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An algebraic model for free rational G -spectra

J. P. C. Greenlees and B. Shipley

ABSTRACT

We show that for any compact Lie group G with identity component N and component group $W = G/N$, the category of free rational G -spectra is equivalent to the category of torsion modules over the twisted group ring $H^*(BN)[W]$. This gives an algebraic classification of rational G -equivariant cohomology theories on free G -spaces and a practical method for calculating the groups of natural transformations between them.

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1. Introduction

1.1. Context

In algebraic topology, one of the basic invariants of G -spaces is an equivariant cohomology theory, $E_G^*(\cdot)$ (one thinks first of various types of equivariant K -theory and equivariant cobordism, but also of Bredon and Borel cohomology). Such a cohomology theory is, in particular, a contravariant functor to an abelian category satisfying the Eilenberg–Steenrod and Milnor axioms (which is to say it is homotopy invariant, is an exact functor, has a Mayer–Vietoris sequence, and takes sums to products). In general, we impose restrictions on behaviour under suspensions, but to simplify the situation somewhat, we restrict the cohomology theory to free G -spaces, which makes the other restrictions unnecessary. A formal stabilization process constructs a category (‘free G -spectra’) in which such cohomology theories are represented by an object E in the sense that, for any based G -space X , we have $E_G^*(X) = [X, E]_G^*$. The category of spectra has the advantage of a much richer structure, and in particular one can do homotopy theory in it. The spectra E correspond to cohomology theories $E_G^*(\cdot)$ and spaces of natural transformations are also calculated as homotopy classes of maps: $\text{Nat}(E_G^*(\cdot), F_G^*(\cdot)) = [E, F]_*^G$.

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However, the category of free G -spectra remains almost as complicated as the category of free G -spaces, so one does not expect to have a complete global understanding in this generality. To simplify things further we assume that the cohomology theories take values in rational vector spaces, and these are correspondingly represented by free rational G -spectra. We prove, in this paper, that this category is simple enough to have a purely algebraic model. Some special cases are relatively elementary. By Serre's early work, if G is trivial, then rational cohomology theories correspond to graded rational vector spaces, and if G is finite, then rational cohomology theories on free G -spaces correspond to graded $\mathbb{Q}G$ -modules, but for infinite compact Lie groups, the situation is more complicated.

1.2. Results

We have previously given a small and concrete model of free rational G -spectra when G is a connected compact Lie group [2]. The main result of the present paper extends this to general compact Lie groups, but perhaps more interesting is the new method (essentially that of [3]), which involves fewer equivalences and better respects multiplicative structures. Furthermore, some readers may find it helpful to see the method of [3] implemented in the present simple context.

The case of free G -spectra has the attraction that it is rather easy to describe both the homotopy category of free G -spectra and also the algebraic model. The homotopy category coincides with the category of rational cohomology theories on free G -spaces; better still, on free G -spaces an equivariant cohomology theory is the same as one in the naive sense (that is, a contravariant functor satisfying the Eilenberg–Steenrod axioms and the wedge axiom).

To describe the algebraic model, we suppose that G has identity component N and component group $W = G/N$. Note that W acts on the polynomial ring $H^*(BN)$ by ring isomorphisms, and we write $H^*(\tilde{B}N)$ to advertise the W action. We may then form the twisted group ring $H^*(\tilde{B}N)[W]$. A module over this ring is said to be a torsion module if it is torsion as a module over the polynomial ring $H^*(\tilde{B}N)$. The algebraic model consists of differential graded torsion modules over $H^*(\tilde{B}N)[W]$.

THEOREM 1.1. *For any compact Lie group G , with identity component N and component group $W = G/N$, there is a Quillen equivalence*

$$\text{free-}G\text{-spectra}/\mathbb{Q} \simeq \text{tors-}H^*(\tilde{B}N)[W]\text{-mod}$$

of model categories. In particular, their derived categories are equivalent

$$\mathbb{Q}\text{-cohomology-theories-on-free-}G\text{-spaces} = Ho(\text{free-}G\text{-spectra}/\mathbb{Q}) \simeq D(\text{tors-}H^*(\tilde{B}N)[W]\text{-mod})$$

as triangulated categories.

Note that the algebraic model does not detect the fact that the extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow W \longrightarrow 1$$

need not be split. For example, both free $O(2)$ -spectra and free $\text{Pin}(2)$ -spectra are equivalent to torsion modules over the twisted polynomial ring $\mathbb{Q}[c][W]$, where W is a group of order 2 acting to negate c . This should not be surprising, since the 2:1 map $\text{Pin}(2) \longrightarrow O(2)$ induces a rational equivalence on categories of free spectra.

1.3. Conventions

Certain conventions are in force throughout the paper. The most important is that *everything is rational*: henceforth all spectra and homology theories are rationalized without comment.

For example, the category of free rational G -spectra will now be denoted ‘free- G -spectra’. We also use the standard conventions that ‘DG’ abbreviates ‘differential graded’. We focus on homological (lower) degrees, with differentials reducing degrees; for clarity, cohomological (upper) degrees are called *codegrees* and are converted to degrees by negation in the usual way. Finally, we write $H^*(X)$ for the unreduced cohomology of a space X with rational coefficients.

2. The proof

2.1. Organization of the paper

Most of the rest of the paper is devoted to establishing the following sequence of Quillen equivalences, several of which are themselves zig-zags. We will repeatedly use cellularization (right localization) to focus attention on the free part of the model, and the Cellularization Principle of [5] (quoted in the relevant special case in Appendix). The cellularizations are all with respect to the images of the cell G_+ ; using the Adams spectral sequence of Section 9, these are recognized by their homotopy, and hence behave as expected under the functors between the different models below

$$\begin{aligned} \text{free-}G\text{-spectra} &\stackrel{(1)}{\simeq} \text{cell-}DEG_+\text{-mod-}G\text{-spectra} \stackrel{(2)}{\simeq} \text{cell-}D\tilde{B}N_+\text{-mod-}W\text{-spectra} \\ &\stackrel{(3)}{\simeq} \text{cell-}C^*(\tilde{B}N)[W]\text{-mod} \stackrel{(4)}{\simeq} \text{cell-}H^*(\tilde{B}N)[W]\text{-mod} \\ &\stackrel{(5)}{\simeq} \text{tors-}H^*(\tilde{B}N)[W]\text{-mod}. \end{aligned}$$

To start with, we use the G_+ -cellularization of spectra as a model for free G -spectra. Now, Equivalence (1) is obtained from the change of rings adjunction arising from the map $\mathbb{S} \rightarrow DEG_+$ of ring G -spectra (where $DEG_+ = F(EG_+, \mathbb{S})$ is the functional dual of EG_+) by cellularizing with respect to G_+ . This is described in Section 3.

Equivalence (2) reduces to a finite group of equivariance. It is obtained by passage to N -fixed points; $D\tilde{B}N_+$ is the W -spectrum $(DEG_+)^N$, with the tilde included as a reminder that W is acting. This is a form of Eilenberg–Moore equivalence and is discussed in Sections 4 and 5 (see [4] for a more general discussion).

Equivalence (3) is the big step from topology to algebra and this step is described in Section 6 calling on the results of [11]. In this case, the ring W -spectrum $D\tilde{B}N_+$ is an algebra over the rational Eilenberg–MacLane spectrum, and hence equivalent to a $\mathbb{Q}W$ -algebra, which we call $C^*(\tilde{B}N)$ because it is a DGA with cohomology $H^*(\tilde{B}N)$. Since we are working over the rationals, this may be taken to be commutative. A $C^*(\tilde{B}N)$ -module in $\mathbb{Q}W$ -modules is the same as a module over the twisted group ring $C^*(\tilde{B}N)[W]$, and we will use language from the latter point of view.

Equivalence (4) moves from a differential graded algebra to an ordinary graded ring by a little formality argument described in Section 7. It is basically the usual argument that commutative polynomial rings are intrinsically formal, but a little care is needed to deal with the representations.

Equivalence (5) is a change of model which means that cellularization at the model category level is replaced by the use of a more economical underlying category. This is described in Section 8.

2.2. Relationship to other results

We should comment on the relationship between the strategy implemented here and that used for free spectra in [2]. Both strategies start with a category of G -spectra and end with the same purely algebraic category, and the connection in both relies on finding an intermediate category

that is visibly rigid in the sense that it is determined by its homotopy category (the archetype of this is the category of modules over a commutative DGA with polynomial cohomology). With some additional effort, the methods of [2] can be applied to prove the main equivalence of Theorem 1.1.

The difference comes in the route taken. Roughly speaking, the strategy in [2] is to move to non-equivariant spectra as soon as possible, whereas that adopted here is to keep working in the ambient category of G -spectra for as long as possible. The present method appears to have several advantages. It uses fewer steps, and (although we do not pursue it here) the monoidal structures are visible throughout.

We present the argument as briefly as possible, so as to highlight the line of argument. The technical ingredients can be found in [4, 5] (a more condensed account all in one place can be found in [3]).

3. Modules over DEG_+

To start we need a model for free G -spectra; there are several Quillen equivalent alternatives (see [2, Section 3] for discussion). For definiteness, we start with orthogonal G -spectra [10], and use the G_+ -cellularization of the category of \mathbb{S} -module G -spectra, where \mathbb{S} is a strictly commutative model for the rationalized sphere spectrum. Next, we explain that since EG is a G -space, we have a strictly co-commutative diagonal $EG \rightarrow EG \times EG$ and hence the functional dual $DEG_+ = F(EG_+, \mathbb{S})$ becomes a commutative ring G -spectrum. The completion map

$$\mathbb{S} = F(\mathbb{S}, \mathbb{S}) \rightarrow F(EG_+, \mathbb{S}) = DEG_+,$$

then gives a map of ring spectra. Accordingly, we have a Quillen adjunction given by extension and restriction of scalars, with counit given by the action map

$$DEG_+ \wedge X \rightarrow X$$

for DEG_+ -modules X and unit

$$Y \rightarrow DEG_+ \wedge Y.$$

Noting that the \mathbb{S} -module G_+ is taken to $DEG_+ \wedge G_+ \simeq G_+$, we continue to write G_+ for the image cell. We note that both the derived unit and counit are non-equivariant equivalences and hence G_+ -equivalences. It follows from the Cellularization Principle A.1 that we have a Quillen equivalence of cellularizations

$$\text{free-}G\text{-spectra} = \text{cell-}\mathbb{S}\text{-mod-}G\text{-spectra} \simeq \text{cell-}DEG_+\text{-mod-}G\text{-spectra}.$$

4. Passage to N -fixed points

Now we note that since the identity component N of G is a normal subgroup, Lewis-May fixed points give a functor from G -spectra to W -spectra. We write

$$D\tilde{B}N_+ := (DEG_+)^N$$

for the image of DEG_+ . It is a ring W -spectrum with underlying non-equivariant spectrum DBN_+ , and we include the tilde in the notation to emphasize that it will typically have a non-trivial W -action.

The Lewis-May fixed point functor takes DEG_+ -module G -spectra to $D\tilde{B}N_+$ -module W -spectra, and we denote this functor Ψ^N . It has a left adjoint given by inflation and extension of scalars, and as usual we suppress the notation for inflation. The adjunction is discussed at greater length in [4] or [3, Section 11].

Once again we have a Quillen adjunction

$$DEG_+ \wedge_{D\tilde{B}N_+} : D\tilde{B}N_+\text{-mod-}W\text{-spectra} \rightleftarrows DEG_+\text{-mod-}G\text{-spectra} : \Psi^N .$$

The Wirthmuller equivalence $(G_+)^N \simeq \Sigma^d W_+$ [9, II.6.3] gives the image of the cell G_+ as a W -spectrum, where d is the dimension of G , and the module structure is unique by Corollary 9.2. By the Cellularization Principle (Proposition A.1), to see that we get a Quillen equivalence after cellularization, we need only check that the unit and counit are derived equivalences on the cells.

For any N -free G -space Y the counit is a map

$$DEG_+ \wedge_{D\tilde{B}N_+} Y^N \simeq DEG_+ \wedge_{D\tilde{B}N_+} \Sigma^d Y/N \longrightarrow Y,$$

and we are interested in the special case $Y = G_+$. This counit is an equivalence for any N -free Y by the Eilenberg–Moore theorem since N is a connected group, but we give the complete proof in Section 5, since it is especially simple in the rational case. It then follows that the unit

$$W_+ \longrightarrow (DEG_+ \wedge_{D\tilde{B}N_+} W_+)^N \simeq (\Sigma^{-d} G_+)^N$$

is also an equivalence. By the Cellularization Principle A.1, we thus have a Quillen equivalence on the cellularizations

$$\text{cell-}DEG_+\text{-mod-}G\text{-spectra} \simeq \text{cell-}D\tilde{B}N_+\text{-mod-}W\text{-spectra}.$$

5. The Eilenberg–Moore argument

We give a self-contained argument for the Eilenberg–Moore equivalence in the previous section.

PROPOSITION 5.1. *For any N -free DEG_+ -module G -spectrum Y the counit of the fixed point adjunction*

$$DEG_+ \wedge_{D\tilde{B}N_+} Y^N \simeq DEG_+ \wedge_{D\tilde{B}N_+} \Sigma^d Y/N \longrightarrow Y$$

is a weak equivalence.

Proof. Consider the map

$$\epsilon : DEG_+ \wedge_{D\tilde{B}N_+} Y^N \longrightarrow Y$$

for arbitrary DEG_+ -modules Y . This is a map of G -spectra, and it is a weak equivalence in our model of free G -spectra, provided that it is a non-equivariant equivalence. This means that it suffices to argue purely N -equivariantly in showing that the counit is a weak equivalence. Accordingly, it is enough to argue entirely with N -spectra, which we do for the remainder of the proof, so that we have the N -map

$$\epsilon : DEN_+ \wedge_{D\tilde{B}N_+} Y^N \longrightarrow Y.$$

Now note that the class of Y for which the counit is an equivalence is closed under cofibre sequences, retracts and arbitrary wedges. The map ϵ is tautologously an equivalence for the DEN_+ -module $Y = DEN_+$ itself. It follows that it suffices to show that DEN_+ builds the free cell N_+ , since all N -free spectra are built from N_+ .

It remains to show that $Y = N_+$ is built from DEN_+ . The basic idea is to use the standard Koszul resolution, but the implementation of the idea is complicated by some dualization. We recall that $R = H^*(BN)$ is a polynomial ring, so that if we choose polynomial generators

x_1, \dots, x_r , then we may form a Koszul complex, and hence an exact sequence

$$0 \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Q} \longrightarrow 0.$$

Here, F_s is a free module on generators corresponding to s -fold products of the generators: $F_s = \bigoplus_{|A|=s} x^A R$ (where A runs over subsets of $\{1, 2, \dots, r\}$, and x^A is simply designed to keep track of the degrees). Of course, since $\pi_*^N(DEN_+) = R$, we can realize the modules with DEN_+ -module N -spectra by taking

$$\mathbb{F}_s = \bigvee_{|A|=s} x^A DEN_+,$$

so that

$$F_s = \pi_*^N(\mathbb{F}_s).$$

It is also easy to realize the maps in the exact sequence in DEN_+ -modules and one may go further to realize an entire filtered spectrum in a standard way.

However, it is convenient to do a little more, and find the pre-dual. In the algebraic world, $F_s \cong (E_s)^\vee$ for a suitable module E_s ; indeed we may take $E_s = (F_s)^\vee$. Taking duals throughout, we have a resolution

$$0 \longleftarrow E_r \longleftarrow E_{r-1} \longleftarrow \dots \longleftarrow E_1 \longleftarrow E_0 \longleftarrow \mathbb{Q} \longleftarrow 0$$

by injective $H^*(BN)$ -modules.

In the topological world $\pi_*^N(N_+) = \Sigma^d \mathbb{Q}$, $\pi_*^N(EN_+) = \Sigma^d R^\vee$, and we may realize $\Sigma^d E_s$ with $\mathbb{E}_s = \bigvee_{|A|=s} x^{-A} EN_+$. In the usual way we form a tower

$$\begin{array}{ccccccc} Y_0 & \longleftarrow & Y_1 & \longleftarrow & \dots & \longleftarrow & Y_{r-1} & \longleftarrow & Y_r & \longleftarrow & Y_{r+1} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ \mathbb{E}_0 & & \Sigma^{-1} \mathbb{E}_1 & & & & \Sigma^{-r+1} \mathbb{E}_{r-1} & & \Sigma^{-r} \mathbb{E}_r & & \end{array}$$

so that $Y_0 = N_+$ and the composites $\Sigma^{-s+1} \mathbb{E}_{s-1} \longrightarrow \Sigma Y_s \longrightarrow \Sigma^{-s+1} \mathbb{E}_s$ are the (d -fold suspension of the) maps in the exact sequence. In more detail, we start with $Y_0 = N_+$ and find a map $N_+ \longrightarrow \mathbb{E}_0$ realizing $\mathbb{Q} \longrightarrow E_0$, and take Y_1 to be the fibre. Now $\pi_*^N(\Sigma Y_1) = \text{im}(E_0 \longrightarrow E_1)$ and we may realize $\pi_*^N(\Sigma Y_1) \longrightarrow E_1 = \pi_*^N(\mathbb{E}_1)$ by a map $\Sigma Y_1 \longrightarrow \mathbb{E}_1$.

Taking functional duals, we obtain the required resolution by DEN_+ -modules. The point of working with pre-duals is that we have a diagram of N -free spectra. Thus, the fact that $\pi_*^N(Y_{r+1}) = 0$ is sufficient to show that $Y_{r+1} \simeq *$ [2, 6.1]. Now we may form the dual tower

$$\begin{array}{ccccccc} Y^0 & \longleftarrow & Y^1 & \longleftarrow & \dots & \longleftarrow & Y^{r-1} & \longleftarrow & Y^r & \longleftarrow & Y^{r+1} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ \Sigma^{1-0} \mathbb{E}_0 & & \Sigma^{1-1} \mathbb{E}_1 & & & & \Sigma^{1-r+1} \mathbb{E}_{r-1} & & \Sigma^{1-r} \mathbb{E}_r & & \end{array}$$

defined by the cofibre sequences

$$Y_s \longrightarrow Y_0 \longrightarrow Y^s.$$

Here, we start with $Y^0 \simeq *$ and build up $Y^{r+1} \simeq Y_0 = N_+$ by the $r + 1$ cofibre sequences

$$\Sigma^{-s} \mathbb{E}_s \longrightarrow Y^{s+1} \longrightarrow Y^s.$$

Dualizing, we have a corresponding construction with $DY^0 \simeq *$ and build up $DY^{r+1} \simeq DY_0 \simeq DN_+ \simeq \Sigma^{-d} N_+$ (using the Wirthmüller isomorphism for the last equivalence) by the $r + 1$ cofibre sequences

$$\Sigma^s D\mathbb{E}_s \longleftarrow DY^{s+1} \longleftarrow DY^s.$$

This shows that N_+ is finitely built from DEN_+ , which completes the proof. □

6. From topology to algebra

We now have a commutative ring W -spectrum $D\tilde{B}N_+$, and by Shipley [11, 1.2], since we are working over the rationals, this corresponds to a commutative monoid in the category of differential graded $\mathbb{Q}W$ -modules. We write $C^*(\tilde{B}N)$ for this DGA since its cohomology is $H^*(\tilde{B}N)$. Furthermore, we have Quillen equivalences between the categories of modules by Shipley [11, 2.15] and between their cellularizations by Proposition A.1

$$\text{cell-}D\tilde{B}N_+\text{-mod-}W\text{-spectra} \simeq \text{cell-}C^*(\tilde{B}N)\text{-mod-}\mathbb{Q}[W]\text{-mod.}$$

Finally, for convenience we re-express this category of modules. Indeed, the category of $C^*(\tilde{B}N)$ -modules in $\mathbb{Q}W$ -modules is the same as the category of $C^*(\tilde{B}N)[W]$ -modules in \mathbb{Q} -modules, where $C^*(\tilde{B}N)[W]$ is the twisted group ring, so that we have an isomorphism.

$$\text{cell-}C^*(\tilde{B}N)\text{-mod-}\mathbb{Q}[W]\text{-mod} \cong \text{cell-}C^*(\tilde{B}N)[W]\text{-mod.}$$

7. Formality

Next, we replace $C^*(\tilde{B}N)$ by its homology. To start with, a classical result of Borel states $H^*(\tilde{B}N)$ is a polynomial algebra on even degree generators. Furthermore, if we regard it as a W -module, then it is a symmetric algebra on the finite-dimensional, evenly graded $\mathbb{Q}W$ -submodule $V = QH^*(\tilde{B}N)$. We will argue that there is a copy of V inside the cycles of $C^*(\tilde{B}N)$. This gives a chain map $V \rightarrow ZC^*(\tilde{B}N) \rightarrow C^*(\tilde{B}N)$ of $\mathbb{Q}W$ -modules. Since $C^*(\tilde{B}N)$ is commutative, the universal property of the symmetric algebra gives a map

$$H^*(\tilde{B}N) = \text{Symm}(V) \rightarrow ZC^*(\tilde{B}N) \rightarrow C^*(\tilde{B}N)$$

of differential graded $\mathbb{Q}W$ -algebras, and it is a homology isomorphism by construction. This then gives a Quillen equivalence

$$H^*(\tilde{B}N)[W]\text{-mod} \simeq C^*(\tilde{B}N)[W]\text{-mod}$$

and hence also an equivalence of cellularized categories by Proposition A.1, where the generating cell $\mathbb{Q}W$ is characterized by its homology by Corollary 9.2.

To construct the map, we work with increasing codegrees. Since V is positively cogenerated, we can start with the zero map in degree 0. When we reach codegree n , we have an epimorphism

$$Z^n C^*(\tilde{B}N) \rightarrow H^n(\tilde{B}N) \rightarrow Q^n H^*(\tilde{B}N) = V^n,$$

of $\mathbb{Q}W$ -modules. By Maschke's theorem this splits to give the required $\mathbb{Q}W$ -map $V^n \rightarrow Z^n C^*(\tilde{B}N)$. Since V is concentrated in finitely many degrees, this process will be complete in finitely many steps.

8. Change of algebraic models

The last equivalence changes from a model with underlying category of DG $H^*(\tilde{B}N)[W]$ -modules (and cellular equivalences as weak equivalences) to a model with underlying category the DG torsion- $H^*(\tilde{B}N)[W]$ -modules (and homology isomorphisms as weak equivalences).

In fact, the previous Quillen equivalence leaves us with the $\mathbb{Q}W$ -cellularization of the projective model structure on $H^*(\tilde{B}N)[W]$ -modules; see [11, 2.9] or [8, Section 7]. For the next step, we need the injective model structure on $H^*(\tilde{B}N)[W]$ -modules with weak equivalences the homology isomorphisms and cofibrations the monomorphisms; see [2, Section 8.C], or more

generally [6, 3.6]. Using the identity functors, there is a Quillen equivalence between the $\mathbb{Q}W$ -cellularizations of the projective and injective model structures on $H^*(\tilde{B}N)[W]$ -modules by the Cellularization Principle A.1. Now if we let \mathfrak{m} denote the maximal ideal of $H^*(\tilde{B}N)$, then the \mathfrak{m} -power torsion functor

$$\Gamma_{\mathfrak{m}}M := \{x \in M \mid \mathfrak{m}^s x = 0 \text{ for } s \gg 0\}$$

is right adjoint to the inclusion of the torsion modules:

$$i : \text{tors-}H^*(\tilde{B}N)[W]\text{-mod} \rightleftarrows H^*(\tilde{B}N)[W]\text{-mod}_{inj} : \Gamma_{\mathfrak{m}}.$$

We next verify that this adjunction induces a Quillen equivalence between the $\mathbb{Q}W$ -cellularized injective model category and the injective model structure on torsion modules from [2, 8.6], with weak equivalences the homology isomorphisms and cofibrations the monomorphisms.

First note that the inclusion of the torsion modules into the injective model structure $H^*(\tilde{B}N)[W]\text{-mod}_{inj}$ (before cellularization) is a left Quillen functor since it preserves homology isomorphisms and monomorphisms. We show in Corollary 9.2 that the generating cell $\mathbb{Q}W$ is characterized by its homology, so we see that the (i, Γ) adjunction preserves $\mathbb{Q}W$ up to equivalence. Hence, by the Cellularization Principle A.1, there is an induced Quillen equivalence between the associated $\mathbb{Q}W$ cellularized model structures

$$\text{cell-tors-}H^*(\tilde{B}N)[W]\text{-mod} \simeq \text{cell-}H^*(\tilde{B}N)[W]\text{-mod}_{inj}.$$

The cellular weak equivalences detected by $\text{Hom}(\mathbb{Q}W, \cdot)$ are precisely the homology isomorphisms on torsion modules, so the cellularized model structure on the torsion modules agrees with the original injective model structure. Thus, we have direct Quillen equivalences

$$\text{tors-}H^*(\tilde{B}N)[W]\text{-mod} \simeq \text{cell-}H^*(\tilde{B}N)[W]\text{-mod}_{inj} \simeq \text{cell-}H^*(\tilde{B}N)[W]\text{-mod}_{proj}.$$

9. The Adams spectral sequence

The homotopy groups π_*^N may be used as the basis of an Adams spectral sequence for calculating maps between free rational G -spectra. If X is a G -space, then W acts on $\pi_*^N(X)$, and if X is N -free, then $H^*(\tilde{B}N)$ acts on $\pi_*^N(X) = H_*(X/N)$ by cap product; these structures interact to give the structure of an $H^*(\tilde{B}N)[W]$ -module. Finally, since homotopy elements are supported on finite subspectra (whose homotopy is bounded below), the module is a torsion module.

THEOREM 9.1. *For any free G -spectra X and Y there is a natural Adams spectral sequence*

$$\text{Ext}_{H^*(\tilde{B}N)[W]}^{*,*}(\pi_*^N(X), \pi_*^N(Y)) \Rightarrow [X, Y]_*^G.$$

It is a finite spectral sequence concentrated in rows 0 to r and strongly convergent for all X and Y .

Proof. The proof is standard. First, we observe that enough torsion modules are realizable, since $\pi_*^N(EG_+ \wedge G/N_+) \cong \Sigma^d H^*(\tilde{B}N)[W]^\vee$. Next, we observe that for any free G -spectrum X

$$\pi_*^N : [X, EG_+ \wedge G/N_+]_*^G \longrightarrow \text{Hom}_{H^*(\tilde{B}N)[W]}(\pi_*^N(X), \pi_*^N(EG_+ \wedge G/N_+))$$

is an isomorphism. Indeed, we may use a change of groups isomorphism on the domain and codomain and reduce to showing that

$$\pi_*^N : [X, EN_+]_*^N \longrightarrow \text{Hom}_{H^*(\tilde{B}N)}(\pi_*^N(X), \pi_*^N(EN_+))$$

is an isomorphism. This is a special case of [2, 6.1].

This is enough to construct the Adams spectral sequence, and identify the E_2 -term. For convergence, we need only show that if X is G -free and $\pi_*^N(X) = 0$, then $X \simeq *$. By Greenlees and Shipley [2, 6.1], we know that $\pi_*^N(X) = 0$ implies that X is N -contractible, or equivalently that $X \wedge G/N_+ \simeq *$. It follows that $X \wedge EG/N_+ \simeq *$ and hence we have equivalences

$$X = X \wedge S^0 \simeq X \wedge \tilde{E}G/N \simeq *.$$

The first equivalence is because $X \wedge EG/N_+ \simeq *$ and the second is because X is G -free and $\tilde{E}G/N$ is non-equivariantly contractible. \square

Apart from giving a calculational tool, this result makes plausible the main theorem of the present paper. Nonetheless, it appears that the only way we explicitly use the Adams spectral sequence is in the fact that cells are characterized by their homology.

COROLLARY 9.2. *If X is a free G -spectrum with $\pi_*^N(X) \cong \pi_*^N(G_+) = \Sigma^d \mathbb{Q}W$, then $X \simeq G_+$.*

Proof. The E_2 -term of the Adams spectral sequence for calculating maps between G_+ and X is

$$\text{Ext}_{H^*(\tilde{B}N)[W]}^{*,*}(\mathbb{Q}W, \mathbb{Q}W) = (\Delta V)[W]$$

with $V = QH^*(\tilde{B}N)$ the indecomposables. A degree $-i$ submodule of V gives rise to a bidegree $(1, -i)$ submodule of the Ext group, and so, by degree, the bottom copy of $\mathbb{Q}[W]$ consists of infinite cycles. It follows that the identity map in π_*^N lifts to a map between spectra. This gives maps $G_+ \rightarrow X$ and $X \rightarrow G_+$ whose composites in either order are isomorphisms in π_*^N . By the convergence of the Adams spectral sequence this is an equivalence. \square

In this paper, we often need to know how our chosen cells behave under functors between model categories. We apply the corollary repeatedly to see that each cell maps to the obvious object up to equivalence.

Appendix. Cellularizations

The basic reference for cellularization (or right localization) in model structures is [7]. Our convention throughout this paper is to refer to the cellularization of a stable model category \mathbb{M} with respect to the set of all suspensions of an object $\{\Sigma^i A\}_{i \in \mathbb{Z}}$ as *the cellularization of \mathbb{M} with respect to A* and write $A\text{-cell-}\mathbb{M}$. In this case, $A\text{-cell-}\mathbb{M}$ is again a stable model category; see [1, 4.6].

We now recall the Cellularization Principle from [5] (or [3, Appendix A]), which we use to produce Quillen equivalences of cellularized model categories.

PROPOSITION A.1 [5] (The Cellularization Principle). *Let \mathbb{M} and \mathbb{N} be right proper, stable, cellular model categories with $L : \mathbb{M} \rightarrow \mathbb{N}$ a Quillen adjunction with right adjoint R . Let \underline{L} and \underline{R} denote the associated derived functors.*

(1) *Let A be a homotopically small object in \mathbb{M} such that $\underline{L}A$ is homotopically small and $A \rightarrow \underline{R}\underline{L}A$ is a weak equivalence. Then L and R induce a Quillen equivalence between the A -cellularization of \mathbb{M} and the $\underline{L}A$ -cellularization of \mathbb{N}*

$$A\text{-cell-}\mathbb{M} \simeq_Q \underline{L}A\text{-cell-}\mathbb{N}.$$

(2) Let B be a homotopically small object in \mathbb{N} such that \underline{RB} is homotopically small and $\underline{LRB} \rightarrow B$ is a weak equivalence. Then L and R induce a Quillen equivalence between the B -cellularization of \mathbb{N} and the \underline{RB} -cellularization of \mathbb{M}

$$\underline{RB}\text{-cell-}\mathbb{M} \simeq_Q B\text{-cell-}\mathbb{N}.$$

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