[U^{\#}(\omega)\right] = \int_{-\infty}^{\infty} U^{\#}(\omega) e^{-j\omega\varpi} d\omega \tag{3.15}$$

where ϖ denotes the argument. Substituting (3.7) into (3.15) yields

$$F\left[U^{\#}(\omega)\right] = \int_{-b}^{-a} e^{-j\omega\varpi} d\omega + \int_{a}^{b} e^{-j\omega\varpi} d\omega = \int_{a}^{b} 2\frac{e^{j\omega\varpi}+e^{-j\omega\varpi}}{2} d\omega = \int_{a}^{b} 2\cos\omega\varpi d\omega = \frac{4}{\varpi}\cos\frac{\varpi(a+b)}{2}\sin\frac{\varpi(b-a)}{2} \tag{3.16}$$

Moreover, substituting (3.16) into (3.14) and replacing $F^{-1}(.)$ with the definition of the inverse Fourier transform gives

$$\overbrace{U^{\#}(\omega)*\cdots*U^{\#}(\omega)}^{n} = F^{-1}\left\{\frac{4^n}{\varpi^n}\cos^n\frac{\varpi(a+b)}{2}\sin^n\frac{\varpi(b-a)}{2}\right\}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4^n}{\varpi^n}\cos^n\frac{\varpi(a+b)}{2}\sin^n\frac{\varpi(b-a)}{2} e^{j\varpi\omega} d\varpi = \frac{2}{2\pi}\int_{0}^{\infty}\frac{4^n}{\varpi^n}\cos^n\frac{\varpi(a+b)}{2}\sin^n\frac{\varpi(b-a)}{2}\cos(\omega\varpi) d\varpi \tag{3.17}$$

Thus from (3.11) and (3.17) $J_3(n)$ can be written as

$$J_3(n)=2\left(\frac{2M_u}{\pi}\right)^n \max_{\omega\in f_{Y_n}}\left|H_n(j\omega_1,\ldots,j\omega_n)\right|_{\omega}^{B} \max_{\omega\in f_{Y_n}}\int_0^{\infty}\frac{1}{\varpi^n}\cos^n\frac{\varpi(a+b)}{2}\sin^n\frac{\varpi(b-a)}{2}\cos(\omega\varpi)d\varpi \tag{3.18}$$

Notice that when n is an even number

$$\frac{1}{\varpi^n}\cos^n\frac{\varpi(a+b)}{2}\sin^n\frac{\varpi(b-a)}{2}\geq 0 \ \text{ for } \forall\omega \tag{3.19}$$

In this case using $\cos(\varpi\omega)\leq 1$ gives

$$\frac{1}{\varpi^n}\cos^n\frac{\varpi(a+b)}{2}\sin^n\frac{\varpi(b-a)}{2}\cos(\omega\varpi)\leq\frac{1}{\varpi^n}\cos^n\frac{\varpi(a+b)}{2}\sin^n\frac{\varpi(b-a)}{2} \tag{3.20}$$

and therefore

$$\begin{aligned}&\int_0^{\infty}\frac{1}{\varpi^n}\cos^n\frac{\varpi(a+b)}{2}\sin^n\frac{\varpi(b-a)}{2}\cos(\omega\varpi)d\varpi\leq\int_0^{\infty}\frac{1}{\varpi^n}\cos^n\frac{\varpi(a+b)}{2}\sin^n\frac{\varpi(b-a)}{2}d\varpi\\&=\left(\int_0^{\infty}\frac{1}{\varpi^n}\cos^n\frac{\varpi(a+b)}{2}\sin^n\frac{\varpi(b-a)}{2}\cos(\omega\varpi)d\varpi\right)_{\omega=0}=\max_{\omega\in f_{Y_n}}\int_0^{\infty}\frac{1}{\varpi^n}\cos^n\frac{\varpi(a+b)}{2}\sin^n\frac{\varpi(b-a)}{2}\cos(\omega\varpi)d\varpi\end{aligned} \tag{3.21}$$

since $\omega=0\in f_{Y_n}$ when n is even. In view of this $J_3(n)$ can eventually be expressed as

$$J_3(n)=\begin{cases}2\left(\frac{2M_u}{\pi}\right)^n \max_{\omega\in f_{Y_n}}\left|H_n(j\omega_1,\ldots,j\omega_n)\right|_{\omega}^{B} \max_{\omega\in f_{Y_n}}\int_0^{\infty}\frac{1}{\varpi^n}\cos^n\frac{\varpi(a+b)}{2}\sin^n\frac{\varpi(b-a)}{2}\cos(\omega\varpi)d\varpi & \text{for odd n}\\ 2\left(\frac{2M_u}{\pi}\right)^n \max_{\omega\in f_{Y_n}}\left|H_n(j\omega_1,\ldots,j\omega_n)\right|_{\omega}^{B}\int_0^{\infty}\frac{1}{\varpi^n}\cos^n\frac{\varpi(a+b)}{2}\sin^n\frac{\varpi(b-a)}{2}d\varpi & \text{for even n}\end{cases} \tag{3.22}$$

Equation (3.22) gives a complete analytical description of the criterion $J_3(n)$ and reveals how $J_3(n)$ depends on the system frequency domain properties and the class of input spectra defined by (3.7) and (3.8). This provides an important foundation for theoretically examining the effects of using $J_3(n)$ to truncate nonlinear system expansions in the frequency domain. In addition, (3.22) also provides insight into how to evaluate $J_3(n)$ in practice.

It can be shown from (3.22) and (3.17) that for even n

$$\begin{aligned}J_3(n)&=\frac{M_u^n}{(2\pi)^{n-1}}\max_{\omega\in f_{Y_n}}\left|H_n(j\omega_1,\ldots,j\omega_n)\right|_{\omega}^{B}\left[\frac{2}{2\pi}\int_0^{\infty}\frac{4^n}{\varpi^n}\cos^n\frac{\varpi(a+b)}{2}\sin^n\frac{\varpi(b-a)}{2}d\varpi\right]\\&=\frac{M_u^n}{(2\pi)^{n-1}}\max_{\omega\in f_{Y_n}}\left|H_n(j\omega_1,\ldots,j\omega_n)\right|_{\omega}^{B}\overbrace{U^{\#}(\omega)*\cdots*U(\omega)}^{n}\Big|_{\omega=0}\end{aligned} \tag{3.23}$$

which indicates that $J_3(n)$ can in this case be calculated directly using the n-dimensional convolution integration result for $U^{\#}(\omega)$ evaluated at $\omega=0$.

In the case of odd n it can be observed from (3.22) that for different a, b, and n the frequency at which the integration in the expression of $J_3(n)$ reaches a maximum will be different. For n odd therefore evaluation of $J_3(n)$ has to be performed based on the original expression given by (3.11).

4. GENERAL METHOD FOR THE TRUNCATION OF VOLTERRA SERIES EXPANSIONS IN THE FREQUENCY DOMAIN

4.1 The Method

According to the analysis in the last section, the criterion $J_3(n)$ provides a significant indication of the effect of both the system properties and characteristics of a wide class of inputs on the nth-order nonlinear system output frequency response functions. A general method for determining the terms that need to be included in truncated frequency domain Volterra series expansions of nonlinear systems can now be proposed based on this criterion. The method requires that the system GFRFs or a time domain model and the input spectrum defined by equations (3.7) and (3.8) are known a priori. The method can be summarised as follows.

(i) If the system GFRFs are known then go to step (ii). Otherwise calculate the GFRFs from the time domain model using the harmonic probing method (Billings and Tsang 1989) or use a recursive computation algorithm (Peyton Jones and Billings 1989, Billings and Peyton Jones 1990, Zhang, Billings and Zhu 1995).

(ii) Determine f_{Y_n} from equation (2.6) based on the input frequency range [a, b] and evaluate $\max_{\omega\in f_{Y_n}} \left|H_n(j\omega_1,\ldots,j\omega_n)\right|_{\omega}^{B}$ for n=1,2,... from the GFRFs.

(iii) Compute $U_n^{\#}(\omega)=\overbrace{U^{\#}(\omega)*\cdots*U^{\#}(\omega)}^{n}$ for $\omega=0,\ldots,n\dfrac{M}{2}$ using a revised version of (2.10) such that

$$
\begin{cases}
U_n^{\#}\left(i\dfrac{2\pi}{MT}\right)=T\tilde{\tilde{U}}\left[i+\left(\dfrac{M}{2}-1\right)n\right]\left(\dfrac{2\pi}{M}\right)^{n-1} \quad i=0,\ldots,n\left(\dfrac{M}{2}\right) \\
\left\{\tilde{\tilde{U}}(0),\ldots,\tilde{\tilde{U}}[n(M-1)]\right\}=Conv\left\{\overbrace{[\tilde{U}(0),\ldots,\tilde{U}(M-1)],\ldots,[\tilde{U}(0),\ldots,\tilde{U}(M-1)]}^{n}\right\} \\
\tilde{U}(i)=\left|U_d^{\#}\left[\dfrac{2\pi}{M}\left(i-\dfrac{M}{2}+1\right)\right]\right|, \quad i=0,1,\ldots,M-1
\end{cases}
\tag{4.1}
$$

where

$$
U_d^{\#}(\omega)=\begin{cases} 1/T & |\omega|\in[aT,bT] \\ 0 & \text{otherwise}\end{cases}
\tag{4.2}
$$

(iv) Evaluate $J_3(n)$ for n=1,2,...using the formula

$$
J_3(n)=\begin{cases}
\dfrac{M_u^n}{(2\pi)^{n-1}}\max_{\omega\in f_{Y_n}}\left|H_n(j\omega_1,\ldots,j\omega_n)\right|_{\omega}^{B}\max_{i\in[0,\ldots,Mn/2)} U_n^{\#}\left(i\dfrac{2\pi}{MT}\right) & \text{for odd n} \\
\dfrac{M_u^n}{(2\pi)^{n-1}}\max_{\omega\in f_{Y_n}}\left|H_n(j\omega_1,\ldots,j\omega_n)\right|_{\omega}^{B} U_n^{\#}(0) & \text{for even n}
\end{cases}
\tag{4.3}
$$

where the computations for even n are based on equation (3.23).

(v) Based on the results of $J_3(n)$ n=1,2,... determine n^* such that

$J_3(n) <$ a priori given small number ε for all $n > n^*$

then take $N = n^*$ and the system can then be expressed by a truncated description in the frequency domain by including up to the Nth-order generalised frequency response functions provided that the input spectra satisfy the constraint given by (3.7).and (3.8).

Generally the result $\max_{\omega \in f_{Y_n}} \left| H_n(j\omega_1,\ldots,j\omega_n) \right|_{\omega}^{B}$ in Step (ii) will depend upon the specific expressions of the system generalised frequency response functions and therefore it is not possible to develop a special algorithm for this. But for nonlinear systems which can be described by the NARX (Nonlinear AutoRegressive with eXogenous) model (Chen and Billings 1989)

$$y(k) = \sum_{m=1}^{\overline{M}} y_m(k) \tag{4.4}$$

where $y_m(k)$, the 'NARX mth-order output' of the system, is given by

$$y_m(k) = \sum_{p=0}^{m} \sum_{l_1, l_{p+q}=1}^{\overline{K}} c_{pq}(l_1,\ldots,l_{p+q}) \prod_{i=1}^{p} y(k-l_i) \prod_{i=p+1}^{p+q} u(k-l_i) \tag{4.5}$$

with

$$p+q=m, \quad l_i = 1,\ldots,\overline{K},\ i=1,\ldots,p+q, \text{ and } \sum_{l_1,l_{p+q}=1}^{\overline{K}} \equiv \sum_{l_1=1}^{\overline{K}} \cdots \sum_{l_{p+q}=1}^{\overline{K}} \tag{4.6}$$

it has been shown by Billings and Lang (1995b) that

$$\max_{\omega \in f_{Y_n}} \left| H_n(j\omega_1,\ldots,j\omega_n) \right|_{\omega}^{B} = H_n^B \tag{4.7}$$

and H_n^B can be obtained using the following recursive algorithm

$$\left\{ \begin{array}{c} H_n^B = \frac{1}{L_n}[0_{1\times(n-1)}, \sum c_{0n}, \sum c_1^n + \sum c_{20}\overline{H}_{n-1},\ldots,\sum c_{n-1}^n + \sum c_{n0}\overline{H}_1][H_0^{B_n}, H_1^{B_n},\ldots,H_{n-1}^{B_n}]^T, \\ H_0^{B_n} = [0_{1\times(n-1)},1], \quad H_k^{B_n} = [\overline{H}_{n-k}(H_{k-1}^{B_{n-1}})^T,\ H_k^{B_{n-1}}],\ k=1,2,\ldots,n-2, \quad H_{n-1}^{B_n} = [\overline{H}_1(H_{n-2}^{B_{n-1}})^T], \\ \sum c_k^n = [\sum c_{k1},\ldots,\sum c_{kn-k}],\ k=1,\ldots,n-1,\ \sum c_{pq} = \sum_{k_1,k_{p+q}=1}^{K} \left| c_{pq}(k_1,\ldots,k_{p+q}) \right|, \\ \overline{H}_{n-k} = [H_1^B,\ldots,H_{n-k}^B],\ k=1,\ldots,n-1. \end{array} \right. \tag{4.8}$$

with

$$L_n = \min_{\omega \in f_{Y_n}} \left| 1 - \sum_{l_1=1}^{\overline{K}} c_{10}(l_1)\exp(-j\omega l_1) \right| \tag{4.9}$$

Thus for nonlinear systems of the NARX model, Steps (i) and (ii) can readily be determined using equations (4.7) and (4.8) and the parameters of the system time domain model directly.

4.2 Illustration of the Method

To illustrate the application of the general method consider an example where the nonlinear system to be examined is described by the continuous time general model illustrated in Fig.1 where

$$H_\alpha(s)=\frac{5.0625}{0.0001s^4+0.0039s^3+0.0768s^2+0.8819s+5.0625}$$
$$H_\beta(s)=\frac{1}{0.0001s^4+0.0026s^3+0.0341s^2+0.2613s+1}$$

and

$$P[u_1(t)]=\sum_{n=1}^{8}p_n u_1^n(t)=-0.0745u_1(t)-0.00585u_1^2(t)+0.92279u_1^3(t)+0.0233u_1^4(t)-0.37014u_1^5(t)$$
$$-0.02081u_1^6(t)+0.04428u_1^7(t)+0.0039u_1^8(t)$$

is a polynomial approximation to a dead zone plus saturation nonlinear characteristic from Lang (1994).

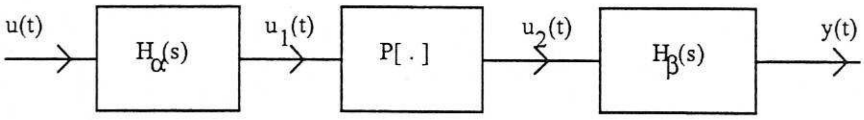


Figure1 A nonlinear continuous time general model.

It is easy to show that in this case the GFRFs of the system are given by

$$H_n(j\omega_1,\ldots,j\omega_n)=p_n\prod_{i=1}^{n}H_\alpha(j\omega_i)H_\beta[j(\omega_1+,\cdots,+\omega_n)],\quad n=1,2,\ldots,8$$

This implies that the result of the second step can be determined based on the relationship

$$\max_{\omega\in f_{Y_n}}\left|H_n(j\omega_1,\ldots,j\omega_n)\right|_\omega^B=p_n\max_{\omega\in f_{Y_n}}\left|H_\beta(j\omega)\right|\left[\prod_{i=1}^{n}\left|H_\alpha(j\omega_i)\right|\right]_\omega^B\le p_n$$

which is obtained by observing from Fig.2 that

$$\max_\omega\left|H_\beta(j\omega)\right|=\max_\omega\left|H_\alpha(j\omega)\right|=1$$

Thus $J_3(n)$ for this situation can be evaluated as follows

$$J_3(n)=\begin{cases}\dfrac{M_u^n}{(2\pi)^{n-1}}|p_n|\max\limits_{i\in\{0,\ldots,Mn/2\}}U_n^\#\left(i\dfrac{2\pi}{MT}\right) & \text{for } n=1,3,5\\ \dfrac{M_u^n}{(2\pi)^{n-1}}|p_n|U_n^\#(0) & \text{for } n=2,4,6,8\end{cases}$$

where $U_n^\#(.)$ is the results calculated in the third step using equations (4.1) and (4.2).

Consider three situations in which the input spectra satisfy (3.7) and (3.8) with

(1) $a=1, b=3.3, M_u=1$
(2) $a=1, b=3.3, M_u=1.6$

and

(3) $a=1, b=3.3, M_u=1.8$

respectively. The results obtained using $T = 0.01$ and $M = 2000$ are shown in Tables 1, 2 and 3.

These results show that for higher input amplitudes higher orders of nonlinearities will exist in the truncated frequency domain Volterra series expansion. Table 1 indicates that if we take $\varepsilon = 0.05$ then since $J_3(2)$, $J_3(4)$, and $J_3(5),\ldots,J_3(8)$ are all less than ε the truncated Volterra series expansion of the system will only involve terms of order 1 and 3. Similarly in the case of $M_u = 1.6$ and $M_u = 1.8$, it can be observed from Tables 2 and 3 that with $\varepsilon = 0.05$, the truncated frequency domain Volterra series model will only contain nonlinear terms of order 1,3,5 and 1,3,5,7, respectively.

Notice that in all three cases the even order nonlinearity terms are all negligible. This is reasonable because P[.] in Fig.1 is an approximation to a dead zone plus saturation nonlinear characteristic which is an odd function.

In order to confirm the above conclusions which were obtained based on the new method, two simulation studies were performed. The first was on the system in Fig.1 and the second on a similar system but with P[.] replaced by

$$\hat{p}[u_1(t)] = -0.0745u_1(t) + 0.92279u_1^3(t) - 0.37014u_1^5(t)$$

In both cases the input excitation was

$$u(t) = \frac{2M_u}{\pi(b-a)t^2}\left[2\cos\left(\frac{a+b}{2}t\right) - \cos bt - \cos at\right], \quad t = -20\ Sec\ ---20\ Sec$$

where a=1, b=3.3, $M_u = 1.6$. The spectrum of the input excitation is shown in Fig.3. This input satisfies condition (2) and the results should therefore correspond to those in Table 2. Fig.4 shows the output frequency response of the original and the truncated (up to fifth order nonlinearity as indicated in Table 2) and the difference between the two, and clearly verifies the effectiveness of the proposed method.

Table 1 Results for the general model (a=1,b=3.3,$M_u = 1$)

n	1	2	3	4	5	6	7	8
$J_3(n)$	<u>0.0745</u>	0.0041	<u>0.2561</u>	0.0041	0.0337	0.0012	0.0015	0.0001

Table 2 Results for the general model (a=1,b=3.3,$M_u = 1.6$)

n	1	2	3	4	5	6	7	8
$J_3(n)$	<u>0.1192</u>	0.0105	<u>1.0489</u>	0.0265	<u>0.3538</u>	0.0208	0.0412	0.0040

Table 3 Results for the general model (a=1,b=3.3, $M_u = 1.8$)

n	1	2	3	4	5	6	7	8
$J_3(n)$	0.1341	0.0133	1.4934	0.0425	0.6376	0.0421	0.0940	0.0102

5. APPLICATION TO SINGLE DEGREE MECHANICAL OSCILLATORS

The single degree mechanical oscillator shown in Fig.5 is a mechanical structure widely used in engineering where y(t) denotes the displacement of mass m, u(t) is the force imposed

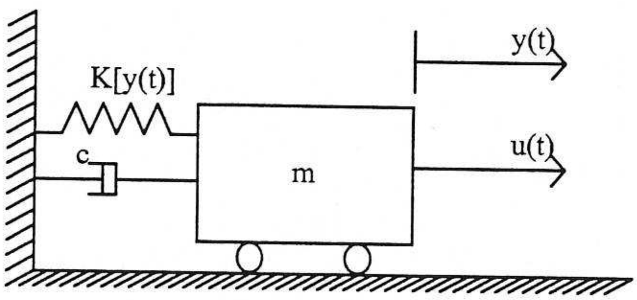


Figure 5 Single degree mechanical oscillator

on the mass , K(.) is the characteristic function of the spring and c is the parameter of the damper.

The equation of motion of the mechanical system can readily be obtained as

$$m\ddot{y}(t) + c\dot{y}(t) + K[y(t)] = u(t) \tag{5.1}$$

When the nonlinearity of the spring characteristic is taken into account,

$$K[y(t)] = K_1 y(t) + K_3 y^3(t) \tag{5.2}$$

equation (5.1) becomes

$$m\ddot{y}(t) + c\dot{y}(t) + K_1 y(t) + K_3 y^3(t) = u(t) \tag{5.3}$$

Discretising using a backward difference method such that

$$\dot{y}(t) = \frac{y(kT_s) - y[(k-1)T_s]}{T_s} = \frac{y(k) - y(k-1)}{T_s} \tag{5.4}$$

$$\ddot{y}(t) = \frac{y[(k+1)T_s] - 2y(kT_s) + y[(k-1)T_s]}{T_s^2} = \frac{y(k+1) - 2y(k) + y(k-1)}{T_s^2} \tag{5.5}$$

gives, after substituting (5.4) and (5.5) into (5.3), a NARX model

$$y(k) = c_{10}(1)y(k-1) + c_{10}(2)y(k-2) + c_{30}(1)y^3(k-1) + c_{01}(1)u(k-1) \tag{5.6}$$

where

$$
\begin{cases}
c_{10}(1) = -\left(\dfrac{c}{m}T_s + \dfrac{T_s^2}{m}K_1 - 2\right) \\
c_{10}(2) = \left(\dfrac{c}{m}T_s - 1\right) \\
c_{30}(1) = -\dfrac{T_s^2 K_3}{m} \\
c_{01}(1) = \dfrac{T_s^2}{m}
\end{cases}
\tag{5.7}
$$

and T_s is the sampling interval. As long as T_s is selected appropriately (5.6) will reproduce the dynamics of the original continuous time system (5.1).

Because there is a cubic nonlinear term in y(.) in the model the number of terms in the Volterra series expansion could be infinite. However it is well-known that in most practical cases the nonlinear effects of the single degree mechanical oscillator can be ignored. By applying the new method for truncating nonlinear system expansions in the frequency domain, this can be explained, and it can be shown that even if the effects of the nonlinearity have to be considered a truncated Volterra series expansion of the system can also be found, under certain conditions on the input spectrum, so that the system can be regarded as a weakly nonlinear plant. In addition the new method can also be used to predict those situations where the input spectrum of the oscillator will induce significant nonlinear effects which will generate considerable oscillations beyond the input frequency band.

Because the mechanical oscillator can be described by the NARX model (5.6), the truncation of the Volterra series expansion of this system can be realised by performing the calculation of $J_3(n)$ using

$$
J_3(n) = \begin{cases}
\dfrac{M_u^n}{(2\pi)^{n-1}} H_n^B \max\limits_{i\in\{0,\dots,M^n/2\}} U_n^{\#}\left(i\dfrac{2\pi}{M}\right) & \text{for odd n} \\
\dfrac{M_u^n}{(2\pi)^{n-1}} H_n^B U_n^{\#}(0) & \text{for even n}
\end{cases}
\tag{5.8}
$$

where $U_n^{\#}(.)$ is obtained from (4.1) and (4.2) with T=1 and H_n^B is determined from (4.8) and (4.9) using the parameters given in (5.6). Because (5.6) represents a class of specific NARX model systems, the computation procedures for H_n^B in (5.8) can be further simplified.

From (4.8) it is known that for the NARX model given by (5.6)

$$
\begin{cases}
H_1^B = \dfrac{\sum c_{01}}{L_1}, \\
H_2^B = \dfrac{1}{L_2}\left[0_{1\times1}, \sum c_{02}, \sum c_1^2 + \sum c_{20}H_1^B\right]\left[0,1,H_1^{B_2}\right]^T = \dfrac{1}{L_2}[0,0,0]\left[0,1,H_1^{B_2}\right]^T = 0
\end{cases}
\tag{5.9}
$$

and for $n \geq 3$,

$$
H_n^B = \frac{1}{L_n}\left[0_{1\times(n-1)}, \sum c_{0n}, \sum c_1^n + \sum c_{20}\overline{H}_{n-1}, \sum c_2^n + \sum c_{30}\overline{H}_{n-2}, \sum c_3^n + \sum c_{40}\overline{H}_{n-3}, \cdots\cdots \sum c_{n-1}^n + \sum c_{n0}\overline{H}_1\right]
$$
$$
\left[H_0^{B_n}, H_1^{B_n}, H_2^{B_n}, H_3^{B_n}, \cdots\cdots, H_{n-1}^{B_n}\right]^T
$$

$$= \frac{1}{L_n}\left[0_{1\times(n-1)}, 0, 0_{1\times(n-1)} + 0\times\overline{H}_{n-1}, 0_{1\times(n-2)} + \sum c_{30}\overline{H}_{n-2}, 0_{1\times(n-3)} + 0\times\overline{H}_{n-3}, \cdots\cdots, 0 + 0\times\overline{H}_1\right]$$

$$\left[H_0^{B_n}, H_1^{B_n}, H_2^{B_n}, H_3^{B_n}, \cdots\cdots, H_{n-1}^{B_n}\right]^T$$

$$= \frac{1}{L_n}\left[\sum c_{30}\overline{H}_{n-2}\right]\left[H_2^{B_n}\right]^T \tag{5.10}$$

where $H_2^{B_n} = \left[\overline{H}_{n-2}(H_1^{B_{n-1}})^T, H_2^{B_{n-1}}\right]$.

Moreover notice that

$$H_2^{B_3} = \left[\overline{H}_1\left(H_1^{B_2}\right)^T\right] = \left[\overline{H}_1\left[\overline{H}_1\left(H_0^{B_1}\right)^T\right]^T\right] = \left[\overline{H}_1\overline{H}_1^T\right] = \left(H_1^B\right)^2 = \overline{H}_1\left(\overline{H}_1^{\#}\right)^T \tag{5.11}$$

where $\overline{H}_1^{\#} = \left[H_1^B\right]$

$$H_2^{B_4} = \left[\overline{H}_2\left(H_1^{B_3}\right)^T, H_2^{B_3}\right] = \left[\overline{H}_2\left[\overline{H}_2\left(H_0^{B_2}\right)^T, H_1^{B_2}\right]^T, \left(H_1^B\right)^2\right] = \left[\overline{H}_2\left[H_2^B, H_1^B\right]^T, \left(H_1^B\right)^2\right] = \left[H_1^B H_2^B + H_2^B H_1^B, \left(H_1^B\right)^2\right] = \left[\overline{H}_2\left(\overline{H}_2^{\#}\right)^T, \overline{H}_1\left(\overline{H}_1^{\#}\right)^T\right] \tag{5.12}$$

and $\overline{H}_2^{\#} = \left[H_2^B, H_1^B\right]$. It can be shown by extending the results of (5.11) and (5.12) into the general case of $H_2^{B_n}$ that

$$H_2^{B_n} = \left[\overline{H}_{n-2}(\overline{H}_{n-2}^{\#})^T, \overline{H}_{n-3}(\overline{H}_{n-3}^{\#})^T, \cdots, \overline{H}_1(\overline{H}_1^{\#})^T\right] \tag{5.13}$$

where

$$\overline{H}_i^{\#} = \left[H_i^B, H_{i-1}^B, \ldots, H_1^B\right] \quad \text{i=1,2,...,n-2} \tag{5.14}$$

Thus substituting (5.13) into (5.10) gives

$$H_n^B = \frac{1}{L_n}\sum c_{30}\left[H_1^B, \ldots, H_{n-2}^B\right]\left[\overline{H}_{n-2}(\overline{H}_{n-2}^{\#})^T, \overline{H}_{n-3}(\overline{H}_{n-3}^{\#})^T, \cdots, \overline{H}_1(\overline{H}_1^{\#})^T\right]^T$$

$$= \left(\frac{\sum c_{30}}{L_n}\right)\sum_{i=1}^{n-2} H_i^B\left(\overline{H}_{n-2-i+1}^T \overline{H}_{n-2-i+1}^{\#}\right) \quad \text{for } n \geq 3 \tag{5.15}$$

Therefore evaluation of $J_3(n)$ for the single degree mechanical oscillator can be implemented by the procedure below

(i) Evaluate H_n^B using (5.15) and (4.9) with the initial values $H_1^B = \frac{\sum c_{01}}{L_1}$ and $H_2^B = 0$.

(ii) Calculate $U_n^{\#}(.)$.

(iii) Compute $J_3(n)$ from (5.8).

Consider a practical single degree mechanical oscillator where the parameters are given by (Chen and Tomlinson 1994)

$m = 39.2\,Kg, \quad c = 39.2\,N\ Sec/meter$

$K_1 = 4.9\times10^5\,N/meter, \quad K_3 = 4.9\times10^{10}\,N/meter^3, \ T_s = 1/150\ Sec$

Substituting these parameters into (5.7) gives the NARX model (5.6) of the oscillator as

$$c_{10}(1) = 1.438334,\ c_{10}(2) = -0.9933334,\ c_{30}(1) = -55555.553,\ c_{01}(1) = 1.1\times10^{-6}$$

Consider four cases of this system in which the input spectra of the oscillator satisfy (3.7) and (3.8) with

(1) a=0.1, b=0.4, M_u=100
(2) a=0.1, b=0.4, M_u=500
(3) a=0.1, b=0.4, M_u=1000

and

(4) a=0.1, b=0.4, M_u=10000

respectively. The results of $J_3(n)$, $n = 1,\ldots,8$, using the procedure in steps (i)-(iii) above with M=1000 are shown in Tables 4,5,6, and 7.

Table 4 Results for the mechanical oscillator (a=0.1, b=0.4, $M_u = 100$)

n	1	2	3	4	5	6	7	8
$J_3(n)$	2.7659 x10^{-4}	0	1.3204 x10^{-6}	0	2.3690 x10^{-8}	0	6.6955 x10^{-10}	0

Table 5 Results for the mechanical oscillator (a=0.1, b=0.4, $M_u = 500$)

n	1	2	3	4	5	6	7	8
$J_3(n)$	0.0014	0	1.6505 x10^{-4}	0	7.4030 x10^{-5}	0	5.2309 x10^{-5}	0

Table 6 Results for the mechanical oscillator (a=0.1, b=0.4, $M_u = 1000$)

n	1	2	3	4	5	6	7	8
$J_3(n)$	0.0028	0	0.0013	0	0.0024	0	0.0067	0

Table 7 Results for the mechanical oscillator (a=0.1, b=0.4, $M_u = 10000$)

n	1	2	3	4	5	6	7	8
$J_3(n)$	0.0277	0	1.3204	0	236.897	0	6.6955 x10^{4}	0

Table 4 with $M_u = 100$ indicates that for $\varepsilon = 10^{-4}$ all $J_3(2),\cdots,J_3(8)$ are less than ε and are therefore negligible so that the truncated frequency domain Volterra series expansion of the system has only a linear term. That is the mechanical oscillator can be adequately represented by a linear model. To confirm this analysis the behaviour of the oscillator was simulated using the NARX model representation with the excitation

$$u(k) = \frac{M_u}{\pi} \frac{\sin(0.4k) - \sin(0.1k)}{k}, \; k = 0, \pm 1, \pm 2, \ldots \tag{5.16}$$

and $M_u = 100$ which satisfies condition (1) above. The simulation results are illustrated in Fig 6 and show that the output spectrum is negligible outside the input frequency band [0.1,0.4] and demonstrates that the effects of nonlinearities on the system output can be ignored.

The results in Table 5 indicate that when M_u is increased from 100 to 500, to accommodate a larger class of inputs, the third order nonlinearity must be included in the truncated frequency domain Volterra series expansion of the system if ε is taken as 10^{-4}. The simulation results for this case are shown in Fig. 7 and indicate that compared to Fig.6 more response beyond the input frequency band [0.1,0.4] can be observed.

Table 6 shows that for the case when $M_u = 1000$ the effects of the system nonlinearities on the output have become comparable to the linear effects and nonlinear terms involving the 3rd, 5th, 7th, and even higher order nonlinearities need to be included in the truncated Volterra series expansion of the system since all these are larger than $\varepsilon = 10^{-4}$. Fig.8 illustrates the simulation results for this case and shows that compared to the case when $M_u = 500$ there are more considerable effects around $\omega = 0.77$ beyond the input frequency band [0.1,0.4].

The results in Table 7 indicate that in the fourth case when $M_u = 10000$ the effects of the system nonlinearity dominate the output response because $J_3(n)$ increases up to 6.6955×10^4 for n=7 and it will generally be impossible to find a truncated Volterra series expansion of this system. The simulation results for this situation are shown in Fig.8 which clearly demonstrate the dominant nonlinear effects outside the input frequency band .

The oscillations at $\omega = 0.77$ in $|\mathbf{Y}(j\omega)|$ must be due to the nonlinear effects in the system because $\omega = 0.77$ is outside the input frequency band [0.1,0.4]. These oscillations are in fact caused by the resonance of the system since the maximum value of

$$\left| \frac{1}{1 - c_{10}(1)e^{-j\omega} - c_{10}(2)e^{-2j\omega}} \right|$$

where $c_{10}(1) = 1.438334$, $c_{10}(2) = -0.9933334$, is at $\omega = 0.77$. This observation implies that when the intrinsic resonance frequencies of a single degree mechanical oscillator are beyond the frequency range of the input excitation, which is often the case in practical situations, the extent of oscillations due to the resonance modes can be predicted by examining $J_3(n)$ for n>1 at different input amplitudes using the algorithm developed. This type of analysis might also be of significance for more complicated engineering systems and will be investigated further in later studies.

6. CONCLUSIONS

Although most practical systems are intrinsically nonlinear, linear models can usually be used to approximate the systems quite well provided the system input varies within a small neighbourhood of the operating point. But when the system input deviates far from the operating point the system nonlinearities will begin to influence the system output. In these circumstances the Volterra series theory of nonlinear systems can be applied to analyse the system based on a truncated Volterra series description. Truncation of nonlinear Volterra series expansions based on both the properties of the system and the input excitation is therefore an important area of analysis.

In the present study recently developed results on the output frequency characteristics of nonlinear systems have been used to investigate the Volterra series expansions and a new algorithm has been introduced to determine which frequency terms are significant. The effectiveness of the new method has been verified using simulation studies including a single degree mechanical oscillator. The new procedure can only be implemented if the system GFRFs or a time domain model and the properties of the input are known. These are reasonable and realistic requirements because the effects of truncating a nonlinear Volterra series expansion is usually investigated based on this information (Rugh 1981, Thapar and Leon 1984) and the main problem is to decide which terms should be included in the truncated Volterra series expansion.

Acknowledgements

SAB gratefully acknowledges that part of this work was supported by EPSRC. ZQL expresses his thanks to Sheffield University for support from a research scholarship.

REFERENCES

Atherton Derek P., 1982, Nonlinear Control Engineering, Van Nostrand Reinhold Company, London.

Boyd Stephen and Chua, Leon O., 1985, Fading memory and the problem of approximating nonlinear operators with Volterra series, IEEE Trans. on Circuit and Systems, Vol.32, No.11, pp1150-1161.

Billings, S.A. and Lang Zi-Qiang, 1995(a), A Bound for the magnitude characteristics of nonlinear output frequency response functions, Part I: Analysis and computation. Int. J. Control (to appear).

Billings, S.A. and Lang Zi-Qiang, 1995(b), A Bound for the magnitude characteristics of nonlinear output frequency response functions, Part II: Practical computation of the bound for systems described by the NARX model. Int. J. Control (to appear).

Billings, S.A. and Peyton Jones, J.C., 1990, Mapping nonlinear integro-differential equations into the frequency domain. Int. J. Control, 54(4), 863-879.

Billings S.A. and Tsang, K.M., 1989, Spectral analysis for nonlinear systems, Part I: Parametric nonlinear spectral analysis, Mechanical Systems and Signal Processing, 3, 319-339.

Chen, S. and Billings, S.A., 1989, Representations of nonlinear systems: the NARMAX model, Int. J. Control, 49, 1013-1032.

Chen, Q., Tomlinson, G.R., 1994, A new type of time series model for the identification of nonlinear dynamical systems, Mechanical Systems and Signal Processing, 8(5), 531-549.

George, D.A., 1959, Continuous nonlinear systems, Tech. Report:335, MIT Research Lab, California.

Lang Zi-Qiang and Billings, S.A., 1994, Output frequency characteristics of nonlinear systems, Int. J. Control (to appear).

Lang Zi-Qiang and Billings, S.A., 1995, Output frequencies of nonlinear systems (submitted for publication).

Lang Zi-Qiang , 1994, On identification of controlled plants described by the Hammerstein system, IEEE Trans. on Automatic Control, Vol.39, No.3, 569-573.

Mitzel, G., Chancy, S., Rugh, W., 1979, On transfer function representations for homogeneous nonlinear systems, IEEE Trans. on Automatic Control, 24, 242-249.
Peyton Jones, J.C. and Billings, S.A., 1990, Interpretation of nonlinear frequency response functions. Int. J. Control, 52(2), 319-346.
Peyton Jones, J.C. and Billings, S.A., 1989, A recursive algorithm for computing the frequency response of a class of nonlinear difference equation models, Int. J. Control, 50(5), 1925-1949.
Rugh, W.J., 1981, Nonlinear system theory: the Volterra/Wiener approach, Johns Hopkins Univ. Press, Baltimore, MD.
Thapar, Hemant K. and Leon Benjamin J., 1984, Transform-domain and time-domain characterisation of nonlinear systems with Volterra series. IEEE Trans. on Circuit and System, Vol.31, No.10, pp906-912.
Zhang, H., Billings, S.A. and Zhu, Q.M., 1995, Frequency response functions for nonlinear rational models, Int. J. Control, 61(5), 1073-1097.

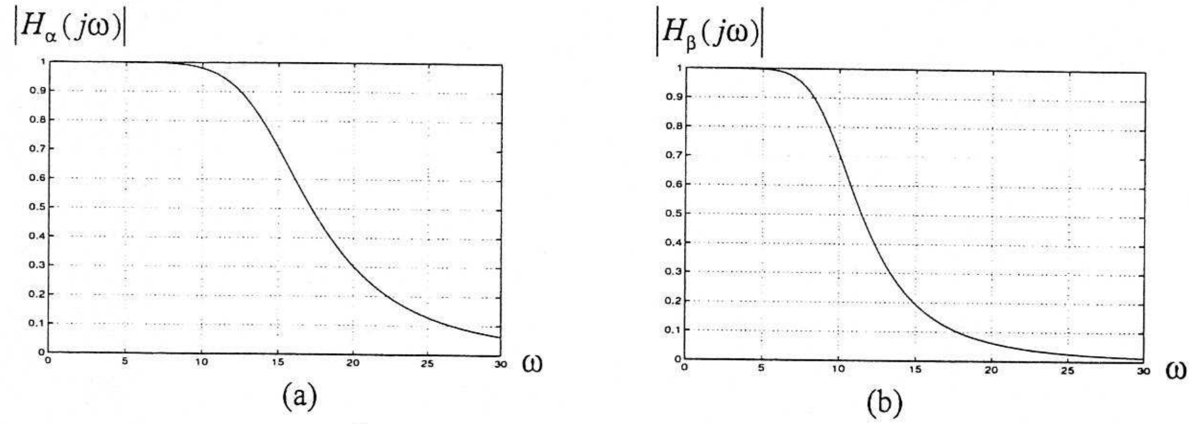


Figure 2 Magnitude characteristics of $H_\alpha(j\omega)$ and $H_\beta(j\omega)$ for the example in Section 4.2

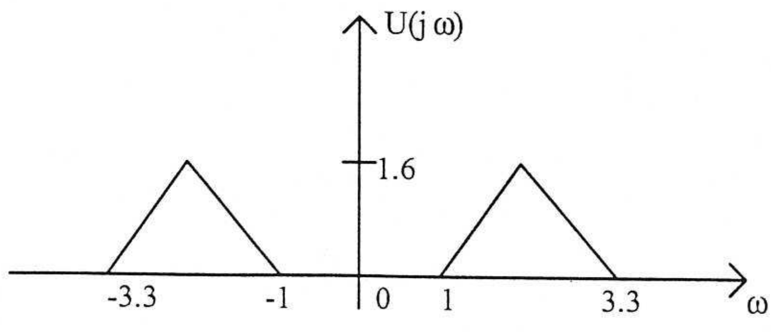
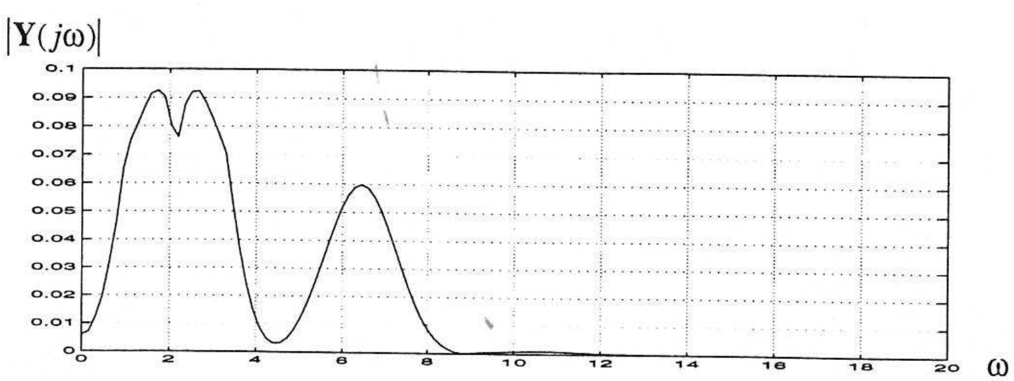
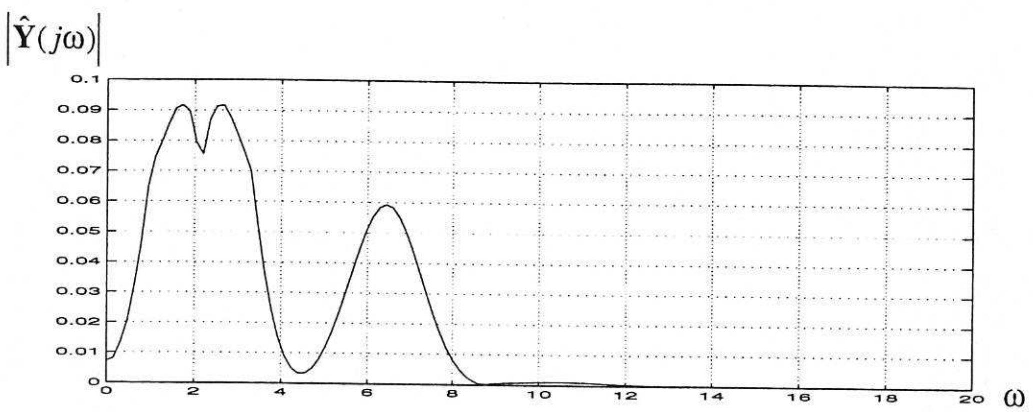


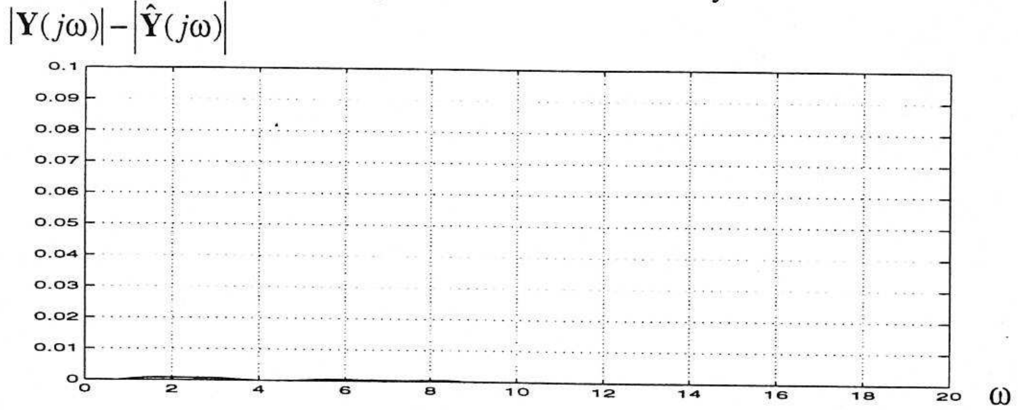
Figure 3 The input spectrum for simulation studies in Section 4.2



(a) Output spectrum of the original system



(b) Output spectrum of the truncated system



(c) The difference between the results in (a) and (b)

Figure 4 Simulation results for the example in Section 4.2

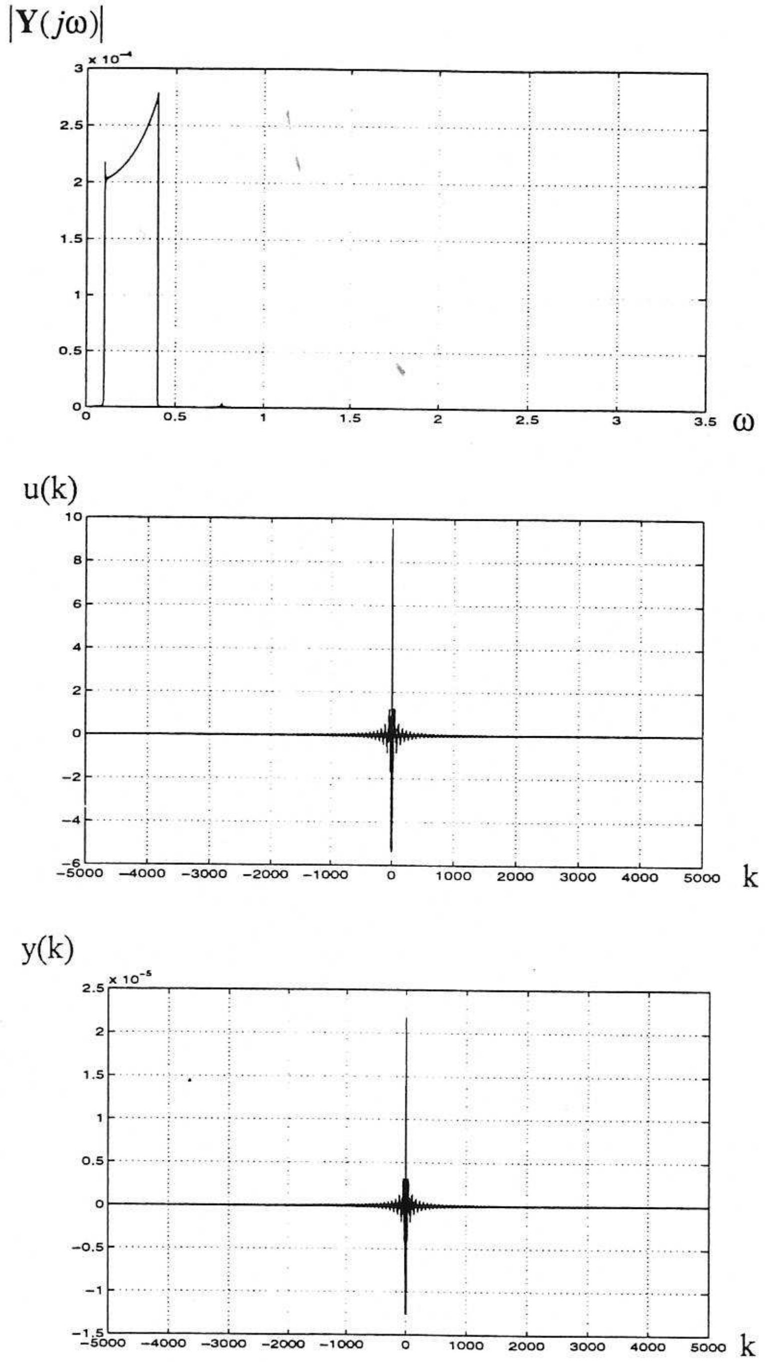


Figure 6. Simulation results for the mechanical oscillator with the input given by equation (5.16) and $M_u = 100$

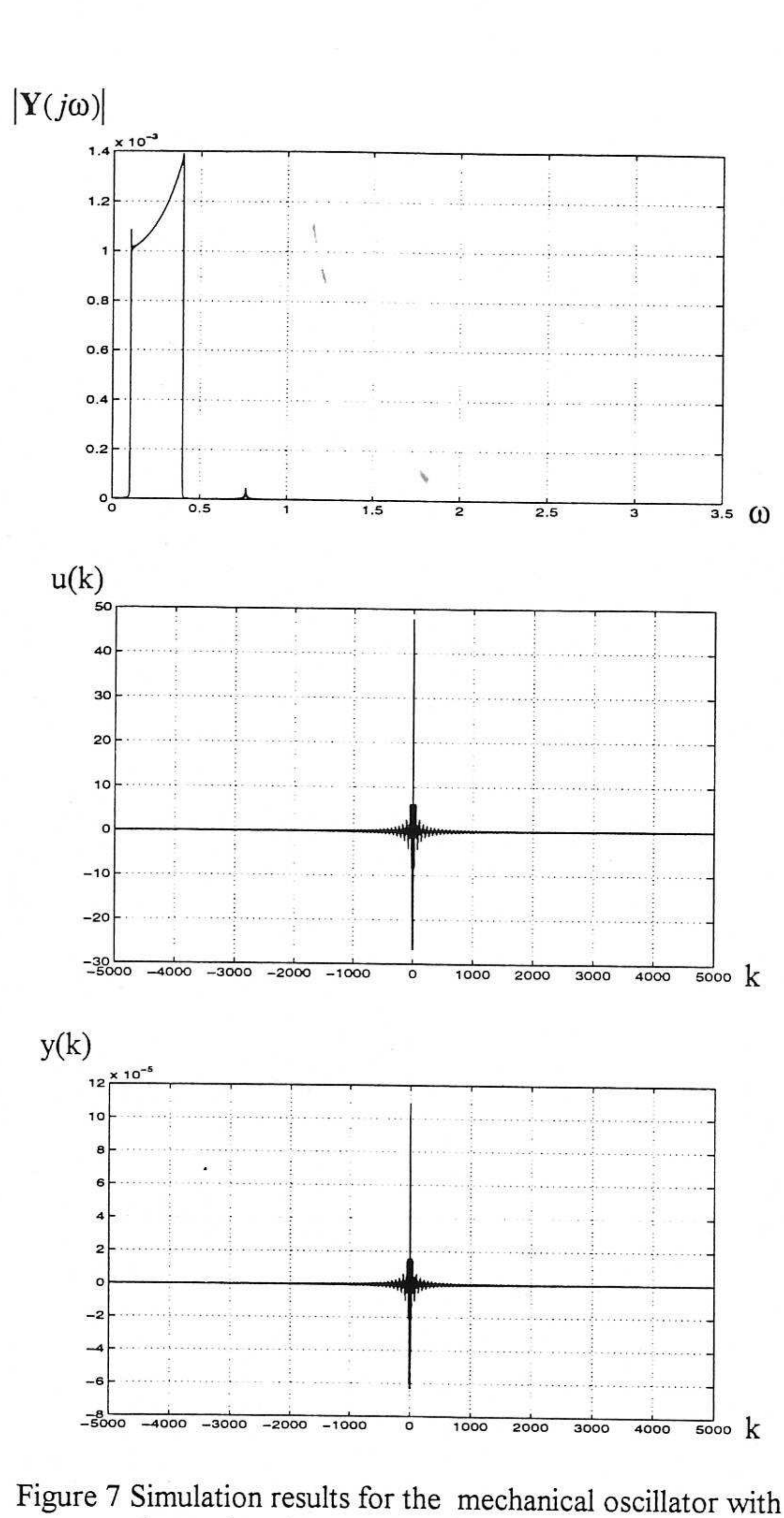


Figure 7 Simulation results for the mechanical oscillator with the input given by equation (5.16) and $M_u = 500$

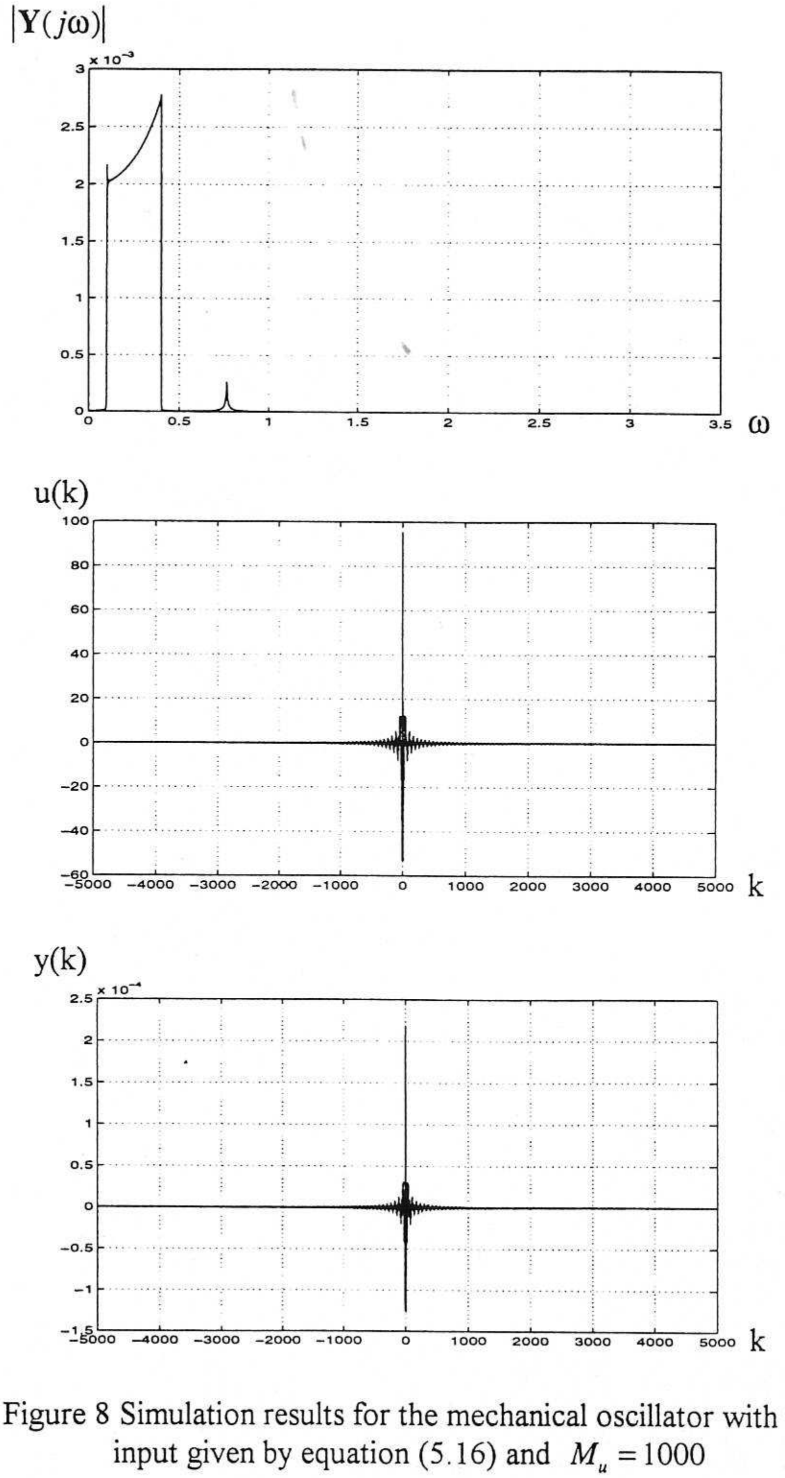


Figure 8 Simulation results for the mechanical oscillator with the input given by equation (5.16) and $M_u = 1000$

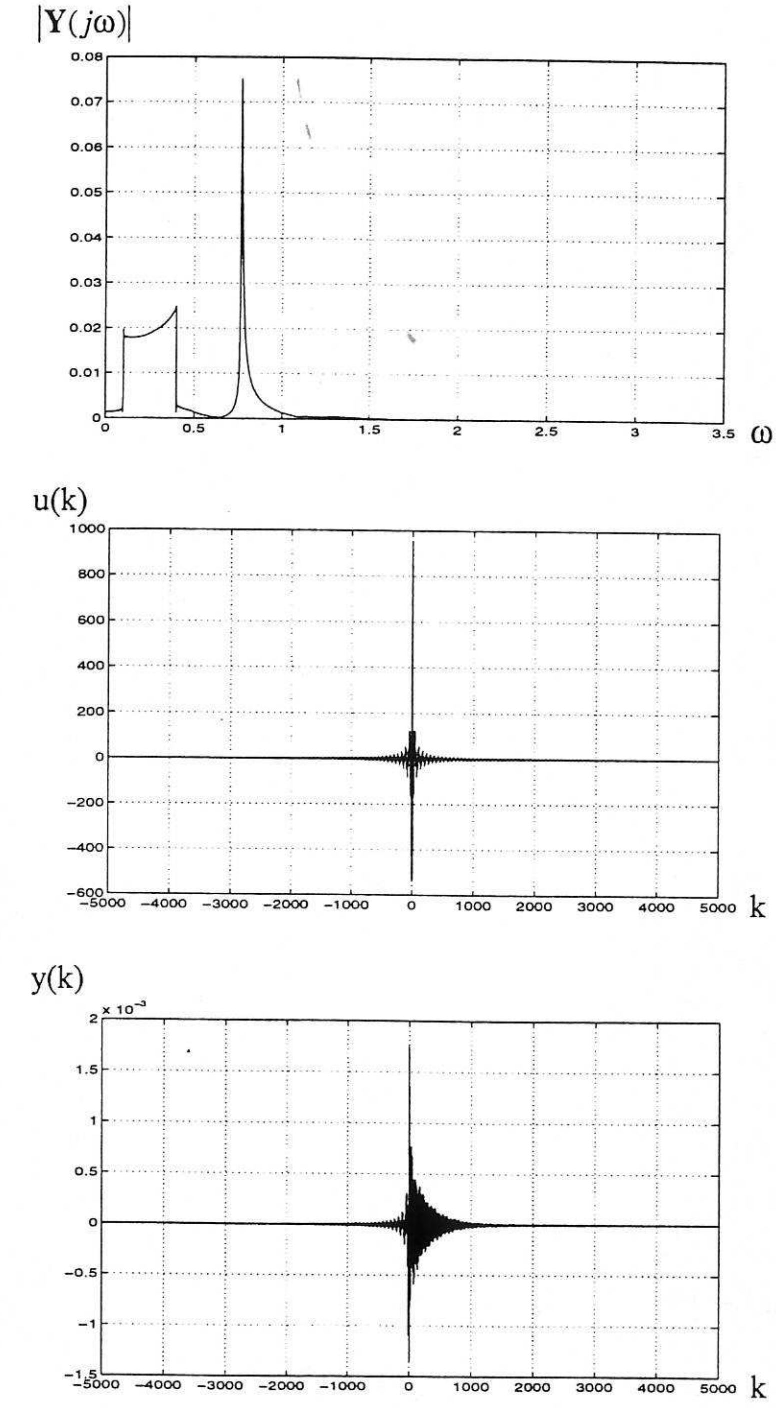


Figure 9 Simulation results for the mechanical oscillator with the input given by equation (5.16) and $M_u = 10000$