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Unified characterisations of resolution hardness measures*

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Abstract. Various "hardness" measures have been studied for resolution, providing theoretical insight into the proof complexity of resolution and its fragments, as well as explanations for the hardness of instances in SAT solving. In this paper we aim at a unified view of a number of hardness measures, including different measures of width, space and size of resolution proofs. Our main contribution is a unified game-theoretic characterisation of these measures. As consequences we obtain new relations between the different hardness measures. In particular, we prove a generalised version of Atserias and Dalmau's result on the relation between resolution width and space from [5].

1 Introduction

Arguably, resolution is the best understood among all propositional proof system, and at the same time it is the most important one in terms of applications. To understand the complexity of resolution proofs, various *hardness measures* have been defined and investigated. Historically the first and most studied measure is the *size of resolution proofs*, with the first lower bounds dating back to Tseitin [55] and Haken [33]. A number of ingenious techniques have been developed to show lower bounds for the size of resolution proofs, among them feasible interpolation [41], which applies to many further systems. In their seminal paper [12], Ben-Sasson and Wigderson showed that resolution size lower bounds can be elegantly obtained by showing lower bounds to the *width* of resolution proofs. Indeed, the discovery of this relation between width and size of resolution proofs was a milestone in our understanding of resolution. Around the same time *(tree) resolution space* was investigated, and first lower bounds were obtained [53,23,24,25,54]. The primary method to obtain lower bounds on resolution space is based on width, and the general bound was shown in the fundamental paper by Atserias and Dalmau [5]. Since then the relations between size, width and space have been intensely investigated, resulting in particular in sharp trade-off results [10,8,11,48,49,**?**]. Independently, in [42,45,46] the concept of "hardness" has been introduced, with an algorithmic focus (as shown in [42], equivalent to tree resolution space; one can also say "tree-hardness"), together with a generalised form of width, which we call "asymmetric width" in this paper.

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One of the prime motivations to understand these measures is their close correspondence to SAT solving. In particular, resolution size and space relate to the running time and memory consumption, respectively, of executions of SAT solvers on unsatisfiable instances. However, size and space are not the only measures which are interesting with respect to SAT solving, and the question what constitutes a good hardness measure for practical SAT solving is a very important one (cf. [4,36] for discussions).

The aim of this paper is to review different hardness measures defined in the literature, and to provide *unified characterisations* for these measures in terms of Prover-Delayer games and sets of partial assignments satisfying some consistency conditions. These unified characterisations allow elegant proofs of basic relations between the different hardness measures. Unlike in the works [8,11,48], our emphasis here is not on trade-off results, but on *exact relations* between the different measures. For a clause-set F we will consider the following measures: *(i) size measures:* the depth $\mathrm{dep}(F)$ and the hardness $\mathrm{hd}(F)$ (of best resolution refutations of F); *(ii) width measures:* the symmetric and asymmetric width $\mathrm{wid}(F)$ and $\mathrm{awid}(F)$; *(iii) clause-space measures:* semantic space $\mathrm{css}(F)$, resolution space $\mathrm{crs}(F)$ and tree-resolution space $\mathrm{cts}(F)$.

Game-theoretic methods have a long tradition in proof complexity, as they provide intuitive and simplified methods for lower bounds in resolution, e.g. for Haken's exponential bound for the pigeonhole principle in dag-like resolution [50], or the optimal bound in tree resolution [13], and even work for very strong systems [9]. Inspired by the Prover-Delayer game of Pudlák and Impagliazzo [51], we devise a game that characterises the hardness measure $\mathrm{hd}(F)$. In contrast to [51] it also works for satisfiable formulas (Theorem 12), due to elimination of the communication between Prover and Delayer. We then explain a more general game, allowing the Prover to also *forget* some information. This game tightly characterises the asymmetric width hardness $\mathrm{awid}(F)$ (Theorem 23); and restricting this game by disallowing forgetting yields the hd-game (Lemma 24).

Characterisations by partial assignments provide an alternative combinatorial description of the hardness measures. In [5] such a characterisation is obtained for $\mathrm{wid}(F)$. Taking this as a starting point, we devise a hierarchy of consistency conditions for sets of partial assignments which serve to characterise asymmetric width $\mathrm{awid}(F)$ (k-consistency, Theorem 22), hardness $\mathrm{hd}(F)$ (weak k-consistency), and depth $\mathrm{dep}(F)$ (bare k-consistency).

Relations between these measures can be easily obtained by exploiting the above characterisations. We obtain a generalised version of Atserias and Dalmau's connection between width and resolution space from [5], replacing symmetric width by the stronger notion of asymmetric width (handling long clauses now), and resolution space by the possibly tighter semantic space (Theorem 27). The full picture is presented in the following diagram, where $F \in \mathcal{CLS}$ has n variables, minimal clause length p, and maximal length q of necessary clauses:

$$\begin{tikzcd}
p \arrow[r] \arrow[dr] & \mathrm{awid}(F) \arrow[r] \arrow[dr] & \mathrm{css}(F) \arrow[r, dash, "\sim *3"] & \mathrm{crs}(F) \arrow[r] & \mathrm{cts}(F) \arrow[r, dash, "=-1"] & \mathrm{hd}(F) \arrow[r] & \mathrm{dep}(F) \arrow[r] & n \\
 & q \arrow[r] & \mathrm{wid}(F) \arrow[urrrr] & & & & &
\end{tikzcd}$$

An arrow "$h(F) \to h'(F)$" means $h(F) \le h'(F)$, and there is a sequence (F_n) of clause-sets with bounded $h(F_n)$ but unbounded $h'(F_n)$; in case of an undirected edge no such separation is possible (crs differs from css at most by a factor of 3, while $\mathrm{cts} - 1 = \mathrm{hd}$). The separation awid $\to$ css is shown in [49], crs $\to$ cts in [36], hd $\to$ dep and wid $\to$ dep use unsatisfiable Horn 3-clause-sets, and dep $\to n$ uses unsatisfiable clause-sets which are not minimally unsatisfiable.

These measures do not just apply to unsatisfiable clause-sets, but are extended to **satisfiable clause-sets**, taking a *worst-case approach* over all unsatisfiable sub-instances obtained by applying partial assignments (instantiations). For a fixed bound these measures allow for polynomial-time SAT solving via "oblivious" SAT algorithms — certain basic steps, applied in an arbitrary manner, are guaranteed to succeed. The sets $\mathcal{UC}_k$ of all clause-sets F with $\mathrm{hd}(F) \le k$ yield the basic hierarchy, and we have SAT decision in time $O(n(F)^{2\,\mathrm{hd}(F)-2} \cdot \ell(F))$. The special case $\mathcal{UC}_1 = \mathcal{UC}$ was introduced in [57] for the purpose of Knowledge Compilation (KC), and in [28,31] it is shown that $\mathcal{UC} = \mathcal{SLUR}$ holds, where $\mathcal{SLUR}$ is the class introduced in [52] as an umbrella class for polynomial-time SAT solving. By [6,31] we get that membership decision for $\mathcal{UC}_k$ with $k \ge 1$ is coNP-complete.

Perhaps the main aim of measuring the complexity of satisfiable clause-sets is to obtain *SAT representations of boolean functions* of various quality ("hardness") and sizes; see [29,32] for investigations into XOR-constraints. The goal is to obtain "good" representations F of boolean functions (like cardinality or XOR-constraints) in the context of a larger SAT problem representations. "Good" means not "too big" and of "good" inference power. The latter means (at least), that all unsatisfiable instantiations of F should be easy for SAT solvers, motivating the worst-case approach (over all unsatisfiable sub-instances). In the diagram above, having low $\mathrm{dep}(F)$ is the strongest condition, having low $\mathrm{awid}(F)$ the weakest. The KC aspects, concerning size-hardness trade-offs, are further investigated in [30]; see Corollary 29 for an application. This study of the "best"choice of a representation, considering size (number of clauses) and hardness (like hd, awid or css) among *all (logically) equivalent* clause-sets, likely could not be carried out using (symmetric) width, but requires *asymmetric width*, so that unbounded clause length can be handled. The traditional method to bound the clause-length, by breaking up clauses via auxiliary variables, introduces unnecessary complexity, and can hardly be applied if we only want to consider (logically) equivalent clause-sets (without auxiliary variables).

This paper is organised as follows. After fixing notation in Sect. 2, we define all hardness measures in Sect. 3 and prove some first results. Our main results then follow in Sect. 4, where we prove the combinatorial characterisations of the measures and infer basic connections. We conclude in Sect. 5 with a discussion and some open questions.

2 Preliminaries

We use the general notions as in [38], but also define all notations.

Clause-sets. $\mathcal{VA}$ is the (infinite) set of variables, while $\mathcal{LIT}$ is the set of literals, where every literal is either a variable v or a complemented (negated) variable $\overline{v}$. For a set $L \subseteq \mathcal{LIT}$ of literals we use $\overline{L} := \{\overline{x} : x \in L\}$. A clause is a finite $C \subset \mathcal{LIT}$ with $C \cap \overline{C} = \emptyset$ (i.e., without conflicting literals), the set of all clauses is $\mathcal{CL}$. A clause-set is a finite set of clauses, the set of all clause-sets is $\mathcal{CLS}$. For $k \in \mathbb{N}_0$ we define k–$\mathcal{CLS}$ as the set of all $F \in \mathcal{CLS}$ where every clause $C \in F$ has length (width) at most k, i.e., $|C| \le k$. We use $\mathrm{var} : \mathcal{LIT} \to \mathcal{VA}$ for the underlying variable of a literal, while $\mathrm{var}(C) := \{\mathrm{var}(x) : x \in C\}$ for a clause C, and $\mathrm{var}(F) := \bigcup_{C \in F} \mathrm{var}(C)$ for a clause-set F. Measures for $F \in \mathcal{CLS}$ are $n(F) := |\mathrm{var}(F)| \in \mathbb{N}_0$ (number of variables) and $c(F) := |F| \in \mathbb{N}_0$ (number of clauses). A special clause is the empty clause $\bot := \emptyset \in \mathcal{CL}$, a special clause-set is the empty clause-set $\top := \emptyset \in \mathcal{CLS}$.

A partial assignment is a map $\varphi : V \to \{0,1\}$ for some finite $V \subset \mathcal{VA}$, the set of all partial assignments is $\mathcal{PASS}$; we use $\mathrm{var}(\varphi) := V$, and the number of variables in a partial assignment is denoted by $n(\varphi) := |\mathrm{var}(\varphi)|$. For a clause C we denote by $\varphi_C \in \mathcal{PASS}$ the partial assignment which sets precisely the literals in C to 0; furthermore we use $\langle x \to \varepsilon \rangle \in \mathcal{PASS}$ for a literal x and $\varepsilon \in \{0,1\}$, while $\langle\rangle \in \mathcal{PASS}$ denotes the empty partial assignment. The natural partial order on $\mathcal{PASS}$ is given by inclusion $\varphi \subseteq \psi$, that is, $\mathrm{var}(\varphi) \subseteq \mathrm{var}(\psi)$ and φ, ψ are compatible (do not assign different values to the same variable). The application (instantiation) of φ to $F \in \mathcal{CLS}$ is denoted by $\varphi * F \in \mathcal{CLS}$, obtained by first removing satisfied clauses $C \in F$ (i.e., containing a literal $x \in C$ with $\varphi(x) = 1$), and then removing all falsified literals from the remaining clauses.

The set of satisfiable clause-sets is $\mathcal{SAT} := \{F \in \mathcal{CLS} \mid \exists\, \varphi \in \mathcal{PASS} : \varphi * F = \top\}$, while $\mathcal{USAT} := \mathcal{CLS} \setminus \mathcal{SAT}$ is the set of unsatisfiable clause-sets. For $F, F' \in \mathcal{CLS}$ the implication-relation is defined as usual: $F \models F' :\Leftrightarrow \forall\, \varphi \in \mathcal{PASS} : \varphi * F = \top \Rightarrow \varphi * F' = \top$. We write $F \models C$ for $F \models \{C\}$. A clause C with $F \models C$ is an *implicate* of F, while a *prime implicate* is an implicate C such that no $C' \subset C$ is also an implicate; $\mathrm{prc}_0(F)$ is the set of prime implicates of F. Finally, by $\mathrm{r}_1 : \mathcal{CLS} \to \mathcal{CLS}$ we denote unit-clause propagation, which is defined recursively by $\mathrm{r}_1(F) := \{\bot\}$ if $\bot \in F$, $\mathrm{r}_1(F) := F$ if F does not contain a unit-clause, while otherwise choose $\{x\} \in F$ and set $\mathrm{r}_1(F) := \mathrm{r}_1(\langle x \to 1 \rangle * F)$.

Resolution. Two clauses C, D are resolvable if $|C \cap \overline{D}| = 1$, i.e., they clash in exactly one variable. For two resolvable clauses C and D the resolvent $C \diamond D := (C \cup D) \setminus \{x, \overline{x}\}$ for $C \cap \overline{D} = \{x\}$ is the union of the two clauses minus the resolution literals. $\mathrm{var}(x)$ is called the resolution variable. The closure of $F \in \mathcal{CLS}$ under resolution has $\mathrm{prc}_0(F)$ as its subsumption-minimal elements.

The set of nodes of a tree T is denoted by $\mathrm{nds}(T)$, the set of leaves by $\mathrm{lvs}(T) \subseteq \mathrm{nds}(T)$. The height $\mathrm{ht}_T(w) \in \mathbb{N}_0$ of a node $w \in \mathrm{nds}(T)$ is the height of the subtree of T rooted at w (so $\mathrm{lvs}(T) = \{w \in \mathrm{nds}(T) : \mathrm{ht}_T(w) = 0\}$). A resolution tree is a pair $R = (T, C)$ such that T is an ordered rooted tree, where every inner node has exactly two children, and where the set of nodes is denoted by $\mathrm{nds}(T)$ and the root by $\mathrm{rt}(T) \in \mathrm{nds}(T)$, while $C : \mathrm{nds}(T) \to \mathcal{CL}$ labels every node with a clause such that the label of an inner node is the resolvent of the labels of its two parents. We use $\mathrm{ax}(R) := \{\mathrm{C}(w) : w \in \mathrm{lvs}(T)\} \in \mathcal{CLS}$ for the "axioms" (or "premisses") of R, $\mathrm{C}(R) := C(\mathrm{rt}(T)) \in \mathcal{CL}$ for the "conclusion", and $\mathrm{cl}(R) := \{C(w) : w \in \mathrm{nds}(T)\} \in \mathcal{CLS}$ for the set of all clauses in R.

A resolution proof R of a clause C from a clause-set F, denoted by $R : F \vdash C$, is a resolution tree $R = (T, C)$ such that $\mathrm{ax}(R) \subseteq F$ and $\mathrm{C}(R) = C$. We use $F \vdash C$ if there exists a resolution proof R of some $C' \subseteq C$ from F (i.e., $R : F \vdash C'$). A resolution refutation of a clause-set F is a resolution proof deriving $\perp$ from F. The tree-resolution complexity $\mathrm{Comp}^*_{\mathrm{R}}(R) \in \mathbb{N}$ is the number of leaves in R, that is, $\mathrm{Comp}^*_{\mathrm{R}}(R) := |\mathrm{lvs}(T)|$. The resolution complexity $\mathrm{Comp}_{\mathrm{R}}(R) \in \mathbb{N}$ is the number of *distinct* clauses in R, that is $\mathrm{Comp}_{\mathrm{R}}(R) := c(\mathrm{cl}(R))$. Finally, for $F \in \mathcal{USAT}$ we set $\mathrm{Comp}^*_{\mathrm{R}}(F) := \min\{\mathrm{Comp}^*_{\mathrm{R}}(R) \mid R : F \vdash \perp\} \in \mathbb{N}$ and $\mathrm{Comp}_{\mathrm{R}}(F) := \min\{\mathrm{Comp}_{\mathrm{R}}(R) \mid R : F \vdash \perp\} \in \mathbb{N}$.

3 Hardness measures

In this section we define the hardness measures $\mathrm{hd}, \mathrm{dep}, \mathrm{wid}, \mathrm{awid}, \mathrm{css}, \mathrm{crs}, \mathrm{cts}$ ("hardness, depth, width, asymmetric width, semantic/resolution/tree space") that we investigate in this article, and observe some first connections.

First we discuss a *general method for extending measures* h_0 for unsatisfiable clause-sets to measures h for arbitrary clause-sets. The basic idea is to consider the hardness of unsatisfiable sub-instances, obtained by partial instantiations. In a probabilistic setting this has been considered e.g. in [1,3]. We however consider the worst-case, which yields precise measurements. The special case of extension of "hardness" was first mentioned (as one of two possibilities) by [4]. Our motivation was that the extension of clause-sets falsifiable by unit-clause propagation yields precisely the class $\mathcal{SLUR}$ ([28,31]).

A measure $h_0 : \mathcal{USAT} \to \mathbb{N}_0$, which is not increased by applying partial assignments, is extended to $h : \mathcal{CLS} \to \mathbb{N}_0$ by $h(\top) := \min_{F \in \mathcal{USAT}} h_0(F)$, while for $F \in \mathcal{CLS} \setminus \{\top\}$ we define $h(F)$ as the maximum of $h_0(\varphi * F)$ for $\varphi \in \mathcal{PASS}$ with unsatisfiable $\varphi * F$. So also h is not increased by applying partial assignments, and $h(F) = h_0(F)$ for $F \in \mathcal{USAT}$, while for $h_0 \leq h'_0$ we get $h \leq h'$. Note that for the computation of $h(F)$, as the maximum of $h_0(\varphi * F)$ for unsatisfiable $\varphi * F$, one only needs to consider minimal φ (since application of partial assignments can not increase the measure), that is, φ_C for $C \in \mathrm{prc}_0(F)$; so for $F \in \mathcal{CLS} \setminus \{\top\}$ we have $h(F) = \max_{C \in \mathrm{prc}_0(F)} h_0(\varphi * F)$. In the following we will define the hardness measure only for unsatisfiable clauses and then extend them via the above method.

3.1 Tree-hardness

We start with what in our opinion is one of the central hardness measures for resolution, which is why we simply call it *hardness* (but for differentiation it might be called *tree-hardness*, then written "thd"). This concept was reinvented in the literature several times. Intuitively, hardness measures the height of the biggest full binary tree which can be embedded into each tree-like resolution refutation of the formula. This is also known as the Horton-Strahler number of a tree (see

[58,22]). In the context of resolution this measure was first introduced in [42,46]. In a more loose sense, based on reduction rules, "hardness classes" are mentioned in [26,27], based on an unpublished manuscript of Stålmarck from 1994.[3)] The equivalent approach via tree-resolution space was introduced in [53,24,25,54]. These approaches concern only unsatisfiable clause-sets; the extension to satisfiable clause-sets considered in [42,46] generalises the reduction-rules-based approach, and is essentially different from the general extension process as discussed above; the extension as in this paper was first considered in [4].

Definition 1. *For $F \in \mathcal{USAT}$ let* $\mathbf{hd}(\boldsymbol{F}) \in \mathbb{N}_0$ *be the minimal* $k \in \mathbb{N}_0$ *such that a resolution tree* $T : F \vdash \bot$ *exists, where the Horton-Strahler number of* T *is at most* k*, that is, for every node in* T *there exists a path to some leaf of length at most* k*. For* $k \in \mathbb{N}_0$ *let* $\boldsymbol{\mathcal{UC}_k} := \{F \in \mathcal{CLS} : \mathrm{hd}(F) \le k\}$*.*

See [42,46,28,31] for equivalent descriptions in this setting, where especially the algorithmic approach, via generalised unit-clause propagation r_k, is notable: hardness is the minimal level k of generalised unit-clause propagation needed to derive a contradiction under any instantiation. As shown in [42, Corollary 7.9], and more generally in [46, Theorem 5.14], we have

$$2^{\mathrm{hd}(F)} \le \mathrm{Comp}^*_{\mathrm{R}}(F) \le (n(F)+1)^{\mathrm{hd}(F)}$$

for $F \in \mathcal{USAT}$.[4)] A simpler measure is the minimum depth ([56,19,20]):

Definition 2. *For* $F \in \mathcal{USAT}$ *let* $\mathbf{dep}(\boldsymbol{F}) \in \mathbb{N}_0$ *be the minimal height of a resolution tree* $T : F \vdash \bot$*.*

Obviously $\mathrm{hd}(F) \le \mathrm{dep}(F)$ for all $F \in \mathcal{CLS}$. For $k \in \mathbb{N}_0$ the class of $F \in \mathcal{CLS}$ with $\mathrm{dep}(F) \le k$ is called $\mathrm{CANON}(k)$ in [18,7]; by definition $\mathrm{CANON}(0) = \mathcal{UC}_0$. See Subsection 7.2 in [31] and Subsection 9.2 in [30] for further results.

3.2 Asymmetric width

The standard resolution-width of an unsatisfiable clause-set is the minimal k such that a resolution refutation using only clauses of length at most k exists:

Definition 3. *For* $F \in \mathcal{USAT}$ *the* ***symmetric width*** $\mathbf{wid}(\boldsymbol{F}) \in \mathbb{N}_0$ *is the smallest* $k \in \mathbb{N}_0$ *such that there is* $T : F \vdash \bot$ *with* $\mathrm{cl}(T) \in k\text{-}\mathcal{CLS}$*.*

Based on the notion of "k-resolution" introduced in [37], the "asymmetric width" was introduced in [42,45,46] (and further studied in [31,30,32]).[5)] Different from the symmetric width, only *one parent clause* needs to have size at most k (while there is no restriction on the other parent clause nor on the resolvent):

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3) We have never seen these fragments called "A proof theoretic concept of tautological hardness", but the ideas circulated amongst some researchers from the formal methods community.

4) Our motivation for the lower bound came from [21]. A similar lower bound is mentioned in [26,27], based on the manuscript of Stålmarck. An equivalent bound is shown in [51] (see Subsection 4.1). In [23,24] the lower bound $2^{\mathrm{crs}(F)-1}$ is shown.

5) In [31] the notation "whd" was used, to emphasise that we have an extension of "hardness"; but now we consider the relation to "width" as more important.

Definition 4. *For a resolution tree T its **asymmetric width** $\mathbf{awid}(\boldsymbol{T}) \in \mathbb{N}_0$ is defined as 0 if T is trivial (i.e., $|\mathrm{nds}(T)| = 1$), while otherwise for left and right children w_1, w_2 with subtrees T_1, T_2 we define $\mathrm{awid}(T)$ as the maximum of $\min(|C(w_1)|, |C(w_2)|)$ and $\max(\mathrm{awid}(T_1), \mathrm{awid}(T_2))$.*

We write $\boldsymbol{R : F \vdash^k C}$ if $R : F \vdash C$ and $\mathrm{awid}(R) \leq k$. Now for $F \in \mathcal{USAT}$ we define $\mathbf{awid}(\boldsymbol{F}) := \min\{k \in \mathbb{N}_0 \mid F \vdash^k \bot\}$. Finally for $k \in \mathbb{N}_0$ let $\boldsymbol{\mathcal{WC}_k} := \{F \in \mathcal{CLS} : \mathrm{awid}(F) \leq k\}$.

The asymmetric width is a natural, but less known generalisation of symmetric width, and these measures can be very different. Namely, for an unsatisfiable Horn clause-set F holds $\mathrm{awid}(F) \leq 1$, since unit-clause resolution (i.e., asymmetric width at most 1) is sufficient to derive unsatisfiability. But $\mathrm{wid}(F)$ is unbounded: if F is minimally unsatisfiable, then $\mathrm{wid}(F)$ equals the maximal clause-length of F. For general minimally unsatisfiable F, the maximal clause-length is a lower bound for $\mathrm{wid}(F)$, but is unrelated to $\mathrm{awid}(F)$. For bounded clause-length of F however, wid and awid can be considered asymptotically equivalent by Lemma 5 below.

In a seminal paper, Ben-Sasson and Wigderson [12] observe a fundamental relation between symmetric width and proof size for resolution refutations, thereby establishing one of the main methods to prove resolution lower bounds. We recall that in [42, Theorem 8.11] and [46, Theorem 6.12, Lemma 6.15] this size-width relation is indeed strengthened to asymmetric width:

$$e^{\frac{1}{8}\frac{\mathrm{awid}(F)^2}{n(F)}} < \mathrm{Comp}_{\mathrm{R}}(F) < 6 \cdot n(F)^{\mathrm{awid}(F)+2}$$

for $F \in \mathcal{USAT} \setminus \{\{\bot\}\}$, where $e^{\frac{1}{8}} = 1.1331484\ldots$ Note that compared to [12] the numerator of the exponent does not depend on the maximal clause-length of F. In [42, Lemma 8.13] it is shown that the partial ordering principle has asymmetric width the square-root of the number of variables, while having a polysize resolution refutation. Comparing asymmetric width to (tree-)hardness, we have $\mathcal{WC}_0 = \mathcal{UC}_0$ and $\mathcal{WC}_1 = \mathcal{UC}_1$, while for all $F \in \mathcal{CLS}$ holds $\mathrm{awid}(F) \leq \mathrm{hd}(F)$. The latter is shown in [46, Lemma 6.8] (for unsatisfiable F), and in Corollary 25 below we provide an alternative proof.

It is an open problem whether for (fixed) $k \geq 3$ we can decide "$F \vdash^k \bot$" in polynomial time. For $k = 1$ there is a linear-time algorithm (since $F \vdash^1 \bot$ iff $\mathrm{r}_1(F) = \{\bot\}$), and for $k = 2$ there is a quartic-time algorithm by [17]. See the underlying report [16] for some partial results. In [42,46] a stronger system was considered (which allows polynomial-time decision). It uses the closure under input resolution, where only the conclusion is restricted to length $\leq k$. Using this system, [42, Lemma 8.5] obtains the connections $\mathrm{wid}(F) - \max(p, \mathrm{awid}(F)) \leq \mathrm{awid}(F)$ for $F \in \mathcal{USAT} \cap p\text{–}\mathcal{CLS}$ (see [46, Lemma 6.22] for a generalisation). We give a freestanding proof in the underlying report [16]:

Lemma 5. *For $F \in p\text{–}\mathcal{CLS}$, $p \in \mathbb{N}_0$, holds $\mathrm{wid}(F) \leq \mathrm{awid}(F) + \max(p, \mathrm{awid}(F))$.*

3.3 Space complexity

The last measures that we discuss in this paper relate to space complexity. We consider three measures: *semantic space*, *resolution space* and *tree space* (all counting clauses to be stored, under different rules). Semantic space was introduced in [2]; a slightly modified definition follows.

Definition 6. *Consider $F \in \mathcal{CLS}$ and $k \in \mathbb{N}$. A* ***semantic k-sequence for*** *F is a sequence $F_1, \dots, F_p \in \mathcal{CLS}$, $p \in \mathbb{N}$, fulfilling the following conditions:*

1. *For all $i \in \{1, \dots, p\}$ holds $c(F_i) \le k$.*
2. *$F_1 = \top$, and for $i \in \{2, \dots, p\}$ either holds $F_{i-1} \models F_i$ (inference), or there is $C \in F$ with $F_i = F_{i-1} \cup \{C\}$ (axiom download).*

A semantic sequence is called ***complete*** *if $F_p \in \mathcal{USAT}$. For $F \in \mathcal{USAT}$ the* ***semantic-space complexity*** *of F, denoted by $\mathbf{css}(\boldsymbol{F}) \in \mathbb{N}$ ("c" for "clause"), is the minimal $k \in \mathbb{N}$ such there is a complete semantic k-sequence for F.*

Different from [2], the elimination of clauses ("memory erasure") is integrated into the inference step, since we want our bound $\mathrm{awid} \le \mathrm{css}$ to be as tight as possible, and the tree-space, a special case of semantic space, shall fulfil $\mathrm{cts} = \mathrm{hd} + 1$. By definition we have $\mathrm{css}(\varphi * F) \le \mathrm{css}(F)$ for $F \in \mathcal{USAT}$ and $\varphi \in \mathcal{PASS}$, and thus $\mathrm{css}(F)$ is naturally defined for all $F \in \mathcal{CLS}$.

We come to the notion of resolution space originating in [39,40] and [53,24]. This measure was intensively studied during the last decade (cf. e.g. [11,48]).

Definition 7. *Consider $F \in \mathcal{CLS}$ and $k \in \mathbb{N}$. A* ***resolution k-sequence for*** *F is a sequence $F_1, \dots, F_p \in \mathcal{CLS}$, $p \in \mathbb{N}$, fulfilling the following conditions:*

1. *For all $i \in \{1, \dots, p\}$ holds $c(F_i) \le k$.*
2. *$F_1 = \top$, and for $i \in \{2, \dots, p\}$ either holds $F_i \setminus F_{i-1} = \{C\}$, where C is a resolvent of two clauses in F_i (removal of clauses and/or addition of one resolvent), or there is $C \in F$ with $F_i = F_{i-1} \cup \{C\}$ (axiom download).*

A resolution k-sequence is ***complete*** *if $\perp \in F_p$. For $F \in \mathcal{USAT}$ the* ***resolution-space complexity*** *of F, denoted by $\mathbf{crs}(\boldsymbol{F}) \in \mathbb{N}$, is the minimal $k \in \mathbb{N}$ such there is a complete resolution k-sequence for F.*

We can also define a variant of space for tree-like resolution refutations.

Definition 8. *A* ***tree k-sequence for*** *F is a resolution k-sequence for F, such that in case of adding an inferred clause via $F_i \setminus F_{i-1} = \{R\}$, for $R = C \diamond D$ with $C, D \in F_{i-1}$, we always have $C, D \notin F_i$. For $F \in \mathcal{USAT}$ the* ***tree-resolution space complexity*** *of F, denoted by $\mathbf{cts}(\boldsymbol{F}) \in \mathbb{N}$, is the minimal $k \in \mathbb{N}$ such there is a complete tree k-sequence for F.*

Both measures $\mathrm{crs}, \mathrm{cts}$ are again not increased by applying partial assignments. By definition we have $\mathrm{css}(F) \le \mathrm{crs}(F) \le \mathrm{cts}(F)$ for $F \in \mathcal{CLS}$. We recall a basic connection between tree space and hardness ([42, Subsection 7.2.1]):

Lemma 9 ([42]). *For $F \in \mathcal{CLS}$ holds $\mathrm{cts}(F) = \mathrm{hd}(F) + 1$.*

We remarked earlier that by definition we have $\mathrm{css}(F) \le \mathrm{crs}(F)$. In fact, the two measures are the same up to a linear factor, as shown by [2]. Our factor is different from [2]; see the underlying report [16] for the proof:

Proposition 10. *For $F \in \mathcal{CLS}$ we have* $\mathrm{crs}(F) \le 3\,\mathrm{css}(F) - 2$.

See the conclusions (Section 5) for further discussions.

4 Combinatorial characterisations

In this section we come to the main topic of this article: the characterisations of the hardness measures introduced in the previous section by Prover-Delayer games and sets of partial assignments.

4.1 Game characterisations for hardness

The game of Pudlák and Impagliazzo [51] is a well-known and classic Prover-Delayer game, which serves as one of the main and conceptually very simple methods to obtain resolution lower bounds for unsatisfiable formulas in CNF. The game proceeds between a Prover and a Delayer. The Delayer claims to know a satisfying assignment for an unsatisfiable clause-set, while the Prover wants to expose his lie and in each round asks for a variable value. The Delayer can either choose to answer this question by setting the variable to 0/1, or can defer the choice to the Prover. In the latter case, Delayer scores one point. This game provides a method for showing lower bounds for tree resolution. Namely, Pudlák and Impagliazzo [51] show that exhibiting a Delayer strategy for a CNF F that scores at least p points against every Prover implies a lower bound of 2^p for the proof size of F in tree resolution. This can now be understood through hardness; by Lemma 9 we know that for unsatisfiable clause-set F holds $\mathrm{cts}(F) = \mathrm{hd}(F)+1$, while in [25] it was shown that the optimal value of the above game plus one equals $\mathrm{cts}(F)$, and thus $\mathrm{hd}(F)$ is the optimal value of that game for F. We remark that thus the game of Pudlák and Impagliazzo does not characterise tree resolution size *precisely*; in [15,13] a modified (asymmetric) version of the game is introduced, which precisely characterises tree resolution size ([14]). We present now the generalised hardness game, also handling satisfiable clause-sets. First we need to determine how hardness is affected when assigning one variable:

Lemma 11. *For clause-sets $F \in \mathcal{CLS}$ and $v \in \mathrm{var}(F)$ either there is $\varepsilon \in \{0,1\}$ with $\mathrm{hd}(\langle v \to \varepsilon\rangle * F) = \mathrm{hd}(F)$ and $\mathrm{hd}(\langle v \to \overline{\varepsilon}\rangle * F) \le \mathrm{hd}(F)$, or we have $\mathrm{hd}(\langle v \to 0\rangle * F) = \mathrm{hd}(\langle v \to 1\rangle * F) = \mathrm{hd}(F) - 1$. If F is unsatisfiable and $\mathrm{hd}(F) > 0$, then there is a variable $v \in \mathrm{var}(F)$ and $\varepsilon \in \{0,1\}$ with $\mathrm{hd}(\langle v \to \varepsilon\rangle * F) < \mathrm{hd}(F)$.*

Proof. The assertion on the existence of v and ε follows by definition. Assume now that neither of the two cases holds, i.e., that there is some $\varepsilon \in \{0,1\}$ such that $\mathrm{hd}(\langle v \to \varepsilon\rangle * F) \le \mathrm{hd}(F) - 1$ and $\mathrm{hd}(\langle v \to \overline{\varepsilon}\rangle * F) \le \mathrm{hd}(F) - 2$. Consider a partial assignment φ such that $\varphi * F \in \mathcal{USAT}$ and $\mathrm{hd}(\varphi * F) = \mathrm{hd}(F)$ (recall Definition 1). Then $v \notin \mathrm{var}(\varphi)$ holds. Now $\mathrm{hd}(\langle v \to \varepsilon\rangle * (\varphi * F)) \le \mathrm{hd}(F) - 1$ and

$\text{hd}(\langle v \to \overline{\varepsilon}\rangle * (\varphi * F)) \le \text{hd}(F) - 2$, so by definition of hardness for unsatisfiable clause-sets we have $\text{hd}(\varphi * F) \le \text{hd}(F) - 1$, a contradiction. ☐

We are ready to present the new game, which characterises $\text{hd}(F)$ for arbitrary F. A feature of this game, not shared by the original game, is that there is just one "atomic action" for both players, the choice of a variable *and* a value, and the rules are just about how this choice can be employed.

Theorem 12. *Consider $F \in \mathcal{CLS}$. The following game is played between Prover and Delayer, where the partial assignments θ all fulfil $\text{var}(\theta) \subseteq \text{var}(F)$:*

1. *The two players play in turns, and Delayer starts. Initially $\theta := \langle\rangle$.*
2. *A move of Delayer extends θ to $\theta' \supseteq \theta$.*
3. *A move of Prover extends θ to $\theta' \supset \theta$ with $\theta' * F = \top$ or $n(\theta') = n(\theta) + 1$.*
4. *The game ends as soon $\bot \in \theta * F$ or $\theta * F = \top$. In the first case Delayer gets as many points as variables have been assigned by Prover. In the second case Delayer gets zero points.*

Now there is a strategy of Delayer which can always achieve $\text{hd}(F)$ many points, while Prover can always avoid that Delayer gets $\text{hd}(F) + 1$ or more points.

Proof. The strategy of Prover is: If $\theta * F$ is satisfiable, then extend θ to a satisfying assignment. Otherwise choose $v \in \text{var}(F)$ and $\varepsilon \in \{0, 1\}$ s.t. $\text{hd}(\langle v \to \varepsilon\rangle * F)$ is minimal. The strategy of Delayer is: Initially extend $\langle\rangle$ to some θ such that $\theta * F \in \mathcal{USAT}$ and $\text{hd}(\theta * F)$ is maximal. For all other moves, and also for the first move as an additional extension, as long as there are variables $v \in \text{var}(\theta * F)$ and $\varepsilon \in \{0, 1\}$ with $\text{hd}(\langle v \to \varepsilon\rangle * (\theta * F)) \le \text{hd}(\theta * F) - 2$, choose such a pair (v, ε) and extend θ to $\theta \cup \langle v \to \overline{\varepsilon}\rangle$. The assertion follows by Lemma 11. ☐

The game of Theorem 12 can be extended to handle asymmetric width (Theorem 23): Delayer in both cases just extends the current partial assignment, while Prover for awid can additionally "forget" assignments.

4.2 Characterising hardness and depth by partial assignments

We now provide an alternative characterisation of hardness of clause-sets F by sets $\mathbb{P}$ of partial assignments, complementing the games. The "harder" F is, the better $\mathbb{P}$ "approximates" satisfying F. The minimum condition is:

Definition 13. *A set $\mathbb{P} \subseteq \mathcal{PASS}$ is **minimal consistent for** $F \in \mathcal{CLS}$ if $\text{var}(\mathbb{P}) = \bigcup_{\varphi \in \mathbb{P}} \text{var}(\varphi) \subseteq \text{var}(F)$, for all $\varphi \in \mathbb{P}$ holds $\bot \notin \varphi * F$, and $\mathbb{P} \neq \emptyset$.*

$\mathbb{P}$ is a partially ordered set (by inclusion). Recall that a *chain* K is a subset constituting a linear order, while the *length* of K is $|K| - 1 \in \mathbb{Z}_{\ge -1}$, and a **maximal chain** is a chain which can not be extended without breaking linearity.

Definition 14. *For $k \in \mathbb{N}_0$ and $F \in \mathcal{USAT}$ let a **weakly k-consistent set of partial assignments for** F be a minimally consistent set $\mathbb{P}$ for F, such that the minimum length of a maximal chain in $\mathbb{P}$ is at least k, and for every non-maximal $\varphi \in \mathbb{P}$, every $v \in \text{var}(F) \setminus \text{var}(\varphi)$ and every $\varepsilon \in \{0, 1\}$ there is $\varphi' \in \mathbb{P}$ with $\varphi \cup \langle v \to \varepsilon\rangle \subseteq \varphi'$.*

There can be gaps between $\varphi \subset \varphi'$ for $\varphi, \varphi' \in \mathbb{P}$, corresponding to the moves of Delayer in Theorem 12, who needs to prevent all "bad" assignments at once.

Proposition 15. *For all $F \in \mathcal{USAT}$ we have $\mathrm{hd}(F) > k$ if and only if there is a weakly k-consistent set for partial assignments for F.*

Proof. If there is a weakly k-consistent $\mathbb{P}$, then Delayer from Theorem 12 has a strategy achieving at last $k+1$ points by choosing a minimal $\theta' \in \mathbb{P}$ extending θ, and maintaining in this way $\theta \in \mathbb{P}$ as long as possible. And a weakly $(\mathrm{hd}(F)-1)$-consistent $\mathbb{P}$ for $\mathrm{hd}(F) > 0$ is given by the partial assignments obtained from those $\varphi \in \mathcal{PASS}$ with $\bot \notin \varphi * F$ by extending φ to $\varphi' := \varphi \cup \langle v \to \varepsilon \rangle$ for such $v \in \mathrm{var}(F) \setminus \mathrm{var}(\varphi)$ and $\varepsilon \in \{0, 1\}$ with $\mathrm{hd}(\varphi' * F) = \mathrm{hd}(\varphi * F)$, and repeating this extension as long as possible. $\square$

A similar characterisation can also be given for the depth-measure $\mathrm{dep}(F)$ (cf. Definition 2). For this we relax the concept of weak consistency:

Definition 16. *For $k \in \mathbb{N}_0$ and $F \in \mathcal{USAT}$ let a **barely k-consistent set of partial assignments for** F be a minimally consistent $\mathbb{P}$ for F such that $\langle\rangle \in \mathbb{P}$, and for every $\varphi \in \mathbb{P}$ with $n(\varphi) < k$ and all $v \in \mathrm{var}(F) \setminus \mathrm{var}(\varphi)$ there is $\varepsilon \in \{0, 1\}$ with $\varphi \cup \langle v \to \varepsilon \rangle \in \mathbb{P}$.*

By [56, Theorem 2.4] we get the following characterisation (we provide a proof due to technical differences):

Proposition 17. *For all $F \in \mathcal{USAT}$ we have $\mathrm{dep}(F) > k$ if and only if there is a barely k-consistent set for partial assignments for F.*

Proof. If F has a resolution proof T of height k, then for a barely k'-consistent $\mathbb{P}$ for F we have $k' < k$, since otherwise starting at the root of T we follow a path given by extending $\langle\rangle$ according to the extension-condition of $\mathbb{P}$, and we arrive at a $\varphi \in \mathbb{P}$ falsifying an axiom of T, contradicting the definition of $\mathbb{P}$. On the other hand, if $\mathrm{dep}(F) > k$, then there is a barely k-consistent $\mathbb{P}$ for F as follows: for $j \in \{0, \dots, k\}$ put those partial assignments $\varphi \in \mathcal{PASS}$ with $\mathrm{var}(\varphi) \subseteq \mathrm{var}(F)$ and $n(\varphi) = j$ into $\mathbb{P}$ which do not falsify any clause derivable by a resolution tree of depth at most $k - j$ from F. Now consider $\varphi \in \mathbb{P}$ with $j := n(\varphi) < k$, together with $v \in \mathrm{var}(F) \setminus \mathrm{var}(\varphi)$. Assume that for both $\varepsilon \in \{0, 1\}$ we have $\varphi \cup \langle v \to \varepsilon \rangle \notin \mathbb{P}$. So there are clauses C, D derivable from T by a resolution tree of depth at most $k - j - 1$, with $v \in C$, $\overline{v} \in D$, and $\varphi * \{C, D\} = \{\bot\}$. But then $\varphi * \{C \diamond D\} = \{\bot\}$, contradicting the defining condition for φ. $\square$

4.3 Characterising symmetric width by partial assignments

We now turn to characterisations of the width-hardness measures, starting with the symmetric width measure wid. It is instructive to review the characterisation for $\mathrm{wid}(F)$ for $F \in \mathcal{USAT}$ from [5], using a different formulation.

Definition 18. *Consider $F \in \mathcal{CLS}$ and $k \in \mathbb{N}_0$. A **symmetrically k-consistent set of partial assignments for** F is a minimally consistent $\mathbb{P}$ for F, such that for all $\varphi \in \mathbb{P}$, all $v \in \mathrm{var}(F) \setminus \mathrm{var}(\varphi)$, and all $\psi \subseteq \varphi$ with $n(\psi) < k$ there exists $\varepsilon \in \{0, 1\}$ and $\varphi' \in \mathbb{P}$ with $\psi \cup \langle v \to \varepsilon \rangle \subseteq \varphi'$.*

Note that a symmetrically k-consistent set is also barely k-consistent. For the (simple) proof of the following lemma see the underlying report [16].

Lemma 19. *Consider $F \in \mathcal{USAT}$ and $k \in \mathbb{N}_0$. Then Duplicator wins the Boolean existential k-pebble game on F in the sense of [5] if and only if there exists a symmetrically k-consistent set of partial assignment for F.*

By [5, Theorem 2] (there "$F \in r\text{–}\mathcal{CLS}$" is superfluous):

Corollary 20. *For $F \in \mathcal{USAT}$ and $k \in \mathbb{N}_0$ holds* $\mathrm{wid}(F) > k$ *if and only if there exists a symmetrically k-consistent set of partial assignments for F.*

4.4 Characterising asymmetric width by partial assignments

Similar to Definition 18, we characterise asymmetric width — the only difference is, that the extensions must work for both truth values.

Definition 21. *Consider $F \in \mathcal{CLS}$ and $k \in \mathbb{N}_0$. A **k-consistent set of partial assignments for** F is a minimally consistent $\mathbb{P}$ for F, such that for all $\varphi \in \mathbb{P}$, all $v \in \mathrm{var}(F) \setminus \mathrm{var}(\varphi)$, all $\psi \subseteq \varphi$ with $n(\psi) < k$ and for both $\varepsilon \in \{0,1\}$ there is $\varphi' \in \mathbb{P}$ with $\psi \cup \langle v \to \varepsilon \rangle \subseteq \varphi'$.*

Similarly to [5, Theorem 2], where the authors provide a characterisation of symmetric width, we obtain a characterisation of asymmetric width:

Theorem 22. *For $F \in \mathcal{USAT}$ and $k \in \mathbb{N}_0$ holds* $\mathrm{awid}(F) > k$ *if and only if there exists a k-consistent set of partial assignments for F.*

Proof. First assume $\mathrm{awid}(F) > k$. Let $F' := \{C \in \mathcal{CL} \mid \exists\, R : F \vdash^k C\}$. Note that by definition $F \subseteq F'$, while by assumption we have $\bot \notin F'$. Let $\mathbb{P}$ be the set of maximal $\varphi \in \mathcal{PASS}$ with $\mathrm{var}(\varphi) \subseteq \mathrm{var}(F)$ and $\bot \notin \varphi * F'$. We show that $\mathbb{P}$ is a k-consistent set of partial assignments for F. Consider $\varphi \in \mathbb{P}$, $v \in \mathrm{var}(F) \setminus \mathrm{var}(\varphi)$ and $\psi \subseteq \varphi$ with $n(\psi) < k$. Due to maximality of φ there are $C, D \in F'$ with $v \in C$, $\overline{v} \in D$ and $(\varphi \cup \langle v \to 0 \rangle) * \{C\} = (\varphi \cup \langle v \to 1 \rangle) * \{D\} = \{\bot\}$. Assume that there is $\varepsilon \in \{0,1\}$, such that for $\psi' := \psi \cup \langle v \to \varepsilon \rangle$ there is no $\varphi' \in \mathbb{P}$ with $\psi' \subseteq \varphi'$. Thus there is $E \in F'$ with $\psi' * \{E\} = \{\bot\}$; so we have $v \in \mathrm{var}(E)$ and $|E| \le k$. Now E is resolvable with either C or D via k-resolution, and for the resolvent $R \in F'$ we have $\varphi * \{R\} = \{\bot\}$, contradicting the definition of $\mathbb{P}$.

For the other direction, assume that $\mathbb{P}$ is a k-consistent set of partial assignments for F. For the sake of contradiction assume there is $T : F \vdash^k \bot$. We show by induction on $\mathrm{ht}_T(w)$ that for all $w \in \mathrm{nds}(T)$ and all $\varphi \in \mathbb{P}$ holds $\varphi * \{C(w)\} \neq \{\bot\}$, which at the root of T (where the clause-label is $\bot$) yields a contradiction. If $\mathrm{ht}_T(w) = 0$ (i.e., w is a leaf), then the assertion follows by definition; so assume $\mathrm{ht}_T(w) > 0$. Let w_1, w_2 be the two children of w, and let $C := C(w)$ and $C_i := C(w_i)$ for $i \in \{1,2\}$. W.l.o.g. $|C_1| \le k$. Note $C = C_1 \diamond C_2$; let v be the resolution variable, where w.l.o.g. $v \in C_1$. Consider $\varphi \in \mathbb{P}$; we have to show $\varphi * \{C\} \neq \{\bot\}$, and so assume $\varphi * \{C\} = \{\bot\}$. By induction hypothesis we know $\bot \notin \varphi * \{C_1, C_2\}$, and thus $v \notin \mathrm{var}(\varphi)$. Let $\psi := \varphi \,|(\mathrm{var}(C_1) \setminus \{v\})$, and $\psi' := \psi \cup \langle v \to 0 \rangle$. There is $\varphi' \in \mathbb{P}$ with $\psi' \subseteq \varphi'$, thus $\psi' * \{C_1\} = \{\bot\}$, contradicting the induction hypothesis. $\square$

4.5 Game characterisation for asymmetric width

The characterisation of asymmetric width by partial assignments from the previous subsection will now be employed for a game-theoretic characterisation; in fact, the k-consistent sets of partial assignments will directly translate into strategies for Delayer (while a strategy of Prover is given by a resolution refutation). We only handle the unsatisfiable case here — the general case can be handled as in Theorem 12. This game extends (in a sense) the Prover-Delayer game from [54] for symmetric width (but again without communication).

Theorem 23. *Consider $F \in \mathcal{USAT}$. The following game is played between Prover and Delayer (as in Theorem 12, always $\mathrm{var}(\theta) \subseteq \mathrm{var}(F)$ holds):*

1. *The two players play in turns, and Delayer starts. Initially $\theta := \langle\rangle$.*
2. *Delayer extends θ to $\theta' \supseteq \theta$.*
3. *Prover chooses some θ' compatible with θ such that $|\mathrm{var}(\theta') \setminus \mathrm{var}(\theta)| = 1$.*
4. *If $\bot \in \theta * F$, then the game ends, and Delayer gets the maximum of $n(\theta')$ chosen by Prover as points (0 if Prover didn't make a choice).*
5. *Prover must play in such a way that the game is finite.*

We have the following:

1. *For a strategy of Delayer, which achieves $k \in \mathbb{N}$ points whatever Prover does, we have $\mathrm{awid}(F) \geq k$.*
2. *For a strategy of Prover, which guarantees that Delayer gets at most $k \in \mathbb{N}_0$ points in any case, we have $\mathrm{awid}(F) \leq k$.*
3. *There is a strategy of Delayer which guarantees at least $\mathrm{awid}(F)$ many points (whatever Prover does).*
4. *There is a strategy of Prover which guarantees at most $\mathrm{awid}(F)$ many points for Delayer (whatever Delayer does).*

Proof. W.l.o.g. $\bot \notin F$. Part 1 follows by Part 4 (if $\mathrm{awid}(F) < k$, then Prover could guarantee at most $k-1$ points), and Part 2 follows by Part 3 (if $\mathrm{awid}(F) > k$, then Delayer could guarantee at least $k+1$ points).

Let now $k := \mathrm{awid}(F)$. For Part 3, a strategy of Delayer guaranteeing k many points (at least) is as follows: Delayer chooses a $(k-1)$-consistent set $\mathbb{P}$ of partial assignment (by Theorem 22). The move of Delayer is to choose some $\theta' \in \mathbb{P}$. If Prover then chooses some θ' with $n(\theta') \leq k-1$, then the possibility of extension is maintained for Delayer. In this way the empty clause is never created. Otherwise the Delayer has reached his goal, and might choose anything.

It remains to show that Prover can force the creation of the empty clause such that Delayer obtains at most k many points. For that consider a resolution refutation $R : F \vdash \bot$ which is a k-resolution tree. The strategy of Prover is to construct partial assignments θ' (from θ as given by Delayer) which falsify some clause C of length at most k in R, where the height of the node is decreasing — this will falsify finally some clause in F, finishing the game. The Prover considers initially (before the first move of Delayer) just the root. When Prover is to move, he considers a path from the current clause to some leaf, such that only clauses of length at most k are on that path. There must be a first clause C (starting from the falsified clause, towards the leaves) on that path not falsified by θ (since θ does not falsify any axiom). It must be the case that θ falsifies all literals in C besides one literal $x \in C$, where $\mathrm{var}(x) \notin \mathrm{var}(\theta)$. Now Prover chooses θ' as the restriction of θ to $\mathrm{var}(C) \setminus \{\mathrm{var}(x)\}$ and extended by $x \to 0$. □

We already remarked in Section 3 that always $\mathrm{awid}(F) \le \mathrm{hd}(F)$. Based on the game characterisations shown here, we provide an easy alternative proof for this fundamental fact for $F \in \mathcal{USAT}$:

Lemma 24. *Consider the game of Theorem 23, when restricted in such a way that Prover must always choose some θ' with $n(\theta') > n(\theta)$. This game is precisely the game of Theorem 12.*

Corollary 25. *For all $F \in \mathcal{CLS}$ we have* $\mathrm{awid}(F) \le \mathrm{hd}(F)$.

4.6 Width hardness versus semantic space

We have already seen in Corollary 25, that our game-theoretic characterisations allow quite easy and elegant proofs on tight relations between different hardness measures. Our next result also follows this paradigm. It provides a striking relation between asymmetric width and semantic space. We recall that Atserias and Dalmau [5, Theorem 3] have shown $\mathrm{wid}(F) \le \mathrm{crs}(F) + r - 1$, where $F \in \mathcal{USAT} \cap r\text{–}\mathcal{CLS}$ (all $r \ge 0$ are allowed; note that now we can drop the unsatisfiability condition). We generalise this result in Theorem 27 below, replacing resolution space $\mathrm{crs}(F)$ by the tighter notion of semantic space $\mathrm{css}(F)$. More important, we eliminate the additional $r-1$ in the inequality, by changing symmetric width $\mathrm{wid}(F)$ into asymmetric width $\mathrm{awid}(F)$ (cf. Lemma 5 for the relation between these two measures). First a lemma similar to [5, Lemma 5]:

Lemma 26. *Consider $F \in \mathcal{CLS}$, a k-consistent set $\mathbb{P}$ of partial assignments for F ($k \in \mathbb{N}_0$), and a semantic k-sequence $(F_1, \dots, F_p)$ for F (recall Definition 6). Then there exist $\varphi_i \in \mathbb{P}$ with $\varphi_i * F_i = \top$ for each $i \in \{1, \dots, p\}$.*

Proof. Set $\varphi_1 := \langle\rangle \in \mathbb{P}$. For $i \in \{2, \dots, p\}$ the partial assignment φ_i is defined inductively. If $\varphi_{i-1} * F_i = \top$, then $\varphi_i := \varphi_{i-1}$; this covers the case where F_i is obtained from F_{i-1} by addition of inferred clauses and/or removal of clauses. So consider $F_i = F_{i-1} \cup \{C\}$ for $C \in F \setminus F_{i-1}$ (thus $c(F_i) < k$), and we assume $\varphi_{i-1} * F_i \ne \top$. So there is a literal $x \in C$ with $\mathrm{var}(x) \notin \varphi_{i-1}$, since φ_{i-1} does not falsify clauses from F. Choose some $\psi \subseteq \varphi_{i-1}$ with $n(\psi) \le c(F_{i-1})$ such that $\psi * F_{i-1} = \top$.[6)] There is $\varphi_i \in \mathbb{P}$ with $\psi \cup \langle x \to 1\rangle \subseteq \varphi_i$, whence $\varphi_i * F_i = \top$. □

We can now show the promised generalisation of [5, Theorem 3]:

Theorem 27. *For $F \in \mathcal{CLS}$ holds* $\mathrm{awid}(F) \le \mathrm{css}(F)$.

6) For every partial assignment φ and every clause-set F with $\varphi * F = \top$ there exists $\psi \subseteq \varphi$ with $n(\psi) \subseteq c(F)$ and $\psi * F = \top$; see for example Lemma 4 in [5], and see Corollary 8.6 in [47] for a generalisation.

Proof. Assume $F \in \mathcal{USAT}$ and $\mathrm{awid}(F) > \mathrm{css}(F)$; let $k := \mathrm{css}(F)$. By Theorem 22 there is a k-consistent set $\mathbb{P}$ for F. Let $(F_1, \dots, F_p)$ be a complete semantic k-sequence for F according to Definition 6. Now for the sequence $(\varphi_1, \dots, \varphi_p)$ according to Lemma 26 we have $\varphi_p * F_p = \top$, contradicting $F_p \in \mathcal{USAT}$. □

We are now in a position to order most of the hardness measures that we investigated here (cf. also the diagram in the introduction):

Corollary 28. $\mathrm{awid}(F) \leq \mathrm{css}(F) \leq \mathrm{crs}(F) \leq \mathrm{cts}(F) = \mathrm{hd}(F) + 1$ *for* $F \in \mathcal{CLS}$.

We conclude by an application of the extended measures $\mathrm{cts}, \mathrm{css} : \mathcal{CLS} \to \mathbb{N}_0$. In [30] it is shown that for every k there are clause-sets in $\mathcal{UC}_{k+1}$ where every (logically) equivalent clause-set in $\mathcal{WC}_k$ is exponentially bigger. This implies, in the language of representing boolean functions via CNFs, that allowing the tree-space to increase by 2 over semantic space allows for an exponential saving in size (regarding logical equivalence):

Corollary 29. *For all constant* $k \in \mathbb{N}$ *there are sequences* (F_n) *of clause-sets with* $\mathrm{cts}(F_n) \leq k+2$ *for all* n*, where all equivalent sequences* (F'_n) *with* $\mathrm{css}(F'_n) \leq k$ *(for all* n*) are exponentially bigger.*

5 Conclusion and open problems

In this paper we aimed at *unified characterisations* for the main hardness measures for resolution, thereby obtaining precise relations between these measures. Continuing this programme, a deeper understanding of the three space measures is required. In terms of the game-theoretic characterisations, the main question left open is whether $\mathrm{crs}, \mathrm{css} : \mathcal{CLS} \to \mathbb{N}$ can be characterised in a similar spirit by games and/or partial assignments (for cts we provided such characterisations).

A further block of questions concerns the *exact relationship between the measures*. We believe that Theorem 27 can be improved:

Conjecture 30. $\mathrm{awid}(F) + 1 \leq \mathrm{css}(F)$ for $F \in \mathcal{CLS}$.

Then in Corollary 29 the "+2" could be replaced by "+1". Note that Corollary 29 shows that such small measurement differences actually matter! Concerning the space measures, it is conceivable that $\mathrm{crs} = \mathrm{css}$ could hold; if not then there could be substantial differences between crs and css regarding expressive power, that is, regarding the power to represent boolean functions. Concerning the precise relation between symmetric and asymmetric width, it appears that Lemma 5 could be improved to $\mathrm{wid}(F) \leq \mathrm{awid}(F) + p - 1$ for $F \in p\text{–}\mathcal{CLS}$, $p \geq 1$ (then Theorem 27 would precisely imply [5, Theorem 3]).

The question on the *expressive power* of the various classes (measures) seems very relevant, and can also be raised when allowing new variables for the representation of boolean functions; see [30] for a thorough discussion of these issues.

Also for practical SAT solving the influence of *blocked clauses elimination/addition* (introduced in [43,44], a generalisation of Extended Resolution) on hardness measures needs to be studied (see [34,35] for recent developments).

Finally, a general theory of *hardness measures* might be possible (which might also be applicable to other proof systems than resolution).

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