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Abstract—The paper is addressed at the problem of developing a static and dynamic model of a single flexible manipulator link which is of sufficient accuracy for use in a multi-flexible-link system. Although flexible link modelling has received much attention in the past, none of the models developed have adequate accuracy for application in a multi-link system. Very high modelling accuracy is necessary because there is significant inter-link coupling in a flexible manipulator and any modelling errors are therefore cumulative. Previous work based on an assumed mode model has made some progress towards improving accuracy by including a correction factor derived from finite element analysis, and the work reported here extends this by including a shear deformation term in the equations. The significant improvements in modelling accuracy thereby achieved are demonstrated by simulations of link motion.

1.INTRODUCTION

Conventional industrial robots have large-mass, rigid links which cause serious control difficulties when they move at anything other than a slow speed because of the large inertia forces which are induced by motion. In response to the need in many production situations for faster robot motion, there is currently much research in progress looking into developing much lighter and hence lower-inertia robot arms. However, a consequence of making the links lighter is that a degree of flexibility is introduced into them. Thus the control problem is not eliminated but only changed: instead of compensating for inertia forces, the controller has to compensate for link flexure and the tendency to oscillate.

In order to analyse and design a controller for a manipulator system containing flexible links, an accurate model of the system is necessary. The fundamental equations of motion for each flexible link in the system and the interactions between links must be derived. However, this task is not as daunting as it may seem because only two links normally contribute significantly to the inertia problem in most industrial robots. Thus a lightweight robot only needs in fact to have two flexible links. Hence, in order to model such a lightweight manipulator, it is sufficient to model a two-flexible-link system and then to subsequently superimpose onto this the motion of the other rigid links in the manipulator.

The starting point for modelling such a two-link system is to develop a model of a single link and then to consider the interactions of two such links connected together. Much work on modelling single flexible links has previously been reported. However, most of this has been concerned only with vibrations rather than with rotational motion of the link about its actuated joint. Even where rotational motion has been modelled, the models developed have insufficient accuracy for use in a two-link system, where the flexure and oscillations of the first link are transmitted to the second link and vice-versa so that modelling errors are cumulative. Because of this inter-link coupling, the accurate modelling of the end-tip slope and velocity as well as its position is required. This paper therefore describes work to produce a single link model of sufficient accuracy which is suitable for inclusion in a model of a two-flexible-link system. The inclusion of a term describing the shear deformation effect in the link will be shown to be an essential element in this.

2.BASIC STATIC MODEL

The assumed mode method (AMM) is a computationally efficient scheme which serves as a useful starting point in formulating a flexible link model. Assuming the magnitude of flexure to be low, the slope and static deflection of a flexible beam bending under gravity are described by:

$$\frac{du_m}{dx} = -\frac{mg}{2EIl}\left(l^2x - lx^2 + \frac{x^3}{3}\right) \;; \qquad u_m = -\frac{mg}{2EIl}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right) \tag{1}$$

where m is the mass of the beam, l is the length of the beam, EI is the flexural stiffness of the beam, g is the gravity vector, x is the position on the beam of the point where the slope and deflection is measured and the subscript m denotes the slope or deflection resulting from the mass of the beam.

For a flexible link with an end-tip load m_t (Figure 1), the mass m_t produces a negative slope and deflection given by:

$$\frac{du_{m_t}}{dx} = -\frac{m_tg}{EI}\left(lx - \frac{x^2}{2}\right) \;; \qquad u_{m_t} = -\frac{m_tg}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) \tag{2}$$

By the principle of superposition the total static slope and deflection for a flexible link are given by:

$$\frac{du}{dx} = \frac{du_m}{dx} + \frac{du_{m_t}}{dx} = -\frac{g}{2EIl}\left((m+2m_t)l^2x - (m+m_t)lx^2 + \frac{mx^3}{3}\right)$$

$$u = u_m + u_{m_t} = -\frac{g}{2EIl}\left((m+2m_t)\frac{l^2x^2}{2} - (m+m_t)\frac{lx^3}{3} + \frac{mx^4}{12}\right) \tag{3}$$

The maximum static slope and deflection of the flexible link occur at the free end, where $x = l$, i.e.

$$\frac{du_{max}}{dx} = -\frac{l^2g}{2EI}\left(\frac{m}{3} + m_t\right) \;; \qquad u_{max} = -\frac{l^3g}{2EI}\left(\frac{m}{4} + \frac{2m_t}{3}\right) \tag{4}$$

The deflection of the link end-tip is calculated in the above equations on the assumption that the end-tip moves vertically downwards instead of in a circular arc. This is clearly only valid if the magnitude of flexure is low. This condition is unlikely to be satisfied in typical industrial flexible manipulator links, and modification of the equations is therefore necessary.

Previous work [1] has shown that the case of large magnitude flexure can be handled by adding a correction factor to the basic equations. This is calculated by considering the link as a body composed of n equal sections and applying finite element analysis.

The corrected coordinates of the end-tip are then given by [1]:

$$x_e = l - s \qquad \text{and} \qquad y_e = u(l-s) \tag{5}$$

where:

$$s = \sum_{i=1}^{n-1} w_i \tag{6}$$

$$v_n = L - l/n \;; \qquad w_n = \frac{v_n l}{nL} \;; \qquad L = \sqrt{\left(u(l-s) - u((n-1)l/n - s)\right)^2 + (l/n)^2} \tag{7}$$

$$u(l - s - w_n) = \frac{w_n}{v_n}\, u(l-s) \tag{8}$$

3.DYNAMIC MODELLING

The equation of motion of an undamped flexible link without payload is described by [2] and [3]:

$$\rho\frac{\partial^2 u(x,t)}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u(x,t)}{\partial x^2}\right) \; ; \qquad u(x,t) = \phi(x)q(t) \tag{9}$$

where ρ is the mass per unit length of the link, $u(x,t)$ is the deflection of the link, $\phi(x)$ is the assumed mode shape function and $q(t)$ is the modal function. Assuming that EI is a constant, allows Eq. (9) to be written as

$$\frac{1}{q(t)}\frac{d^2q(t)}{dt^2} = -\frac{EI}{\rho}\frac{1}{\phi(x)}\frac{d^4\phi(x)}{dx^4} \tag{10}$$

which leads to the two following differential equations:

$$\frac{d^4\phi(x)}{dx^4} - \beta^4\phi(x) = 0 \; ; \qquad \frac{d^2q(t)}{dt^2} + \omega^2 q(t) = 0 \tag{11}$$

where w is a constant and $\beta^4 = \rho\omega^2/EI$.
The solution as found in [2] and [3] is:

$$\phi_i(x) = C_i(\cos\beta_i x - \cosh\beta_i x) + (\sin\beta_i x - \sinh\beta_i x) \tag{12}$$

and

$$q_i(t) = A_i\cos\omega_i t + B_i\sin\omega_i t \tag{13}$$

where A_i, B_i, C_i and ω_i are constants, i denotes the number of modes of vibration. The deflection is then given by

$$u(x,t) = \sum_{i=1}^{\infty}\phi_i(x)q_i(t) \tag{14}$$

From the boundary conditions $(u(0,t) = u(l,t) = \frac{\partial u}{\partial x}(0,t) = \frac{\partial^2 u}{\partial x^2}(l,t) = 0)$ we obtain

$$C_i = \frac{\cos\ \beta_i l + \cosh\ \beta_i l}{\sin\ \beta_i l - \sinh\ \beta_i l} \tag{15}$$

and β_i as a solution to

$$\cos\ \beta_i l\ \cosh\ \beta_i l = -1 \tag{16}$$

Solving Eq. (16) for the first four modes gives $\beta_1 l = 1.875$, $\beta_2 l = 4.694$, $\beta_3 l = 7.854$ and $\beta_4 l = 10.995$. From here, using the definition that $\beta_i^4 = \rho\omega_i^2/EI$, we can deduce the values of the natural frequencies ω_i of the flexible link for the first four modes. This means that, given an initial excitation F, the link is going to oscillate according to a combination of these four natural frequencies.

The equation of motion can be generalised as an eigenvalue problem linking the two parts of the system (the assumed mode shape functions $\phi_i(x)$ and the modal functions $q_i(t)$). Subsequent analysis [1] taking into account the first three modes (i =1,2 and 3) leads to the following equations for the vertical displacement $u(x,t)$ of any point x on the link at any time t, the slope $u'(x,t)$ of the link at any point x and any time t and the velocity $\dot{u}(x,t)$ of any point x on the link at any time t:

$$u(x,t) = \phi_1(x)q_1(0)\cos\omega_1 t + \phi_2(x)q_2(0)\cos\omega_2 t + \phi_3(x)q_3(0)\cos\omega_3 t \tag{17}$$

$$u'(x,t) = \frac{\partial u(x,t)}{\partial x} = \phi_1'(x)q_1(0)\cos\omega_1 t + \phi_2'(x)q_2(0)\cos\omega_2 t + \phi_3'(x)q_3(0)\cos\omega_3 t \tag{18}$$

$$\dot{u}(x,t) = \frac{\partial u(x,t)}{\partial t} = -\phi_1(x)q_1(0)\omega_1\sin\omega_1 t - \phi_2(x)q_2(0)\omega_2\sin\omega_2 t - \phi_3(x)q_3(0)\omega_3\sin\omega_3 t \tag{19}$$

subject to the initial conditions:

$$u(x,0) = \sum_{i=1}^{\infty}\phi_i(x)q_i(0) = f(x) \; ; \qquad \dot{u}(x,0) = \sum_{i=1}^{\infty}\phi_i(x)\dot{q}_i(0) = g(x) \tag{20}$$

Using the orthogonal relation, the corresponding initial conditions in the *normal coordinates* (the normalisation or weighting is operated on all modes) are

$$q_i(0) = \frac{\rho l}{m_{ii}} \int_0^l f(x)\phi_i(x)dx \; ; \qquad \dot{q}_i(0) = \frac{\rho l}{m_{ii}} \int_0^l g(x)\phi_i(x)dx \tag{21}$$

Similarly, $q_i(0)$ and $\dot{q}_i(0)$ can be obtained from the normalised flexural stiffness as

$$q_i(0) = \frac{EI}{k_{ii}} \int_0^l f''(x)\phi_i''(x)dx \; ; \qquad \dot{q}_i(0) = \frac{EI}{k_{ii}} \int_0^l g''(x)\phi_i''(x)dx \tag{22}$$

When an end-tip load is added to the link, an extra eigenvalue will appear in the boundary conditions and it can be shown [1] that the effect is to cause the link to vibrate at a slower frequency and for vibrations to persist for a longer period of time.

4.INCLUSION OF SHEAR DEFORMATION EFFECT

The assumed mode method neglects shear deformation effects and calculates link deformation on the assumption that this is due only to the bending moment created by the mass and end-tip load of the link. This assumption appears to have been made in all flexible link models previously reported. However, a shear force also exists which acts in the opposite direction to the bending moment and produces motion which the later analysis in the paper will show to be significant. Thus, in the interests of accurate modelling, the shear deformation effect must be included in both static and dynamic models of a flexible link.

It is known that the shear force of flexible arms depends on the shape of the cross-section of the arm. Therefore, a physical quantity called the *numerical factor*, representing the geometric characteristics of the link cross-section, is required in the dynamic formulation of the manipulator.

Referring to [4], the numerical factor of a flexible beam is defined as

$$K = \frac{AQ}{I_a d} \tag{23}$$

where, I_a is the moment of inertia of the cross-sectional shape of the link computed with respect to its neutral axis, Q denotes the first moment about the neutral axis of the area contained between an edge of the cross-section of the beam parallel to the main axis and the surface at which the shear stress is to be computed, A is the cross-sectional area and d is the width of the cross-sectional area at which the shear deformation is required. For a uniform link of square cross section, the factor K is given by:

$$K = \frac{AQ}{I_a d} = \frac{d^2(\rho l)l^2/2}{(\rho l)l^2 d/3} = \frac{3d}{2} \tag{24}$$

An element dx of the flexible link is deformed by the shear force V and the bending moment M shown in Figure 2. When the shear force is zero, the centre line of dx is normal to the face of the cross-section. If $\partial u(x,t)/\partial x$ is the slope due to the bending moment M, neglecting the interaction between the shear and the moment, the shear force will cause a rectangular element to become a parallelogram without a rotation of the faces.

Thus, the slope of the deflection curve is decreased by the shear angle as formulated in the following equation:

$$\frac{\partial u_c(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial x} - \frac{V}{KAG} \tag{25}$$

where V is the value of the shear force and is equal to $EI\rho$, G the shear modulus of the material the link is made of and $\partial u_c/\partial x$ the total slope cause by both shear and moment.