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A CIRCULAR ORDER ON EDGE-COLOURED TREES AND RNA m-DIAGRAMS

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Abstract. We study a circular order on labelled, m-edge-coloured trees with k vertices, and show that the set of such trees with a fixed circular order is in bijection with the set of RNA m-diagrams of degree k, combinatorial objects which can be regarded as RNA secondary structures of a certain kind. We enumerate these sets and show that the set of trees with a fixed circular order can be characterized as an equivalence class for the transitive closure of an operation which, in the case $m = 3$, arises as an induction in the context of interval exchange transformations.

1. Introduction

Interval exchange transformations are, roughly speaking, a generalization of a rotation of a circle. More precisely, given a partition of the unit interval into smaller segments, an interval exchange transformation is a map from the unit interval to itself where the segments are rearranged according to a choice of permutation. Recently there has been particular interest in the class of interval exchange transformations induced by the permutation $(k\ k-1\ \cdots\ 1)$. In [2] a combinatorial interpretation in terms of an induction proccess on labelled trees with edges coloured with 3 possible colours, called trees of relations, has proven a fruitful tool leading to new results in the understanding of the associated languages. In this paper we extend this combinatorics by defining a generalization of the induction process to the case of labelled trees with edges coloured with m possible colours and study its properties.

Given positive integers k and m, an *m-edge-coloured* tree with k vertices is a tree whose edges are labelled with m possible colours in such a way that no vertex is incident with two edges of the same colour. It is said to be *labelled* if its vertices are labelled with $\{1, 2, \dots, k\}$. Such trees were shown to possess a circular order in the case $m = 3$ in [2] corresponding to the permuation of the interval exchange transformation. By giving a new proof of this fact, we show that the circular order can be generalized to arbitrary m and we study its relation to the generalized induction proccess we define.

Firstly, we show that the set $\mathcal{T}_{k,m}$ of labelled m-edge-coloured trees with k vertices and a fixed circular order is in bijection with a collection of combinatorial objects consisting of collections of noncrossing arcs in a disk with coloured marked points on its boundary, satisfying a certain matching rule. Such diagrams can be regarded as generalized RNA secondary structures of a certain kind, in the sense of [13], drawn in the Nussinov circle representation [10], so we refer to them as *RNA m-diagrams of degree k*. Such a bijection was given for the case $m = 3$ in [2].

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We compute the cardinality of $\mathcal{T}_{k,m}$ (and thus also the number of RNA m-diagrams of degree k) using the well-known correspondence between trees and m-angulations of a polygon which induces a bijection with a collection of appropriately labelled m-angulations

We prove the existence of a transformation of labelled m-edge-coloured trees preserving the circular order. We call this transformation *generalized induction* as it generalizes the induction in the case $m = 3$, studied combinatorially in [2]. In [4] it was shown that this induction (for $m = 3$) gives rise to languages naturally generalizing Sturmian languages and arising from interval exchange transformations. We also give an example showing that the most straightforward generalization of this induction to the case $m > 3$ does not in general preserve the circular order.

We go on to show that the equivalence classes of the transitive closure of the generalized induction are characterized by the circular order. Thus the set of equivalence classes is in bijection with the set of k-cycles in the symmetric group of degree k.

Finally, motivated by snake-triangulations and m-snakes in cluster theory, we give an interpretation of the generalized induction process in terms of labelled m-angulations of polygons.

This article is structured as follows. In Section 2, we introduce the combinatorial objects considered in the paper. In Section 3 we give the bijections mentioned above and compute the cardinality of the set of labelled m-edge-coloured trees with a fixed circular order. In Section 4 we show that the circular order characterizes the equivalence classes of the transitive closure of the generalized induction and in section 5 we show that induction can be given by a composition of flips of diagonals in an m-angulation.

2. Some combinatorial objects

We first introduce the main combinatorial objects we will be considering: edge-coloured trees, RNA m-diagrams, and m-angulations.

Given positive integers k, m, we consider m-edge-coloured trees on k vertices, i.e. trees whose edges are coloured with one of the m symbols $S_1, S_2, \ldots, S_m$ in such a way that no two edges incident with the same vertex have the same colour. We say that such a tree is *labelled* if its vertices are labelled with $\{1, 2, \ldots, k\}$. As usual, if the connectedness assumption is not satisfied, we refer to the corresponding objects as forests. See Figure 3(b) for an example of a (rooted) m-edge-coloured tree and (c) for an example of a labelled m-edge-coloured tree.

Given an m-edge coloured tree $\mathcal{S}$ with k vertices, each symbol S_r determines a map (with the same name) from the set of vertices of $\mathcal{S}$ to itself. A vertex is fixed by S_r unless it is incident with an edge coloured S_r, in which case it is sent to the vertex at the other end of that edge. Let $\sigma_{\mathcal{S}}$ be the composition $S_m S_{m-1} \cdots S_1$: this is a permutation of the vertices of $\mathcal{S}$. We use the same definition for a labelled m-edge coloured tree, G, obtaining a permutation σ_G in the symmetric group of degree k. We refer to $\sigma_{\mathcal{S}}$ (respectively, σ_G) as the *circular order* of $\mathcal{S}$ (respectively, G). In the 3-symbol case these maps were considered in [2]. The circular order is always a k-cycle:

Lemma 2.1. *Let G be a labelled m-edge-coloured tree with k vertices. Then the circular order σ_G of G is a k-cycle.*

Proof. This is clearly true if $k = 1$ or 2. Suppose it holds for smaller values of k. Let v be a vertex of G which is not a leaf. Suppose that v is incident with edges $e_1, e_2, \ldots, e_d$ in G, coloured with $S_{r_1}, S_{r_2}, \ldots, S_{r_d}$ respectively, where $r_1 < r_2 < \cdots < r_d$. Let the end-points of these edges (other than v) be $v_1, v_2, \ldots, v_d$. Removing v from G leaves d subtrees $G_1', G_2', \ldots, G_d'$ incident with $v_1, v_2, \ldots, v_d$ respectively. Let G_i be the subtree G_i' with v and e_i reattached to v_i.

By the inductive hypothesis the σ_{G_i} are all cycles. Hence, repeatedly applying σ_{G_1} to v cycles through the vertices of G_1. Since $\sigma_G = \sigma_{G_1}$ on all vertices of G_1 except $w = \sigma_{G_1}^{-1}(v)$, repeatedly applying σ_G also cycles through all the vertices of G_1. Since r_2 is minimal such that S_{r_2} is a symbol colouring an edge incident with v with $r_2 > r_1$, $\sigma_G(w)$ will lie in G'_2. In fact $\sigma_G(w) = \sigma_{G_2}(v)$, since in G_2, v is not incident with any edge with symbol other than S_{r_2}. Repeatedly applying σ_G then cycles through the vertices of G'_2 before coming back to $\sigma_{G'_2}^{-1}(v)$. Repeating this argument, we see that repeatedly applying σ_G to v first cycles through G'_1, then through $G'_2, G'_3, \ldots, G'_d$ in order before eventually returning to v. □

Next, we define an *RNA m-diagram of degree k* as follows. We take k vertices $1, 2, \ldots, k$, numbered clockwise around a circle. Each vertex contains the symbols $S_1, S_2, \ldots, S_m$ written in clockwise order. A collection of arcs connect equal symbols at different vertices. A symbol at a vertex can be incident to at most one arc. Such a diagram is *noncrossing* if it can be drawn in such a way that there are no crossings between the arcs. We say that it is *connected* provided there is a path between any two vertices (where moving between symbols at a vertex is allowed). See Figure 3(a) for an example of a connected noncrossing RNA 4-diagram of degree 10.

Diagrams of this kind (but with different rules) have been considered in [1] in the context of Fuss-Catalan algebras. Such diagrams can be regarded as a certain kind of abstract RNA secondary structures in the sense of [13] (see also [8, §5]), drawn in the Nussinov circle representation [10].

Finally, let P be an $(m-2)k+2$-sided regular polygon. An m-angulation of P is a collection of diagonals dividing P into k m-gons. We say that an m-angulation of P is *diagonal-coloured* if each diagonal in P is coloured with a symbol from the set $\{S_1, S_2, \ldots, S_m\}$ in such a way that the colours on the sides of each m-sided polygon in the m-angulation are $S_1, S_2, \ldots, S_m$ in clockwise order. We say that it is *rooted* if there is a distinguished m-sided polygon in the m-angulation. We say that it is *m-gon-labelled* if the m-gons are labelled $1, 2, \ldots, k$.

3. Bijections

Our main aim in this section is to compute the number of connected noncrossing RNA m-diagrams of degree k and to give a bijective proof that this is the same as the number of labelled m-edge-coloured trees with k vertices and circular order $(k\ k-1\ \cdots\ 1)$.

Theorem 3.1. *There is a bijection between the following sets:*

(I) *The set of degree k noncrossing RNA m-diagrams.*

(II) *The set of labelled m-edge-coloured forests G with k vertices such that, on writing $G = \sqcup_i G_i$ as a union of connected components, we have the following:*
  - (a) *If $i \neq j$ and $a_1, a_2 \in G_i$, $b_1, b_2 \in G_j$, we cannot have that $a_1 > b_1 > a_2 > b_2$.*
  - (b) *If $a \in G_i$ for some i, then $\sigma_G(a)$ is the maximal vertex of G less than a lying in G_i (or, if no such vertex exists, it is the largest vertex of G lying in G_i).*

Proof. Let Σ be a noncrossing RNA m-diagram of degree k on m symbols as in (I).

Let G be the labelled m-edge-coloured graph with k vertices whose edges given by the arcs of Σ. That is, there is an edge between vertices i and j of G coloured with S_k if and only if there is an arc in Σ between the instances of the symbol S_k in vertices i and j in Σ.

Claim: G is a forest.

We prove the claim. Suppose, for a contradiction, there is a cycle

$$a_1 \overset{S_{i_1}}{\longrightarrow} a_2 \overset{S_{i_2}}{\longrightarrow} \cdots \qquad a_{p-1} \overset{S_{i_{p-1}}}{\longrightarrow} a_p \overset{S_{i_p}}{\longrightarrow} a_1$$

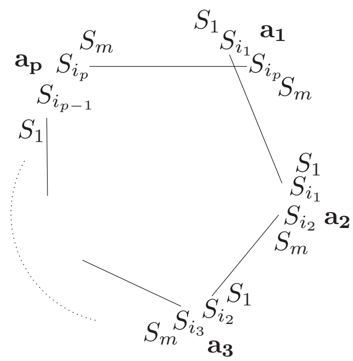


FIGURE 1. A cycle in G leads to a crossing

in G, and thus a corresponding cycle in Σ. Without loss of generality, we may assume that $i_1 < i_2$. For vertices a, b, we denote by (a, b) the set of vertices c of Σ lying strictly clockwise of a and strictly anticlockwise of b.

Then $a_3 \in (a_2, a_1)$, since the arc in Σ corresponding to the edge in G between a_2 and a_3 cannot cross the arc in Σ corresponding to the edge in G between a_1 and a_2.

By assumption on Σ, $i_2 \neq i_3$. If $i_2 > i_3$, there can be no path in Σ from the symbol S_{i_3} in vertex a_3 of Σ back to vertex a_1 of Σ without crossings, a contradiction, hence $i_2 < i_3$.

Repeating this argument, we see that, moving clockwise on Σ from a_1 we meet vertices $a_2, a_3, \ldots, a_p$ in order before returning to a_1, and that $i_1 < i_2 < \cdots < i_p$. But then the arc between a_p and a_1 (on symbol S_{i_p} crosses the arc between a_1 and a_2 (on symbol S_{i_1}), since $i_1 < i_p$; see Figure 1. Hence G has no cycles, and must be a tree. The claim is shown.

We next prove that (a) holds. Suppose that $i \neq j$, $a_1, a_2 \in G_i$, $b_1, b_2 \in G_j$, and $a_1 > b_1 > a_2 > b_2$. Then a_1, b_1, a_2, b_2 follow each other anticlockwise around the circle. Then, since a_1, a_2 are in the same connected component of G, there is a path in Σ from a_1 to a_2, and similarly from b_1 to b_2. The arrangement of a_1, a_2, b_1 and b_2 implies that these two paths cross, a contradiction. Hence no such arrangement can occur, and (a) is shown.

We next prove that (b) holds. It is enough to prove the following claim:

Claim: Let $a \in G_i$. Then $\sigma_G(a)$ is the next vertex of G_i (considered as a vertex of Σ) anticlockwise on the circle from a.

To prove the claim, we note that, by the definition of σ_G, $\sigma_G(a)$ is connected to a by a path in G:

$$a = a_1 \overset{S_{i_1}}{\overline{}} a_2 \overset{S_{i_2}}{\overline{}} \cdots \qquad \cdots \overset{S_{i_{p-1}}}{\overline{}} a_p = \sigma_G(a) \ ,$$

where $i_1 < i_2 < \cdots < i_{p-1}$. Furthermore, a_1 is not incident with any arc with symbol S_r for $r < i_1$, a_p is not incident with any arc with symbol S_r for $r > i_{p-1}$, and, for $2 \leq j \leq p-1$, a_j is not incident with any symbol S_r for $i_{j-1} < r < i_j$.

The existence of the above path implies that $\sigma_G(a)$ lies in G_i. Since $i_1 < i_2 < \cdots < i_p$ and there are no crossings, the path must go clockwise around the circle; see Figure 2. The conditions above and the fact there are no crossings imply that no vertex in $(\sigma_G(a), a)$ has an arc with a vertex in $(a, \sigma_G(a))$ or with a or $\sigma_G(a)$, so these vertices are connected only amongst themselves. It follows that they do not lie in G_i and we see that (II)(b) holds. Thus G is a labelled m-edge-coloured forest satisfying (II).

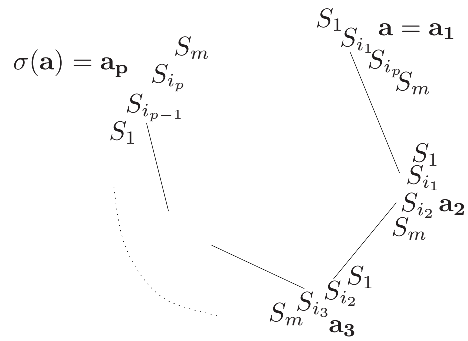


FIGURE 2. The part of Σ between a and $\sigma_G(a)$.

Conversely, suppose that we have a labelled m-edge-coloured forest with k vertices satisfying (II). Let Σ be the RNA m-diagram of degree k with an arc joining S_r in vertex i with S_r in vertex j if and only if there is an edge in G between vertices i and j coloured with symbol S_r. We must check that Σ can be drawn with no crossing arcs, i.e. that it is noncrossing.

We do this by induction on the number of vertices. Suppose first that G has more than one connected component, i.e. that G is not connected. By induction, each component G_i corresponds to a noncrossing RNA m-diagram (on the vertices of G_i).

Suppose that we had $a_1 > a_2 \in G_i$ and $b_1 > b_2 \in G_j$ for two distinct components G_i and G_j, with arcs between a_1 and a_2 and b_1 and b_2 which cross in Σ. Then, going around the circle anticlockwise, starting at vertex k, we must encounter a_1, b_1, a_2, b_2 in order, or b_1, a_1, b_2, a_2 in order. Swapping G_i and G_j if necessary, we can assume we are in the first case. But then $a_1 > b_1 > a_2 > b_2$, contradicting (II)(a). Hence Σ is noncrossing.

So we are reduced to the case in which G has exactly one connected component, i.e. G is connected. Suppose that vertex k is incident with edges $e_1, e_2, \ldots, e_d$ in G, coloured with symbols $S_{r_1}, S_{r_2}, \ldots, S_{r_d}$ where $r_1 < r_2 < \ldots < r_d$. Let the endpoints of these edges (other than k) be $v_1, v_2, \ldots, v_d$. Removing vertex k from G leaves precisely d trees $T_1, T_2, \ldots, T_d$ containing vertices $v_1, v_2, \ldots, v_d$ respectively. By (b), we know that $\sigma_G = S_m S_{m-1} \cdots S_1$ induces the permutation $(k\ k-1\ \cdots 1)$ on the vertices of G.

We apply σ_G to vertex k, and then repeatedly apply σ_G. By its definition, each application of σ_G corresponds to following a certain path through G, i.e. passing along the edges corresponding to the symbols in the sequence $S_1, S_2, \ldots, S_m$ in that order, when such incident edges exist. Since the edge e_1 has symbol S_{r_1}, and no edge incident with k has smaller symbol, it follows that, after the first application of σ_G, we obtain vertex $k_1 := k-1$ in T_1.

Since σ_G is a k-cycle, after repeated application of σ_G, we must leave T_1. Suppose that k_2 is the number of the first vertex reached outside T_1. Since r_2 is the minimum number of a symbol adjacent to k greater than r_1, k_2 will lie in T_2. Repeating this argument, we will obtain $k > k_1 > k_2 > \cdots > k_d \geq 1$ such that vertices $k_{i+1}+1, \ldots, k_i$ lie in tree T_i for $i = 1, 2, \ldots, d-1$. At the final step, the first vertex reached on leaving T_d must be k. Since σ_G is a k-cycle, all vertices must have been visited.

Let $k_{d+1} = 0$. It follows from the above that tree T_i contains precisely vertices $k_{i+1}+1, \ldots, k_i$ for each i. Thus, the numbering of the vertices of G is first the vertices of T_d in some order, then the vertices of T_{d-1} in some order, then the vertices of T_{d-2}, and so on, ending with the vertices of T_1 and then finally k. Each T_i will correspond (by the inductive hypothesis) to a noncrossing RNA m-diagram on its vertices. Thus the vertices $v_1, v_2, \ldots, v_d$ in G will be numbered in decreasing order. The arcs in Σ

from k to these vertices are numbered by symbols $S_{r_1}, S_{r_2}, \ldots, S_{r_d}$, respectively, with $r_1 < r_2 < \cdots < r_d$. It follows these arcs do not cross each other or any of the other arcs in Σ. See Figure 3(a) for an example, where $v_1 = 9$, $r_1 = 1$, and $v_2 = 8$, $r_2 = 4$. Hence, Σ is noncrossing and thus an object in (I) as required.

It is clear that the two maps we have constructed are inverse to each other, so the theorem is proved. □

The following lemma follows easily from the definitions.

Lemma 3.2. *A noncrossing RNA m-diagram is connected if and only if the corresponding labelled m-edge-coloured forest is connected, i.e. is a tree.*

Remark 3.3.
(1) Since a forest on k vertices is connected if and only if it has exactly $k-1$ edges, a noncrossing RNA m-diagram with k vertices is connected if and only if it has $k-1$ arcs.
(2) In the connected case, the circular order of G is just the permutation $(k\ k-1\ \cdots\ 1)$ and we have a bijection between the following sets:

The set of connected noncrossing RNA m-diagrams of degree k.

$\updownarrow$

The set of labelled m-edge-coloured trees with k vertices and circular order $(k\ k-1\ \cdots\ 1)$.

(3) The labelling on the vertices for a tree in the latter set in (2) is determined by a distinguished vertex, that labelled k, say, since σ_G then determines the labels on all the other vertices.

By Remark 3.3 and Lemma 2.1 we have:

Corollary 3.4. *The bijection in Theorem 3.1 induces a bijection between the following sets:*

(a) *The set of connected noncrossing RNA m-diagrams of degree k.*
(b) *The set of rooted m-edge-coloured trees with k vertices.*
(c) *The set of labelled m-edge-coloured trees with k vertices and circular order $(k\ k-1\ \cdots\ 1)$.*

See Figure 3(a)-(c) for an illustration of this bijection. The following is well-known; see, for example, [12, Sect. 6.2] (adding the labelling is straightforward).

Theorem 3.5. *There is a bijection between the following sets:*

(a) *The set of m-edge-coloured trees with k vertices.*
(b) *The set of diagonal-coloured m-angulations of an $(m-2)k+2$-sided regular polygon up to rotation.*

Corollary 3.6. *There is a bijection between the following sets:*

(a) *The set of labelled m-edge-coloured trees with k vertices and circular order $(k\ k-1\ \cdots\ 1)$.*
(b) *The set of rooted diagonal-coloured m-angulations of an $(m-2)k+2$-sided regular polygon up to rotation.*

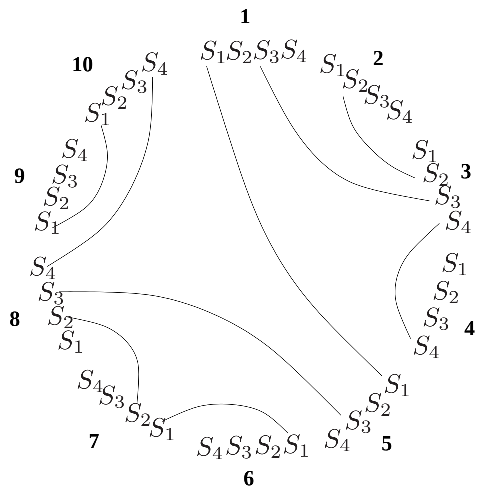
Proof. This follows from Theorem 3.5 and Corollary 3.4. □

Corollary 3.7. *The cardinality of each of the following sets:*

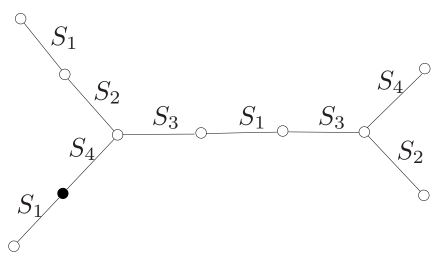
(a) *The set of labelled m-edge-coloured trees with k vertices and circular order $(k\ k-1\ \cdots\ 1)$;*
(b) *The set of connected noncrossing RNA m-diagrams of degree k;*

is given by

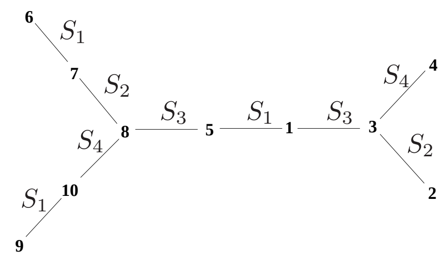
$$T_{k,m} = \frac{m}{(m-2)k+2}\binom{(m-1)k}{k-1}.$$



(a) A connected noncrossing RNA 4-diagram of degree 10.



(b) A rooted 4-edge-coloured tree with 10 vertices.



(c) A labelled 4-edge-coloured tree with 10 vertices and circular order $(10\ 9\ \cdots 1)$.

FIGURE 3. Objects corresponding to each other under the bijections in Corollary 3.4.

Proof. Both sets have the same cardinality by Corollary 3.4 and, by Corollary 3.6, they have the same cardinality as the set of rooted diagonal-coloured m-angulations of an $(m-2)k+2$-sided regular polygon up to rotation. The number $S_{k,m}$ of such m-angulations without a root, with no labelling of diagonals and ignoring rotational equivalence is well-known (see e.g. [7]). Let C_k^m be the kth Fuss-Catalan number of degree m:

$$C_k^m = \frac{1}{k}\binom{mk}{k-1} = \frac{1}{(m-1)k+1}\binom{mk}{k}.$$

Then

$$S_{k,m} = C_k^{m-1} = \frac{1}{(m-2)k+1}\binom{(m-1)k}{k}.$$

Since there are k m-sided polygons in an m-angulation, there are k possibilities for the root. There are m possibilities for a labelling since once one diagonal is coloured, all other diagonals in the m-angulation have determined colours using the rule that each m-gon must have its edges coloured $S_1, S_2, \ldots, S_m$ clockwise around the boundary. Each orbit of diagonal-coloured rooted m-angulations under the action of the rotation group of the polygon contains $(m-2)k+2$ elements (the number of sides of P). Hence, we have:

$$T_{k,m} = \frac{kmS_{k,m}}{(m-2)k+2}$$

and the result follows. □

It is interesting to note that by [9, §3], the sequence $T_{1,m}$, $T_{2,m}, \ldots$ is the m-fold convolution of the sequence $S_{0,m}$, $S_{1,m}, \ldots$.

Example 3.8. For $m = 3, 4, 5, 6$, the first few values of $T_{k,m}$ are given in the following table:

k	0	1	2	3	4	5	6
$T_{k,3}$	1	1	3	9	28	90	297
$T_{k,4}$	1	1	4	18	88	455	2448
$T_{k,5}$	1	1	5	30	200	1425	10626
$T_{k,6}$	1	1	6	45	380	3450	32886

The cases $m = 3, 4$ are sequences A071724 and A006229, respectively, in [11]; the cases $m = 5, 6$ do not appear. The case $m = 3$ appears in [2, Prop. 7.5].

Corollary 3.9. *The total number of labelled m-edge-coloured trees with k vertices is:*

$$U_{k,m} = \frac{m((m-1)k)!}{((m-2)k+2)!}$$

Proof. By Lemma 2.1, the circular order of any labelled m-edge-coloured tree with k vertices is a k-cycle. By Corollary 3.7 the number of such trees with a given circular order is $T_{k,m}$. Since any k-cycle can arise, we have $U_{k,m} = \frac{1}{(k-1)!}T_{k,m}$, giving the result. □

Note that this result is already known [3], [12, 5.28, p124].

4. Generalized Induction

In this section we give the definition of a generalized induction on labelled m-edge-coloured trees with k vertices. Generalized induction generates new labelled m-edge-coloured trees with the same number of vertices starting with a given such tree, and the transitive closure is an equivalence relation. We show that the circular order is an invariant, giving rise to a classification of the equivalence classes by k-cycles in the symmetric group of degree k.

Given a labelled m-edge-coloured tree G and integers $i, j \in \{1, \ldots, m\}$ we define a maximal S_i - S_j chain B in G to be a (linear) subtree of G whose edges are only coloured S_i and S_j such that no other edges incident to B are coloured by S_i or S_j.

Definition 4.1. Let G be a labelled m-edge-coloured tree with k vertices. Fix $i, j \in \{1, \ldots, m\}$ with $i < j$. Let B be a maximal $S_i - S_j$ chain in G. Define $R^B_{i,j}(G)$ to be the labelled m-edge-coloured tree with k vertices obtained from G by

- first removing all subtrees in the complement of the maximal chain B
- interchanging the vertices of each edge of B coloured by S_j
- interchanging the symbols S_i and S_j on the whole maximal chain B
- reattaching the previously removed subtrees to B at the vertices with the same label they were removed from.

Similarly, define $L^B_{i,j}(G)$, where in the second bullet point in the above definition we interchange the vertices of each edge coloured by S_i rather than those coloured by S_j. We also set $R^B_i := R^B_{i,i+1}$ and $L^B_i := L^B_{i,i+1}$ and we will write R_i and L_i if B is clear from the context.

Remark 4.2. (1) Induction can also be defined on m-edge-coloured trees: For an m-edge-coloured tree with k vertices, choose an arbitrary vertex-labelling, apply induction, and then remove the vertex labelling. It is clear that this is independent of the vertex-labelling chosen.

(2) The inductions $R^B_{i,j}$ and $L^B_{i,j}$ are mutually inverse maps.

Lemma 4.3. *Let $i, j \in \{1, \ldots, m\}$ with $i < j$ and B be a maximal $S_i - S_j$-chain with no incident edges coloured by $S_{i+1}, \ldots, S_{j-1}$. Then we have*

$$\begin{aligned} R^B_{i,j}(G) &= L_i L_{i+1} \cdots L_{j-2} R_{j-1} R_{j-2} R_{j-3} \cdots R_i \\ &= R_{j-1} R_{j-2} \cdots R_{i+1} R_i L_{i+1} L_{i+2} \cdots L_{j-1}; \\ L^B_{i,j}(G) &= L_i L_{i+1} \cdots L_{j-2} L_{j-1} R_{j-2} R_{j-3} \cdots R_i \\ &= R_{j-1} R_{j-2} \cdots R_{i+1} L_i L_{i+1} L_{i+2} \cdots L_{j-1}, \end{aligned}$$

where, in each case, the inductions of form R_p and L_p are applied to all maximal chains contained in B. In particular, the induction $R^B_{i,j}(G)$ can be written as a product of inductions of the form R_p or L_p for $p = i, i+1, \ldots, j-1$.

Proof. Suppose first that B has the following form:

$$a_1 \overset{S_i}{\text{———}} a_2 \overset{S_j}{\text{———}} a_3 \overset{S_i}{\text{———}} a_4 \qquad \cdots \qquad a_{r-1} \overset{S_i}{\text{———}} a_r \ .$$

Then, in $R^B_{i,j}(G)$, B' becomes the chain

$$B' = \ a_1 \overset{S_j}{\text{———}} a_3 \overset{S_i}{\text{———}} a_2 \overset{S_j}{\text{———}} a_5 \overset{S_i}{\text{———}} a_4 \qquad \cdots \qquad a_{r-2} \overset{S_j}{\text{———}} a_r \ .$$

On the other hand, applying $R_{j-2} R_{j-3} \cdots R_{i+1} R_i$ (with each induction applying to all the maximal chains of the appropriate type contained in B), we obtain the maximal $S_{j-1} - S_j$-chain

$$a_1 \overset{S_{j-1}}{\text{———}} a_2 \overset{S_j}{\text{———}} a_3 \overset{S_{j-1}}{\text{———}} a_4 \qquad \cdots \qquad a_{r-1} \overset{S_{j-1}}{\text{———}} a_r \ .$$

Next, apply R^B_{j-1} to get the maximal $S_{j-1} - S_j$ chain:

$$a_1 \overset{S_j}{\text{———}} a_3 \overset{S_{j-1}}{\text{———}} a_2 \overset{S_j}{\text{———}} a_5 \qquad \cdots \qquad a_{r-2} \overset{S_j}{\text{———}} a_r \ .$$

It is then clear that subsequently applying $L_i L_{i+1} \cdots L_{j-2}$ (in each case applying the induction to all the maximal chains of appropriate type contained in the full subgraph on the vertices $a_1, a_2, \ldots, a_r$) gives the chain B'.

The proof works in a similar way for the other configurations of maximal $S_i - S_j$-chains, i.e. for the above case with S_i and S_j switched and for the chains of the form

$$a_1 \overset{S_i}{\text{———}} a_2 \overset{S_j}{\text{———}} a_3 \overset{S_i}{\text{———}} a_4 \qquad \cdots \qquad a_{r-1} \overset{S_j}{\text{———}} a_r$$

and

$$a_1 \overset{S_j}{\text{———}} a_2 \overset{S_i}{\text{———}} a_3 \overset{S_j}{\text{———}} a_4 \qquad \cdots \qquad a_{r-1} \overset{S_j}{\text{———}} a_r \ .$$

The other identities are proved similarly. □

Definition 4.4. Call two labelled m-edge-coloured trees with k vertices G, G' *induction equivalent* if there is a sequence of inductions taking G to G', either of the form L_i, for $1 \leq i \leq m-1$ or of the form R_i, for $1 \leq i \leq m-1$. Clearly this is a reflexive relation. It is symmetric since L_i and R_i are inverse maps and it is easy to see that it is transitive. Hence it is an equivalence relation. We write $\Gamma(G)$ for the equivalence class containing G.

We next show that two labelled m-edge-coloured trees with k vertices are induction equivalent if and only if they have the same circular order. Firstly, Lemmas 4.5 and 4.6 show that the circular order is invariant under an induction of the form R_i or L_i. Key to showing the converse is Proposition 4.8, which shows that every labelled m-edge coloured tree is induction equivalent to one whose labels are only S_1 and S_m, which is necessarily a line. This, together with a calculation of the order of induction (Lemma 4.9) allows a counting argument for proving the converse; see Theorem 4.11.

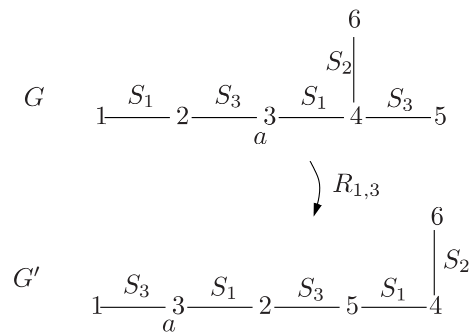


FIGURE 4. The induction $R_{13}^{2,3,4,5}$ does not preserve the circular order: $\sigma_G(3) = 6$ while $\sigma_{G'}(3) = 5$

Lemma 4.5. *Let G be a labelled m-edge-coloured tree with k vertices containing a maximal S_i-S_j chain B with no incident edges coloured S_l, $i < l < j$. Then the circular order, σ_G, is unchanged after an induction of the form $R^B_{i,j}$ or $L^B_{i,j}$ is applied.*

Proof. We show the result for $R_{i,j}$-induction (the proof for $L_{i,j}$-induction follows a similar argument).

Let G be a labelled m-edge-coloured tree with k vertices and circular order σ_G and containing a maximal S_i-S_j-chain B such that no edges coloured S_l, for $i < l < j$ are incident to B. Let $G' = R^B_{i,j}(G)$ with circular order $\sigma_{G'}$. Let a be a vertex in B. Let $S'_1, \dots S'_m$ denote the maps corresponding to the symbols $S_1, \dots, S_m$ in the labelled m-edge-coloured tree G'. There are two possible situations to consider.

Case 1: Suppose first that a has edges incident to it which are coloured with labels S_t with $1 \le t < i$. Consider the edge e incident with a coloured with label S_r with r minimal. Let T be the subtree of $G \setminus B$ connected to a via e. Then $\sigma_G(a)$ lies in T. Applying the induction $R^B_{i,j}$ to G results in an m-edge-coloured tree G' in which a is reconnected to T by edge e, still coloured with S_r. We see that $\sigma_G(a) = \sigma_{G'}(a)$.

Case 2: Suppose that a is not incident with any edges coloured with S_l for $l < i$. An easy case-by-case check verifies that $S'_j S'_i(a) = S_j S_i(a)$, so $S'_j \cdots S'_1(a) = S_j \cdots S_1(a)$. Any edges incident with $S_j S_i(a)$ with label S_l for $l > j$ are reattached to the same vertex in G' with the same label. It follows that $\sigma_G(a) = \sigma_{G'}(a)$. □

Corollary 4.6. *The circular orders of two induction equivalent labelled m-edge-coloured trees with k vertices are the same.*

Proof. In Lemma 4.5 assume that $j = i + 1$. □

Remark 4.7. In general, the circular order of a labelled m-edge-coloured tree with k vertices is not invariant under general inductions of the form $R_{i,j}$ and $L_{i,j}$ with $i < j+1$: in the above proof suppose that we have an edge e coloured S_l, for $i < l < j$ connected to $S_i(a)$ in the chain B of G and let T be a subtree of G connected to e. Then $\sigma_G(a)$ lies on the subtree T. On the other hand in $R^B_{i,j}(G)$, the edge S_k is connected to $S_i(a) = S'_i S'_j S'_i(a)$ and thus $\sigma_{G'}(a) \neq \sigma_G(a)$ in general. For an example of this with $i = 1$ and $j = 3$ (and T just consisting of the vertex 6), see Figure 4.

Note that this implies that it is not possible, in general, to write $R_{i,j}$ as a composition of inductions of the form R_i and L_i, since such inductions preserve the circular order (by Lemma 4.5). However, Lemma 4.3 says that sometimes this is possible, however.

Proposition 4.8. *Let G be a labelled m-edge-coloured tree with k vertices. Then there exists a labelled m-edge-coloured tree, induction equivalent to G, whose edges are only coloured by S_1 and S_m.*

Proof. We first show that there exists a labelled m-edge-coloured tree G_2 with k vertices that is induction equivalent to G, none of whose edges is coloured with S_2, by removing the symbols S_2 one by one. Firstly, remove all edges coloured with symbols $S_4, \ldots, S_m$ and call the resulting labelled 3-edge-coloured tree $\widetilde{G}$. By [2, Prop. 5.2] there exists a sequence of inductions of the form R_1 and L_2 taking $\widetilde{G}$ to a labelled 3-edge-coloured tree $\widetilde{G}_2$ with no edge coloured S_2. Let G_2 be the labelled m-edge-coloured tree $\widetilde{G}_2$ with the detached edges reattached (to the vertices with the same label). Since none of the detached edges are coloured with S_1, S_2, or S_3 this sequence of inductions also takes G to G_2 by identifying maximal chains in $\widetilde{G}$ with corresponding maximal chains in G.

Suppose we have shown that G is induction equivalent to G_{l-1}, where G_{l-1} has no edges coloured $S_2, \ldots, S_{l-1}$.

Then detach all edges coloured $S_{l+2}, \ldots, S_m$ from G_{l-1}. Call the resulting labelled m-edge-coloured tree $\widetilde{G}$. By [2, Prop. 5.2] there is a sequence of inductions of the form $R_{1,l}$ and $L_l = L_{l,l+1}$ taking $\widetilde{G}$ to $\widetilde{G}_l$, where $\widetilde{G}_l$ has no edges coloured $S_2, \ldots, S_l$. Reattach the detached edges and call the resulting labelled m-edge-coloured tree G_l. Since none of the reattached edges are coloured by S_1, S_l, or S_{l+1}, this sequence of inductions takes G_{l-1} to G_l by identifying the maximal chains in $\widetilde{G}$ with the maximal chains in G_{l-1}. By Lemma 4.3, each application of $R_{1,l}$ can be written as a composition of inductions R_p, L_p, $p = 1, 2, \ldots, l-1$, since G_{l-1} has no edges coloured $S_2, \ldots, S_{l-1}$.

Note that none of the symbols $S_2, \ldots, S_l$ appears in G_l. Hence, by induction on l, we can construct $G_{m-1} = G'$ with no symbols $S_2, \ldots, S_{m-1}$ and a sequence of inductions, each of the form R_p or L_p, with $1 \leq p \leq m-1$, taking G to G'. □

Lemma 4.9. (a) *Let k be odd, let $i \neq j$ and let G be a labelled m-edge-coloured tree with k vertices whose edges are coloured with S_i and S_j only. Let $\mathcal{S}$ be the underlying unlabelled tree of G. Then $R_{i,j}$ and $L_{i,j}$ have order k on G, producing k distinct trees with shape $\mathcal{S}$.*

(b) *Let k be even, $i \neq j$ and let G be a labelled m-edge-coloured tree with k vertices and whose edges are coloured with S_i and S_j only. Let $\mathcal{S}$ be the underlying unlabelled tree of G. Then $R_{i,j}$ and $L_{i,j}$ have order k on G, producing $k/2$ distinct labelled trees whose underlying unlabelled tree is $\mathcal{S}$ in each case and $k/2$ distinct labelled trees whose underlying unlabelled tree is $R_{i,j}(\mathcal{S})$.*

Proof. (a) Since $\mathcal{S}$ contains only the symbols S_i and S_j, it is a line. Suppose the line is drawn horizontally and suppose the leftmost edge has colour S_j and vertices a_1 and a_2 from left to right. Then in $(R^G_{i,j})^d(G)$, if it is drawn with orientation given by $\cdot \overset{S_i}{\text{———}} a_1 \overset{S_j}{\text{———}} \cdot$ (where one of the edges may not exist), the vertex a_1 is the d^{th} vertex from the left. It is clear that all the induced trees $(R^G_{i,j})^d(G)$ have the same underlying unlabelled tree (those for d odd should be read from right to left).

(b) The proof is similar to the one in (a), except that for d odd, the underlying unlabelled tree of $(R^G_{i,j})^d(G)$ is $R_{i,j}(\mathcal{S})$. □

Remark 4.10. Since in the context of Lemma 4.9 the underlying unlabelled tree of G is a line, we can replace $R_{i,j}$ or $L_{i,j}$ in Lemma 4.9 by one of the products in Lemma 4.3.

We finally obtain:

Theorem 4.11. *Let G and G' be labelled m-edge-coloured trees with k vertices. Then G' is in $\Gamma(G)$ if and only if $\sigma_G = \sigma_{G'}$.*

Proof. Suppose that G' is in $\Gamma(G)$. We have, by Lemma 4.6, that $\sigma_G = \sigma_{G'}$. Conversely, suppose that $\sigma_G = \sigma_{G'}$. By Proposition 4.8 there exist labelled m-edge-coloured trees with k vertices, G_*, in $\Gamma(G)$ and G'_* in $\Gamma(G')$ where the underlying unlabelled trees of G_* and G'_* contain symbols S_1 and S_m only.

Suppose that k is odd and that G_* and G'_* are as follows:

$$G_* :\ a_1 \overset{S_1}{\text{———}} a_2 \overset{S_m}{\text{———}} a_3 \overset{S_1}{\text{———}} a_4 \quad \cdots \quad a_{k-1} \overset{S_m}{\text{———}} a_k$$

$$G'_* :\ a'_1 \overset{S_1}{\text{———}} a'_2 \overset{S_m}{\text{———}} a'_3 \overset{S_1}{\text{———}} a'_4 \quad \cdots \quad a'_{k-1} \overset{S_m}{\text{———}} a'_k\ .$$

Then

$$\sigma_{G_*} = (a_1a_3a_5\dots a_ka_{k-1}a_{k-4}\dots a_2) = (a'_1a'_3a'_5\dots a'_ka'_{k-1}a'_{k-4}\dots a'_2) = \sigma_{G'_*}.$$

Because the underlying unlabelled tree of G'_* is a line, G'_* is determined by a'_1 and its circular order. Thus, given G_* there are at most k possibilities for G'_* such that $\sigma_{G_*} = \sigma_{G'_*}$ holds.

By Lemma 4.9 and Lemma 4.6, repeatedly applying either $R_{i,j}$-induction or $L_{i,j}$-induction to G_* gives k distinct labelled m-edge-coloured trees H with k vertices satisfying $\sigma_{G_*} = \sigma_H$. Hence any G'_* such that $\sigma_{G_*} = \sigma_{G'_*}$ must be one of these trees H. Therefore, using Remark 4.10, G'_* is in $\Gamma(G_*)$ and thus G' is in $\Gamma(G)$.

Suppose that k is even. In this case, there are at most $k/2$ distinct possibilities for G'_* because of the symmetry of the underlying unlabelled tree. The result then follows from Lemma 4.9, Remark 4.10, and Lemma 4.6 as above. □

Lemma 4.12. *Let G be a labelled m-edge-coloured tree with k vertices containing a maximal S_i-S_j chain B with l vertices and no incident edges coloured S_l, $i < l < j$. Then $R^B_{i,j}$ has order l on G. In particular, R_i and L_i have finite order on any maximal chain in G.*

Proof. This follows easily from Lemma 4.9. □

Let G be a labelled 3-edge-coloured tree with k vertices. Let $\Gamma'(G)$ be the smallest set of labelled 3-edge-coloured trees with k vertices closed under R_1 and L_2 (as defined in [2, §4], identifying the symbols $+, =, -$ with S_1, S_2 and S_3 respectively). Then:

Corollary 4.13. (a) *Two labelled m-edge-coloured trees with k vertices are induction equivalent if and only if there is a sequence of inductions only of the form R_p (respectively, only of the form L_p) taking one to the other.*
(b) *If G is a 3-edge-coloured tree with k vertices, then $\Gamma(G) = \Gamma'(G)$.*

Proof. Part (a) follows from Lemma 4.12, using the fact that L_p (on a given maximal chain) is the inverse of R_p. For part (b), note that (on a given maximal chain), L_1 is the inverse of R_1 and R_1 has finite order. Similarly, R_2 is the inverse of L_2 and L_2 has finite order. □

Finally in this section, we note the interesting fact that every induction equivalence class of labelled m-edge coloured trees contains a labelling of any given unlabelled m-edge coloured tree. We first have:

Lemma 4.14. *Any two (unlabelled) m-edge coloured trees are induction equivalent.*

Proof. Let $\mathcal{S}$ and $\mathcal{S}'$ be arbitrary m-edge-coloured trees with k vertices. By Lemma 4.8, $\mathcal{S}$ is induction equivalent to a labelled m-edge-coloured tree with k vertices containing only the symbols S_1 and S_m; similarly for $\mathcal{S}'$. If k is odd there is only one such tree:

$$\bullet \overset{S_m}{\text{———}} \bullet \overset{S_1}{\text{———}} \bullet \overset{S_m}{\text{———}} \bullet \quad \cdots \quad \bullet \overset{S_1}{\text{———}} \bullet\ ,$$

and it follows that $\mathcal{S}$ and $\mathcal{S}'$ are induction equivalent. If k is even, there are two such trees:

$$\mathcal{S}_m :\ \bullet \overset{S_m}{\text{———}} \bullet \overset{S_1}{\text{———}} \bullet \overset{S_m}{\text{———}} \bullet \quad \cdots \quad \bullet \overset{S_m}{\text{———}} \bullet$$

$$\mathcal{S}_1 :\ \bullet \overset{S_1}{\text{———}} \bullet \overset{S_m}{\text{———}} \bullet \overset{S_1}{\text{———}} \bullet \quad \cdots \quad \bullet \overset{S_1}{\text{———}} \bullet\ .$$

Then $R_{1,m}^{\mathcal{S}_m}(\mathcal{S}_m) = \mathcal{S}_1$, so $\mathcal{S}_m$ is induction equivalent to $\mathcal{S}_1$ by Lemma 4.3. It follows that $\mathcal{S}$ and $\mathcal{S}'$ are induction equivalent in this case also. □

Corollary 4.15. *Let G be a labelled m-edge-coloured tree with k vertices. Then every possible m-edge-coloured tree with k vertices appears as the underlying unlabelled m-edge-coloured tree of a labelled m-edge-coloured tree in $\Gamma(G)$.*

Proof. Given a labelled m-edge-coloured tree with k vertices, G, whose underlying unlabelled tree is $\mathcal{S}$, and an arbitrary m-edge-coloured tree with k vertices, $\mathcal{S}'$, Lemma 4.14 shows that $\mathcal{S}$ and $\mathcal{S}'$ are induction equivalent. It follows that G and a vertex-labelling of $\mathcal{S}'$ are induction equivalent, and the result follows. □

5. Induction on m-Angulations of Polygons

By Theorem 3.5, the set of labelled m-edge-coloured trees with k vertices is in bijection with the set of m-gon-labelled diagonal-coloured m-angulations of a polygon P_n with $n = (m-2)k+2$ sides, up to rotation, where m-gon-labelled means that the m-gons are labelled with $1, 2, \ldots, k$. The colours satisfy a *boundary rule*, i.e. those bounding each m-gon in the m-angulations must be the symbols $S_1, \ldots, S_m$ in clockwise order.

Our aim in this section is to rewrite induction in the language of $(m-2)$-clusters. The set of m-angulations of P_n containing k m-gons is in bijection with the set of $(m-2)$-clusters of type A_{k-1} by Fomin-Reading [5]. Mutation corresponds to rotating a diagonal one step anticlockwise in the subpolygon obtained when the diagonal is removed.

We shall see that induction has a description as a composition of such mutations and appropriate recolourings of diagonals and vertex relabellings. An easy induction argument based on cutting an m-angulation along one of its sides shows that:

Lemma 5.1. *Every m-angulation $\mathcal{M}$ of a polygon has at least two m-gons with $m-1$ boundary edges or is an m-angulation of an m-gon.*□

Remark 5.2. If all the diagonals in an m-angulation are incident with one vertex, then by moving them one after another, we get the same m-angulation rotated through $2\pi/n$.

We label the vertices of P_n $1, 2, \ldots, n$ clockwise around the boundary, and use the notation $[i, j]$ to denote a diagonal in the polygon connecting vertex i with vertex j.

Lemma 5.3. *Let $\mathcal{M}$ be an m-angulation of P_n with k m-gons. Then there is an explicit sequence of mutations taking $\mathcal{M}$ to its rotation through $2\pi/n$ anticlockwise.*

Proof. We induct on the number k of m-gons in the m-angulation. The result is trivial if $k = 1$, so consider the general case. By Lemma 5.1, there is an m-gon M with a unique internal edge e, joining vertices $[i, i+(m-1)]$ for some i. Let R be the subpolygon of P_n given by the union of the m-gons incident with i. Applying Remark 5.2 to the induced m-angulation of R to rotate it one step anticlockwise, we obtain a new m-angulation of R, and hence of P_n, containing an m-gon M' with vertices $i-1, i, i+1, \ldots, i+(m-1)-1$. Let R' be the subpolygon of R with M' removed. Applying Remark 5.2 repeatedly we may rotate the induced m-angulation of R' one step clockwise.

We obtain a new m-angulation of P_n. By induction we may rotate the subpolygon with M' removed anticlockwise by an explicit sequence. It is easy to check that the total effect of the above is to rotate the original m-angulation of P_n one step anticlockwise. □

Definition 5.4. Let $\mathcal{M}$ be a diagonal-coloured m-angulation of P_n. If there is at least one internal diagonal in $\mathcal{M}$ and all of the internal diagonals of $\mathcal{M}$ are coloured only with S_i or S_{i+1} for fixed i, we call $\mathcal{M}$ a *snake m-angulation*. A subpolygon of P_n with this property is called a *snake* subpolygon. Note that in any snake subpolygon the internal diagonals must be of the form $[i_1, i_2]$, $[i_2, i_3]$, and so on.

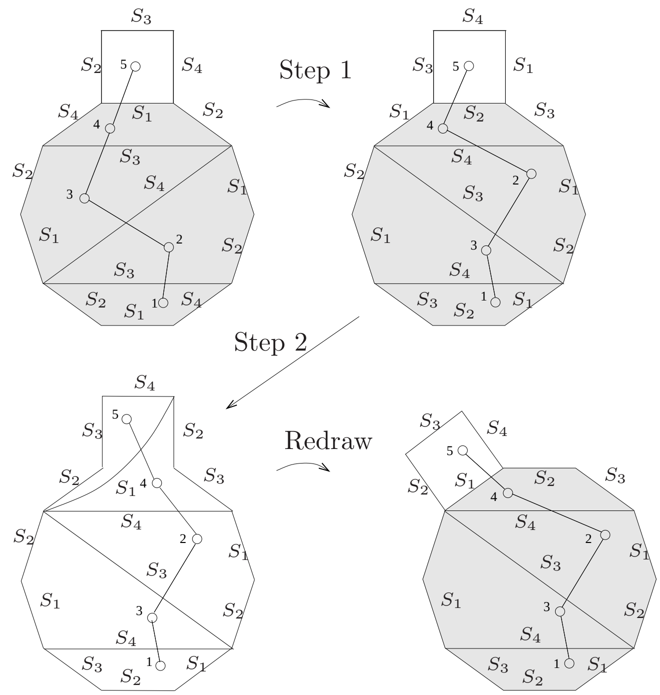


FIGURE 5. Mutations giving rise to R_3 induction (see Proposition 5.5).

In a cluster algebra context, snake triangulations appeared in [6, Sect. 3.5] and for general m in [5, Sect. 5.1] as *m-snakes*.

Let $\mathcal{M}$ be an m-gon-labelled diagonal-coloured m-angulation of P_n and let $1 \leq i \leq m$. Choose a maximal snake subpolygon $\mathcal{B}$ of $\mathcal{M}$ with internal diagonals coloured S_i or S_{i+1}. Let $\mathcal{R}_i^{\mathcal{B}}(\mathcal{M})$ be defined using the following procedure.

Step 1: Let M_1, M_2 be the two m-gons in $\mathcal{B}$ with $m-1$ boundary edges in $\mathcal{B}$, and let $e = e_1$ be the internal edge of M_1 in $\mathcal{B}$. Let $e_2, \ldots, e_N$ be the other internal edges in $\mathcal{B}$, numbered so that e_{i-1} and e_i are incident for all i. If e_1 has colour S_i (respectively, S_{i+1}), mutate edges $e_2, e_4, \ldots$ (respectively, $e_1, e_3, \ldots$), recolouring the new diagonals with S_i and using the boundary rule to recolour the rest of the polygon. The labelling of an m-gon is given by the number of the m-gon before mutation whose intersection with the boundary of $\mathcal{B}$ was the same.

Step 2: Let C_1, C_2 be the union of the connected components of the complement of $\mathcal{B}$ in P_n incident with M_1, M_2 respectively, and let $D_i = C_i \cup M_i$. Applying Lemma 5.3 to D_1, D_2 we get a sequence of mutations rotating the induced m-angulation of each D_i anticlockwise one step around its boundary. We recolour the diagonals according to the boundary rule. The label of the image of an m-gon under this rotation is the same.

Note that Step 2 has the same effect as detaching each component of C_i in the original m-angulation from an edge of M coloured S_j, $j \neq i, i+1$, and reattaching it after Step 1 to a boundary edge of M with the same symbol S_j (now one step anticlockwise around the boundary of M), keeping the original colouring of the diagonals of C_i (i.e. from before Step 1). Such a boundary edge of M always exists, since $j \neq i, i+1$.

It is easy to see that the above procedure does not depend on the initial choice of M_1. Note also that the procedure commutes with any rotational symmetry of P_n and so gives a well defined induction on a diagonal-coloured m-angulation of P_n up to rotation.

Comparing the definitions of R_i^B and $\mathcal{R}_i^{\mathcal{B}}$ we see that:

Proposition 5.5. *Suppose that* $1 \leq i \leq m$ *and let* G *be a labelled* m*-edge-coloured tree with* k *vertices containing a maximal* $S_i - S_{i+1}$ *chain* B*. Let* $\mathcal{M}$ *be the corresponding* m*-gon-labelled diagonal-coloured* m*-angulation of* P_n *up to rotation with maximal snake subpolygon* $\mathcal{B}$ *corresponding to* B*. Then* $R_i^B(G)$ *corresponds to* $R_i^{\mathcal{B}}(\mathcal{M})$*.* □

For an example of Proposition 5.5, see Figure 5, with the snake subpolygon shaded.

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