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The Complexity of Theorem Proving in Autoepistemic Logic*

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Abstract. Autoepistemic logic is one of the most successful formalisms for nonmonotonic reasoning. In this paper we provide a proof-theoretic analysis of sequent calculi for credulous and sceptical reasoning in propositional autoepistemic logic, introduced by Bonatti and Olivetti [5]. We show that the calculus for credulous reasoning obeys almost the same bounds on the proof size as Gentzen's system LK. Hence proving lower bounds for credulous reasoning will be as hard as proving lower bounds for LK. This contrasts with the situation in sceptical autoepistemic reasoning where we obtain an exponential lower bound to the proof length in Bonatti and Olivetti's calculus.

1 Introduction

Autoepistemic logic is one of the most popular nonmonotonic logics which is applied in a diversity of areas as commonsense reasoning, belief revision, planning, and reasoning about action. It was introduced by Moore [19] as a modal logic with a single modal operator L interpreted as "is known". Semantically, autoepistemic logic describes possible views of an ideally rational agent on the grounds of some objective information. Autoepistemic logic has been intensively studied, both in its semantical as well as in its computational aspects (cf. [18]). The main computational problems in autoepistemic logic are the credulous and sceptical reasoning problems, formalising that a given formula holds under some, respectively all, views of the agent. Thus these problems can be understood as generalisations of the classical problems SAT and TAUT. However, in autoepistemic logic, these tasks are presumably harder than their propositional counterparts as they are complete for the second level of the polynomial hierarchy [12].

In this paper we target at understanding the complexity of autoepistemic logic in terms of theorem proving. Traditionally, the main objective in proof complexity has been the investigation of propositional proofs [7, 16]. During the last decade there has been growing interest in proof complexity of non-classical logics, most notably modal and intuitionistic logics [14, 15], and strong results have been obtained (cf. [1] for an overview and further references). For autoepistemic logic, Bonatti and Olivetti [5] designed elegant sequent calculi for both credulous and sceptical reasoning. In this paper we provide a proof-theoretic analysis of these calculi. Our main results show that *(i)* the calculus for credulous

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autoepistemic reasoning obeys almost the same bounds to the proof size as the classical sequent calculus LK and *(ii)* the calculus for sceptical autoepistemic reasoning has exponential lower bounds to the size and length of proofs.

These results are interesting to compare with previous findings for default logic—another principal approach in nonmonotonic logic. In a wider attempt to a proof-theoretic formalisation of nonmonotic logics, Bonatti and Olivetti [5] also devise calculi for default logic which were proof-theoretically analysed in [2, 10]. Default logic is known to admit a very close relation to autoepistemic logic via translations [13], but these are not directly applicable to transfer proof complexity results from default logic to the autoepistemic calculi. Our findings on autoepistemic logic in the present paper confirm results from [2] where the authors establish a similar polynomial dependence between proof lengths in LK and credulous default reasoning. Combining results from [2] with Theorem 4 of this paper, we can infer that credulous reasoning in default and in autoepistemic logic have the same complexity in theorem proving. On the other hand, [2] also provides an unconditional exponential lower bound for sceptical default reasoning. This reveals an interesting general picture for nonmonotonic logics: while credulous reasoning is equivalent to classical reasoning in terms of lengths of proofs, lower bounds are easier to obtain for sceptical reasoning. We comment further on the broader picture in Section 5.

This paper is organised as follows. In Sect. 2 we start with some background information on autoepistemic logic and proof systems. Our results on the proof complexity of credulous and sceptical autoepistemic reasoning follow in Sects. 3 and 4, respectively. In Sect. 5, we conclude with a discussion and open questions.

2 Preliminaries

We assume familiarity with propositional logic and basic notions from complexity theory (cf. [16]). By $\mathcal{L}$ we denote the set of all propositional formulas over some fixed standard set of connectives. By $\top$ and $\bot$ we denote the logical constants true and false, respectively. For formulas φ, σ, θ, the notation $\varphi[\sigma/\theta]$ means that all occurrences of subformulas σ in φ are replaced by θ.

Autoepistemic Logic. Autoepistemic logic is an extension of classical logic that has been proposed by Moore [19]. The logic is non-monotone in the sense that an increase in information may decrease the number of consequences. The language of autoepistemic logic $\mathcal{L}^{ae}$ consists of the language $\mathcal{L}$ of classical propositional logic augmented by an unary modal operator L. Intuitively, for a formula φ, the formula $L\varphi$ means that φ is *believed* by a rational agent. We emphasize that L-operators could be nested. Classical propositional formulas without occurrence of L are called *objective* formulas. A set of *premises* is a finite set of $\mathcal{L}^{ae}$ formulas.

Propositional assignments are extended to assignments for autoepistemic logic by considering all formulas of the form $L\varphi$ as propositional atoms, *i.e.*, in autoepistemic logic an assignment is a mapping from all propositional variables and formulas $L\varphi$ to $\{0,1\}$. This yields an immediate extension of the consequence relation to autoepistemic logic: if $\Phi \subseteq \mathcal{L}^{ae}$ and $\varphi \in \mathcal{L}^{ae}$, then $\Phi \models \varphi$ iff φ is true under every assignment which satisfies all formulas from Φ. As in classical logic we define $Th(\Phi) = \{\varphi \in \mathcal{L}^{ae} \mid \Phi \models \varphi\}$.

The main semantical notion in autoepistemic logic are *stable expansions* which correspond to all possible views an ideally rational agent might adopt on the knowledge of some set of premises $\Sigma \subseteq \mathcal{L}^{ae}$. Formally, a stable expansion of $\Sigma \subseteq \mathcal{L}^{ae}$ was defined by Moore [19] as a set $\Delta \subseteq \mathcal{L}^{ae}$ satisfying the fixed-point equation

$$\Delta = Th\left(\Sigma \cup \{L\varphi \mid \varphi \in \Delta\} \cup \{\neg L\varphi \mid \varphi \notin \Delta\}\right).$$

Informally, a stable expansion corresponds to a possible view of an agent, allowing him to derive all statements of his view from the given premises Σ together with his believes and disbelieves.

A set of premises Σ can have none or several stable expansions. A sentence $\varphi \in \mathcal{L}^{ae}$ is *credulously* entailed by Σ if φ holds in *some* stable expansion of Σ. If φ holds in *every* expansion of Σ, then φ is *sceptically* entailed by Σ. We give some examples which will be important later on.

Example 1. (a) If the premises Σ only consist of objective formulas, then Σ has exactly one stable expansion, namely the deductive closure of Σ (together with closure under L) if Σ is consistent and $\mathcal{L}^{ae}$ if Σ is inconsistent. *(b)* The set $\{p \leftrightarrow Lp\}$ has two stable expansions, one containing p and Lp and the other containing both $\neg p$ and $\neg Lp$. *(c)* The set $\{Lp\}$ has no stable expansion.

Proof Systems. Cook and Reckhow [8] defined the notion of a *proof system* for an arbitrary language L as a polynomial-time computable function f with range L. A string w with $f(w) = x$ is called an *f-proof* for $x \in L$. Proof systems for $L = \mathrm{TAUT}$ are called *propositional proof systems*. The sequent calculus LK of Gentzen [11] is one of the most important and best studied propositional proof systems. It is well known that LK and Frege systems mutually p-simulate each other (cf. [16] for background information on proof systems and definitions of LK and Frege).

There are two measures which are of primary interest in proof complexity. The first is the minimal *size* of an f-proof for some given element $x \in L$. To make this precise, let $s^*_f(x) = \min\{|w| \mid f(w) = x\}$ and $s_f(n) = \max\{s^*_f(x) \mid |x| \leq n\}$. We say that the proof system f is *t-bounded* if $s_f(n) \leq t(n)$ for all $n \in \mathbb{N}$. If t is a polynomial, then f is called *polynomially bounded*. Another interesting parameter of a proof is the *length* defined as the number of proof steps. This measure only makes sense for proof systems where proofs consist of lines containing formulas or sequents. This is the case for LK and most systems studied in this paper. For such a system f, we let $t^*_f(\varphi) = \min\{k \mid f(\pi) = \varphi \text{ and } \pi \text{ uses } k \text{ steps}\}$ and $t_f(n) = \max\{t^*_f(\varphi) \mid |\varphi| \leq n\}$. Obviously, it holds that $t_f(n) \leq s_f(n)$, but the two measures are even polynomially related for a number of natural systems as extended Frege (cf. [16]).

The Antisequent Calculus. Bonatti and Olivetti's calculi for autoepistemic logic use three main ingredients: classical propositional sequents and rules of LK, antisequents to refute formulas, and autoepistemic rules. In this section we introduce Bonatti's *antisequent calculus* AC from [4]. In AC we use *antisequents* $\Gamma \nvdash \Delta$, where $\Gamma, \Delta \subseteq \mathcal{L}$. Semantically, $\Gamma \nvdash \Delta$ is true if there exists an assignment satisfying $\bigwedge \Gamma$ and falsifying $\bigvee \Delta$. Axioms of AC are all sequents $\Gamma \nvdash \Delta$, where Γ and Δ are disjoint sets of propositional variables. The inference rules of AC are shown in Fig. 1. Bonatti [4] shows soundness and completeness of the calculus

$$\frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma, \neg\alpha \nvdash \Sigma}\ (\neg \nvdash) \qquad \frac{\Gamma, \alpha \nvdash \Sigma}{\Gamma \nvdash \Sigma, \neg\alpha}\ (\nvdash \neg)$$

$$\frac{\Gamma, \alpha, \beta \nvdash \Sigma}{\Gamma, \alpha \wedge \beta \nvdash \Sigma}\ (\wedge \nvdash) \qquad \frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma \nvdash \Sigma, \alpha \wedge \beta}\ (\nvdash \bullet\wedge) \qquad \frac{\Gamma \nvdash \Sigma, \beta}{\Gamma \nvdash \Sigma, \alpha \wedge \beta}\ (\nvdash \wedge\bullet)$$

$$\frac{\Gamma \nvdash \Sigma, \alpha, \beta}{\Gamma \nvdash \Sigma, \alpha \vee \beta}\ (\nvdash \vee) \qquad \frac{\Gamma, \alpha \nvdash \Sigma}{\Gamma, \alpha \vee \beta \nvdash \Sigma}\ (\bullet\vee \nvdash) \qquad \frac{\Gamma, \beta \nvdash \Sigma}{\Gamma, \alpha \vee \beta \nvdash \Sigma}\ (\vee\bullet \nvdash)$$

$$\frac{\Gamma, \alpha \nvdash \Sigma, \beta}{\Gamma \nvdash \Sigma, \alpha \rightarrow \beta}\ (\nvdash\rightarrow) \qquad \frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma, \alpha \rightarrow \beta \nvdash \Sigma}\ (\bullet \rightarrow\nvdash) \qquad \frac{\Gamma, \beta \nvdash \Sigma}{\Gamma, \alpha \rightarrow \beta \nvdash \Sigma}\ (\rightarrow \bullet \nvdash)$$

Fig. 1. Inference rules of the antisequent calculus AC.

AC. Proofs in the antisequent calculus are always short as observed in [2] (the bounds are not stated explicitly, but are implicit in the proof):

Proposition 2 (contained in [2]). *$s_{AC}(n) \leq n^2$ and $t_{AC}(n) \leq n$.*

The polynomial upper bounds on the complexity of AC are not surprising, since, to prove $\Gamma \nvdash \Delta$ we could alternatively guess assignments to the propositional variables in Γ and Δ and thereby verify antisequents in NP.

3 Proof Complexity of Credulous Autoepistemic Reasoning

We can now describe the calculus *CAEL* of Bonatti and Olivetti [5] for credulous autoepistemic reasoning. A *credulous autoepistemic sequent* is a 3-tuple $\langle \Sigma, \Gamma, \Delta \rangle$, denoted by $\Sigma; \Gamma \mid\!\sim \Delta$, where Σ, Γ, and Δ are sets of $\mathcal{L}^{ae}$-formulas. Moreover, all formulas of Σ are of the form $L\alpha$ or $\neg L\alpha$ and are called *provability constraints*. Semantically, the sequent $\Sigma; \Gamma \mid\!\sim \Delta$ is true, if there exists a stable expansion E of Γ which satisfies all of the constraints in Σ (*i.e.*, $E \models \Sigma$) and $\bigvee \Delta \in E$. The calculus *CAEL* uses credulous autoepistemic sequents and extends LK and AC by the inference rules shown in Fig. 2. Bonatti and Olivetti [5] show soundness and completeness of *CAEL*.

$$(\mathbf{cA1})\ \frac{\Gamma \vdash \Delta}{;\ \Gamma \mid\!\sim \Delta}\ (\Gamma \cup \Delta \subseteq \mathcal{L})$$

$$(\mathbf{cA2})\ \frac{\Gamma \vdash \alpha \qquad \Sigma;\ \Gamma \mid\!\sim \Delta}{L\alpha,\ \Sigma;\ \Gamma \mid\!\sim \Delta}\ (\alpha \in \mathcal{L})$$

$$(\mathbf{cA3})\ \frac{\Gamma \not\vdash \alpha \qquad \Sigma;\ \Gamma \mid\!\sim \Delta}{\neg L\alpha,\ \Sigma;\ \Gamma \mid\!\sim \Delta}\ (\Gamma \cup \{\alpha\} \subseteq \mathcal{L})$$

$$(\mathbf{cA4})\ \frac{\neg L\alpha,\ \Sigma;\ \Gamma[L\alpha/\bot] \mid\!\sim \Delta[L\alpha/\bot]}{\Sigma;\ \Gamma \mid\!\sim \Delta} \qquad (\mathbf{cA5})\ \frac{L\alpha,\ \Sigma;\ \Gamma[L\alpha/\top] \mid\!\sim \Delta[L\alpha/\top]}{\Sigma;\ \Gamma \mid\!\sim \Delta}$$

In rules (**cA4**) and (**cA5**) $L\alpha$ is a subformula of $\Gamma \cup \Delta$ and $\alpha \in \mathcal{L}$.

Fig. 2. Inference rules for the credulous autoepistemic calculus *CAEL*

Theorem 3 (Bonatti, Olivetti [5]). *A credulous autoepistemic sequent is true if and only if it is derivable in CAEL.*

We now investigate the complexity of proofs in *CAEL*, showing a very tight connection to proof size and length in the classical sequent calculus *LK*.

Theorem 4. *CAEL obeys almost the same bounds on proof size and number of proof steps as LK, more precisely:* $s_{LK}(n) \leq s_{CAEL}(n) \leq n(s_{LK}(n) + n^2 + n)$ *and* $t_{LK}(n) \leq t_{CAEL}(n) \leq n(t_{LK}(n) + n + 1)$.

Proof. In the following we will explain all sequent proofs "backwards", i.e., we start the description with the rule that is immediately applied to derive the proven sequent and progress bottom up until we reach initial sequents or axioms. For the first inequality $s_{LK}(n) \leq s_{CAEL}(n)$ (and similarly $t_{LK}(n) \leq t_{CAEL}(n)$) it suffices to observe that each *CAEL*-proof of a sequent $\Gamma \mid\!\sim \Delta$ with $\Gamma \cup \Delta \subseteq \mathcal{L}$ consists of one application of rule (**cA1**) followed by an *LK*-derivation of $\Gamma \vdash \Delta$. This holds as rules (**cA2**) to (**cA5**) are only applicable if $\Gamma \mid\!\sim \Delta$ contain at least one occurrence of the L-operator.

We will now prove the remaining upper bounds, starting with $t_{CAEL}(n) \leq n(t_{LK}(n)+n+1)$. For $\alpha \in \mathcal{L}^{ae}$ we denote by $LC(\alpha)$ the number of occurrences of L in α. We extend this notation to $\Delta \subseteq \mathcal{L}^{ae}$ by defining $LC(\Delta) = \sum_{\alpha \in \Delta} LC(\alpha)$. Let $\Sigma; \Gamma \mid\!\sim \Delta$ be a true credulous autoepistemic sequent of total size n (as a string). We will construct a *CAEL*-derivation of $\Sigma; \Gamma \mid\!\sim \Delta$ starting from the bottom with the given sequent. We first claim that we can normalise the proof such that we start (always bottom-up) by eliminating all subformulas $L\alpha$ in $\Gamma \cup \Delta$ by using rules (**cA4**) and (**cA5**) and then use rules (**cA2**) and (**cA3**) to eliminate all provability constraints. Finally, one application of rule (**cA1**) follows. Thus the normalised proof will look as in Fig. 3. Let us argue that this normalisation is possible. By Theorem 3 there exists a proof Π of $\Sigma; \Gamma \mid\!\sim \Delta$. At its top Π must contain exactly one application of (**cA1**). The rest of the proof are applications of (**cA2**) to (**cA5**). As (**cA2**) and (**cA3**) do not alter the part $\Gamma \mid\!\sim \Delta$ of the sequent, they can be freely interchanged with applications of (**cA4**) and (**cA5**). This yields a normalised proof of the same size as Π.

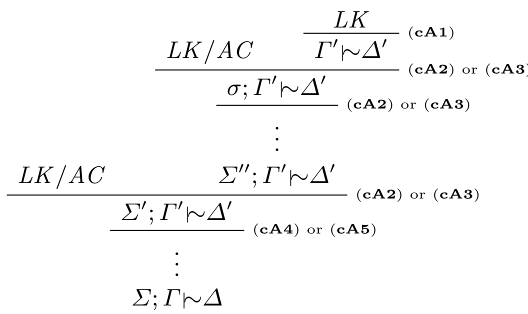

Fig. 3. The structure of the *CAEL*-proof in Theorem 4. LK/AC denotes a proof in either LK or AC, LK denotes an LK-derivation, and σ is the last remaining constraint from Σ' after applications of (**cA2**) and (**cA3**).

We now estimate the length of this normalised proof. Eliminating all subformulas $L\alpha$ in $\Gamma \cup \Delta$ needs at most $LC(\Gamma \cup \Delta)$ applications of rules (**cA4**) and (**cA5**). The number of steps needed could be less than $LC(\Gamma \cup \Delta)$ as one step might delete several instances of $L\alpha$. After this process we obtain a sequent $\Sigma'; \Gamma' \mid\!\sim \Delta'$ with $\Gamma' \cup \Delta' \subseteq \mathcal{L}$ and $|\Sigma'| \leq |\Sigma| + LC(\Gamma \cup \Delta) < n$. From this point on we use rules (**cA2**) and (**cA3**) until we have eliminated all constraints and then finally apply rule (**cA1**) once. This will result in $|\Sigma'| + 1 \leq n$ applications of rules (**cA1**) to (**cA3**). Each of these applications will invoke either an LK or an AC derivation of the left premise, but all these derived formulas are either from Σ or subformulas of Γ or Δ. Therefore all these LK and AC-derivations are used to prove formulas of size $\leq n$. To estimate the lengths of AC-proofs we use Proposition 2. In total this gives $\leq n(t_{LK}(n) + n + 1)$ steps to prove $\Sigma; \Gamma \mid\!\sim \Delta$.

The bound for s_{CAEL} follows as each of the $< n$ applications of (**cA4**) and (**cA5**) leads to a sequent of size $\leq n$ and therefore this part of the proof is of size $\leq n^2$. Each of the (**cA2**) and (**cA3**) applications shortens the sequent $\Sigma'; \Gamma' \mid\!\sim \Delta'$ which is of size $\leq n$ and incurs an LK or AC-derivation of a sequent of size $\leq n$. Using Proposition 2 and taking account of the final (**cA1**) application this contributes at most $n(s_{LK}(n) + n^2)$ to the size of the overall proof. □

In the light of this result, proving either non-trivial lower or upper bounds to the proof size of *CAEL* seems very difficult as such a result would directly imply a corresponding bound for LK which is know to be equivalent with respect to proof size to Frege systems. Showing any non-trivial lower bound for Frege is one of the hardest challenges in propositional proof complexity and this problem has been open for decades (cf. [4, 16]).

The connection between proof size in classical LK and credulous autoepistemic logic has further consequences. In particular, it allows to transfer intractability results from classical logic to autoepstemic reasoning. *Automatizability* asks whether proofs can be efficiently constructed, *i.e.*, whether a proof of φ in a proof system P can be found in polynomial time in the length of the shortest P-proof of φ [6]. Of course automatizability of a proof system is very desirable from a practical point of view. However, most known classical proof systems are not automatizable under cryptographic or complexity-theoretic assumptions. In particular, Bonet, Pitassi, and Raz [6] showed that Frege systems are not automatizable unless Blum integers can be factored in polynomial time (a Blum integer is the product of two primes which are both congruent 3 modulo 4). Frege systems are known to be equivalent to LK [8]. As credulous autoepistemic reasoning extends LK this result easily transfers to credulous autoepistemic reasoning:

Corollary 5. *CAEL is not automatizable unless factoring integers is possible in polynomial time.*

The same result also holds for the sceptical autoepistemic calculus analysed in the next section.

4 Lower Bounds for Sceptical Autoepistemic Reasoning

Bonatti and Olivetti [5] also introduce a calculus for sceptical autoepistemic reasoning. In contrast to the credulous calculus, sequents are simpler as they only consist of two components $\Gamma, \Delta \subseteq \mathcal{L}^{ae}$. An *SAEL* sequent is such a pair $\langle \Gamma, \Delta \rangle$, denoted by $\Gamma \mid\!\sim \Delta$. Semantically, the *SAEL* sequent $\Gamma \mid\!\sim \Delta$ is true, if $\bigvee \Delta$ holds in *all* expansions of Γ.

To give the definition of the *SAEL* calculus of Bonatti and Olivetti [5] we need some notation. An L-subformula of an $\mathcal{L}^{ae}$-formula φ is a subformula of φ of the form $L\theta$. By $LS(\varphi)$ we denote the set of all L-subformulas of φ. $ELS(\varphi)$ denotes the set of all *external* L-subformulas of φ, *i.e.*, all L-subformulas of φ that do not occur in the scope of another L-operator. The notation is extended to sets of formulas Φ by $LS(\Phi) = \bigcup_{\varphi \in \Phi} LS(\varphi)$ and $ELS(\Phi) = \bigcup_{\varphi \in \Phi} ELS(\varphi)$. We say that a set $\Gamma \subseteq \mathcal{L}^{ae}$ is *complete* with respect to $\Sigma \subseteq \mathcal{L}^{ae}$ if for all $\varphi \in \Sigma$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Bonatti and Olivetti's [5] calculus *SAEL* consists of the defining axioms and inference rules of LK and AC together with the rules shown in Fig. 4. Bonatti and Olivetti show soundness and completeness of this calculus for sceptical autoepistemic reasoning:

Theorem 6 (Bonatti, Olivetti [5]). *An SAEL sequent $\Gamma \mid\!\sim \Delta$ is derivable in SAEL if and only if it is true.*

Let us comment a bit on the rules in Fig. 4. In rule (**sA2**), if $\neg L\alpha, \Gamma \mid\!\sim \alpha$ is true, then $\neg L\alpha, \Gamma$ has no stable expansion and thus $\neg L\alpha, \Gamma \mid\!\sim \Delta$ vacuously holds. The same applies to rule (**sA3**). If $L\alpha, \Gamma \not\mid\!\approx \alpha$ holds, then $L\alpha, \Gamma$ does not have any stable expansion and $L\alpha, \Gamma \mid\!\sim \Delta$ is true (cf. [5, Theorem 5.14] for a detailed argument). Thus, to derive a sequent $\Gamma \mid\!\sim \Delta$ where the antecedent Γ has a stable

$$(\mathbf{sA1})\ \frac{\Gamma \vdash \Delta}{\Gamma \mathrel{|\!\sim} \Delta} \qquad (\mathbf{sA2})\ \frac{\neg L\alpha, \Gamma \mathrel{|\!\sim} \alpha}{\neg L\alpha, \Gamma \mathrel{|\!\sim} \Delta} \qquad (\mathbf{sA3})\ \frac{L\alpha, \Gamma \not\vdash \alpha}{L\alpha, \Gamma \mathrel{|\!\sim} \Delta}$$

where $\Gamma \cup \{L\alpha\}$ is complete wrt. $ELS(\Gamma \cup \{\alpha\})$ in rule (**sA3**)

$$(\mathbf{sA4})\ \frac{L\alpha, \Gamma \mathrel{|\!\sim} \Delta \qquad \neg L\alpha, \Gamma \mathrel{|\!\sim} \Delta}{\Gamma \mathrel{|\!\sim} \Delta}\ (L\alpha \in LS(\Gamma \cup \Delta))$$

Fig. 4. Inference rules for the sceptical autoepistemic calculus *SAEL*.

expansion, we can only use one of the rules (**sA1**) or (**sA4**) to immediately get $\Gamma \mathrel{|\!\sim} \Delta$. Note that rule (**sA1**) is quite powerful. Not only can it be used to derive sequents $\Gamma \mathrel{|\!\sim} \Delta$ with Γ and Δ comprising of only classical formulas, but it also applies to autoepistemic sequents $\Gamma \mathrel{|\!\sim} \Delta$ if L-subformulas are treated as propositional atoms. We give an example.

Example 7. Let Γ_n be the sequence $p_1 \leftrightarrow Lp_1, \ldots, p_n \leftrightarrow Lp_n, q$ and $\Delta_n = p_1 \vee Lq$. We obtain the derivation in Fig. 5. Neither Γ_n nor Δ_n consist of classical formulas and $\Gamma_n \vdash \Delta_n$ are no true classical sequents, but still (**sA1**) together with the omitted LK-derivations guarantee short proofs. Note that Γ_n has 2^n stable expansions (cf. also Example 1), but still the overall proofs of $\Gamma_n \mathrel{|\!\sim} \Delta_n$ are of linear length.

$$\dfrac{\dfrac{\dfrac{LK}{Lq, (p_i \leftrightarrow Lp_i)_{i\in[n]}, q \vdash p_1 \vee Lq}}{Lq, (p_i \leftrightarrow Lp_i)_{i\in[n]}, q \mathrel{|\!\sim} p_1 \vee Lq}\,(\mathbf{sA1}) \qquad \dfrac{\dfrac{\dfrac{LK}{\neg Lq, (p_i \leftrightarrow Lp_i)_{i\in[n]}, q \vdash q}}{\neg Lq, (p_i \leftrightarrow Lp_i)_{i\in[n]}, q \mathrel{|\!\sim} q}\,(\mathbf{sA1})}{\neg Lq, (p_i \leftrightarrow Lp_i)_{i\in[n]}, q \mathrel{|\!\sim} p_1 \vee Lq}\,(\mathbf{sA2})}{(p_i \leftrightarrow Lp_i)_{i\in[n]}, q \mathrel{|\!\sim} p_1 \vee Lq}\,(\mathbf{sA4})$$

Fig. 5. Derivation of $\Gamma_n \mathrel{|\!\sim} \Delta_n$ in Example 7

In our next result we will show an exponential lower bound to the proof length (and therefore also to the proof size) in the sceptical calculus *SAEL*.

Theorem 8. *There exist sequents S_n of size $\mathcal{O}(n)$ such that every SAEL-proof of S_n has $2^{\Omega(n)}$ steps. Therefore, $s_{SAEL}(n), t_{SAEL}(n) \in 2^{\Omega(n)}$.*

Proof. Let Γ_n consist of the formulas $p_i \leftrightarrow Lp_i$, $p_i \leftrightarrow q_i$ with $i = 1, \ldots, n$ and $\Delta_n = \bigwedge_{i=1}^{n} (Lp_i \leftrightarrow Lq_i)$. We will prove that each *SAEL*-proof of $\Gamma_n \mathrel{|\!\sim} \Delta_n$ contains 2^n applications of rule (**sA4**). Consider now sequents

$$(Lp_i : i \in I_p^+), (\neg Lp_i : i \in I_p^-), (Lq_i : i \in I_q^+), (\neg Lq_i : i \in I_q^-), \Gamma_n \mathrel{|\!\sim} \Delta_n \qquad (1)$$

where $I_p^+, I_p^-, I_q^+, I_q^- \subseteq [n]$ and $I_p^+ \cap I_p^- = I_q^+ \cap I_q^- = \emptyset$. If additionally $I_p^+ \cap I_q^- = I_p^- \cap I_q^+ = \emptyset$, we call a sequent of the form (1) a *k-sequent* for $k = \big|[n] \setminus (I_p^+ \cup I_p^- \cup I_q^+ \cup I_q^-)\big|$.

For a variable p let us denote by p^1 the variable p while p^{-1} stands for $\neg p$. We first note that each antecendent Γ of a k-sequent $\Gamma \mid\!\sim \Delta$ has exactly 2^k stable expansions. Let $J = [n] \setminus (I_p^+ \cup I_p^- \cup I_q^+ \cup I_q^-)$ be the index set corresponding to the L-subformulas which are not already fixed by the antecedent. Then the stable expansions of Γ are generated by $p_j^{e_j}, q_j^{e_j}$ with $j \in J$ (together with $(p_i, q_i : i \in I_p^+ \cup I_q^+)$ and $(p_i^{-1}, q_i^{-1} : i \in I_p^- \cup I_q^-)$) where the variables $(e_j)_{j \in J}$ range over all 2^k elements of $\{-1, 1\}^k$.

We will now prove the following claim:

Claim. For all $k = 1, \dots, n$, each *SAEL*-proof Π of $\Gamma_n \mid\!\sim \Delta_n$ contains at least 2^k $(n-k)$-sequents. Moreover, all of these $(n-k)$-sequents appear as a premise of an application of (**sA4**) which has a $(n-k+1)$-sequent as its consequence.

For $k = n$ this claim yields the desired lower bound.

We prove the claim by induction on k. For the base case $k = 1$ observe that $\Gamma_n \mid\!\sim \Delta_n$ is an n-sequent. We first determine which rule which was used in the proof Π to derive $\Gamma_n \mid\!\sim \Delta_n$. The antecedent Γ_n has 2^n stable expansions. Therefore, $\Gamma_n \mid\!\sim \Delta_n$ cannot have been derived by either rule (**sA2**) or (**sA3**) (cf. the discussion before Example 7). Likewise, $\Gamma_n \mid\!\sim \Delta_n$ is not derivable by (**sA1**). This is so because even considering all subformulas Lp_i, Lq_i as propositional atoms, $\Gamma_n \mid\!\sim \Delta_n$ is not a true propositional sequent. Therefore $\Gamma_n \mid\!\sim \Delta_n$ is derived by an application of (**sA4**) by branching over some L-subformula Lp_i or Lq_i. This yields two distinct $(n-1)$-sequents.

For the inductive step let $\Gamma' \mid\!\sim \Delta'$ be a $(n-k)$-sequent in Π which appears as a premise of an application of (**sA4**) and has a $(n-k+1)$-sequent as its consequence. Let us determine which rule which was used in the proof Π to derive $\Gamma' \mid\!\sim \Delta'$. As $\Gamma' \mid\!\sim \Delta'$ is a $(n-k)$-sequent, its antecedent Γ' has 2^{n-k} stable expansions (see above). Therefore, $\Gamma' \mid\!\sim \Delta'$ cannot have been derived by either rule (**sA2**) or (**sA3**) (cf. the discussion before Example 7). Likewise, $\Gamma' \mid\!\sim \Delta'$ is not derivable by (**sA1**). This is so because even considering all subformulas Lp_i, Lq_i as propositional atoms, $\Gamma' \mid\!\sim \Delta'$ is not a true propositional sequent. Its succedent Δ' contains subformulas $Lp_i \leftrightarrow Lq_i$, $i \in [n] \setminus (I_p^+ \cup I_p^- \cup I_q^+ \cup I_q^-)$ which are not propositionally implied by the antecedent Γ'. Therefore $\Gamma' \mid\!\sim \Delta'$ is derived by an application of (**sA4**) branching over an L-subformula Lx_i of $\Gamma' \cup \Delta'$ where x_i stands for either p_i or q_i. There are three cases according to the choice of variable x_i.

Case 1: $i \in I_p^+ \cap I_q^+$ or $i \in I_p^- \cap I_q^-$. In this case applying (**sA4**) yields two sequents, one of them a sequent with contradictory formulas in the antecedent, the other one again a $(n-k)$-sequent which deviates from $\Gamma' \mid\!\sim \Delta'$ only in that Lx_i occurs repeatedly in Γ'. As this only increases the size of the overall proof, Case 1 does not occur in proofs of minimal size.

Case 2: $i \in I_p^+ \triangle I_q^+$ or $i \in I_p^- \triangle I_q^-$.[1] As both cases are symmetric let us assume $i \in I_p^+ \triangle I_q^+$. Then (**sA4**) yields the two sequents $Lx_i, \Gamma' \mid\!\sim \Delta'$ and $\neg Lx_i, \Gamma' \mid\!\sim \Delta'$. The latter sequent $\neg Lx_i, \Gamma' \mid\!\sim \Delta'$ contains either both $\neg Lp_i$ and Lq_i (if $x_i = p_i$) or both of Lq_i and $\neg Lp_i$ (if $x_i = q_i$) in its antecedent. Therefore the antecedent is even propositionally unsatisfiable and hence the sequent $\neg Lx_i, \Gamma' \mid\!\sim \Delta'$ can be proven by an LK-derivation followed by (**sA1**).

The first sequent $Lx_i, \Gamma' \cup \Delta'$ is again a $(n-k)$-sequent (which, however, does not fulfil the second sentence of the inductive claim). We apply again our previous argument to this sequent: it must have been derived by (**sA4**). This application might fall again under Case 2, but this can only occur a constant number of times and eventually we will get an application of (**sA4**) to a $(n-k)$-sequent according to the only remaining Case 3.

Case 3: $i \in [n] \setminus (I_p^+ \cup I_p^- \cup I_q^+ \cup I_q^-)$. In this case (**sA4**) produces two ancestor sequents $Lx_i, \Gamma' \mid\!\sim \Delta'$ and $\neg Lx_i, \Gamma' \mid\!\sim \Delta'$. Both of these are $(n-k-1)$-sequents and also fulfil the second condition of the inductive claim.

As we have seen, all three cases start with a $(n-k)$-sequent and lead to two $(n-k-1)$-sequents, and all of these sequents fulfil the second condition of the inductive claim. By the induction hypothesis, Π contains 2^k many $(n-k)$-sequents. All of these are derived by one or more applications of (**sA4**) from prerequisite $(n-k-1)$-sequents which are mutually distinct. Thus Π contains 2^{k+1} many $(n-k-1)$-sequents, completing the argument. □

We point out that our argument does not only work against tree-like proofs, but also rules out sub-exponential dag-like derivations for $\Gamma_n \mid\!\sim \Delta_n$. Thus, while dag-like derivations are typically shorter we also obtain an exponential lower bound in this stronger model.

5 Conclusion and Discussion

In this paper we have shown that with respect to lengths of proofs, proof systems for credulous autoepistemic reasoning and for propositional logic are very close to each other. On the other hand, we demonstrated exponential bounds for sceptical autoepistemic reasoning in the natural calculus of [5]. Such bounds are completely out of reach for the calculus LK in propositional logic. This situation closely resembles our findings for propositional default logic [2]. Credulous reasoning is Σ_2^{p}-complete for both default logic and autoepistemic logic while the sceptical reasoning tasks are both Π_2^{p}-complete as shown by Gottlob [12] (cf. also [3, 9] for a refined analysis). Can this common underlying complexity of the decision problems serve as explanation for the similarities in proof complexity of these logics?

Let us dwell a bit on this theme. Although deciding credulous autoepistemic sequents is presumably harder than deciding tautologies (the former is Σ_2^{p}-complete [12], while the latter is complete for coNP), the difference disappears when we want to prove these objects. This becomes most apparent when

[1] Here $\triangle$ denotes symmetric difference, defined as $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

we consider polynomially bounded proof systems: by the classical theorem of Cook and Reckhow [8], polynomially bounded propositional proof systems exist if and only if $\mathsf{NP} = \mathsf{coNP}$, while credulous autoepistemic reasoning (or any logic with a Σ_2^{p}-complete decision problem) has polynomially bounded proof systems if and only if $\mathsf{NP} = \Sigma_2^{\mathsf{p}}$. However, the assertions $\mathsf{NP} = \mathsf{coNP}$ and $\mathsf{NP} = \Sigma_2^{\mathsf{p}}$ are equivalent and this also extends to other proof lengths:

Proposition 9. *Let L be a language in Σ_2^{p} and let f be any monotone function. Then* TAUT $\in \mathsf{NTIME}(f(n))$ *implies* $L \in \mathsf{NTIME}(p(n)f(p(n)))$ *for some polynomial p. In other words, for each propositional proof system P with $s_P(n) \leq f(n)$ there exists a proof system P' for L with $s_{P'}(n) \leq p(n)f(p(n))$.*

Proof. If $L \in \Sigma_2^{\mathsf{p}}$, then there exists a polynomial-time nondeterministic oracle Turing machine M which decides L under oracle access to TAUT. Assume now TAUT $\in \mathsf{NTIME}(f(n))$ via NTM N. We build an NTM N' for L by simulating M and replacing each oracle query θ to TAUT by the following nondeterministic procedure. Guess the answer to query θ. If the answer is yes, then simulate $N(\theta)$ and check that it accepts. Otherwise, if the answer is no, then guess an assignment α and verify that α satisfies $\neg\theta$. If p is the polynomial bounding the running time of M, then each oracle query is of size $\leq p(n)$ and there can be at most $p(n)$ such queries. Therefore the running time of N' is bounded by $p(n)f(p(n))$. The second claim follows as each nondeterministic machine for L can be converted into a proof system for L (and vice versa). □

This observation implies that from each propositional proof system P we can obtain a proof system for credulous autoepistemic logic which obeys almost the same bounds on the proof size. Theorem 4 tells us that the proof system for credulous autoepistemic reasoning constructed by this general method from LK is essentially the sequent calculus $CAEL$ of Bonatti and Olivetti [5].

For *sceptical* autoepistemic (or default) reasoning—both of them Π_2^{p}-complete [12]—the situation is less clear. To the best of our knowledge it is not known whether a similar result as Proposition 9 holds for $L \in \Pi_2^{\mathsf{p}}$. While sceptical autoepistemic reasoning has polynomially bounded proof systems if and only if this holds for TAUT (because $\mathsf{NP} = \mathsf{coNP}$ iff $\mathsf{NP} = \Pi_2^{\mathsf{p}}$), we leave open whether this equivalence between extends to other bounds. Thus it is conceivable that lower bounds for sceptical reasoning are generally easier to obtain. This phenomenon particularly occurs with non-classical logics of even higher complexity as modal and intuitionistic logics which typically are PSPACE-complete and where exponential lower bounds are known for Frege and even extended Frege systems in these logics [14, 15].

In conclusion, the sequent calculi of Bonatti and Olivetti for credulous reasoning (both default and autoepistemic) are as good as one can hope for from a proof complexity perspective, whereas the calculi for sceptical reasoning call for stronger versions. This presents the double challenge of designing systems which are both natural and elegant and allow concise proofs. We remark that Krajíček and Pudlák [17] introduced very elegant sequent calculi G_i for quantified propositional logic, thus for logics with decision complexity ranging from

Σ_2^{p} and Π_2^{p} through all the polynomial hierarchy up to PSPACE. However, no nontrivial lower bounds are known for these systems. As sceptical autoepistemic reasoning is Π_2^{p}-complete one could translate *SAEL*-sequents into propositional $\forall\exists$-formulas and use the sequent calculus G_2 from [17] (cf. also [7]).

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