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A note on coefficient of restitution models including the effects of impact induced vibration

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1 Introduction

Modelling impacts in mechanical systems is a classical problem which continues to engage many researchers from different fields. Comprehensive discussions of the subject and reviews of the associated literature can be found in a range of texts [1–4]. Despite the large amount of work in the area, the intuitively simple approach defined by Newton [5] of the *coefficient of restitution* as the ratio of pre and post impact velocities, is still widely used in modelling today. However, limitations in the Newtonian definition of the coefficient of restitution have led to several redefinitions of this quantity — comprehensive discussions are given by Brogliato [6] and Stronge [4], see also [7, 8].

A relatively recent definition of the coefficient of restitution is in a form which resolves the effect of additional energy losses due to impact — primarily vibrations induced in the contacting bodies. Hurmuzlu [9] introduced this concept for the Panlevé type of problem [6] of a rigid rod striking a horizontal surface. In this approach an energy balance is used to relate the pre and post impact velocities to the energy dissipation during contact. A related approach developed by Wagg [10, 11] for flexible bodies impacting against a rigid constraint uses an energy balance *be-*

tween subsequent impacts to account for energy dissipated due to impact induced vibrations in the flexible body. In the approach described by Hurmuzlu [9], the analysis is for a single impact and the effect of friction during contact is included. The example discussed by Wagg [11] is for periodic vibro-impact motion, where friction effects are not included in the model. In both examples an energy balance is applied between a pre and post impact velocity state in the system, and the coefficient of restitution is redefined as an energy loss factor.

In this work we consider the case of a multi-modal system subject to impact. An energy balance is developed for an arbitrary contact interval which includes the effects of modal vibration. The energy balance is used to obtain a relationship between the coefficient of restitution and the modal energy during the contact period. The subsequent analytical relationships demonstrate that increasing contact duration and excitation of higher modes can reduce the effective value of the coefficient of restitution. We relate this approach to the work by Stronge [4] on energetically consistent impacts.

2 Multi-modal systems subject to impact

In this work we restrict our attention to flexible systems with uniformly distributed parameters which can be modelled by

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}_I(t), \tag{1}$$

where $\mathbf{M}, \mathbf{C}, \mathbf{K}$ are the mass, damping and stiffness matrices respectively, $\mathbf{x} = \{x_1, x_2 \ldots, x_N\}^T$ and $\mathbf{f}_I(t)$ is the impact force vector. It is assumed that a single of the x_i coordinates is constrained by a compliant motion-limiting constraint at a distance, $x_s \geq 0$. The area of contact is assumed to be small, and related to the x_i coordinate alone.

The analysis presented here is for systems with uniformly distributed parameters, with the property that $\mathbf{M} = m\mathbf{I}$, $\mathbf{C} = c\mathbf{D}$ and $\mathbf{K} = k\mathbf{E}$, where $\mathbf{E}$ is the stiffness coupling matrix as defined by [12], $\mathbf{D}$ is the damping coupling matrix $\mathbf{I}$ is the identity matrix, and m, c and k are scalars representing the mass, stiffness and damping. This restriction still includes a wide class of discretised systems including *lumped mass* systems, and some discretised models of continuous systems.

Equation 1 can be decoupled in the normal manner [13] to give

$$\mathbf{I}\ddot{\mathbf{q}} + \mathbf{\Xi}\dot{\mathbf{q}} + \mathbf{\Omega}\mathbf{q} = \frac{1}{m}\mathbf{\Psi}^T\mathbf{f}_I(t), \tag{2}$$

where $\mathbf{q} = \{q_1, q_2, \dots q_N\}^T$, $\mathbf{x} = \mathbf{\Psi}\mathbf{q}$, $\mathbf{\Psi}$ is the orthogonal modal matrix, $\mathbf{\Xi} = \text{diag}\{....2\zeta_j\omega_{nj}....\}$, $\mathbf{\Omega} = \text{diag}\{....\omega_{nj}^2....\} = (k/m)\mathbf{\Lambda}$, where $\mathbf{\Lambda}$ is the diagonal eigenvalue matrix and $\zeta_j = \sum_n a_n\omega_{nj}^{2n}$. The coefficients a_n are determined to give an appropriate relationship (polynomial fit) between the N ζ_j and ω_{nj} values. i.e. extended Rayleigh damping [14]. The choice of damping model is significant, in that it allows the system to be decoupled. However, as we will see from the energy analysis it also has an effect on the impact modelling when higher modes are excited by impact.

When an impact occurs, the pre and post impact velocities can be related via a coefficient of restitution matrix written as

$$\dot{\mathbf{x}}(t_f) = \mathbf{R}\dot{\mathbf{x}}(t_i) \quad x_i = x_s \tag{3}$$

where t_i is the start of impact time, t_f is the end of impact time, $\mathbf{R} = \text{diag}\{1, 1, ..., -e, ...1, 1\}$ and $e \in [0, 1]$ is the coefficient of restitution. The position of e in matrix $\mathbf{R}$ corresponds to the x_i coordinate. The impact is assumed to be effectively collinear (i.e. no frictional component), so that e can be taken as either the Newtonian, Poisson or Stronge definition [4].

3 Analysis of energy during the contact period

For the type of multi-modal system considered here, energy loss occurs primarily due to the impact process and, to a lesser extent, due to vibration damping. In [11] an energy balance for this system was obtained, by exploiting the periodicity of vibro-impact motion. An energy balance for the contact period of a rigid bar impacting on a horizontal surface has been considered by [9]. In both cases the objective of the study was to explain the effect of impact induced vibration on the value of coefficient of restitution used in modelling an experimental system. Here we consider a modal energy analysis of the contact period $t_i \leq t \leq t_f$, (as in [9]) for the system described in section 2.

In modal coordinates the coefficient of restitution rule, equation 3, becomes

$$\mathbf{\Psi}\dot{\mathbf{q}}(t_f) = \mathbf{R}\mathbf{\Psi}\dot{\mathbf{q}}(t_i). \tag{4}$$

This leads to the relation for the modal velocities after impact

$$\dot{\mathbf{q}}(t_f) = \hat{\mathbf{R}}\dot{\mathbf{q}}(t_i), \tag{5}$$

where $\hat{\mathbf{R}} = \mathbf{\Psi}^{\mathbf{T}}\mathbf{R}\mathbf{\Psi}$ is the matrix which represents the relationship between modal velocities before impact to modal velocities after impact.

Premultiplying the (modal) equation of motion for an N degree of freedom system, equation 2, by $m\dot{\mathbf{q}}^T$ and integrating (term by term) with respect to t gives an expression for the energy (excluding rigid body modes) between t_i and t_f, which can be written as

$$\begin{aligned} &\frac{m}{2}(\dot{\mathbf{q}}^T\mathbf{I}\dot{\mathbf{q}}(t_f) - \dot{\mathbf{q}}^T\mathbf{I}\dot{\mathbf{q}}(t_i)) + \frac{k}{2}(\mathbf{q}^T\mathbf{\Lambda}\mathbf{q}(t_f) - \mathbf{q}^T\mathbf{\Lambda}\mathbf{q}(t_i)) \\ &= \int_{t_i}^{t_f} \dot{\mathbf{q}}^T\mathbf{\Psi}^T\mathbf{f}_I(t)\mathrm{d}t - m\int_{t_i}^{t_f} \dot{\mathbf{q}}^T\mathbf{\Xi}\dot{\mathbf{q}}\mathrm{d}t. \end{aligned} \tag{6}$$

This represents the energy balance during an impact and can be expressed as

$$KE + PE = FE_i - DE \tag{7}$$

where KE is kinetic energy, PE is potential (or strain) energy, FE_i the impact force energy and DE the modal damping energy. All these quantities are the final values, taken at the end of the contact period, and therefore representing the change of the respective energetic quantity at the end of the impact. The kinetic energy term (first term in equation 6) can be evaluated using the relations $\dot{\mathbf{q}}(t_i) = \mathbf{\Psi}^T\dot{\mathbf{x}}(t_i)$, $\dot{\mathbf{q}}^T(t_i) = \dot{\mathbf{x}}^T(t_i)\mathbf{\Psi}$, $\dot{\mathbf{q}}(t_f) = \mathbf{\Psi}^T\mathbf{R}\dot{\mathbf{x}}(t_i)$ and $\dot{\mathbf{q}}^T(t_f) = \dot{\mathbf{x}}^T(t_i)\mathbf{R}\mathbf{\Psi}$, to give

$$KE = \frac{m}{2}(\dot{\mathbf{x}}^T(t_i)\mathbf{R}\mathbf{R}\dot{\mathbf{x}}(t_i) - \dot{\mathbf{x}}^T(t_i)\mathbf{I}\dot{\mathbf{x}}(t_i)), \tag{8}$$

which reduces to

$$KE = -\frac{m}{2}v_0^2(1 - e^2) \tag{9}$$

where v_0 denotes the velocity at impact (t_i). Equation 9 represents the change in kinetic energy over the contact period — thus a negative quantity ($m > 0$ always).

The assumptions that $PE \approx 0$ and $DE \approx 0$ are equivalent to the assumptions required for rigid body impact theory (as discussed in detail by [4]). However, the theoretical formulation developed above allows for cases when $PE \neq 0$ and $DE \neq 0$ which can be the situation in flexible body impact problems.

Consider first the general case when $PE \neq 0$ and $DE \neq 0$. In this case the energy balance over the contact period can be written as

$$\frac{m}{2}v_0^2(1 - e^2) = PE + DE - FE_i, \tag{10}$$

By rearranging equation 10 we can obtain an expression for the coefficient of restitution including contact displacement and modal vibration damping as

$$\hat{e} = \sqrt{1 - \frac{2}{mv_0^2}(PE + DE - FE_i)} \tag{11}$$

where $\hat{e}$ now represents the coefficient of restitution including vibration effects. This expression indicates that the coefficient of restitution is a function of system parameters and impact velocity and the relative energy input/output during contact [1]. Equation 11 can be written as

$$\hat{e} = \sqrt{1 - \frac{RE}{KE_i}} \tag{12}$$

where $KE_i = mv_0^2/2$ is the kinetic energy at the start of the contact period, and $RE = PE + DE - FE_i$ is the residual energy for all modes at the end of the contact period. In fact RE will vary dependant on the assumptions made:

- Rigid body impact theory, $PE = DE = 0$, $RE = -FE_i$. Note that if the impact is assumed to be instantaneous, it automatically follows that $PE = DE = 0$.
- $PE \approx 0$, $DE \neq 0$ — the case, for example, in structures where low velocity impacts and approximately elastic indentation occurs, but vibration is significant; $RE = DE - FE_i$. This will be called the intermediate case.
- Full flexible impact $RE = PE + DE - FE_i$ and $PE + DE \neq 0$.

It is clear from this analysis that the relative value of $\hat{e}$ will be affected depending on which modelling assumptions are used. We note that DE is always positive, and PE would normally be negative — for permanent post impact displacement. This means that it's possible for certain flexible body impacts that $PE + DE \approx 0$, which could correspond to a situation where rigid body theory can give a good approximation to the flexible problem.

We also note that equation 12 defines a class of physically realisable models assuming that $0 \leq \hat{e} \leq 1$. Then from equation 12, $0 \leq RE \leq KE_i$ where $KE_i > 0$ is a strictly positive quantity [11] for all v_0. It is clear then that the condition on RE means that the choice of both the impact force model and the damping model are important in order to obtain a physically realistic overall model.

Of the three cases listed, rigid body theory is well developed, the intermediate case is of current interest, and the full flexible case an area for future work (and therefore will not be considered in detail here).

3.1 Analysis of the rigid body case

For the rigid body case $PE \approx 0$, and the energy damped due to vibrations in the flexible body during impact is negligible, $DE \approx 0$. Then in this case

$$\frac{m}{2}v_0^2(1-e^2) = -\int_{t_i}^{t_f} \dot{\mathbf{q}}^T \mathbf{\Psi}^T \mathbf{f}_I(t)\mathrm{d}t = -\int_{t_i}^{t_f} \dot{\mathbf{x}}^T \mathbf{f}_I(t)\mathrm{d}t = -\int_{t_i}^{t_f} \dot{x}_i f_i(t)\mathrm{d}t, \quad (13)$$

where f_i is the impact force which occurs when x_i comes into contact with the motion constraint, with velocity $\dot{x}_i$. Note that the right hand side of equation 13 reduces to a scalar integral in this analysis because the vector $\mathbf{f}_I(t)$ has only a single non-zero component — in this case, f_i.

Now we can use the analysis presented by Stronge [4] which relates the impact force to impulse via the relation $f_i(t)\mathrm{d}t = \mathrm{d}p$, where p is impulse. This gives

$$\frac{m}{2}v_0^2(1-e^2) = -\int_{p(t_i)=0}^{p(t_f)=p_f} v_i(p)\mathrm{d}p, \quad (14)$$

where $v_i(p) = \dot{x}_i$ is the velocity during impact, which as Stronge points out (for scalar systems) can be approximated as a linear function of impulse of the form $v_i(p) = v_0 + p/m$ [4]. Evaluating the right hand side of equation 14 using $p_f = -mv_0(1+e)$ [4], gives the kinetic energy lost during impact $mv_0^2(1-e^2)/2$.

3.2 Analysis of the intermediate case

The intermediate case, when $PE \approx 0$, $DE \neq 0$ and $RE = DE - FE_i$, includes a wide class of vibration and impact problems with low velocity impacts. This is the primary case of interest as the vibration induced by impact can have a significant effect on the the coefficient of restitution, as discussed by both [9] and [11].

In this case the energy balance can be written $DE = FE_i - KE$, which gives

$$m \int_{t_i}^{t_f} \dot{\mathbf{q}}^T \mathbf{\Xi} \dot{\mathbf{q}} \mathrm{d}t = \int_{t_i}^{t_f} \dot{x}_i f_i(t) \mathrm{d}t - \frac{m}{2} v_0^2 (1 - e^2), \qquad (15)$$

or

$$m \sum_{j=1}^{N} \int_{t_i}^{t_f} \dot{q}_j 2\zeta_j \omega_{nj} \dot{q}_j \mathrm{d}t = \int_{t_i}^{t_f} \dot{x}_i f_i(t) \mathrm{d}t - \frac{m}{2} v_0^2 (1 - e^2). \qquad (16)$$

This expression together with the condition $0 \leq RE \leq KE_i$ defines an energetically consistent impact-damping model for the intermediate case. For equation 16 to hold, appropriate values of N, ζ_j, e and f_i are required.

Therefore, in order to compute energetically consistent simulations of the impact process three key parts of the model need to be identified: (i) the impact force model, f_i (ii) the number of modes, N and (iii) the modal damping coefficients, ζ_i. If experimental data is available, it should be possible to estimate (i)–(iii), however it is worth noting that the presence of impacts can have a significant effect of modal damping values — see for example the difference between impacting and non-impact power spectra shown in [15]. In this case the use of extended Rayleigh damping allows the modal damping coefficients to be selected appropriately to ensure energetic consistency in the model. We note that if suitable experimental data is available, the number of modes could be estimated using proper orthogonal decomposition which has already been considered for vibro-impact systems by [16]. Impact force models have been the subject of intensive research over many years (see [1, 4] for detailed summaries) and one of several standard models can be selected for $f_i(t)$ as appropriate.

In figure 1 an example is shown for a cantilever beam impacting a constraint. Numerical simulations of this example were computed using the collocation techniques for a cantilever beam described in [17]. The procedure is to decompose the governing equation for the beam into a finite set of modal equations in the form of equation 2. These equations are then put into first order form and iterated forward in time using a Rosenbrock method. Rosenbrock is required because the large difference in the beam and impact stiffness leads to a stiff set of first order ordinary differential equations. The impact force model used in this case was a Simon impact model of the form

$$f_i = -k_w|\delta|^{1/2}(\delta + c_w|\delta|\dot{\delta}) \tag{17}$$

where $\delta = u(B,t) - a$ for $u(B,t) > a$ is the indentation — see [4], chapter 5. The impact force is evaluated at each time-step during contact, when $t_i \leq t \leq t_f$. The beam simulation is started in free vibration from an initial deflection away from the impact stop. The data from the first impact to occur is then recorded.

Figure 2 shows typical results for a intermediate case behaviour where $PE \approx 0$ and $DE \neq 0$. Physical and energetic quantities over the contact period are shown schematically, to demonstrate typical behaviour for this type of impact-damping system.

The velocity-time (and also velocity-impulse) relationship in this case is now typically nonlinear, as shown in figure 2 (b) — as opposed to the linear assumption used in the rigid body case [4]. From figure 2 (h), we see that increasing the contact interval (or number of modes) will typically increase the final DE value because DE increases as a (weakly) monotonic function of time. This would then typically reduce the effective value of the coefficient of restitution compared to the rigid body case, where DE is effectively zero. The final energy balance represented by equation 7 would be found from the computing the values in figure 2 (e)–(h) at time t_f — note these are not shown to scale in figure 2.

4 Conclusion

The main motivation for this work has been to model impact induced vibration effects on the coefficient of restitution value during impact. The analysis presented here leads to an analytical relationship that relates the coefficient of restitution as a function of impact velocity v_0, and energy terms PE, DE, and FE_i. For the intermediate case, the following observations can be made:

(1) Accurate modelling requires the appropriate choice of number of modes of vibration, damping model and impact force model.
(2) Increased contact duration and/or excitation of higher modes increase DE, which in turn typically reduces RE and the effective coefficient of restitution.
(3) For energetically consistent impacts $0 \leq RE \leq KE_i$ and $KE + PE = FE_i - DE$ at time t_f.

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Figure Captions

- Figure 1: Schematic diagram of cantilever beam example. Parameters used in this simulation are beam length, L, 300mm, beam width 25.5mm, beam thickness 0.486mm, Young's modulus 2.05×10^{11}N/m^2, density 8500kg/m^3, degrees of freedom, $N = 8$, $\zeta_1 = 0.0007$, $\zeta_2 = 0.1164$, $\zeta_3 = 0.2046$, $\zeta_4 = 0.2469$, $\zeta_5 = 0.3174$, $\zeta_6 = 0.3526$, $\zeta_7 = 0.2116$, $\zeta_8 = 0.14107$, stop distance $a = 0.01m$.
- Figure 2: Schematic representation of a rate dependent compliant impact during the contact interval $t_i \leq t \leq t_f$: (a) displacement; (b) velocity; (c) impact force; (d) impulse; (e) kinetic energy; (f) potential energy; (g) impact force energy; (h) modal damping energy. Note the plotting convention for force and impulse ((c) and (d)), is to use absolute values $||f_I||$ and $||p||$. This simulation was computed using the same example as described in [17] combined with a Simon impact model described in [4], chapter 5 with stiffness $k_w = 1 \times 10^5$ and damping $c_w = 0.0485$.

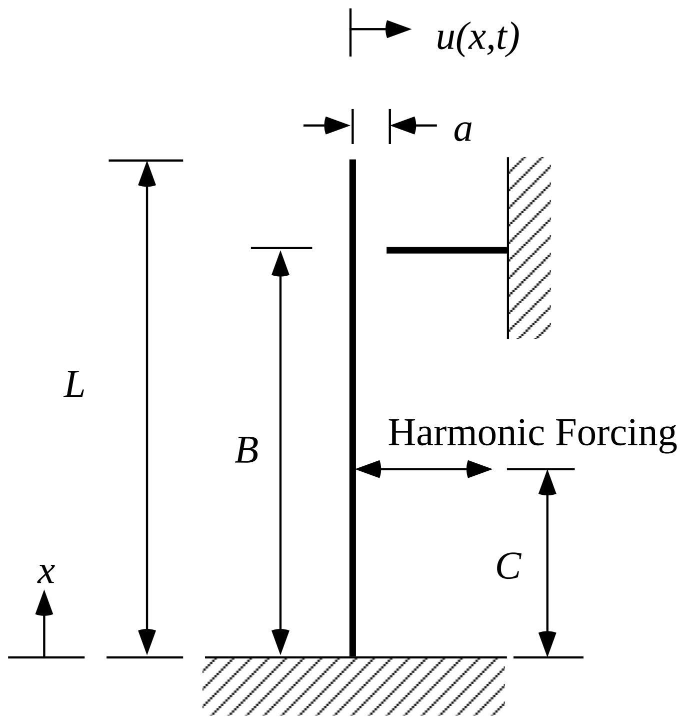

Fig. 1.

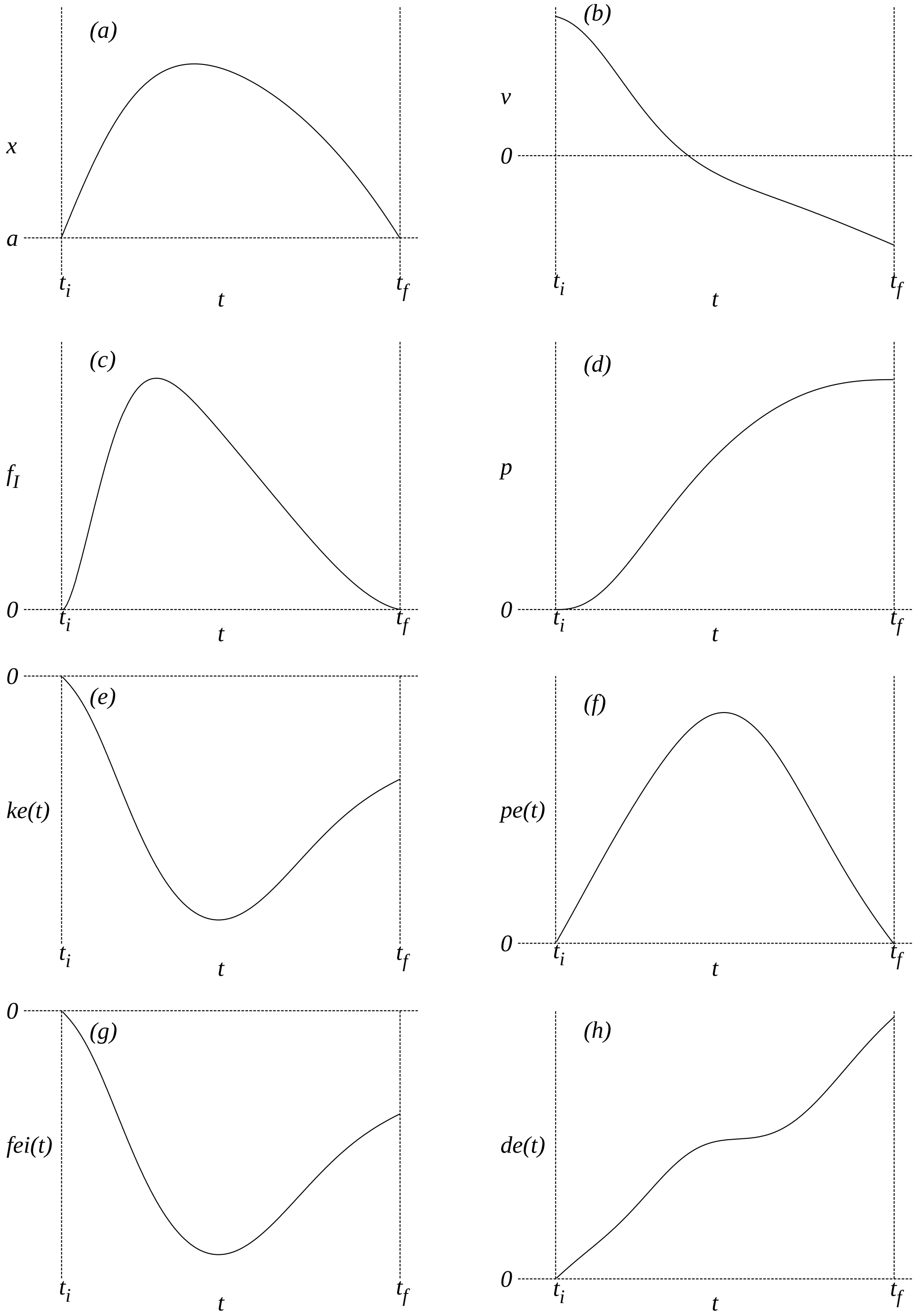
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Fig. 2.