The Reconstruction of Cycle-free Partial Orders from their Automorphism Groups

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The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

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Abstract

Is a cycle-free partial order recognisable from its abstract automorphism group? This thesis resolves that question for two disjoint families: those cycle-free partial orders which share an automorphism group with a tree; and those which satisfy certain transitivity conditions, before giving a method for combining the two.

Chapter 1, the introduction, as well as introducing some notation and defining the cycle-free partial orders (CFPOs), gives a list of the results that this thesis calls upon.

Chapter 2 gives a structure theorem for $\aleph_0$-categorical trees, which is of particular interest here as their reconstruction problem is completely solved, and for the $\aleph_0$-categorical CFPOs, which when combined with the results in Chapter 3, gives a complete reconstruction result for $\aleph_0$-categorical CFPOs.

Chapter 3 asks which CFPOs have an automorphism group isomorphic to one of a tree. It gives conditions on the CFPO and the automorphism group that allow the invocation of the work done by Rubin on the reconstruction of trees. In a brief epilogue the results are also used to show that many of the model theoretic properties of the trees are also properties of the CFPOs.

The second family is addressed in Chapter 4 using a method used by Shelah and Truss on the symmetric groups of cardinals, which uses the alternating group on five elements.

In Chapter 5 I give a method of attaching structures of the first kind to structures of the second, which admits a second order definition in the abstract automorphism group of the automorphism groups of the components.

The last chapter is a discussion of how the work done here can be made more complete. I have included an appendix, which lists the formulas used in Chapters 4 and 5, which the reader can tear out and keep at hand to save flicking between pages.

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Chapter 1

Introduction

How much information about a structure is contained in its group of automorphisms?

There are two closely related approaches to this problem: finding collections of structures whose automorphism groups are isomorphic to a given group; and 'defining' the original structure using the maps. The first approach involves looking for 'pathologies', easily described objects that cannot be recognised from the maps. The second asks for an interpretation of the original structure inside the group using some formal logic.

1.1 Summary

This thesis is concerned with the reconstruction of cycle-free partial orders (CFPOs) from their automorphism groups, as abstract groups, a problem posed by Matatyahu Rubin in his memoir about the reconstruction of trees [23]. Intuitively, a cycle-free partial order is a generalisation of a tree, in that one is allowed to branch as one moves down the order as well as up. The number of times one can 'change direction' in this way is referred to as the 'width' of the CFPO.

I take two different approaches to this problem, working with two very different subclasses of CFPOs: those which share an automorphism group with a tree (is *treelike*), and those which possess a certain degree of transitivity.

In Chapter 3, I give two kinds of condition that show when a CFPO is treelike. The first kind is related to the properties of the CFPO under the action of its automorphism group, two key results of which are:

Theorem 1.1.1 *(Theorem 3.1.11) If a CFPO has a point that is fixed by every automorphism then it is treelike.*

Theorem 1.1.2 *(Theorem 3.1.14 and Corollary 3.1.15) If a CFPO has 'finite width' then it is treelike.*

The second kind gives conditions on the abstract automorphism group, related to the presence of a subgroup isomorphic to the infinite dihedral group which is not contained in a supergroup which is contained in a certain family of groups, which I call 'dendromorphic groups' .

Theorem 1.1.3 *(Theorem 3.2.13) A CFPO is not treelike if and only if its automorphism group contains a copy of D_∞ which is not contained in a dendromorphic group .*

These results show when we may appeal directly to Rubin's results and methods. However many CFPOs are not treelike, in particular if we assume that a CFPO is 1-transitive and is not a tree, then it is not treelike.

In Chapter 4 we use a method used by Shelah in his work on the reconstruction of the permutation groups of cardinals from the corresponding abstract group ([31], [30] and [33]). We take a very different class of CFPOs, that of 'cone transitive' CFPOs where both the upward and downward ramification order are at least 5. Cone transitivity implies 1-transitivity, and ensures that the automorphism group is rich, while the assumption on the ramification orders guarantees the existence of finite subgroups isomorphic to A_5, the alternating group on five elements, chosen because it is the smallest non-abelian simple group.

We show that tuples of automorphisms whose elementary diagram is that of A_5 must fix a common point, and then use various first-order formulas exploiting the non-abelian simple nature of A_5 to express, among other things, disjointness of support. The culmination of this is:

Theorem 1.1.4 *(Section 4.2) If M is a cone transitive CFPO where both the upward and downward ramification orders are at least 5 then the family of first order formulas described in Chapter 4 gives an interpretation of M with the betweenness relation in its abstract automorphism group.*

I fully recover the order with first order formulas in limited circumstances, and everywhere with $L_{\omega_1,\omega}$ formulas, but reconstructing betweenness is sufficient for most purposes.

These two subclasses of CFPOs far from cover the whole class, and Chapter 5 seeks to address that by attaching treelike CFPOs to cone transitive ones in a process called decoration. The main result of Chapter 5 is:

Theorem 1.1.5 *(Theorems 5.2.8 and 5.2.16) Given an abstract automorphism group of a decorated CFPO, we can use second order formulas to define groups isomorphic to the automorphism groups of the components of the decorated CFPO.*

Rubin gives a stronger result in [23] for the $\aleph_0$-categorical trees, so in Chapter 2 I give a classification of first the $\aleph_0$-categorical trees and then the $\aleph_0$-categorical CFPOs. These classifications work by looking at important substructures, maximal chains for the trees and CFPOs of shorter width for the CFPOs, with some extra structure added to them.

While too technical to give here, the classification guarantees that all $\aleph_0$-categorical CFPOs are treelike, and hence Rubin's stronger result applies directly to them.

The remainder of this introduction is devoted to various historical notes, a brief introduction to CFPOs, and a short proof-less summary of Rubin's results about trees.

1.2 History of CFPOs and Related Results

In order to pose the question about the reconstruction of cycle-free partial orders, Rubin had to define them. However there were a few problems with the definition he gave in [23], and so Warren gave in [39] a different version that better matches our intuition. The technical details of this can be found in Subsection 1.3.1.

Warren's memoir, a polishing of his Ph.D. thesis [38], is concerned with various homogeneity and transitivity questions concerning CFPOs.

Definition 1.2.1 *A model M is said to be k-**transitive** if for every finite $A, B \in M$ such that $A \cong B$ and $|A| = k$ there is an automorphism of M that maps A to B. If M is k-transitive for all k then M is called **transitive**.*

*A model M is said to be k-**homogeneous** if for every finite $A, B \in M$ such that $A \cong B$ and $|A| = k$ then every isomorphism from A to B extends to an automorphism of M. If M is k-homogeneous for all k then M is called **homogeneous**.*

Rather than 'vanilla' homogeneity and transitivity, Warren is primarily concerned with CS-homogeneity and transitivity, where the CS stands for 'connecting set'. This means that the conditions of homogeneity and transitivity are weakened; the conditions only apply to the finite substructures which are connecting sets, defined in Definition 1.3.13.

1.2.1 Homogeneous and Transitive Orders

There is a large body of work dedicated to studying the homogeneity and transitivity properties of ordered structures. Morel produced the first result in this direction, classifying the countable transitive linear orders in [16]. A considerable amount is known about transitive linear orders of larger cardinalities, and I would recommend reading Chapter 8 of [21] to learn more, but there is one result in that work that I wish to draw special attention to, especially for a lemma used in the proof of it.

That result is Rosenstein's classification of the $\aleph_0$-categorical linear orders, first published in [20]. The Ryll-Nardzewski Theorem allows us to recast $\aleph_0$-categoricity as a transitivity property.

Definition 1.2.2 *A model M is said to be* ***almost*** k***-transitive*** *if M has only finitely many k-orbits. M is called* ***almost transitive*** *if it is almost k-transitive for all k.*

If a countable linear order is almost transitive, then its theory is $\aleph_0$-categorical (see [15], Theorem 7.3.1). In the production of this result, Rosenstein proves the following lemma:

Lemma 1.2.3 *If a linear order is almost 2-transitive then it is almost transitive.*

This is a very useful result, which inspired an analogue result for the CFPOs (Lemma 2.2.12), which gets used in many places in this thesis. It is arguably a quantifier elimination-style result, if stated in the following form:

Lemma 1.2.4 *Let $\bar{x}$ be an n-tuple from CFPO M. Then*

$$\mathrm{qftp}(\bar{x}) \cup \bigcup \mathrm{tp}(x_i, x_j) \vdash \mathrm{tp}(\bar{x})$$

Many similar results exist, and not just for almost transitivity. In particular, Droste and Macpherson in [11] show that if a partial order is both 1- and 4-homogeneous then it is homogeneous, and in the same paper find for each n an n-homogeneous graph which is not $(n + 1)$-homogeneous, showing the total absence of an analogous result for homogeneity in graphs.

Schmerl in [29] gave a classification of all countable homogeneous partial orders, and subsequently structure theorems for countable transitive, k-transitive, homogeneous and k-homogeneous trees were given by Droste in [6] and [7], and Chicot gave in her Ph.D. thesis [4] a classification of the 1-transitive trees. Accounts of these results and sketches of their proofs can be found in a survey by Truss, [37], if one feels disinclined to read the original papers.

As well as considering different types of partial orders, we may also add restrictions to the type of finite substructures we use in our definitions of homogeneity and transitivity. As mentioned earlier, Warren considered CS-transitivity, finding the class of k-CS-transitive CFPOs much richer class than the class of k-transitive CFPOs.

Droste, Holland and Macpherson considered in [8], [9] and [10] weakly 2-transitive trees, i.e. trees whose automorphism groups are transitive on related pairs. Chicot, while working towards her classification of 1-transitive trees in [4], gave a classification of the lower 1-transitive linear orders, the linear orders which have only one initial segment up to isomorphism.

Instead of weakening our transitivity and homogeneity conditions, we could add additional structure to our orders by adding colour predicates to the language. Examples of this kind of result include Campero-Arena and Truss' [2], which classifies the countable 1-transitive coloured linear orders and Mwesigye and Truss' classification of the coloured $\aleph_0$-categorical linear orders in [17].

1.2.2 CFPOs

Warren's definition of the CFPOs takes place in what is known as the Dedekind-MacNeille completion. Since this is not necessarily definable, this leads to the question of whether the class of CFPOs is axiomatisable or not. Truss answered this question in [36], an account of which can be found in Section 1.3.1.

Warren's study of k-CS-transitive CFPOs was extended by two papers in 1998, [35] by Truss and [5] by Creed, Truss and Warren, both of which add to cases not fully dealt with by Warren in [39].

Gray and Truss in [14] examine the relationship between ends of a graph and CFPOs, and extract a number of results from this relationship. This viewpoint is rather illuminating, even if one is not familiar with ends of graphs.

1.3 Preliminaries

Definition 1.3.1 *A tree (also called a semi-linear order) is a partial order that satisfies the two additional axioms:*

- $\forall x, y, z(x, y \leq z \rightarrow (x \leq y \vee y \leq x))$
- $\forall x, y \exists z(z \leq x, y)$

Definition 1.3.2 *A λ-coloured tree is a structure $\langle T, \leq, C_i \, : \, i \leq \lambda \rangle$ such that $\langle T, \leq \rangle$ is a tree, while the C_i are mutually exclusive unary predicates.*

Definition 1.3.3 *If $f \in \mathrm{Aut}(M)$ then the **support** of f is the following set:*

$$\mathrm{supp}(f) := \{x \in M \, : \, f(x) \neq x\}$$

If $F \subseteq \mathrm{Aut}(M)$ *and* $x \in M$ *then*

$$F(x) := \{f(x) \;:\; f \in F\}$$

and the support of F *is the following set:*

$$\mathrm{supp}(F) := \bigcup_{f \in F} \mathrm{supp}(f)$$

Definition 1.3.4 *Let* $\mathcal{L}_0$ *and* $\mathcal{L}_1 = \langle P_i, f_j, c_k \;:\; i \in I \; j \in J \; k \in K\rangle$ *be two languages, where the* P_i *are predicate symbols, the* f_j *are function symbols and the* c_k *are constant symbols. Let* α_i *and* β_j *be the arities of* P_i *and* f_j *respectively. Let* K_0 *and* K_1 *be classes of models in* $\mathcal{L}_0$ *and* $\mathcal{L}_1$ *respectively and let* $R \subseteq K_0 \times K_1$. *If* $\bar{x}$ *is a tuple, let* $l(\bar{x})$ *be the length of the tuple.*

We say that K_1 *is* ***interpretable*** *in* K_0 *relative to* R *if there are the following:*

- *an* $\mathcal{L}_0$*-formula* $\phi_{Dom}(\bar{x})$*;*
- *an* $\mathcal{L}_0$*-formula* $\phi_{Eq}(\bar{y}_0, \bar{y}_1)$*;*
- *for each* $i \in I$ *an* $\mathcal{L}_0$*-formula* $\phi_{P_i}(\bar{z}^i_0, \ldots, \bar{z}^i_{\alpha_i})$*;*
- *for each* $j \in J$ *an* $\mathcal{L}_0$*-formula* $\phi_{f_j}(\bar{w}^j_0, \ldots, \bar{w}^j_{\beta_j}, \bar{v}^j)$*;*
- *for each* $k \in K$ *an* $\mathcal{L}_0$*-formula* $\phi_{c_k}(\bar{v}^k)$*;*

such that:

$$\begin{array}{rcll} l(\bar{x}) & = & l(\bar{y}_0) = l(\bar{y}_1) & \\ & = & l(\bar{z}^i_0) = \ldots = l(\bar{z}^i_{\alpha_{i-1}}) & \text{for all } i \in I \\ & = & l(\bar{w}^j_0) = \ldots = l(\bar{w}^j_{\beta_{j-1}}) = l(\bar{v}^j) & \text{for all } j \in J \\ & = & l(\bar{z}^k) & \text{for all } k \in K \end{array}$$

and if $M_0 \in K_0$ and $M_1 \in K_1$ are such that $R(M_0, M_1)$ then there is a surjection $\tau : \phi_{Dom}(M_0) \to M_1$ such that:

$$\begin{aligned} \tau(\bar{a}) = \tau(\bar{b}) &\Leftrightarrow M_1 \models \phi_{Eq}(\bar{a}, \bar{b}) \\ M_0 \models \phi_{P_i}(\bar{a}_0, \ldots, \bar{a}_{\alpha_i}) &\Leftrightarrow M_1 \models P_i(\tau(\bar{a}_0), \ldots, \tau(\bar{a}_{\alpha_i})) \\ M_0 \models \phi_{f_j}(\bar{a}_0, \ldots, \bar{a}_{\beta_j}) = \bar{b} &\Leftrightarrow M_1 \models f_j(\tau(\bar{a}_0), \ldots, \tau(\bar{a}_{\beta_j})) = \tau(\bar{b}) \\ M_0 \models \phi_{c_k}(\bar{a}) &\Leftrightarrow M_1 \models \tau(\bar{a}) = c_k \end{aligned}$$

The collection of formulas

$$\{\phi_{Dom}(\bar{x}), \phi_{Eq}(\bar{y}_0, \bar{y}_1), \phi_{P_i}(\bar{z}^i_0, \ldots, \bar{z}^i_{\alpha_i}), \\ \phi_{f_j}(\bar{w}^j_0, \ldots, \bar{w}^j_{\beta_j}, \bar{v}^j), \phi_{c_k}(\bar{v}^k) \ : \ i \in I, j \in J, k \in K\}$$

*is called an **interpretation**.*

*If all the formulas in an interpretation are first order, second order, or $L_{\omega_1,\omega}$, then we call that interpretation a **first order interpretation**, a **second order interpretation** or an $L_{\omega_1,\omega}$**-interpretation** respectively.*

*We say that K_1 and K_2 are **bi-interpretable** if:*

1. *there is Φ, a first order interpretation of K_1 in K_2 such that Φ is an interpretation of K_1 in $\langle \psi_{Dom}(K_2), \psi_{Eq}, \ldots, \rangle$;*
2. *there is Ψ, a first order interpretation of K_2 in K_1 such that Ψ is an interpretation of K_1 in $\langle \phi_{Dom}(K_2), \phi_{Eq}, \ldots \rangle$;*
3. *the interpretation of K_1 inside the interpretation of K_2 in K_1 is $\emptyset$-definable; and*
4. *the interpretation of K_2 inside the interpretation of K_1 in K_2 is $\emptyset$-definable.*

Remark 1.3.5 *If P is a partial order, then $x \leq_P y$ means that $P \models x \leq y$. The symbols $<_P$, $\geq_P$ and $>_P$ are defined similarly.*

1.3.1 Cycle-free Partial Orders

This section will deal with the definition of CFPOs, as well as certain useful concepts relating to them. Observations about the properties of the CFPOs as first order models depend on results proved in Chapter 3, so will be made then, in Section 3.3.

The problem that this thesis seeks to address was posed before the objects of study were fully defined. In the introduction to his memoir on the reconstruction of trees, Rubin suggested extending the problem to cycle-free partial orders, which he defined as follows:

Definition 1.3.6 (Definition 0.19 of [23]) *Let $M = \langle A, < \rangle$ be a poset. M is cycle-free, if for every $a_0, \ldots a_{n-1} \in A$: if for every $i < n$ either $a_i < a_{i+1 \bmod n}$ or $a_i > a_{i+1 \bmod n}$, then there are $i, j < n$ such that $j \neq i, j \neq i + 1 (\operatorname{mod} n)$ and a_j belongs to the closed interval whose end points are a_i and $a_{i+1 (\operatorname{mod} n)}$.*

This seeks to define the class by forbidding 'diamonds' and the class of structure called 'n-crowns':

Definition 1.3.7 *The four-element partial order $\{p_0, p_1, p_2, p_3\}$ is called a diamond if $p_0 \leq p_1 \leq p_3$ and $p_0 \leq p_1 \leq p_3$ but $p_1 \not\leq p_2$ and $p_2 \not\leq p_1$. This is depicted in Figure 1.1.*

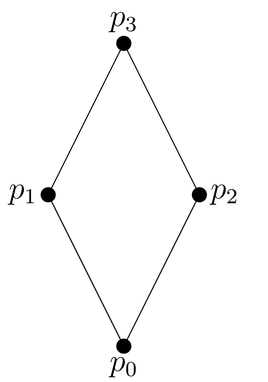

Figure 1.1: A Diamond

Definition 1.3.8 *The finite partial order P is said to be a n-**crown** if:*

- $P = \{p_1, \dots p_{2n}\}$;
- *for all $i \in \mathbb{Z}$, if $j = 2i - 1 \bmod 2n$ and $k = 2i + 1 \bmod 2n$*

$$p_j \leq p_{2i} \geq p_k$$

- *otherwise $p_i \not\leq p_j$ and $p_i \not\geq p_j$.*

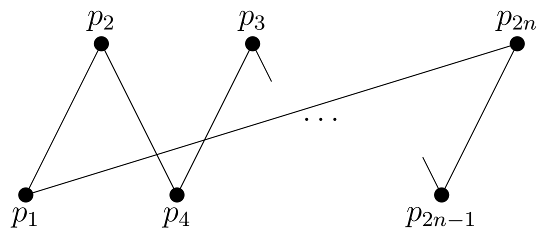

Figure 1.2: An n-crown

CFPOs are a generalisation of trees. Unfortunately Definition 1.3.6 does not behave well with taking substructures. This observation is due to Richard Warren in his Ph.D. thesis [39], who gives the example of the 2-crown and one of its superstructures.

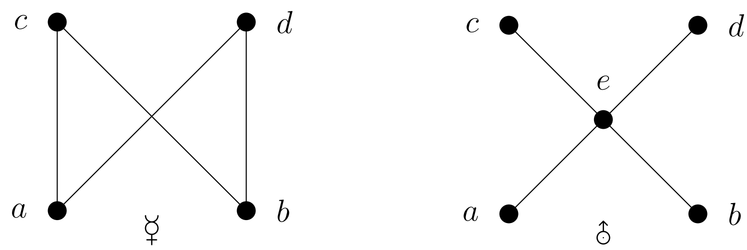

Figure 1.3: Warren's Example

Rubin's definition views neither of these orders as cycle-free, but this seems counter-intuitive. While the notion of a, b, c and d forming a cycle in ☿ (Mercury) seems acceptable, in ⛢ (Uranus) this cycle repeatedly passes through e. If we were to insist that e must be contained in that tuple, Rubin's definition would count ⛢ as cycle-free.

However, such a definition would produce a class not closed under taking substructures (☿ $\subseteq$ ⛢). These worries were resolved by Warren, whose definition goes via the concept of Dedekind-MacNeille completion. The following are taken from Warren's memoir [39]. They introduce some notation which will be used throughout:

Definition 1.3.9 *Let P be partially ordered by $\leq$, and let $a, b \in P$ and $X, Y, I, F \subseteq P$.*

1. *a and b are **comparable** (written as $a \leq\geq b$) if $a \leq b$ or $a \geq b$ (if $a \leq\geq b$ and $a \neq b$ then we write $a <> b$);*
2. *a and b are **incomparable** (written as $a \parallel b$) if a and b are not comparable;*
3. *X is a **chain** if it is linearly ordered by $\leq$;*
4. *Y is an **antichain** if it is pairwise incomparable;*
5. $\inf_P(X)$ *is the infimum of X in P if it exists;*
6. $\sup_P(Y)$ *is the supremum of Y in P if it exists;*
7. *$a \leq X$ if $\forall x \in X(a \leq x)$; ($a < X$, $a \geq X$, $a > X$, $X \leq Y$ and $X < Y$ are defined similarly);*
8. *$\bigvee^P X := \{p \in P : p \geq X\}$; $\bigwedge^P X := \{p \in P : p \leq X\}$;*
9. *I is a **Dedekind Ideal**, written $I \in I^D(P)$ if $I \neq \emptyset$, $\bigwedge^P I \neq \emptyset$ and*

$$\bigwedge{}^P \bigvee{}^P I = I$$

10. *F is a **Dedekind Filter**, written $F \in F^D(P)$ if $F \neq \emptyset$, $\bigvee^P F \neq \emptyset$ and*

$$\bigvee^P \bigwedge^P F = F$$

11. *X is **downwards closed** in P if for all $p \in P$, if $p \leq x$ for some $x \in X$ then $p \in X$; (upwards closed is defined similarly);*

12. *$P^\ominus$ is $I^D(P)$ ordered by inclusion; $P^\oplus$ is $F^D(P)$ ordered by reverse inclusion;*

13. *$P^{\leq x} := \{p \in P \ : \ p \leq x\}$; ($P^{<x}$, $P^{\geq x}$ and $P^{>x}$ are defined similarly);*

14. *if $I \in P^\ominus$ and there exists an $x \in P$ such that $I = I^{\leq x}$ then I is said to be a **principal ideal**; (**principal filters** are defined similarly)*

Facts 1.3.10

1. *P can be embedded into both $P^\ominus$ and $P^\oplus$;*

2. *there is an isomorphism from $P^\ominus$ to $P^\oplus$ which fixes P pointwise;*

3. *$\forall a, b \in P^\ominus$, if $a \parallel b$ and $\exists c\,(c \leq \{a, b\})$ then $\inf_{P^D}(\{a, b\})$ exists;*

4. *$\forall a, b \in P^\ominus$, if $a \parallel b$ and $\exists c\,(c \geq \{a, b\})$ then $\sup_{P^D}(\{a, b\})$ exists;*

Definition 1.3.11

1. *We call $P^D := P^\ominus$ the **Dedekind-MacNeille completion** of P.*

2. *If $P^D \cong P$ then we say that P is **Dedekind-MacNeille complete**.*

Facts 1.3.12

1. *For all partial orders, $(P^D)^D \cong P^D$. i.e. Dedekind-MacNeille completions are Dedekind-MacNeille complete.*

2. *If P and Q are partial orders such that $P \subseteq Q$ and Q is Dedekind-MacNeille complete then there is an embedding $\beta : P^D \to Q$ such that $\beta|_P = id$.*

Warren introduces the notion of path and connecting set in the Dedekind-MacNeille completion in order to define cycle-freeness. Both he and I routinely use the notion of path when talking about CFPOs; it is an extremely useful concept, and provides a good definition of the class.

Definition 1.3.13 (2.3.2 of [39]) *If M is a partial order and $a, b \in M$, then the n-tuple $C = \langle c_1, c_2, \ldots, c_n \rangle$ (for $n \geq 2$) is said to be a **connecting set** from a to b in M, written $C \in C^M \langle a, b \rangle$, if the following hold:*

1. $c_1 = a, c_n = b, c_2, \ldots, c_{n-1} \in M^D$
2. *if* $1 \leq i \leq n-1$, *then* $c_i \not\parallel c_{i+1}$
3. *if* $1 < i < n$, *then* $c_{i-1} < c_i > c_{i+1}$ *or* $c_{i-1} > c_i < c_{i+1}$

Definition 1.3.14 (2.3.3 of [39]) *Let M be a partial order, $a, b \in M$, and let $C = \langle c_1, c_2, \ldots, c_n \rangle$ be a connecting set from a to b in M. Let σ_k (for $1 < k < n$) be maximal chains in M^D with endpoints $c_k, c_{k+1} \in \sigma_k$, such that if $x \in \sigma_i \cap \sigma_j$ for some $i < j$, then $j = i + 1$ and $x = c_{i+1}$. Then we say that $P = \bigcup_{0<k<n} \sigma_k$ is a **path** from a to b in M.*

Definition 1.3.15 *A partial order M is said to be a **cycle-free partial order** (CFPO) if for all $x, y \in M$ there is at most one path between x and y in M^D. If it exists, this unique path is denoted by* $\mathrm{Path}\langle x, y \rangle$

Definition 1.3.16 *A partial order is said to be **connected** if there is a path between any two points, i.e.* $\mathrm{Path}\langle x, y \rangle$ *exists for all $x, y \in M$, and is said to be disconnected otherwise.*

*Let M be a partial order and let $C \subseteq M$. We say that C is a **connected component** of M if it is a maximal connected subset of M, i.e. for all $x, y \in M$ if $x \in C$ and* Path$\langle x, y\rangle$ *exists then $y \in C$.*

We will also need the concept of paths between sets, as well as between points.

Definition 1.3.17

$$\begin{aligned}
\mathrm{Path}\langle x, Y\rangle &:= \bigcap_{y \in Y} \mathrm{Path}\langle x, y\rangle \\
\mathrm{Path}\langle X, y\rangle &:= \bigcap_{x \in X} \mathrm{Path}\langle x, y\rangle \\
\mathrm{Path}\langle X, Y\rangle &:= \bigcap_{\substack{x \in X \\ y \in Y}} \mathrm{Path}\langle x, y\rangle
\end{aligned}$$

Since this definition is about the Dedekind-MacNeille completion of a partial order, rather than the partial order itself, we may be concerned about whether the class of CFPOs is axiomatisable or not. Truss finds an axiomatisation in [36], but also shows that there is no axiomatisation of the class of connected CFPOs.

We are able to refer to paths in a CFPO's Dedekind-MacNeille completion as we do not require the full completion, just points that are definable. These points are the 'meet' and the 'join' of pairs of elements, which are defined below.

Definition 1.3.18 *Let M be a partial order and let $x, y \in M$. The **meet** of x and y is written and defined as:*

$$x \wedge y := \sup\{t \in T \ : \ t \leq x, y\}$$

*The **join** of x and y is written and defined as:*

$$x \vee y := \inf\{t \in T \ : \ t \geq x, y\}$$

Facts 3 and 4 of Facts 1.3.10 show that while $x \wedge y$ and $x \vee y$ may not exist in M, they will always exist in M^D.

Connectedness is not a first order property as it would require us to ask for arbitrarily long paths, something that would require infinite disjunction.

Definition 1.3.19

1. *An* ***upwards cone*** *of a point t is a maximal set C such that*

$$\forall c \in C\ t < c \quad \text{and} \quad t < c_0 \wedge c_1 \text{ for all } c_0, c_1 \in C$$

2. *An* ***downwards cone*** *of a point t is a maximal set C such that*

$$\forall c \in C\ t > c \quad \text{and} \qquad t > c_0 \vee c_1 \text{ for all } c_0, c_1 \in C$$

3. *If C is a upwards or downwards cone of t then C's* ***extended cone*** *is the following set:*

$$\{x \in M\ :\ \text{Path}\langle x, t\rangle \cap C \neq \emptyset\}$$

1.4 Some Reconstruction Results

One may think of an automorphism group in a number of ways. It can be considered as an abstract group, i.e. a model of the theory of groups in the language of groups ($\mathcal{L}_G = \langle \circ, {}^{-1}, \mathrm{id}\rangle$). This abstract group can be added to the original model to form a two sorted structure, i.e. from model M in language $\mathcal{L}_M$ we obtain

$$\langle M, \text{Aut}(M), \mathcal{L}_M, \mathcal{L}_G, \mathrm{Op}\rangle$$

where

$$\mathrm{Op} : \begin{cases} \text{Aut}(M) \times M & \rightarrow & M \\ (f, x) & \mapsto & f(x) \end{cases}$$

We may strip the structure of the original language to obtain $\langle M, \text{Aut}(M), \mathcal{L}_G, \text{Op}\rangle$, the permutation group. In this structure, we may define a topology on the group, where the basic open neighbourhoods of id are the point-wise stabilisers of finite sets. The topological group is $\langle \text{Aut}(M), \mathcal{L}_G, \tau\rangle$, where τ is the topology just mentioned.

With all these different notions of automorphism group there is a risk that when reading '$\text{Aut}(M) \cong \text{Aut}(N)$', one will be left wondering in what way are the two isomorphic? Therefore I introduce the following notation.

Definition 1.4.1

$$\begin{aligned}
\text{Aut}(M) \cong_A \text{Aut}(N) &\Leftrightarrow \langle \text{Aut}(M), \mathcal{L}_G\rangle \cong \langle \text{Aut}(N), \mathcal{L}_G\rangle \\
\text{Aut}(M) \cong_T \text{Aut}(N) &\Leftrightarrow \langle \text{Aut}(M), \mathcal{L}_G, \tau_M\rangle \cong \langle \text{Aut}(N), \mathcal{L}_G, \tau_N\rangle \\
\text{Aut}(M) \cong_P \text{Aut}(N) &\Leftrightarrow \langle \text{Aut}(M), M, \mathcal{L}_G, \text{Op}\rangle \cong \langle \text{Aut}(N), N, \mathcal{L}_G, \text{Op}\rangle
\end{aligned}$$

The subscript A stands for 'abstract', T for 'topology' and P for 'permutation'

Ahlbrandt and Ziegler showed in [1] that if M and N are countable and $\aleph_0$-categorical in a countable language then $\text{Aut}(M) \cong_T \text{Aut}(N)$ implies that M and N are bi-interpretable. Previously it was already known that if the automorphism groups of M and N are isomorphic as permutation groups then not only are they bi-interpretable, but that we may take the domain of the interpretation in either direction to be M^1 or N^1 as appropriate.

The corresponding result for abstract automorphism groups is false, but the situation is still very well understood thanks to Rubin, who examined the reconstruction of $\aleph_0$-categorical structures from their abstract automorphism groups in [24]. He showed that if M and N are $\aleph_0$-categorical structures which have an '$\forall\exists$-interpretation', and no algebraicity (i.e. for all finite A, there is no $a \notin A$ algebraic over A) then

$$\text{Aut}(M) \cong_A \text{Aut}(N) \Leftrightarrow \langle M, \mathcal{L}_M\rangle \text{ and } \langle N, \mathcal{L}_N\rangle \text{ are bi-interpretable}$$

Rubin also considered Boolean algebras and their automorphism groups. He managed to get a strong reconstruction result concerning groups that act in a certain way on complete atomless Boolean algebras. He uses this result in several other settings. The reconstruction of linear orders and reducts of linear orders from their abstract automorphism groups; manifolds from their abstract autohomeomorphism groups; and the reconstruction of trees; these all depend on his results about Boolean algebras.

There is a large body of work concerning the properties of the automorphism group of an ordered structure, a particularly pertinent example being [12]. Unfortunately I am insufficiently familiar with this work to give a good account of it, but I can say that it concerns the reconstruction of 2-homogeneous linear orders with countable cofinality from quotients of the automorphism group.

In this thesis, I have borrowed a method for reconstruction from Shelah in [31, 30] and from Shelah and Truss in [33]. This utilises the alternating group on five elements to define the permutation structure of the symmetric groups of cardinals and their quotients inside those groups as abstract groups. This approach works well for CFPOs with a certain degree of transitivity, as we shall see in Chapter 4.

1.4.1 The Reconstruction of Trees

This section, apart from some minor comment and narrative on my part, is taken from [22, 23, 25, 26]. While the comment is my own, the results belong to Rubin and Rubin and McCleary. The reference [23] is the chief source for the results, but the powerful methods used have their history throughout the other three references.

Definition 1.4.2 *Let $\mathcal{K}$ be a class of first-order models. $\mathcal{K}$ is said to be* ***faithful*** *if*

$$\forall K_0, K_1 \in \mathcal{K}\ (\mathrm{Aut}(K_0) \cong_A \mathrm{Aut}(K_1) \Rightarrow K_0 \cong K_1)$$

Definition 1.4.3 *$\mathcal{K}' \subset \mathcal{K}$ is said to be* ***canonical*** *if $\mathcal{K}'$ is faithful and*

$$\forall K \in \mathcal{K} \exists L \in \mathcal{K}' \, (\mathrm{Aut}(L) \cong_A \mathrm{Aut}(K))$$

If the larger $\mathcal{K}$ is not faithful, then there may be many possible canonical subfamilies. While the axiom of choice implies that there is a canonical subfamily of every family of first order models, this would teach us nothing about the structure of the models involved. Besides, we are looking for a way of defining a canonical model inside its automorphism group. The chances of defining a member of a class we obtained using non-constructive methods are rather slim!

Therefore we must make some moral decision as to which first order models we add to a canonical class. A great deal more can be said by adding colour predicates, so Rubin works with the class of coloured trees, rather than the non-coloured trees.

Proposition 1.4.4 *Let $\langle P, \leq \rangle$ be a partial order, and let I be a unary predicate such that*

$$\langle P^D, \leq, I \rangle \models I(x) \quad \Leftrightarrow \quad x \in P^D \setminus P$$

Then $\mathrm{Aut}(P, \leq) \cong_A \mathrm{Aut}(P^D, \leq, I)$.

Proof

The I predicate ensures that any automorphism of P^D preserves P, and since P is dense in P^D every automorphism of P extends uniquely to an automorphism of P^D. □

However, this is not the end of it. Many different notions of completion also exhibit this property.

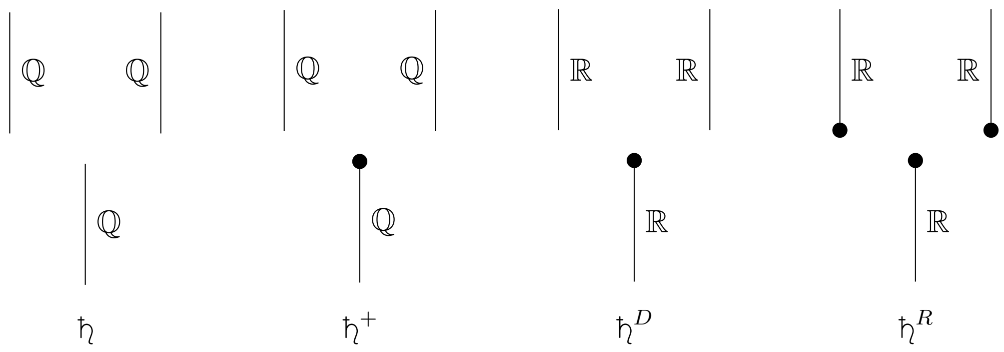

Figure 1.4: Saturn (♄) and its Completions

♄ is the partial order with domain $\{0, 1, 2\} \times \mathbb{Q}$ and order

$$(i, x) \leq_{♄} (j, y) \text{ iff } \begin{cases} i = 0 \quad \text{and} \quad j = 1, 2 & \text{or} \\ i = j \quad \text{and} \quad x \leq_{\mathbb{Q}} y \end{cases}$$

$♄^+$ is obtained by adding a single point a such that

$$((i, x) \leq_{♄^+} a \Leftrightarrow i = 0) \quad \text{and} \quad (a \leq_{♄^+} (i, x) \Leftrightarrow i = 1, 2)$$

This is known as the ramification completion or path completion, and is defined for any tree in Definition 2.2.3, and any CFPO in Definition 2.5.1.

$♄^D$ is the standard Dedekind-MacNeille completion of ♄, while $♄^R$ is given by adding two points b and c to $♄^D$ such that

$$((i, x) \leq_{♄^R} b \Leftrightarrow i = 0) \qquad \text{and} \qquad (b \leq_{♄^R} (i, x) \Leftrightarrow i = 1)$$
$$\text{and}$$
$$((i, x) \leq_{♄^R} c \Leftrightarrow i = 0) \qquad \text{and} \qquad (c \leq_{♄^R} (i, x) \Leftrightarrow i = 2)$$

This $\natural^R$ is called the Rubin completion of a tree. If the I predicate is defined for each of these completions in the same way as in Proposition 1.4.4 then

$$\text{Aut}(\natural, \leq) \cong_A \text{Aut}(\natural^+, \leq, I) \cong_A \text{Aut}(\natural^D, \leq, I) \cong_A \text{Aut}(\natural^R, \leq, I)$$

Definition 1.4.5 *Let T be a Dedekind-MacNeille complete tree. T is said to be **Rubin complete** if for all $t \in T$, if there is more than one cone above t then all the cones above t have a least element.*

If T is any tree then T^R is defined to be the least Rubin complete tree that contains T.

The Rubin completion has many of the same properties as the Dedekind-MacNeille completion. Every tree has a unique minimal Rubin completion, and every automorphism of a tree extends uniquely to an automorphism of the completion. Please refer to Chapter 3 of [23] for details and proofs.

Proposition 1.4.6 *Let T be any tree, and let I be a unary predicate such that*

$$T^R \models I(x) \Leftrightarrow x \in T^R \setminus T$$

Then

$$\text{Aut}(T, \leq) \cong_A \text{Aut}(T, \leq, I)$$

Completions are not the only way two trees can share an automorphism group!

Proposition 1.4.7 *Let $\langle T, \leq \rangle$ be a tree. We say that $a_0, \ldots, a_{n-1} \in T$ is an n-chain of unique successors if for all i:*

- *a_i is the unique predecessor of a_{i+1}; and*
- *a_{i+1} is the unique successor of a_i.*

Let $\langle S, \leq, C_i \,:\, i \in \omega\rangle$ be the coloured tree obtained by replacing every n-chain of unique successors with a single point for each n, where $\langle S, \leq, C_i \,:\, i \in \omega\rangle \models C_n(x)$ if and only if x was added to replace an n-chain of unique successors.

Then $\mathrm{Aut}(\langle T, \leq\rangle) \cong_A \mathrm{Aut}(\langle S, \leq, C_i \,:\, i \in \omega\rangle)$.

Proposition 1.4.8 *Let ♀ and ♂ be the trees pictured in Figure 1.5.*

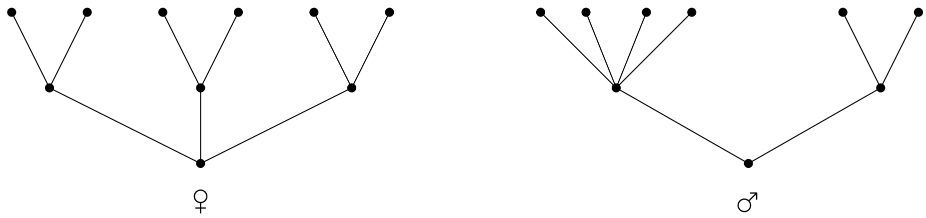

Figure 1.5: Trees ♀ and ♂

Then Aut(♀) $\cong_A$ Aut(♂).

Proposition 1.4.9 *Let $\langle T, \leq_T\rangle$ be a tree. Let $a \in T$. Let $\{C_i \,:\, i \in I\}$ be the family of cones above a and let $C_i \sim C_j$ if and only if there is an automorphism of T that maps C_i to C_j. This is an equivalence relation, and we let $\{c_\alpha \,:\, \alpha \in \beta\}$ be the set of equivalence classes. We define $\langle S(a), \leq_S, P_2\rangle$ to be the tree on the domain*

$$\{a\} \cup \{c_\alpha \,:\, \alpha \in \beta\} \cup \bigcup_{i \in I} C_i$$

where P_2 is a unary predicate given by $S(a) \models P_2(x)$ if and only if $x = c_\alpha$ for some α. The ordering is given by:

$$x \leq_S y \quad \Leftrightarrow \quad \begin{cases} x \leq_T y & \text{or} \\ x = a & \text{or} \\ x = c_\alpha \wedge y \in C_i \in c_\alpha & \end{cases}$$

Let $\langle \tilde{T}, \leq, P_2 \rangle$ *be the tree obtained by taking* T *and at every* a*, replacing* $T^{\geq a}$ *by* $S(a)$*.*

Then $\mathrm{Aut}(T) \cong_A \mathrm{Aut}(\tilde{T})$.

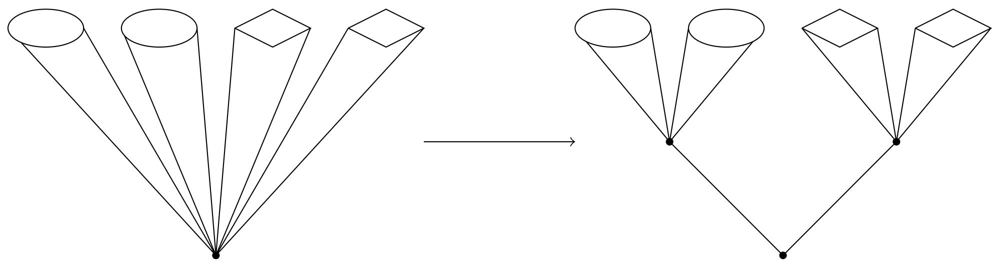

Figure 1.6: An example of $T^{\geq a}$ and $S(a)$

These four propositions are what we call pathologies; they describe behaviour that leads to trees sharing automorphism groups. However, these pathologies let us make canonical choices. We prefer coloured Rubin-complete trees to incomplete trees. We prefer coloured singletons to chains of unique successors with arbitrary length. We prefer ♀ to ♂ as Aut(♀) acts transitively on the levels of ♀, while the same cannot be said about ♂. We prefer trees where either all or none of the cones above a point can be swapped.

There is, however, a pathology where it is not clear how a choice can be made:

Proposition 1.4.10 $(\mathbb{Z}_2)^{\aleph_0} \wr S_3 \cong_A ((\mathbb{Z}_2)^{\aleph_0} \wr S_3) \times \mathbb{Z}_2$*. For a definition of* $\wr$*, the wreath product, please see Definition 3.2.3 or 5.1.7.*

$((\mathbb{Z}_2)^{\aleph_0} \wr S_3) \times \mathbb{Z}_2$ is the automorphism group of the tree pictured below, while $(\mathbb{Z}_2)^{\aleph_0} \wr S_3$ is the automorphism group of the tree obtained by only taking the vertices drawn as circles. The diamond and square ornaments indicate that there are unary predicates that prevent the second highest level from being switched while fixing the third highest level.

How $\mathbb{Z}_2$ moves through various kinds of product is the source of this pathology, and is unfortunately not fully understood, for example:

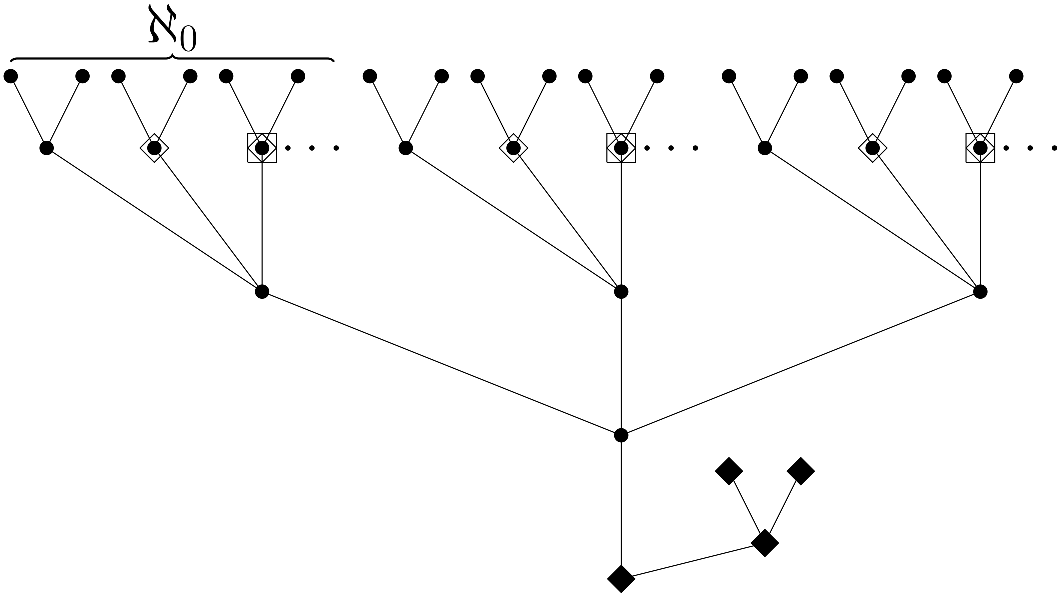

Figure 1.7: Resolved pathology involving $\mathbb{Z}_2$

Question 1.4.11 *The trees in question are drawn in Figure 1.8.*

The empty rectangles are intended to show that a copy of the tree depicted in the filled rectangle is inside, the bottom vertex identified with the top vertex the rectangle touches. This process continues forever. Once again, we let M_0 be the tree of circular vertices, while M_1 is the tree depicted by all the vertices. Does $\mathrm{Aut}(M_0) \cong_A \mathrm{Aut}(M_1)$ *hold?*

These problems force Rubin to exclude all trees T such that there is an s such that s has a successor t which is a maximal element, and the orbit of t under the action of $\mathrm{Aut}_{(s)}(T)$ has exactly two elements. Very loosely, in the abstract group context, if $\mathbb{Z}_2$ occurs in an uncontrolled way, then we are unable to find a canonical tree.

With all the pathologies discussed, here is the class of trees where Rubin's methods work, and his canonical class of trees.

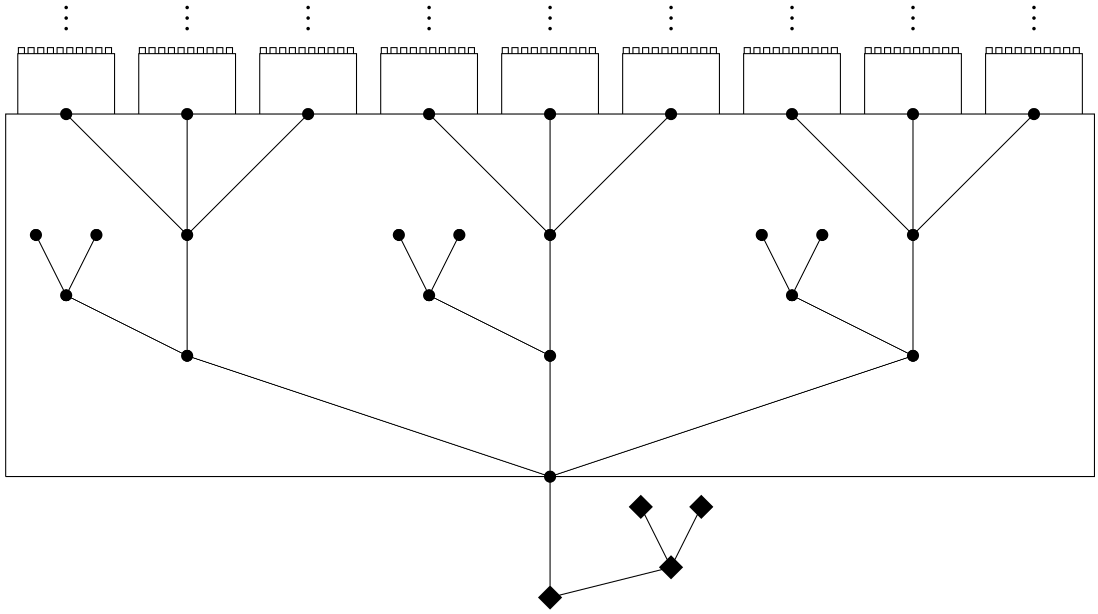

Figure 1.8: Unresolved Pathology involving $\mathbb{Z}_2$

Definition 1.4.12 *Let T be a tree, let $G \subseteq \mathrm{Aut}(T)$, and let $t \in T$.*

$$\mathrm{Or}(t : G) := \{s \in T \ : \ f(t) = s \text{ for some } f \in G\}$$

Definition 1.4.13 *Let T be a tree, $s, t \in T$ and $s < t$. Then $\mathrm{Or}(t : \mathrm{Aut}_{(s)}(T))$ is called* ***rigid*** *if for all $f, g \in \mathrm{Aut}_s(T)$*

$$f(t) = g(t) \Rightarrow f|_{\mathrm{Or}(t:\mathrm{Aut}_{(s)}(T))} = g|_{\mathrm{Or}(t:\mathrm{Aut}_{(s)}(T))}$$

$\mathrm{Or}(t : \mathrm{Aut}_{(s)}(T))$ *is called a* ***bad orbit*** *if it is rigid and* $|\mathrm{Or}(t : \mathrm{Aut}_{(s)}(T))| > 2$

$\mathcal{T}_{Good}$ *is the class of trees that have no bad orbits.*

Example 1.4.14 *Let $\langle \mathbb{Z} \times \{z\}, \leq \rangle$ be the Cartesian product of the integers with itself with the lexicographic order. If a and b lie in different copies of $\mathbb{Z}$ then $\mathrm{Or}(a : \mathrm{Aut}_{(b)}(T))$ is a bad orbit.*

Definition 1.4.15 *Let $\mathcal{T}_{CAN} \subseteq \mathcal{T}_{Good}$ be the class of good trees T such that each of the following hold:*

1. *T is Rubin-complete;*
2. *for all $s \in T$ there is no $t \in T$ with $t > s$ such that $u > s \Rightarrow u \geq t$;*
3. *for all $t \in T$, if t is a maximal element and the successor of some $s \in T$ then there is a $\phi \in \mathrm{Aut}_s(T)$ such that $\phi(t) \neq t$;*
4. *for all $t > s$, if all $\phi \in \mathrm{Aut}_s(T)$ fix t then t is either a successor of s or is a maximal element of T which is the successor of no element of T;*
5. *for all $t \in T$ if one of the cones above t can be mapped by an automorphism to another cone above t, then there is an automorphism that takes that cone to any of the cones above t;*
6. *$T^{\geq t} \not\cong$ ♀ for all $t \in T$.*

$\mathcal{T}_{Rub} \subset \mathcal{T}_{CAN}$ is the class which also satisfy the additional axioms:

7. *for all $t > s$, if all $\phi \in \mathrm{Aut}_s(T)$ fix t then t is a successor of s;*
8. *if t and u are the only successors of s, if x and y are successors of t then there is a $\phi \in \mathrm{Aut}(T)$ such that $\phi(x) = y$;*
9. *for all $s, t \in T$, if t is the successor of s and $|\mathrm{Or}(t : \mathrm{Aut}_{(s)}(T))| = 2$ then t is not maximal.*

Conditions 2., 3., 4. and 7. are all related to the pathology caused by successors (Proposition 1.4.7). Conditions 2. and 3. forbid aberrant behaviour caused by successors. 4. and 9. are ensure that 'small' orbits are caused by manageable behaviour.

Theorem 1.4.16 *There is a function* CAN : $\mathcal{T}_{Good} \to \mathcal{T}_{CAN}$ *such that:*

- *if* $\sim$ *is the binary relation*

$$T \models x \sim y \Leftrightarrow \exists \phi \in \text{Aut}(T)\ (\phi(x) = y)$$

  then $\langle T, \leq, \sim \rangle \cong \langle \text{CAN}(T), \leq, \sim \rangle$*;*

- K_{CAN} *is second order interpretable in* K_{Good} *with respect to* CAN.

Theorem 1.4.17 K_{Rub} *is faithful.*

Rubin reconstructs the trees by building complete atomless Boolean algebras around them, and then using his powerful reconstruction results about Boolean algebras to reconstruct them. As we will see in the next chapter, trees have the property that the behaviour of tuples is controlled by the behaviour of adjacent pairs (Theorem 2.2.14). Much of the machinery of this method relies on this property, and since the CFPOs share this property, it seems reasonable that one would be able to adapt the Boolean algebraisation method to CFPOs.

Chapter 2

$\aleph_0$-categorical Trees and CFPOs

This chapter gives a structure theorem for the $\aleph_0$-categorical trees and cycle-free partial orders. As well as being of intrinsic interest, this leads to a more complete reconstruction result for the $\aleph_0$-categorical trees.

2.1 Motivation

Definition 2.1.1 *A tree T is said to be* ***definably complete*** *if for all non-empty subsets $A \subseteq T$ which are definable over some finite $B \subseteq T$ both* $\inf(A)$ *and* $\sup(A)$ *exist in T. In particular, this means that for all $a, b \in T$ their meet, $a \wedge b$, is an element of T.*

This following definition and theorem are rephrasings of Definition 11.1 and Theorem 11.2 of Rubin's memoir [23].

Definition 2.1.2 *K_{CAT} is the set of $\aleph_0$-categorical trees such that $T \in K_{CAT}$ if and only if all of the following hold:*

1. *T is definably complete;*

2. *no $t \in T$ has exactly one successor;*

3. *for every $t \in T$, either all of the successors of t can be switched by automorphisms, or none of them can;*

4. *for all $s, t \in T$, if $s < t$ and t is definable over s then t is either a successor of s or a maximal element of T, but not both;*

5. *$T^{\geq t} \ncong$ ♀ for all $t \in T$ (see Proposition 1.6.7).*

Theorem 2.1.3 *Let $\sim$ be the relation as defined in Theorem 1.4.16, i.e.*

$$T \models x \sim y \Leftrightarrow \exists \phi \in \mathrm{Aut}(T)\ (\phi(x) = y)$$

Then:

1. *for every $\aleph_0$-categorical tree S there is a $T \in K_{CAT}$ such that $\mathrm{Aut}(S) \cong \mathrm{Aut}(T)$; and*

2. *for every $S, T \in K_{CAT}$ if $\mathrm{Aut}(S) \equiv \mathrm{Aut}(T)$ then $\langle S, \leq \rangle \cong \langle T, \leq \rangle$*

Thus the reconstruction of $\aleph_0$-categorical trees is better understood than the reconstruction of trees in general. Part of this greater understanding stems from the work of Schmerl in [27], whose paper has a number of important results concerning the decidability of the theory of the $\aleph_0$-categorical trees.

Therefore the structure theorem for $\aleph_0$-categorical trees in this chapter is of slight interest to those interested in reconstruction, but it has intrinsic value. Corollary 2.5.11 together with Subsection 3.1.1 show that all $\aleph_0$-categorical CFPOs share an automorphism group with a tree, so Theorem 2.1.3 applies directly.

2.2 Ramification Completions

Definition 2.2.1 *The* ***ramification order*** *of a point t is the number of cones above t.*

Definition 2.2.2 *A tree T is said to be* ***ramification complete*** *if it contains the meet of any two points, i.e. $x \wedge y$ exists for every $x, y \in T$.*

Definition 2.2.3 *The* ***ramification completion*** *of a tree T is the intersection of all S such that:*

- $T \subseteq S \subseteq T^D$*; and*
- $\forall x, y \in S\ x \wedge y \in S$.

It is written as T^+.

From now until Section 2.6, if the word 'completion' is used without qualification it is used to mean 'ramification completion'.

Definition 2.2.4 *$t \in T^+$ is said to be* ***irrational*** *if $t \in T^+ \setminus T$.*

Proposition 2.2.5 *The completion of a countable tree is countable.*

Proof
Every ramification point corresponds to at least one 2-element antichain, of which there are countably many. □

Definition 2.2.6 *The n-**orbits** of a tree are the equivalence classes of n-tuples with no repeated elements under the relation $x \sim y$, which is given by "x can be mapped to y by an automorphism".*

Definition 2.2.7 *A tree is said to be* ***almost*** n***-transitive*** *if it has only finitely many* n*-orbits.*

Theorem 2.2.8 *A tree* T *is* $\aleph_0$*-categorical if and only if it is almost* n*-transitive for all* n*.*

This is a reformulation of the Ryll-Nardzewski Theorem. A proof for this context can be found in [15], Theorem 7.3.1.

The next few lemmas and definitions allow us to reduce to the case $n = 2$ when considering almost n-transitivity.

Definition 2.2.9 *The* ***completion of an*** n***-tuple*** p *is a tuple of least length which contains* p *and is closed under* $\wedge$*.*

If p is a complete tuple, i.e. p is its own completion, then we label the points of p in a canonical way; $p_{i,j}$ is the i^{th} point that is a successor of a point labelled as $p_{k,j-1}$ for some k. This is pictured in Figure 2.1.

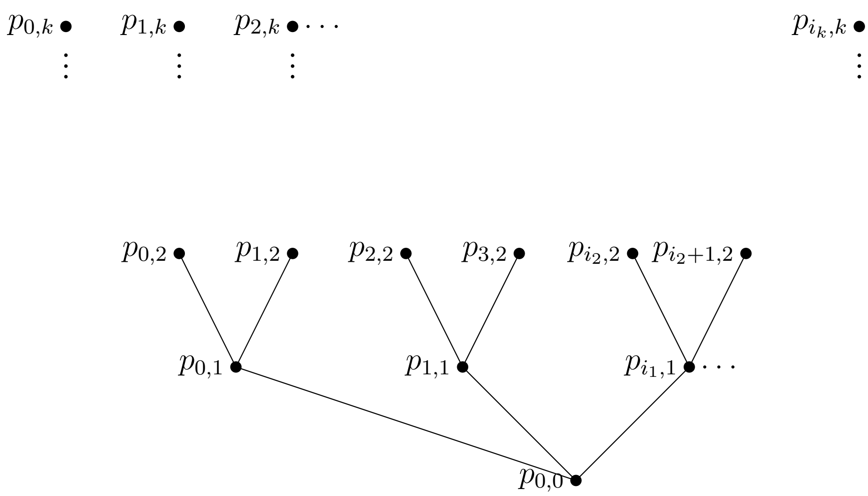

Figure 2.1: The canonical labelling of a complete tuple

Definition 2.2.10 *A **complete** n**-orbit** of T is the orbit of some complete n-tuple.*

Definition 2.2.11 *T is said to be almost n**-complete transitive** if it has finitely many complete n-orbits.*

Lemma 2.2.12 *A complete tree T is almost n-complete transitive for every $n \geq 2$ if and only if T is almost m-transitive for each $m \geq 2$.*

Proof

If T is almost m-transitive for every $m \geq 2$ then T is automatically almost n-complete transitive for every $n \geq 2$. To see the other direction, first notice the following three facts:

1. If p is a complete n-tuple then there are at most 2^n orbits of tuples with p as their completion;
2. If q is an n-tuple then the length of the completion of q is at most $2n$;
3. If p and q lie in the same orbit then their completions lie in the same orbit.

1. holds since every tuple which has p as a completion is a subtuple of p, and p has 2^n subtuples.

To see that 2. is true let $q = (q_0, \ldots, q_{n-1})$. Every time a point is introduced to complete q the ramification point of at least two points is added. This process is repeated finitely many times, so the maximum number of points added is $n - 1$, therefore the maximum length of a completion of an n-tuple is bounded by $2n$.

3. is true as the automorphism between p and q will also carry p's completion to q's completion.

Suppose T is almost n-complete transitive for every $n \geq 2$. Let s_n be the number of n-complete orbits. The number of m-orbits is bounded by $\Sigma_{i=m}^{2m} 2^{s_i}$ as each m-orbit contains a tuple with a completion of length between m and $2m-1$ (say i), and this completion corresponds to at most 2^{s_i} non-isomorphic m-tuples. Hence T is almost m-transitive for each $m \geq 2$. $\square$

Lemma 2.2.13 *In any finitely coloured partial order there are finitely many coloured order types of n-tuples.*

Proof

There are four possible order types that a pair (x, y) can satisfy

$$x < y \qquad x > y \qquad x = y \qquad x \parallel y$$

and since which of these each pair realises determines the order type there can only be finitely many order types. Since there are only finitely many colours, this extends to coloured order types. $\square$

The following theorem was first proved by Pierre Simon, who published it as proposition 4.5 in [34], where it is used to show that coloured trees are dp-minimal. Since the concept of ramification completion is used in the main theorem as well as underpinning this version of the result, and since this proof is more combinatorial in flavour, I have included it despite its greater length than Simon's proof.

Theorem 2.2.14 *Let T be a tree with $T = T^+$. If T is almost 2-transitive then T is almost n-transitive for each $n \geq 2$.*

Proof

Let T be an almost 2-transitive tree such that $T = T^+$ with finitely many 2-orbits. This proof makes use of the canonical labelling of a complete tuple, as pictured in Figure 2.1, so if I refer to $p_{i,j}$, please recall that $p_{0,0}$ is the least element of the tuple $\bar{p}$, and $p_{i,j}$ is the i^{th} point that is a successor of a point labelled as $p_{k,j-1}$ for some k.

By Lemma 2.2.12 it suffices to prove 'if T is almost 2-transitive then T is almost n-complete transitive for all n'. We shall proceed by induction, so we assume that T is almost m-transitive for $m < n$. Let p be a complete n-tuple and let q be another complete-n-tuple such that q has the same coloured order type as p and if $p_{i,j+1}$ is a successor of $p_{k,j}$ then $(p_{i,j+1}, p_{k,j})$ lies in the same 2-orbit as $(q_{i,j+1}, q_{k,j})$.

Note that corresponding adjacent pairs of p and q must have the same colourings, as p and q have the same coloured order type. This assumption is valid since the number of possible coloured order types of p is finite and we are assuming almost transitivity. The number of n-orbits will be determined by the number of 2-types and the number of coloured order types for n-tuples.

Since for all $j \leq i_1$ (recall that i_1 is the maximum value that $p_{-,1}$ can be labelled with) the pairs $(p_{0,0}, p_{j,1})$ and $(q_{0,0}, q_{j,1})$ lie in the same 2-orbit, there is an automorphism f_j carrying the first to the second. Now $p_{j,1}$ is the least element of a complete m_j-tuple, and this subtuple and its corresponding part of q satisfy the theorem, so there is an automorphism g_j carrying the first to the second.

Now let h be the partial function from T to itself given by

$$h(t) = \begin{cases} g_j(t) & \text{if} \quad t \geq p_{j,1} \\ f_j(t) & \text{if} \quad p_0 < (t \wedge p_{j,1}) \leq p_{j,1} \\ f_0(t) & \text{if} \quad t \wedge p_0 \leq p_0 \end{cases}$$

Since h maps p_i to q_i and h consists of automorphisms patched together and these automorphisms agree at their meeting points, h is a partial automorphism. We now seek to extend h to an automorphism by finding automorphisms that send the points not in the domain of h, i.e. the cones of p_0 that do not contain any member of p, to the points not in the image of h. We first extend h by taking $h(t) = f_0(t)$ when $t \notin Dom(h)$ and $f(t) \notin Im(h)$. However this might not fully extend h as f_0 might map some cones that do not contain a member of p to cones that do contain a member of p.

Since p is finite and f_0 is an automorphism, f_0 can map only finitely many cones that do not contain a member of p to a cones that do contain a member of p. Let $q_{\alpha,1}$ be an element of q that lies in a cone whose preimage under f_0 does not contain an element of p. Either $f_0(p_{\alpha,1})$ lies in the image of h or it does not. If it does not lie in the image of h then, since f_0 is an automorphism which maps p_0 to q_0 and by assumption $(p_0, p_{\alpha,1})$ and $(q_0, q_{\alpha,1})$ lie in the same 2-orbit, $(p_0, f_0^{-1}(q_{\alpha,1}))$ lies in the same 2-orbit as $(q_0, f_0(p_{\alpha,1}))$ and so there is an automorphism h_1 that maps $(p_0, f_0^{-1}(q_{\alpha,1}))$ to $(q_0, f_0(p_{\alpha,1}))$, which we extend h by.

If $f_0(p_{\alpha,1})$ is in the image of h then it must lie in a cone that contains a element of q (which we will denote by $q_{\alpha_0,1}$) and we may repeat the process until we find a $q_{\alpha_k,1}$ such that $f_0(p_{\alpha_k,1})$ does not lie in the image of h and we are in the situation dealt with in the previous paragraph. This process can be repeated for each of the cones that are not in the image of h until we have extended h to a total automorphism. □

2.3 Linear Orders and Maximal Chains

Since trees are built up from linear orders, this section will deal with the properties of linear orders and shows what kinds of linear orders can occur in an $\aleph_0$-categorical tree. We first present the known results about $\aleph_0$-categorical linear orders, as well as the definitions required to understand them.

Definition 2.3.1 *If $\langle L_0, <_0\rangle$ and $\langle L_1, <_1\rangle$ are linear orders then their **concatenation**,* *denoted by* $L_0{}^\wedge L_1$ *is the linear order* $\langle L_0 \cup L_1, < \rangle$*, where*

$$x < y \quad \text{iff} \quad \begin{cases} (x, y \in L_0 \quad \text{and} \quad x <_0 y) & \text{or} \\ (x, y \in L_1 \quad \text{and} \quad x <_1 y) & \text{or} \\ \quad (x \in L_0 \quad \text{and} \quad y \in L_1) & \end{cases}$$

Definition 2.3.2 $\langle \mathbb{Q}_n, <_{\mathbb{Q}_n}, C_1 \dots C_n \rangle$ *a countable dense linear order where the colours occur interdensely, i.e. for all* x *and* y *there are* $z_1, \dots z_n$ *between* x *and* y *such that* $C_i(z_i)$ *holds for each* i*.*

$\mathbb{Q}_n$ can also be described as the Fraïssé limit of n-coloured linear orders, and is therefore countably categorical.

Definition 2.3.3 *Let* $\langle L_1, <_1 \rangle, \dots, \langle L_n, <_n \rangle$ *be linear orders. For each* $q \in \mathbb{Q}_n$ *we define* $L(q)$ *to be a copy of* $\langle L_i, <_i \rangle$ *if* $C_i(q)$*. The* $\mathbb{Q}_n$*-**shuffle** of*

$$\langle L_1, <_1 \rangle, \dots, \langle L_n, <_n \rangle$$

denoted by $\mathbb{Q}_n(L_1, \dots L_n)$*, is the linear order* $\langle \bigcup_{q \in \mathbb{Q}_n} L(q), < \rangle$*, where*

$$x < y \quad \text{iff} \quad \begin{cases} \quad x, y \in L(q) & \text{and} \quad x <_i y \quad \text{or} \\ x \in L(q)\,,\, y \in L(p) & \text{and} \quad q <_{\mathbb{Q}_n} p \end{cases}$$

Theorem 2.3.4 (Rosenstein [20], [21]) *If* L *is an* $\aleph_0$*-categorical linear order then* L *is built up from singletons by a finite number of concatenations or shuffles.*

This result was extended to the coloured linear orders by Mwesigye and Truss in the following theorem.

Theorem 2.3.5 (Mwesigye, Truss [17]) *A finite or countable coloured linear order* $(A, \leq, C_0, \dots)$ *is* $\aleph_0$*-categorical if and only if it can be built up in finitely many steps from coloured singletons using concatenations or shuffles.*

Rosenstein's theorem leads to a natural method of describing the countably categorical linear orders.

Definition 2.3.6 *A **term** is built as follows:*

Singleton	*The singleton 1 is a term.*
Concatenation	*If t_0, t_1 are terms then $t_0 {}^\wedge t_1$ is a term.*
$\mathbb{Q}_n$***-shuffle***	*If $t_0, \ldots t_{n-1}$ are terms then $\mathbb{Q}_n(t_0, \ldots, t_{n-1})$ is a term.*

Where $\mathbb{Q}_n$-shuffle is allowed for all $n \in \mathbb{N}$

A finite term is a term that represents a finite linear order. Similarly, an infinite term is one that represents an infinite linear order.

The terms correspond to linear orders in the obvious way, and I will not be particularly careful about distinguishing the two. That every $\aleph_0$-categorical linear order is represented by a term is Theorem 2.3.4, however we will see that it is possible for a linear order to have many different representations.

Lemma 2.3.7 *If f is a permutation of n then the two terms*

$$\mathbb{Q}_n(t_0, t_1, \ldots t_{n-1}) \quad \text{and} \quad \mathbb{Q}_n(t_{f(0)}, t_{f(1)}, \ldots t_{f(n-1)})$$

represent isomorphic linear orders.

Proof

This immediately follows from the fact that relabelling the colours of $\mathbb{Q}_n$ does not affect its isomorphism type. □

Remark 2.3.8 *Since there is no natural way of choosing an order of shuffled terms, we shall now work with equivalence classes of terms, given by permuting shuffled terms.*

Lemma 2.3.9 *The linear order expressed by the term*

$$\mathbb{Q}_{n+1}(t_0, \ldots, t_{n-1}, t_i)$$

where $i < n$ is also expressible by $\mathbb{Q}_n(t_0, \ldots, t_{n-1})$.

Proof

Both of the terms are characterised by the fact that $t_0, \ldots t_{n-1}$ occur interdensely (i.e. each t_i occurs between any two points not contained in the same t_j), and so are isomorphic. □

Lemma 2.3.10 *The linear order expressed by a term of the form*

$$\mathbb{Q}_{m+1}(t_0, \ldots t_{m-1}, \mathbb{Q}_n(t_0, \ldots t_{n-1}))$$

where $m < n$, can also be expressed as $\mathbb{Q}_n(t_0, \ldots t_{n-1})$.

Proof

The first expression is obtained by colouring $\mathbb{Q}$ interdensely with $m + 1$ colours and replacing the points coloured by the i^{th} with one of $t_0, \ldots t_{m-1}$ or $\mathbb{Q}_n(t_0, \ldots t_{n-1})$. Therefore this linear order is characterised by the fact that between any two points there occurs a copy of every t_i and $\mathbb{Q}_n(t_0, \ldots t_{n-1})$. However every t_i for $i < n$ occurs interdensely in $\mathbb{Q}_n(t_0, \ldots t_{n-1})$, so the two terms represent the same linear order. □

Lemma 2.3.11 *The linear order expressed by the term*

$$\mathbb{Q}_m(t_0, \ldots, t_{m-1})^\wedge \tau^\wedge \mathbb{Q}_m(t_0, \ldots, t_{m-1})$$

where τ is either the empty set or one of the t_i for $0 \leq i \leq m-1$, can also be expressed as $\mathbb{Q}_m(t_0, \ldots t_{m-1})$.

Proof

This lemma is an obvious consequence of the facts that $\mathbb{Q}_m(t_0, \ldots t_{m-1})$ is obtained by taking a copy of $\mathbb{Q}$ that is interdensely coloured with m colours and replacing the i^{th} colour with t_i and that both $\mathbb{Q}^\wedge\mathbb{Q}$ and $\mathbb{Q}^\wedge 1^\wedge\mathbb{Q}$ are isomorphic to $\mathbb{Q}$. $\square$

Using these lemmas it is possible to derive a unique representation of not only $\aleph_0$-categorical linear orders, but also $\aleph_0$-categorical coloured linear orders (by allowing coloured singletons to occur in our terms) and infinite concatenations of $\aleph_0$-categorical linear orders. Such representations have certain properties that facilitate a proof regarding the maximal chains of trees.

Definition 2.3.12 *We use induction over the formation of terms to define when a term is in* ***normal form*** *(n.f.).*

1. *All finite terms are in n.f..*
2. *A term of the form $\mathbb{Q}_m(t_0, \ldots, t_{m-1})$ is in n.f. if all the t_i are in n.f. and it does not satisfy the conditions of Lemma 2.3.9 or 2.3.10. As stated in Lemma 2.3.7, if the t_i are permuted then the linear order the terms represent are the same, so we consider shuffles where the terms are permuted to be the same, as in Remark 2.3.8.*
3. *A term of the form $t_0^\wedge \ldots t_{n-1}$ is in n.f. if all the t_i are in n.f. and no $t_{i-1}{}^\wedge t_i{}^\wedge t_{i+1}$ or $t_{i-1}{}^\wedge\emptyset{}^\wedge t_{i+1}$ satisfies the conditions of Lemma 2.3.11.*
4. *If t_i is finite then t_{i+1} is infinite.*

A possibly infinite sequence of terms (s_i) is said to be in normal form if:

1. *each s_i is in normal form;*

2. *no $s_{i-1}{}^\wedge s_i{}^\wedge s_{i+1}$ or $s_{i-1}{}^\wedge\emptyset{}^\wedge s_{i+1}$ satisfies the conditions of Lemma 2.3.11;*

3. *if s_j is finite either:*

   (a) *s_{j+1} is infinite; or*

   (b) *(s_i) is an infinite sequence and $s_j = s_k = 1$ for all $k \geq j$.*

The process of showing that such representations are unique and can describe every $\aleph_0$-categorical linear order is both tedious and unilluminating, consisting of the statement 'Open intervals of $\mathbb{Q}$ are isomorphic to $\mathbb{Q}$' repeated dozens of times, so we shall not provide the proof, and simply state the pertinent facts about normal form representations:

Facts 2.3.13

1. *For every sequence of terms (t_i) there is a sequence in normal form (t_i') that represents the same linear order as (t_i).*

2. *If (t_i) and (s_i) are both in normal form and represent the same linear order then $(t_i) = (s_i)$*

3. *If (t_i) is in normal form then all contiguous subsequences of (t_i) are also in normal form (excluding the case where (t_i) ends in a tail of 1 and the contiguous subsequence contains only part of this tail).*

Lemma 2.3.14 *If L and K be $\aleph_0$-categorical linear orders such that*

$$\forall x \in L \exists y \in K\, L^{\leq x} \cong K^{\leq y}$$

then L is an initial segment of K.

Proof

Let L and K be $\aleph_0$-categorical linear orders such that

$$\forall x \in L \exists y \in K \, L^{\leq x} \cong K^{\leq y}$$

Let $\sigma_0,^\wedge \ldots^\wedge \sigma_n$ be the normal form representation of L.

Suppose that σ_n is finite. Let x be the maximal element of L, and let $y \in K$ be such that $L^{\leq x} \cong K^{\leq y}$. Since x is the maximal element, $L^{\leq x} \cong L$, so L is an initial segment of K.

Suppose that σ_n is infinite and let $x \in \sigma_n$, and let $y \in K$ be such that there is an isomorphism $\phi : L^{\leq x} \to K^{\leq y}$.

Since σ_n is infinite, it is of the form $\mathbb{Q}_k(\tau_0, \ldots, \tau_{k-1})$ for some n.f. terms $\tau_0, \ldots, \tau_{k-1}$. Let $\psi := \sigma_n \to \mathbb{Q}_k$ be the map that sends $x \in \sigma_n$ to $z \in \mathbb{Q}_k$ if x is in the copy of τ_i that replaced z in the construction of σ_n.

Let $J := \psi^{-1}(\mathbb{Q}_k^{<\psi(x)})$. Since $\mathbb{Q}_k^{<\psi(x)}$ is an initial segment of $\mathbb{Q}_k^{\leq\psi(x)}$, this J is also an initial segment of $\sigma_n^{\leq x}$. Additionally, $\mathbb{Q}_k^{<\psi(x)} \cong \mathbb{Q}_k$ and J can obtained from $\mathbb{Q}_k^{<\psi(x)}$ using the same construction that we used to build σ_n from $\mathbb{Q}_k$.

Thus $J \cong \sigma_n$ and

$$L \cong \sigma_0{}^\wedge \ldots{}^\wedge \sigma_{n-1}{}^\wedge J$$

$\phi(\sigma_0{}^\wedge \ldots{}^\wedge \sigma_{n-1}{}^\wedge J)$ is an initial segment of K, so L is an initial segment of K. $\square$

These facts are required to show the following theorem about the possible maximal chains of an $\aleph_0$-categorical tree.

Theorem 2.3.15 *If T is an $\aleph_0$-categorical coloured tree then every maximal chain of T is an $\aleph_0$-categorical coloured linear order.*

Proof

Let L be a maximal chain of T which is not $\aleph_0$-categorical as a linear order. We will consider separately the cases where L has a maximal element and where L does not.

Let l be the maximal element of L. Since L is not $\aleph_0$-categorical there must be an infinite list of pairs (x_i, y_i) such that each pair lives in a different 2-orbit of L, so each triple (x_i, y_i, l) lives in a different 3-orbit of L. Since T is $\aleph_0$-categorical there must be (x_i, y_i, l) and (x_j, y_j, l) such that they lie in the same 3-orbit of T and $i \neq j$. Since they lie in the same 3-orbit of T there is an automorphism of T that carries (x_i, y_i, l) to (x_j, y_j, l), which preserves l and therefore preserves L setwise, so restricts to an automorphism of L which carries (x_i, y_i) to (x_j, y_j), resulting in a contradiction.

Now suppose L does not have a maximal element. As in the previous paragraph every initial section of L is an $\aleph_0$-categorical linear order. Therefore L is expressible as the concatenation of an infinite list of $\aleph_0$-categorical linear orders (L_i). Since L is not $\aleph_0$-categorical the normal form of this sequence is also infinite (as Rosenstein's classification of the countable $\aleph_0$-categorical linear orders, Theorem 2.3.4, states that a countable linear order is $\aleph_0$-categorical if and only if it is represented by a finite term), and so we assume that (L_i) is in normal form.

For each i let x_i be a member of L_i. If T is $\aleph_0$-categorical then it only has finitely many orbits of pairs, and so there must be an automorphism ϕ that sends (x_0, x_{n+1}) to (x_0, x_{m+1}) for some $m < n$.

The restriction of ϕ to $T^{\leq x_{n+1}}$ can be viewed as an isomorphism that maps $L^{\leq x_{n+1}}$ to $L^{\leq x_{m+1}}$. When used in this role, we denote ϕ as $\tilde{\phi}$. Since it is an isomorphism, $\tilde{\phi}$ must send the set of predecessors of x_{n+1} to the predecessors of x_{m+1}.

We suppose that L_{n+1} is finite and therefore $\tilde{\phi}$ maps $L_0{}^\wedge \ldots {}^\wedge L_n$ to $L_0{}^\wedge \ldots {}^\wedge L_m$. Thus the finite sequences $(L_i)_{i=0}^n$ and $(L_i)_{i=0}^m$ are isomorphic and Fact 3. from Facts 2.3.13 shows that these sequences are in normal form. This shows that $(L_i)_{i=0}^n = (L_i)_{i=0}^m$, and so $m = n$, giving a contradiction.

Suppose that L_{n+1} is a shuffle, which we denote by $\mathbb{Q}_n(\tau_0, \ldots, \tau_i)$. We also suppose that x_{n+1} is contained in a copy of τ_0, and we use z to label the point in $\mathbb{Q}_n$ that is replaced by that particular copy of τ_0, and let L'_{n+1} be the initial section of L_{n+1} that corresponds to $(-\infty, z)$, the interval of $\mathbb{Q}_n$.

Since $(-\infty, z) \cong \mathbb{Q}_n$:

$$L_0 {}^\wedge \ldots {}^\wedge L_n {}^\wedge L_{n+1} \cong L_0 {}^\wedge \ldots {}^\wedge L_n {}^\wedge L'_{n+1}$$

Since $L'_{n+1} \cong L_{n+1}$, the normal form representation of L'_{n+1} is equal to the n.f. representation of L_{n+1}. The function $\tilde{\phi}$ is an isomorphism, so the n.f. representation of $\tilde{\phi}(L_0 {}^\wedge \ldots {}^\wedge L_n {}^\wedge L'_{n+1})$ is $L_0 {}^\wedge \ldots {}^\wedge L_n {}^\wedge L_{n+1}$.

Therefore $\tilde{\phi}$ maps L_i to itself for $i \leq n+1$, and thus the n.f. representation of $\tau_0^{\leq x_0}$ is also the n.f. representation of $L_{n+1} {}^\wedge \ldots {}^\wedge L_{m+1}^{\leq x_{m+1}}$.

Since we are assuming that T is $\aleph_0$-categorical, we may also assume that for all $k \in \mathbb{N}$ there is an $m_k \in \mathbb{N}$ such that there is an automorphism mapping (x_0, x_{n+1}) to (x_0, x_{m_k}). Again, we conclude that the n.f. representation of $\tau_0^{\leq x_0}$ is also the n.f. representation of $L_{n+1} {}^\wedge \ldots {}^\wedge L_{m_k+1}^{\leq x_{m_k+1}}$ for all k.

This is a contradiction, as the n.f representation of $\tau_0^{\leq x_0}$ is of fixed length. $\square$

Theorem 2.3.16 *If T is a countable $\aleph_0$-categorical tree then T has only finitely many maximal chains up to isomorphism.*

Proof

Let T be a tree with infinitely many non-isomorphic maximal chains, countably many of which we call L_n for $n \in \omega$. For each $I \subseteq \omega$ we introduce colour predicate C_I such that $T \models C_I(a)$ if and only if

$$I = \{i \in \omega \ : \ a \text{ is contained in a maximal chain isomorphic to } L_i\}$$

We introduce the following notation:

$$\mathcal{I} := \{I \subseteq \omega \ : \ T \models \exists x C_I(x)\}$$

If $I \neq J$ and $T \models C_I(a) \wedge C_J(b)$ then there is a maximal chain A such that A passes through a, but no maximal chain passing through b is isomorphic to A. Any automorphism of T that maps a to b will have to map A to a maximal chain that contains b, showing that a and b lie in different orbits of Aut(T), and hence

$$\text{Aut}(T) = \text{Aut}(\langle T, \leq, \{C_I\}\rangle)$$

So if $\mathcal{I}$ is infinite then there are infinitely many 1-orbits of T, and T cannot be $\aleph_0$-categorical. We therefore assume that $\mathcal{I}$ is finite. Since T has infinitely many maximal chains, $\bigcup \mathcal{I}$ is infinite, so there must be an infinite element contained in $\mathcal{I}$.

If $a \leq b$ and $T \models C_I(a) \wedge C_J(b)$ then $J \subsetneq I$, so if I_0 is a least element of

$$\{I \subseteq \omega \ : \ T \models \exists x C_I(x)\}$$

then there exists an $a_0 \in T$ such that $T^{\geq a_0}$ is mono-chromatically coloured by C_{I_0}.

T is an $\aleph_0$-categorical tree, and so has finitely many orbits. The addition of the C_I predicates do not alter the automorphism group of T, so only finitely many of these C_I can be realised. Since T has infinitely many non-isomorphic maximal chains, there is a $J \in \mathcal{I}$ such that J is infinite and $T \models C_J(x)$. Let $I_0, \ldots I_{k-1}$ be the minimal elements of

$\mathcal{I}$ such that there is a $y \geq x$ such that $T \models C_{I_j}(y)$.

$$J \subseteq \bigcup_{j<k} I_j$$

Since J is infinite, at least one of the I_j is infinite. We assume that I_0 is. Let $y \in T$ realise C_{I_0}, and let $S := T^{\geq y}$. Since I_0 is minimal S is monochromatic.

In short, from our T we have found another tree, S which has infinitely many maximal chains up to isomorphism and every element of S lies on a copy of each of these maximal chain. Theorem 2.3.15 shows that each of these maximal chains is an $\aleph_0$-categorical linear order. Let $\{L_0, \ldots\}$ be the infinite set of pairwise non-isomorphic $\aleph_0$-categorical linear orders which occur as maximal chains of S.

Let K_0 be a maximal chain of S which is isomorphic to L_0 and pick an arbitrary $s \in K_0$. Since there is a maximal chain of S that contains s which is isomorphic to L_i for all $i \in \mathbb{N}$, this s can be regarded as an element of each of the L_i.

S is a tree, so $S^{\leq s}$ is a linear order, and it is also an initial segment of L_i for all $i \in \mathbb{N}$. Therefore every initial segment of L_0 is also an initial segment of L_i for all $i \in \mathbb{N}$.

Therefore for all $i \in \mathbb{N}$

$$\forall x \in L_0 \exists y \in L_i \, L_0^{\leq x} \cong L_i^{\leq y}$$

Lemma 2.3.14 shows that L_i is an initial segment of L_0 for all i, but since L_0 is $\aleph_0$-categorical it has finitely many 1-orbits, and hence finitely many initial segments up to isomorphism, giving a contradiction. $\Box$

2.4 Trees

2.4.1 Ramification Predicates

Trees contain more information than which linear orders occur as their maximal chains, so in order to classify the $\aleph_0$-categorical trees using them we need a way to encode that extra information.

Definition 2.4.1 *Let T be an $\aleph_0$-categorical tree, and let $\{L_i \ : \ i \leq n\}$ be the maximal chains of T. We define R^i_m for $m \in \omega \cup \{\omega\}$ to be a unary predicate that is only realised by $x \in T$ if there are exactly m copies of L_i which contain x.*

Lemma 2.4.2 *If (x_0, y_0) and (x_1, y_1) lie in the same orbit of $\langle T, \leq\rangle$ then they lie in the same orbit of $\langle T, \leq, R^i_m \ : \ i \leq n\,, m \in \omega \cup \{\omega\}\rangle$.*

Proof
If $\phi \in \mathrm{Aut}(T)$ maps (x_0, y_0) to (x_1, y_1), then ϕ maps the maximal chains that contain x_0 (resp. x_1) to the ones that contain y_1 (resp. y_1), so x_0 realises the same R^i_m as y_0, and x_1 realises the same R^i_m as y_1. In other words, introducing the R^i_m does not kill any automorphisms. $\square$

Theorem 2.4.3 *If $\langle T, \leq\rangle$ is $\aleph_0$-categorical then so is $\langle L, \leq, R^i_m \ : \ i \leq n,\ m \in \omega \cup \{\omega\}\rangle$ for any maximal chain L.*

Proof
Suppose that $\langle L, \leq, R^i_m \ : \ i \leq n,\ m \in \omega \cup \{\omega\}\rangle$ is not $\aleph_0$-categorical. Let (a_n, b_n) be an infinite list of pairs from L such that each pair lies in a different 2-orbit. Since T is $\aleph_0$-categorical there are k and j such that (a_k, b_k) and (a_j, b_j) lie in the same orbit of T.

As we saw in the proof of Theorem 2.3.15 we can find an automorphism of T, which we will call ϕ, that preserves L and carries (a_k, b_k) to (a_j, b_j). Lemma 2.4.2 shows that ϕ preserves the R^i_m, and hence so does ϕ's restriction to L, which carries (a_k, b_k) to (a_j, b_j), giving a contradiction. □

2.4.2 Classification

Proposition 2.4.4 *If $(T, <)$ is $\aleph_0$-categorical then $(T^+, <)$ is $\aleph_0$-categorical.*

Proof

Suppose that $(T^+, <)$ is not $\aleph_0$-categorical but $(T, <)$ is $\aleph_0$-categorical. T^+ is not almost 2-transitive, so there is an infinite list of pairs (α_i, β_i) such that

- for all i, j there is no automorphism that carries (α_i, β_i) to (α_j, β_j)
- for all i either α_i or β_i is not contained in T

If α_i is not in T then there must be a pair a_i and b_i such that $a_i \wedge b_i = \alpha_i$ and they are both contained in T. If α_i is in T then set $a_i, b_i = \alpha_i$. Repeat this procedure for β_i to obtain c_i and d_i. Now we have an infinite list of quartets (a_i, b_i, c_i, d_i) in T. Since T is $\aleph_0$-categorical it has finitely many 4-orbits. This means that for some distinct j and k there is an automorphism between (a_j, b_j, c_j, d_j) and (a_k, b_k, c_k, d_k), but any automorphism of T extends to an automorphism of T^+, giving a contradiction. □

Unfortunately T^+ being $\aleph_0$-categorical is not enough to ensure that T is $\aleph_0$-categorical, as the next example shows.

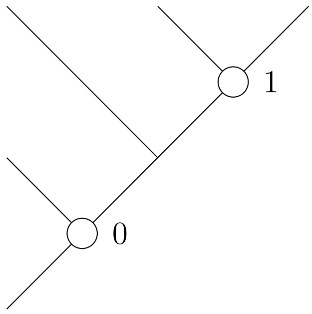

Figure 2.2: The tree from Example 2.4.5

Example 2.4.5 *T is obtained by first taking a copy of $\mathbb{Q}$ and deleting ω then attaching to every point (including the deleted points) another copy of $\mathbb{Q}$. While $(T^+, <)$ is $\aleph_0$-categorical, as the theorem in the next section shows, it is apparent that $(T, <)$ has infinitely many 2-orbits.*

This suggests that we need a way of restricting how points in $(T^+, <)$ can be deleted to ensure that the remaining structure is still $\aleph_0$-categorical. Recall from Definition 2.2.4 that an irrational point of T^+ is a point in $T^+ \setminus T$.

Theorem 2.4.6 *Let I be a unary predicate such that $I(t)$ if and only if t is an irrational point in T. Then $(T, <)$ is $\aleph_0$-categorical if and only if $(T^+, <, I)$ is $\aleph_0$-categorical.*

Proof

The argument that a tree is $\aleph_0$-categorical if and only if it is almost 2-transitive is valid in this expanded language, because I is a unary predicate and does not interfere with the piecing together of automorphisms.

In the $\Rightarrow$ direction, notice that the inclusion of a new unary predicate increases the number of possible n-orbits, but since there are finitely many 2-orbits in $(T, <)$ there are finitely many in $(T^+, <)$.

In the $\Leftarrow$ direction, note that since I is a predicate any isomorphism preserves I, so any automorphism restricts to an automorphism of $(T, <)$. Since $(T^+, <, I)$ is $\aleph_0$-categorical, it only has finitely many 2-orbits. The number of 2-orbits of pairs where neither of the elements satisfy I equals the number of 2-orbits in $(T, <)$, so $(T, <)$ is almost 2-transitive and so $\aleph_0$-categorical. $\Box$

Lemma 2.4.7 *If $(T^+, <, I)$ is $\aleph_0$-categorical then if L is a maximal chain of T^+ then the linear order $(L, <, I)$ is $\aleph_0$-categorical.*

Proof

The proof of Proposition 2.4.3 is easily adapted to this lemma. $\Box$

We are now ready to prove our main theorem about trees.

Theorem 2.4.8 $(T^+, I, <, R^i_m \ : \ i \leq k,\ m \in \omega \cup \{\omega\})$ *is $\aleph_0$-categorical if and only if:*

1. *only finitely many of the R^i_m are realised;*
2. *if L is a maximal chain of T^+ then $\langle L, <, I, R^i_m \ : \ i \leq k,\ m \in \omega \cup \{\omega\})$ is $\aleph_0$-categorical; and*
3. *there are only finitely many maximal chains of T^+ up to isomorphism in the language $\langle <, I, R^i_m \ : \ i \leq n,\ m \in \omega \cup \{\omega\}\rangle$.*

Proof

$\Rightarrow$: Since $(T^+, I, <, R^i_m : i \leq k,\ m \in \omega \cup \{\omega\})$ is $\aleph_0$-categorical it only has finitely many 2-orbits. This means that only finitely many of the R^i_m's can be realised. Theorem 2.3.15

shows that $(L, <, I, R_0^0, \ldots)$ is $\aleph_0$-categorical and Condition 3 is shown by Theorem 2.3.16.

$\Leftarrow$: If two trees, T and S, satisfy the required conditions, have isomorphic maximal chains in the language $\langle <, I, R_m^i \ : \ i \leq k, \, m \in \omega \cup \{\omega\} \rangle$ then we may build an isomorphism from T to S using back-and-forth. Let t_α and s_β for $\alpha, \beta \in \mathbb{N}$ be enumerations of T and S respectively

Base Case Pick T_0, a maximal chain of T that contains t_0 and let $\phi_0 : T_0 \to S_0$ be an isomorphism from T_0 to S_0, a suitable maximal chain of S. This ϕ_0 is also a partial isomorphism from T to S.

Odd Step Let n be odd, let $T_n := \mathrm{Dom}(\phi_n)$ and let $t \in T_n \setminus T_{n-1}$. For each cone of t that is disjoint from T_n, pick a maximal chain that contains the element of the cone which is enumerated with the smallest number. We denote these maximal chains as $L_i(t)$, where $i \in I(t)$, an indexing set for each t.

Since ϕ_n is a partial isomorphism of the language $\langle <, I, R_m^i : i \leq k, \, m \in \omega \cup \{\omega\} \rangle$, the image $\phi_n(t)$ satisfies all of the same R_m^i, therefore there is a maximal chain $K_i(t)$ of S that passes through $\phi_n(t)$ such that:

$$\langle L_i(t), <, I, R_m^i \ : \ i \leq k, \, m \in \omega \cup \{\omega\} \rangle$$
$$\cong$$
$$\langle K_i(t), <, I, R_m^i \ : \ i \leq k, \, m \in \omega \cup \{\omega\} \rangle$$

ϕ_n maps t to $\phi_n(t)$ and is a partial isomorphism, so $T^{\leq t} \cong S^{\leq \phi_n(t)}$, and:

$$\langle L_i(t)^{\leq t}, <, I, R_m^i \ : \ i \leq k, \, m \in \omega \cup \{\omega\} \rangle$$
$$\cong$$
$$\langle K_i(\phi_n(t))^{\leq \phi_n(t)}, <, I, R_m^i \ : \ i \leq k, \, m \in \omega \cup \{\omega\} \rangle$$

Therefore there is an isomorphism $\psi_{i,t}$ that maps

$$\langle L_i(t), t, <, I, R^i_m \ : \ i \leq k,\ m \in \omega \cup \{\omega\}\rangle$$
$$\text{to}$$
$$\langle K_i(\phi_n(t)), \phi_n(t), <, I, R^i_m \ : \ i \leq k,\ m \in \omega \cup \{\omega\}\rangle$$

This $\psi_{i,t}$ is also a partial automorphism from T to S. Since $\psi_{i,t}(t) = \phi_n(t)$ the union $\phi_n \cup (\psi_{i,t}|_{L_i(t)^{>t}})$ is also a partial isomorphism. Indeed, since each $L_i(t)$ lies in a different cone to any other $L_i(t)$,

$$\phi_{n+1} := \phi_n \cup \bigcup_{t \in T_n \setminus T_{n-1}} \bigcup_{i \in I(t)} \psi_{i,t}|_{L_i(t)^{>t}}$$

is a partial isomorphism.

Even Step Let n be even, let $S_n := \mathrm{Im}(\phi_n)$ and let $s \in S_n \setminus S_{n-1}$. For each cone of s that is disjoint from S_n, pick a maximal chain that contains the element of the cone which is enumerated with the smallest number. We denote these maximal chain as $K_i(s)$, where $i \in I(s)$, an indexing set for each t.

Since ϕ_n^{-1} is a partial isomorphism of the language $\langle <, I, R^i_m \ : \ i \leq k,\ m \in \omega \cup \{\omega\}\rangle$, the pre-image $\phi_n^{-1}(s)$ satisfies all of the same R^i_m, therefore there is a maximal chain $L_i(s)$ of T that passes through $\phi_n^{-1}(s)$ such that:

$$\langle K_i(s), <, I, R^i_m \ : \ i \leq k,\ m \in \omega \cup \{\omega\}\rangle$$
$$\cong$$
$$\langle L_i(\phi_n^{-1}(s)), <, I, R^i_m \ : \ i \leq k,\ m \in \omega \cup \{\omega\}\rangle$$

Since ϕ_n^{-1} maps s to $\phi_n^{-1}(s)$ and is a partial isomorphism, $T^{\leq s} \cong S^{\leq \phi_n^{-1}(s)}$, so

$$\langle K_i(s)^{\leq s}, <, I, R_m^i \ : \ i \leq k,\ m \in \omega \cup \{\omega\}\rangle$$
$$\cong$$
$$\langle L_i(\phi_n^{-1}(s))^{\leq \phi_n^{-1}(s)}, <, I, R_m^i \ : \ i \leq k,\ m \in \omega \cup \{\omega\}\rangle$$

Therefore there is an isomorphism $\chi_{i,s}$ that maps

$$\langle K_i(s), s, <, I, R_m^i \ : \ i \leq k,\ m \in \omega \cup \{\omega\}\rangle$$
$$\text{to}$$
$$\langle L_i(\phi_n^{-1}(s)), \phi_n^{-1}(s), <, I, R_m^i \ : \ i \leq k,\ m \in \omega \cup \{\omega\}\rangle$$

$\chi_{i,t}$ is also a partial automorphism from S to T. Since $\chi_{i,t}(s) = \phi_n^{-1}(s)$, the union $\phi_n^{-1} \cup (\chi_{i,t}|_{K_i(s)^{>s}})$ is also a partial isomorphism. Indeed, since each $K_i(s)$ lies in a different cone to any other $K_i(s)$,

$$\phi_{n+1}^{-1} := \phi_n^{-1} \cup \bigcup_{s \in S_n \setminus S_{n-1}} \bigcup_{i \in I(s)} \chi_{i,s}|_{K_i(s)^{>s}}$$

is a partial isomorphism.

Let $\phi := \bigcup \phi_n$. This ϕ is an isomorphism from T to S. To show $\aleph_0$-categoricity, we have some more work to do. Let T be a tree that satisfies the conditions of the theorem.

Suppose that there are infinitely many 2-orbits, and let (x_j, y_j) be such that each lies in a different 2-orbit. We're going to try and build an automorphism that takes some (x_0, y_0) to some other (x_1, y_1), so we need to throw out a lot of unsuitable candidates and do some relabelling.

Only finitely many of the R_m^i are realised, so we may assume that infinitely many of the (x_j, y_j) satisfy the same suite of them, and restrict our attention to those (x_j, y_j). There are only finitely many maximal chains up to isomorphism, so we may assume that infinitely many of the (x_j, y_j) lie on a copy of L_0. We further restrict our attention to those (x_j, y_j).

Since the (x_j, y_j) lie on a copy of L_0, which is $\aleph_0$-categorical, we may assume that (x_0, y_0) and (x_1, y_1) lie in the same 2-orbit of L_0. Thus, if we go through the procedure mentioned at the beginning of this proof, but specifying that the isomorphism that maps T_0 to S_0 takes (x_0, y_0) to (x_1, y_1), then we obtain an automorphism of T.

Therefore T has only finitely many 2-orbits and is $\aleph_0$-categorical. $\square$

This gives us necessary and sufficient conditions for $(T, \leq)$ to be $\aleph_0$-categorical. A description of the coloured $\aleph_0$-categorical trees is contained in the proof of Theorem 2.4.8, as we will now show.

Corollary 2.4.9 *A coloured tree $(T, <, C_0, \ldots)$ is $\aleph_0$-categorical iff*

- *only finitely many of the R^i_m are realised;*
- *$\langle L, <, I, R^0_0, \ldots, C_0, \ldots \rangle$ is $\aleph_0$-categorical for every maximal chain L; and*
- *there are only finitely many such maximal chains up to isomorphism in the language $\langle <, I, R^0_0, \ldots, C_0, \ldots \rangle$.*

where the C_i are the colour predicates.

Proof

Since being finitely coloured is an obvious requirement of $\aleph_0$-categoricity, the proof of Theorem 2.4.8 is easily adapted, by considering $(I \wedge R^i_m \wedge C_j)$ and $(I \wedge \neg R^i_m \wedge C_j)$ instead of $(I \wedge R^i_m)$ and $(I \wedge \neg R^i_m)$. $\square$

2.5 Cycle-free Partial Orders

The aim of this section is to extend the above result to the cycle-free partial orders. We shall develop the analogue of ramification completeness for the CFPOs.

2.5.1 Setup

Definition 2.5.1 *A CFPO M is said to be* ***path complete*** *if for every pair $(a, b) \in M$ the connecting set that witnesses* $\mathrm{Path}\langle a, b\rangle$ *is contained in M. The* ***path completion*** *of M, written as M^+, is obtained by only taking the points of M^D that are elements of a connecting set of a path connecting points in M. This M^+ is countable as it is the countable union of finite sets.*

A tuple is said to be ***path complete*** *it contains every element of the connecting set of every path between elements of the tuple. The* ***path completion*** *of a tuple $p \in M$ is a tuple $q \in M^+$ such that the following are true:*

- $p \subseteq q$
- *q is path complete*
- *if $q' \subsetneq q$ is path complete then $p \not\subseteq q'$*

Note that M is path complete if and only if M is ramification complete, i.e. if u and v are incomparable and u, v are contained in a substructure of M that is isomorphic to a semi-linear order then $u \wedge v$ or $u \vee v$ is contained in M, and the ramification completion of a tree is the same structure as the path completion.

From now on, we will be working in $(M^+, <, I)$, where $I(x)$ holds if and only if $x \in M^+ \setminus M$, as in Theorem 2.4.6. First we will show that we are able to move between $(M, <)$ and $(M^+, <, I)$ in the same manner as in the trees.

Proposition 2.5.2 *$\langle M, < \rangle$ is $\aleph_0$-categorical iff $\langle M^+, <, I \rangle$ is.*

Proof

$\Rightarrow$: $x_0 \vee x_1$ and $y_0 \vee y_1$ lie in different 1-orbits of $\langle M^+, <, I \rangle$ if and only if (x_0, x_1) and (y_0, y_1) lie in different 2-orbits of M. The same remark holds for $x_0 \wedge x_1$ and $y_0 \wedge y_1$.

Let $\bar{a} \in M^+$. If $a_i \in M$ then let $b(a_i) := a_i$. If $a_i \in M^+ \setminus M$ then there is a pair $b(a_i) := (b_i, b'_i) \in M$ such that $a_i = (b_i \wedge b'_i)$ or $(b_i \vee b'_i)$, and let $b(\bar{a})$ be the tuple $(b(a_0), \ldots) \in M$. The length of $b(\bar{a})$ is at most twice the length of $\bar{a}$.

$\bar{a}, \bar{a}' \in M^+$ lie in the same orbit if and only if $b(\bar{a})$ and $b(\bar{a}')$ lie in the same orbit of M. Therefore the number of n-orbits of M^+ is bounded by the number of m-orbits of M where $m \leq 2n$. Since $\langle M, < \rangle$ is $\aleph_0$-categorical the number of m-orbits is finite, so $\langle M^+, \leq, I \rangle$ is $\aleph_0$-categorical.

$\Leftarrow$: All automorphisms of $(M^+, <, I)$ preserve M set-wise, so M cannot have more n-orbits than $(M^+, <, I)$. $\Box$

We are able to prove the familiar lemma about almost 2-transitivity in this context.

Lemma 2.5.3 *A CFPO is almost n-transitive only if it is almost 2-transitive.*

Proof

Let M be an almost 2-transitive CFPO. We may assume that M is path complete (Proposition 2.5.2). Let (x_i) and (y_i) be path complete n-tuples with the same order type such that if $x_k \not\parallel x_j$ then (x_k, x_j) is in the same orbit as (y_k, y_j). We will now prove that there is an automorphism from (x_i) to (y_i) by induction on n. This is obviously true for the case when $n = 2$.

We assume that M is almost k-transitive for $k < n$. Let $\bar{x}$ be a path complete n-tuple. In $\bar{x}$ there must be a x_i such that there is another x_j where $x_j \in \mathrm{Path}\langle x_i, x_k \rangle$ for any $k \neq i, j$, and we can rearrange the tuple so that $i = n-1$ and $j = n-2$. Let $\bar{y}$ be another n-tuple such that if x_i and x_j are 'adjacent' then (x_i, x_j) lies in the same 2-orbit as (y_i, y_j). By the inductive hypothesis the $n-1$ tuples $(x_0, \ldots x_{n-2})$ and $(y_0, \ldots y_{n-2})$ lie in the same $n-1$ orbit, witnessed by ϕ_1. We also know that (x_{n-2}, x_{n-1}) lies in the same 2-orbit as (y_{n-2}, y_{n-1}), witnessed by ϕ_2. We define

$$\phi(m) := \begin{cases} \phi_1(m) & m \in \{t \in M \ : \ x_0 \in \mathrm{Path}\langle t, x_2 \rangle\} \\ \phi_2(m) & m \in \{t \in M \ : \ x_0 \notin \mathrm{Path}\langle t, x_2 \rangle\} \end{cases}$$

This is a valid definition as the two sets only 'touch' at x_{n-2} they partition M, and $\phi_1(x_{n-2}) = \phi_2(x_{n-2})$. This ϕ carries (x_i) to (y_i). Since there are only finitely many order types that an n-tuple can satisfy and there are only finitely many choices for the orbits that each pair lie in there can be only finitely many n-orbits. $\Box$

Definition 2.5.4 Alt *is the partial order with the domain* $\{a_i \ : \ i \in \mathbb{Z}\}$ *ordered by*

- *if i is odd then* $a_{i-1} > a_i < a_{i+1}$
- *if i is even then* $a_{i-1} < a_i > a_{i+1}$

Alt_n *is defined to be* Alt *restricted to* $\{a_0, \ldots a_{n-1}\}$*. Note that flipping the order does not affect the definition of* Alt*, but does affect* Alt_n*. We will write* Alt_n^* *for the reverse ordering of* Alt_n*.*

Alt_ω *is defined to be* Alt *restricted to* $\{a_i \ : \ i \in \omega\}$*. Again, the reverse ordering is denoted by* Alt_ω^*

That Alt is a CFPO is readily apparent.

Proposition 2.5.5 *Let M be a CFPO. If* $\mathrm{Alt}_n \subseteq M$ *for all* $n \in \mathbb{N}$ *then M is not* $\aleph_0$*-categorical.*

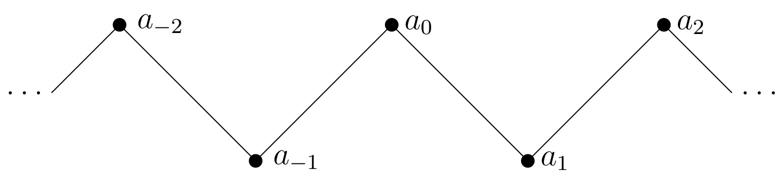

Figure 2.3: The Alternating Chain

Proof

For each $n \in \mathbb{N}$, let $a_n, b_n \in M$ be such that the connecting set of $\text{Path}\langle a_n, b_n\rangle$ is isomorphic to Alt_n. Since paths are preserved by automorphisms, if (a_n, b_n) and (a_m, b_m) lie in the same 2-orbit then $n = m$. Therefore there are infinitely many 2-orbits so M is not $\aleph_0$-categorical. $\square$

Definition 2.5.6 *We say that M, a CFPO is a CFPO_n if M embeds Alt_n but not Alt_{n+1}.*

We may therefore restrict our attention to the CFPO_ns. What will now be useful is a concept of a maximal CFPO_m in a CFPO_n for $m < n$, analogous to the idea of a maximal chain in a semi-linear order.

Definition 2.5.7 *If N is a CFPO_n then M is said to be a maximal CFPO_m in N if the following hold:*

1. *M is a substructure of N;*
2. *M is a CFPO_m; and*
3. *if $\alpha \notin M$ then there is a $\beta \in M$ such that the path $\text{Path}\langle\alpha, \beta\rangle$ is at least $m+1$ long.*

Proposition 2.5.8 *If N is a CFPO_n with maximal CFPO_{n-1} M then the connected components of $N \setminus M$ are trees or reverser ordered trees.*

Proof

Let $a, b \in N \setminus M$ be connected in $N \setminus M$. Since M is maximal there is a copy of Alt_n with the last point a and all the other points contained in M. We will denote the final point of the section contained in M by the letter c_{n-1}. We may assume that $c_{n-1} < a$ (we can reverse the order if not) and this assumption shows that $c_{n-1} < b$ because a and b are connected in $N \setminus M$ and if $c_{n-1} \not< b$ then we arrive at a method of embedding a longer alternating chain than possible.

It is not possible to embed $\bigwedge$ in the connected component that contains a as this would lead to a cycle, as the antichain of $\bigwedge$ would be above c_{n-1}. This means that the path between a and b is a $\bigvee$, and since b is a general point the connected components of $N \setminus M$ are either trees or reverse ordered trees. $\Box$

Definition 2.5.9 *Let N be a CFPO_n and $M \subseteq N$ a maximal CFPO_m for $m < n$. We say that T is an **attached connected component** (ACC) of M if the following hold:*

1. $\exists! t \in T \cap M$
2. *if $x, y \in T$ then $\mathrm{Path}\langle x, y\rangle \cap M \subseteq \{t\}$*
3. *if T' satisfies 1. and 2. and $T \cap M = T' \cap M$ then $T' \subseteq T$.*

Definition 2.5.10 *Let N be an $\aleph_0$-categorical CFPO_n with M, a maximal CFPO_{n-1}. If S_i is an ACC of M then we also use S_i to be a unary predicate on N that is realised by $x \in M$ if and only is x is contained in an ACC isomorphic to S_i.*

2.5.2 Classification

Corollary 2.5.11 *If N is an $\aleph_0$-categorical CFPO then N is a CFPO_n for some n.*

Proof

If N embeds Alt_n then N also embeds Alt_i for all $i < n$. Therefore if N is not a CFPO_n for some n, then N embeds Alt_i for all $i \in \mathbb{N}$, and Proposition 2.5.5 shows that any such N would not be $\aleph_0$-categorical. □

Theorem 2.5.12 *If N is a CFPO_n with M, a maximal CFPO_{n-1} (resp. CFPO_{n-2}) such that*

1. *If S is an ACC of M then it is a member of a finite list of $\aleph_0$-categorical trees or reverse ordered trees; and*
2. *$(M, <, S_0, \ldots S_{i-1})$ is an $\aleph_0$-categorical coloured CFPO_{n-1} (resp. CFPO_{n-2});*

then $\langle N, \leq, S_0, \ldots, S_{i-1}\rangle$ is $\aleph_0$-categorical.

Proof

Let M be a maximal CFPO_{n-1} or CFPO_{n-2} of N which satisfy Conditions 1. and 2. To show that there can only be finitely many 2-orbits we consider where the elements of a representative of the orbit can lie.

- M has only finitely many 2-orbits so only finitely many 2-orbits can have representatives entirely contained in M.
- There are only finitely many non-isomorphic ACCs, each of which have only finitely many 2-orbits so only finitely many of the 2-orbits of N can have representatives entirely contained in an ACC.
- We now consider 2-orbits with a representative of the form (a, b) where a is an ACC and $b \in M$. Too see this let c be the root of the ACC that contains a. The previous two cases show that there are only finitely many orbits that (a, c) and (b, c)

can lie in, hence there are only finitely many 3-orbits of the form (a, b, c). If there were infinitely many 2-orbits with representatives of the form (a, b) there would be infinitely many of these 3-orbits, so there must be finitely many 2-orbits with representatives of this form.

- The final case to consider is that a and b lie in different ACCs, in which case let c (resp. d) be the root of the ACC that contains a (resp. b). By the first two cases there are only finitely many orbits that (a, c), (c, d) and (b, d) can lie in, so there are only finitely many possible orbits of the form (a, b, c, d).

Since all orbits have representatives in one of the forms in the above list there can be only finitely many 2-orbits, and so $(N, <, S_0, \ldots, S_{i-1})$ is $\aleph_0$-categorical. $\Box$

Lemma 2.5.13 *Let N be a $\aleph_0$-categorical* CFPO_n*, and let M be a maximal* CFPO_{n-1}*. Every ACC of M is an $\aleph_0$-categorical tree or reverse ordered tree.*

Proof

Let S be an ACC of M, let $\{s\} = S \cap M$ and let $t \in S$. There are $a_0, \ldots, a_{n-2} \in M$ such that $\{a_0, \ldots, a_{n-2}, s, t\}$ is a copy of either Alt_n or Alt_n^*. We assume that $s < t$.

If S is not a tree then there is a $x \in S$ such that $s \not< x$ but $x < t$. Then $\{a_0, \ldots, a_{n-2}, s, t, x\}$ is a copy of either Alt_{n+1}, contradicting the assumption that N is a CFPO_n. If $s > t$ then we conclude that S is a reverse ordered tree.

We assume that S is a tree. For all $t \in S$ there are $a_0, \ldots, a_{n-3}$ such that

$$\{a_0, \ldots, a_{n-3}, s, t\}$$

is isomorphic to Alt_n. Given any $\phi \in \mathrm{Aut}(N)$,

$$\{\phi(a_0), \ldots, \phi(a_{n-3}), \phi(s), \phi(t)\}$$

Therefore if $\phi(t) \in S$ then $\phi(s) = s$. Therefore if there is an automorphism between $\bar{x}, \bar{y} \in S$ then there is a map $\psi \in \mathrm{Aut}_{\{S\}}(N)$ that maps $\bar{x}$ to $\bar{y}$.

Since $\{\phi|_S \ : \ \phi \in \mathrm{Aut}_{\{S\}}(N)\} \leq \mathrm{Aut}(S)$, this means that since there are only finitely many n-orbits in N, there can be only finitely many n-orbits in S. $\square$

Lemma 2.5.14 *Let N be a $\aleph_0$-categorical* CFPO_n*, and let M be a maximal* CFPO_{n-1}. *There are only finitely many trees and reversed trees that occur as ACCs of M.*

Proof

Suppose $n \geq 4$ and let S_0, S_1 be ACCs of M such that S_i is a tree and $\{s_i\} = S_i \cap M$. Let $x, y \in M$. We consider the extended cones of s_0 (Definition 1.3.19).

Since $n \geq 4$ and S_0 is a tree, there are $a_0 \ldots a_{n-3} \in M$ such that $\{a_0 \ldots a_{n-3}, s_0, t\}$ is a copy of Alt_n for every $t \in S_0$. Therefore there is an upwards cone of s_0 whose extended cone is not a tree. Indeed there is only one such extended cone, as otherwise we could find a copy of Alt_{n+1}. This is depicted in Figure 2.4, labelled as $a_0, \ldots, a_{n-3}, s_0, b_0, b_1$.

If $\phi \in \mathrm{Aut}(N)$ maps s_0 to s_1, then ϕ must map the extended cones of s_0 to the extended cones of s_1. As before, there is only one upwards cone of s_1 whose extended cone is not a tree. Therefore $\phi(S_0) = S_1$. Since there are only finitely many 1-orbits, this means that there can be only finitely many ACCs which are trees up to isomorphism. The argument for reverse ordered trees is almost identical.

Suppose $n = 3$. Then N consists of a tree which is above a linear order, which in turn is above a reverse ordered tree (depicted in Figure 2.5). M is a maximal chain of N. Let S_0, S_1 be ACCs of M such that S_i is a tree and $\{s_i\} = S_i \cap M$.

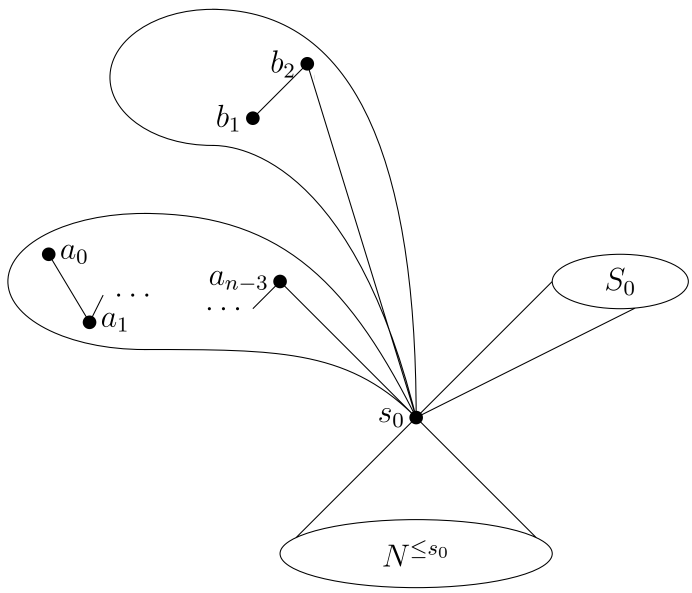

Figure 2.4: The Extended Cones of s_0

We are going to define an N', which will extend N. If M has a maximal element then let $N' = N$. If M does not have a maximal element then we insert one, extending the domain of N as follows:

$$N' := N \cup \{\phi(M) \ : \ \phi \in \text{Aut}(N)\}$$

with the order is

$$x \leq_{N'} y \Leftrightarrow \begin{cases} x \leq_N y & x, y \in N \\ x \in \phi(M) & y = \phi(M) \end{cases}$$

Note that N' has at most one more 1-orbit than N (the orbit that contains $\mathrm{Id}(M)$), and at most twice as many 2-orbits as N (the orbits of N and where the greater of the pair is some $\phi(M)$), so N' is also $\aleph_0$-categorical, and that every automorphism of N' preserves N.

So if $\phi \in \text{Aut}(N')$ maps $(s_0, \mathrm{Id}(M))$ to $(s_1, \mathrm{Id}(M))$ then ϕ maps S_0 to S_1, and thus there are only finitely many ACCs which are trees up to isomorphism. The argument for

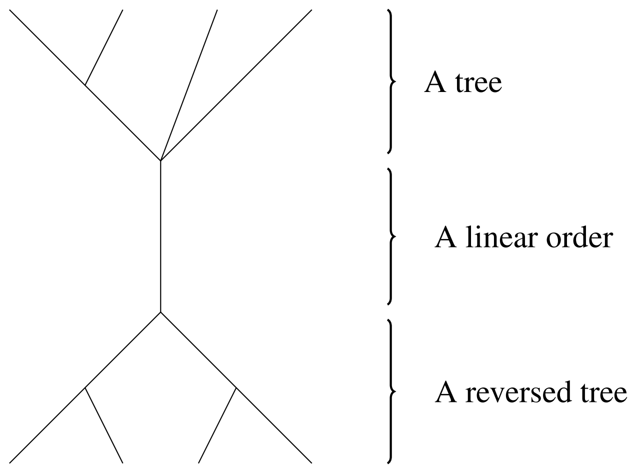

Figure 2.5: A typical CFPO_3

reverse ordered trees is almost identical. □

Theorem 2.5.15 ***Theorem 2.5.16*** *If $\langle N, \leq\rangle$ is an $\aleph_0$-categorical CFPO_n then there is an M, a maximal CFPO_{n-1} of CFPO_{n-2} such that:*

1. *If S is an ACC of M then it is a member of a finite list of $\aleph_0$-categorical trees or reverse ordered trees; and*
2. *$(M, <, S_0, \dots S_{i-1})$ is an $\aleph_0$-categorical coloured CFPO_{n-1}.*

Proof

Let N be a CFPO_n and let K be a maximal CFPO_{n-1} of N and suppose $\mathrm{Aut}(\langle N, \leq\rangle)$ preserves K setwise. We set $M := K$. Lemma 2.5.13 and Lemma 2.5.14 shows that the ACCs of M are members of a finite list of $\aleph_0$ categorical trees or reverse ordered trees.

Now suppose that K is not preserves set-wise. We define L to be the set:

$$\left\{ x \in M \ : \ \exists a_1, \dots, a_{n-2} \in M \exists t \in N \setminus K \left(\begin{array}{c} (x, a_1, \dots a_{n-2}, t) \cong \mathrm{Alt}_n \\ (x, a_1, \dots a_{n-2}, t) \cong \mathrm{Alt}^*_n \end{array} \vee \right) \right\}$$

$N \setminus L$ is a maximal CFPO_{n-1} of N. If $a \in L$ then a is an element on the end of a copy of Alt_n. Therefore Aut(N) can either map a into L or into $N \setminus K$, so Aut(N) preserves M setwise. Let S_0 be an ACC of M and let $\{s_i\} = S_i \cap M$.

We set $M := K \setminus L$, which is a maximal CFPO_{n-2} of N. Every ACC of M is also an ACC of K or of $N \setminus L$, so Lemma 2.5.13 shows that the ACCs of M are $\aleph_0$ categorical trees or reverse ordered trees.

Lemma 2.5.14 shows that there are finitely many ACCs of K up to isomorphism and that there are finitely many ACCs of $N \setminus L$ up to isomorphism. Since every ACC of M is also an ACC of K or of $N \setminus L$, this means that the ACCs of M are $\aleph_0$ categorical trees or reverse ordered trees.

We now have an M, which satisfies Condition 1. whether or not if Aut(N) preserves K setwise. We turn our attention to 2.

Let S_0 be an ACC of M, and let $\{s_i\} = S_i \cap M$. Since Aut(N) preserves M setwise, given any $\phi \in$ Aut(N), the image $\phi(s_0)$ must also have an ACC attached to it, and this ACC must be isomorphic to S_0. Therefore

$$\mathrm{Aut}(\langle N, \leq \rangle) \cong_P \mathrm{Aut}(\langle N, \leq, S_0, \dots S_{i-1} \rangle)$$

Since M is preserved setwise, this means that $\langle M, \leq, S_0, \dots S_{i-1} \rangle$ is also $\aleph_0$-categorical.

□

Chapter 3

Treelike CFPOs

The work of Rubin in [23] is impressive in its scope and complexity. This chapter seeks to appeal to that work directly by saying when a CFPO shares its automorphism group with a tree (treelike). Conditions that guarantee this will be given on the order first.

We will also give conditions on the abstract automorphism group of a CFPO that will let us recognise when there is a tree which has that group as its automorphism group.

While these conditions were initially studied for the purposes of reconstruction, they can be used to deduce that the CFPOs share a number of model theoretic properties with the trees, which is how we end this chapter.

3.1 Treelike CFPOs

The concept of path is the central tool for much of this thesis, and since it gets its first good work out here, I will give a reminder of some notation, as well as some new concepts.

Definition 3.1.1 *This is Definition 1.4.1 repeated*

$$\begin{aligned}
\text{Aut}(M) \cong_A \text{Aut}(N) &\Leftrightarrow \langle \text{Aut}(M), \mathcal{L}_G \rangle \cong \langle \text{Aut}(N), \mathcal{L}_G \rangle \\
\text{Aut}(M) \cong_T \text{Aut}(N) &\Leftrightarrow \langle \text{Aut}(M), \mathcal{L}_G, \tau_M \rangle \cong \langle \text{Aut}(N), \mathcal{L}_G, \tau_N \rangle \\
\text{Aut}(M) \cong_P \text{Aut}(N) &\Leftrightarrow \langle \text{Aut}(M), M, \mathcal{L}_G, \text{Op} \rangle \cong \langle \text{Aut}(N), N, \mathcal{L}_G, \text{Op} \rangle
\end{aligned}$$

The subscript A stands for 'abstract', T for 'topology' and P for 'permutation'

Definition 3.1.2 *CFPO M is said to be **treelike** if there is a coloured tree T such that*

$$\text{Aut}(M) \cong_A \text{Aut}(T)$$

If $G \leq \text{Aut}(M)$ then the action of G is said to be treelike if there is a tree T such that

$$(M, G) \cong_A \text{Aut}(T)$$

We start with CFPOs which have points which are fixed by every automorphism (which we call **fixed points**), and for the rest of this subsection, M will denote a (possibly coloured) CFPO with a fixed point. We will take from the midst of M our fixed point and plant it in the ground, before straightening out the paths of M into branches.

The colouring of M is largely irrelevant for this work, and so takes a very back-seat role. Indeed, for the rest of this subsection the term 'monochromatic' will mean 'monochromatic with respect to U', where U is the predicate introduced in the next definition.

Definition 3.1.3 *Let $\langle M, \leq_M \rangle$ be a connected CFPO whose automorphism group fixes the point r. We will construct $T(M)$ by specifying a new order on $|M|$. Let r be the fixed point of M, which will become the root of $T(M)$. The colour of $r \in M$ is the same in $T(M)$.*

We denote the order on T by $\leq_T$ and define it as follows:

- $r \leq_{T(M)} s$ *for all* $s \in M$
- $s \leq_{T(M)} t$ *if and only if* $s \in \text{Path}\langle r, t\rangle$

We also add a new unary predicate, which we call U. We define the following sets:

$$\begin{aligned}
X_0 &:= \{t \in M \;:\; r \leq_M t\} \\
Y_0 &:= \{t \in M \;:\; t <_M r\} \\
&\vdots \\
X_n &:= \{t \in M \;:\; y \leq_M t \text{ for some } y \in Y_{n-1}\} \setminus \textstyle\bigcup_{i<n}(X_i \cup Y_i) \\
Y_n &:= \{t \in M \;:\; t <_M x \text{ for some } x \in X_{n-1}\} \setminus \textstyle\bigcup_{i<n}(X_i \cup Y_i) \\
&\vdots
\end{aligned}$$

We also define $X := \bigcup X_i$ and say that $U(t)$ holds whenever t $\in X$, and

$$\mathcal{X} := \{X_i, Y_i \;:\; i \in \omega\}$$

.

Lemma 3.1.4 *$\mathcal{X}$ partitions $|M|$.*

Proof

By construction

$$\begin{aligned}
X_i \cap X_j \neq \emptyset &\;\;\Rightarrow\;\; i = j \\
Y_k \cap Y_l \neq \emptyset &\;\;\Rightarrow\;\; k = l
\end{aligned}$$

so it remains to show that $\mathcal{X}$ covers $|M|$. We pick an arbitrary $z \in |M|$ and consider Path$\langle z, r\rangle$, which exists as all CFPOs considered are connected.

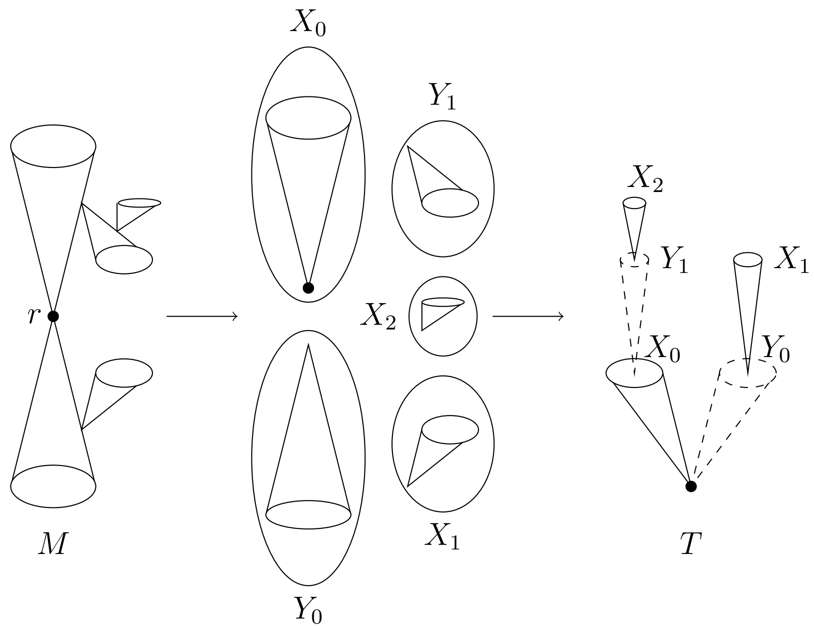

Figure 3.1: Turning M with fixed point r into $T(M)$

Let $z_0(= z), z_1, \dots z_n(= r)$ be the endpoints of Path$\langle z, r\rangle$. We know that $z_n \in X_0$ as $z_n = r$, and hence $z_{n-1} \not\parallel z_n$ implies that $z_{n-1} \in \bigcup \mathcal{X}$. Similarly $z_{n-2} \not\parallel z_{n-1}$ implies that $z_{n-1} \in \bigcup \mathcal{X}$ and so on along Path$\langle z, r\rangle$ until we deduce that $z \in \bigcup \mathcal{X}$. □

Note that if we start with a rooted tree, and use the root for our procedure, our construction returns the original structure with an additional predicate which is realised everywhere. Our eventual goal is to say that the canonical representative of M is the canonical representative of $T(M)$, and to do so we must show that $T(M)$ is a tree with the same automorphism group as M.

This construction has the curious and unfortunate property that we may have to make a choice of fixed point, and the resulting structures depend on this choice. However, since our claim is that $T(M)$ is a tree, rather than a canonical tree, we may sweep this difficulty under the carpet of Rubin's work.

Proposition 3.1.5 *$\langle T(M), \leq_{T(M)}, U\rangle$ is a tree.*

Proof

M is connected so $\leq_{T(M)}$ is defined everywhere.

If $s_0, s_1 \leq_{T(M)} t$ then $\{s_0, s_1\} \subseteq$ Path$\langle t, r\rangle$, and since M is cycle-free this means that either $s_0 \in$ Path$\langle s_1, r\rangle$ or $s_1 \in$ Path$\langle s_0, r\rangle$, showing that $s_0 \not\parallel s_1$, and thus all initial sections of $T(M)$ are linearly ordered. Finally, $r \in$ Path$\langle r, t\rangle$ for all t, so every pair from T has a common lower bound, showing that $\langle T(M), \leq_{T(M)}, U\rangle$ is a tree. □

Of course, this construction is without merit if it does not preserve the automorphism group. We work towards that goal with the following lemmas.

Lemma 3.1.6 *$\langle M, \leq_M, r\rangle$ is interpretable in $\langle T(M), \leq_{T(M)}, U\rangle$.*

Proof

If you require a recap, the definition of interpretation can be found at Definition 1.3.4. The following formulas form an interpretation of $\langle M, \leq_M, r\rangle$ in $\langle T(M), \leq_{T(M)}, U\rangle$.

1. $\phi_{Dom}(x)$, which defines the domain of the interpretation. We take

$$x = x$$

2. $\phi_{Eq}(x)$, which defines equivalence classes on the domain of the interpretation. Again, we take

$$x = x$$

3. A formula $\phi_{\leq_M}(x, y)$. We take the disjunction of the following clauses:

   (a) $(x \leq_T y \wedge \forall z(x \leq_T z \leq_T y \rightarrow U(z)))$

(b) $(y \leq_T x \wedge \forall z(y \leq_T z \leq_T x \rightarrow \neg U(z)))$

(c) $(U(y) \wedge \neg U(x)) \wedge$

$$\exists z \left(\begin{array}{l} z \leq_{T(M)} \{x, y\} \wedge \\ \forall w(z \leq_{T(M)} w \leq_{T(M)} y \rightarrow U(w)) \wedge \\ \forall w(z \leq_{T(M)} w \leq_{T(M)} x \rightarrow \\ \qquad \left(\begin{array}{l} (U(w) \rightarrow \forall v(z \leq_{T(M)} v \leq w \rightarrow U(v))) \wedge \\ (\neg U(w) \rightarrow \forall v(w \leq_{T(M)} v \leq x \rightarrow \neg U(v))) \end{array} \right) \end{array} \right)$$

4. A formula $\phi_r(x)$. We take

$$\forall z \neg (z \leq x)$$

While ϕ_{Dom}, ϕ_{Eq} and ϕ_r are self-explanatory, to show that $\phi_{\leq_M}$ does what is required of it, we examine it clause by clause.

Clause (a) shows that when both x and y lie in the same X_i for some i and $x \leq_{T(M)} y$ then $x \leq_M y$. Clause (b) shows that when both x and y lie in the same Y_i for some i and $y \leq_{T(M)} x$ then $x \leq_M y$. Clause (c) covers when $y \in X_i$ and $x \in Y_{i+1} \cup Y_{i-1}$ for some i, one instance of which is depicted in Figure 3.2. No clause is required for $y \in Y_i$ and $x \notin Y_i$, because if $x \leq_M y$ then $x \in Y_i$ $\square$

Lemma 3.1.7 *Suppose M_0 and M_1 are connected CFPOs with fixed points r_0 and r_1 respectively. Then $\langle M_0, \leq_{M_0}, r_0 \rangle \cong \langle M_1, \leq_{M_1}, r_1 \rangle$ if and only if*

$$\langle T(M_0), \leq_{T(M_0)}, U_{T(M_0)} \rangle \cong \langle T(M_1), \leq_{T(M_1)}, U_{T(M_1)} \rangle$$

Proof

Since we constructed $\leq_T$ and U using path-betweenness and $\leq_M$, both of which are

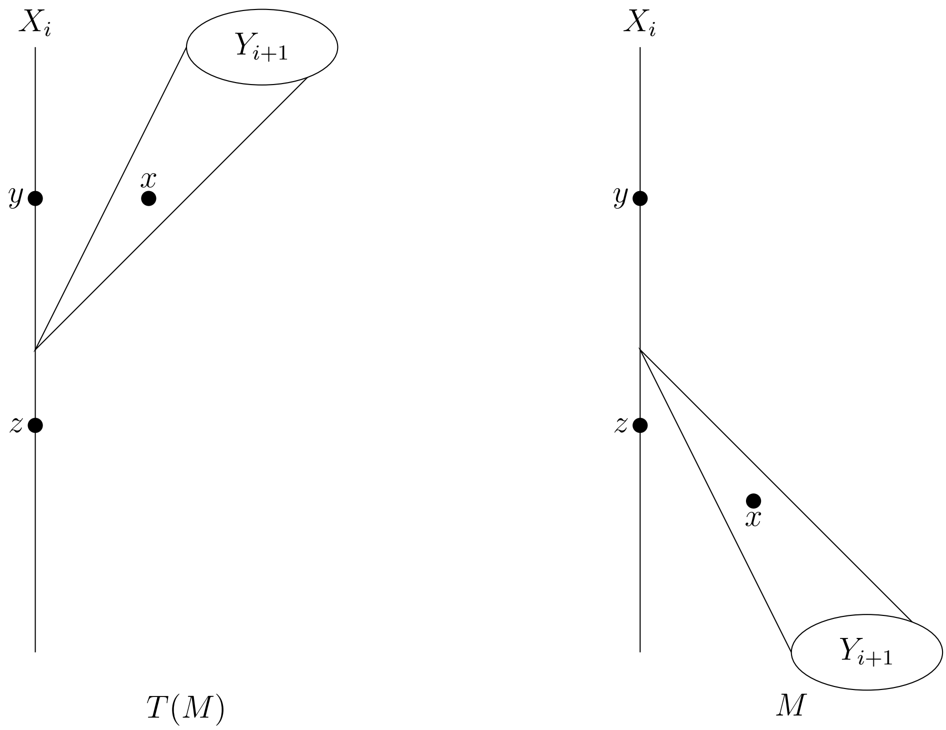

Figure 3.2: Clause (c) of $\phi_{\leq_M}$ in Lemma 3.1.6

preserved by isomorphism,

$$\langle M_0, \leq_{M_0}, r_0 \rangle \cong \langle M_1, \leq_{M_1}, r_1 \rangle \Rightarrow$$
$$\langle T(M_0), \leq_{T(M_0)}, U_{T(M_0)} \rangle \cong \langle T(M_1), \leq_{T(M_1)}, U_{T(M_1)} \rangle$$

The other direction of the isomorphism is a consequence of the fact that in Lemma 3.1.6 the domain of the interpretation is $T(M)$ itself. □

This second lemma shows that the construction behaves when we take certain substructures. We will take from M an extended cone C, and show that $T(C)$ is isomorphic to either the corresponding substructure of $T(M)$, or the corresponding substructure with the roles of U and $\neg U$ reversed.

Lemma 3.1.8 *Let r be a fixed point of M and let $x \in M$. We define*

$$N := \{y \in M \ : \ x \in \text{Path}\langle y, r\rangle\}$$

If we add a colour to N which is only realised by x (to ensure that x is a fixed point of N as a structure in its own right), and use x to construct

$$\langle T(N), \leq_{T(N)}, U_{T(N)}\rangle$$

then if $x \in X$ (recall Definition 3.1.3) then

$$\langle N, \leq_{T(M)}, U_{T(M)}\rangle \cong \langle T(N), \leq_{T(N)}, U_{T(N)}\rangle$$

otherwise $x \in M \setminus X$ (recall Definition 3.1.3) implies that

$$\langle N, \leq_{T(M)}, U_{T(M)}\rangle \cong \langle T(N), \leq_{T(N)}, \neg U_{T(N)}\rangle$$

Proof

This is a simple consequence of the fact that $\text{Path}\langle y, r\rangle = \text{Path}\langle y, x\rangle \cup \text{Path}\langle x, r\rangle$ for all $y \in N$ $\square$

Lemma 3.1.9 *The members of $\mathcal{X}$ are preserved setwise by* Aut(M).

Proof

All automorphisms fix r, so X_0, the points greater than r, and Y_0, the points less than r, are fixed setwise.

Let $x_n \in X_n$ and let $y_{n-1} \in Y_{n-1}$ with $y_{n-1} \leq_M x_n$, and assume as an induction hypothesis that for $i < n$ both X_i and Y_i are fixed setwise by Aut(M). Let $\phi \in$ Aut(M)

be arbitrarily chosen. By the induction hypothesis $\phi(y_{n-1}) \in Y_{n-1}$, and since ϕ is an automorphism $\phi(y_{n-1}) \leq_M \phi(x_n)$. If $\phi(x_n) \in \bigcup_{i<n}(X_i \cup Y_i)$ then ϕ^{-1} violates the induction hypothesis, so X_n is preserved by Aut(M). The argument for Y_n is identical. □

Lemma 3.1.10 Aut(T) *preserves the members of* $\mathcal{X}$ *setwise.*

Proof

Let $x \in X_n$. Since $T \models U(x)$ and $T \models \neg U(y)$ for all $y \in \bigcup Y_i$, we cannot map x to any member of $\bigcup Y_i$. By taking a witness that $x \in X_n$, and a witness that that witness lies in Y_{n-1} and so on, we obtain a maximal chain $x_1 \leq_{T(M)} x_2 \leq_{T(M)} \ldots x_n(= x)$ such that $U(x_i)$ if and only if $\neg U(x_{i-1})$ and $\neg U(x_{i+1})$, with the additional property that for all $x_i \leq_{T(M)} t \leq_{T(M)} x_{i+1}$ either $[x_i, t]$ or $[t, x_i]$ is monochromatic.

Any automorphism would have to send this chain to a similar chain below the image of x, but the length of this chain is determined by n, thus all images of x lie in X_n. A similar argument shows the same for Y_n, and so we conclude that Aut$(T(M))$ preserves the members of $\mathcal{X}$ setwise. □

Theorem 3.1.11 Aut$(\langle M, \leq_M\rangle) \cong_P$ Aut$(\langle T(M), \leq_{T(M)}, U\rangle)$

Proof

In the proof of Lemma 2.5.3 we proved that if $\bar{x}$ and $\bar{y}$ are path complete n-tuples such that they have the same order type and if (x_i, x_j) is an adjacent pair then (x_i, x_j) and (y_i, y_j) lie in the same orbit.

Thus if all the 1- and 2-orbits of M coincide with the 1- and 2-orbits of $T(M)$ then Aut$(T(M)) \cong_P$ Aut(M). We will start with the 1-orbits, which we will prove by induction on $\mathcal{X}$.

Since $\langle X_0, \leq_M\rangle$ is a tree

$$\langle X_0, \leq_M\rangle = \langle X_0, \leq_{T(M)}\rangle$$

and since $\langle X_0, \leq_{T(M)}, U_{T(M)}\rangle$ is monochromatic,

$$\text{Aut}(\langle X_0, \leq_M\rangle) \cong_P \text{Aut}(\langle X_0, \leq_{T(M)}, U_{T(M)}\rangle)$$

From this we conclude that for all $a, b \in X_0$, if a and b lie in different orbits of M but the same orbits of T then

$$\langle\{t \in M \ : \ a \in \text{Path}\langle t, r\rangle\}, \leq_M\rangle \ncong \langle\{t \in M \ : \ b \in \text{Path}\langle t, r\rangle\}, \leq_M\rangle$$

and

$$\langle\{t \in M \ : \ a \leq_{T(M)} t\}, \leq_{T(M)}, U_{T(M)}\rangle$$
$$\cong$$
$$\langle\{t \in M \ : \ b \leq_{T(M)} t\}, \leq_{T(M)}, U_{T(M)}\rangle$$

However, this contradicts Lemma 3.1.8, so if a and b lie in the same orbit of $T(M)$ then they lie in the same orbit of M. By symmetry, we also conclude that if a and b lie in the same orbit of M then they lie in the same orbit of $T(M)$. Similarly, if $a, b \in Y_0$ then a and b lie in the same orbit of M if and only if they lie in the same orbit of $T(M)$.

So now suppose that for $i < n$ the 1-orbits on X_i and Y_i from Aut(M) and Aut$(T(M))$ coincide and let $x, y \in X_n$. We define, as we did in Lemma 3.1.10, $x_1, \ldots x_n$ and $y_1, \ldots, y_n$, which are linearly ordered by $\leq_{T(M)}$, are the connecting sets of Path$\langle x, r\rangle$ and Path$\langle y, r\rangle$ in $\leq_M$.

If x_n and y_n belong to the same orbit of M then the automorphism that witnesses this also witnesses that x_{n-1} and y_{n-1} lie in the same orbit of M, and hence by our induction hypothesis, the same orbit of T. Since there is an automorphism that maps x_{n-1} to y_{n-1},

$$\langle \{z \in M \;:\; x_{n-1} \in \text{Path}\langle r, z\rangle\}, \leq_M\rangle \cong \langle \{z \in M \;:\; y_{n-1} \in \text{Path}\langle r, z\rangle\}, \leq_M\rangle$$

and hence (using Lemmas 3.1.7 and 3.1.8)

$$\begin{gathered}\langle \{z \in M \;:\; x_{n-1} \in \text{Path}\langle r, z\rangle\}, \leq_{T(M)}, U_{T(M)}\rangle \\ \cong \\ \langle \{z \in M \;:\; y_{n-1} \in \text{Path}\langle r, z\rangle\}, \leq_{T(M)}, U_{T(M)}\rangle\end{gathered}$$

And so there is an isomorphism of T that maps x_n to y_n. The arguments for x_n, y_n being in the same orbit of T, and for $x_n, y_n \in Y_n$ are, again, extremely similar, and so omitted.

We now turn out attention to the 2-orbits. Since r is fixed by both Aut(M) and Aut(T), the 1-orbits can be thought of as 2-orbits where one of the elements is r, and the 2-orbits can be thought of as 3-orbits where r is one of the elements. This viewpoint is exploited to show the coincidence of the 2-orbits of Aut(M) and Aut(T).

Suppose (x_0, x_1) and (y_0, y_1) lie in the same orbit of M. We need only consider the case when $x_0 \in \text{Path}\langle x_1, r\rangle$ as otherwise we can take x_2 to be the intersection of Path$\langle x_0, r\rangle$, Path$\langle x_0, x_1\rangle$ and Path$\langle x_1, r\rangle$, and patch automorphisms together around x_2. Note that x_2 would be the meet of x_0 and x_1 in $T(M)$.

There is an automorphism of M that maps x_0 to y_0, and as we have just seen, this means that

$$\langle \{z \in M \;:\; x_0 \in \text{Path}\langle r, z\rangle\}, \leq_M\rangle \cong \langle \{z \in M \;:\; y_0 \in \text{Path}\langle r, z\rangle\}, \leq_M\rangle$$

Since (x_0, x_1) and (y_0, y_1) lie in the same orbit of M, there is an isomorphism from

$$\langle \{z \in M \;:\; x_0 \in \text{Path}\langle r, z\rangle\}, \leq_M\rangle \text{ to } \langle \{z \in M \;:\; y_0 \in \text{Path}\langle r, z\rangle\}, \leq_M\rangle$$

that maps x_1 to y_1. By Lemmas 3.1.7 and 3.1.8 this results in an isomorphism from

$$\langle \{z \in M \ : \ x_0 \leq_{T(M)} z\}, \leq_{T(M)}, U_{T(M)} \rangle$$

to

$$\langle \{z \in M \ : \ y_0 \leq_{T(M)} z\}, \leq_{T(M)}, U_{T(M)} \rangle$$

which maps x_1 to y_1. We call this isomorphism ϕ, and we take any automorphism that takes x_0 to y_0 and call it ψ. The function

$$\theta(t) := \begin{cases} \phi(t) & t \geq_{T(M)} x_0 \\ \psi(t) & \text{otherwise} \end{cases}$$

is an automorphism of T which maps (x_0, x_1) to (y_0, y_1), and thus the 2-orbits of T contain the 2-orbits of M.

Once again, the argument to show that the 2-orbits of M contain the 2-orbits of T is extremely similar, due to the symmetric nature of Lemmas 3.1.7 and 3.1.8, and thus we conclude that the 2-orbits of M and T coincide, and so

$$\text{Aut}(\langle M, \leq_M \rangle) \cong_P \text{Aut}(\langle T(M), \leq_{T(M)}, U \rangle)$$

□

Lots of CFPOs have fixed points, but the CFPOs of the kind discussed in the next lemma reoccur frequently.

Lemma 3.1.12 *Let M be a connected, Rubin complete CFPO. If there are connected $A, B \subsetneq M$ which are disjoint and fixed setwise by* Aut(M) *then there are c, d which are fixed points of M and* Path$\langle A, B \rangle$ = Path$\langle c, d \rangle$.

Proof

Let M be a Rubin complete CFPO, and let A, B be connected proper subsets of M which are disjoint and fixed setwise by Aut(M). We use the notation

$$\text{Path}\langle x,y\rangle^- := \{z \in \text{Path}\langle x,y\rangle \ : \ \exists a,b \in \text{Path}\langle x,y\rangle \ (z = (a \wedge b) \vee z = (a \vee b))\}$$

In words, Path$\langle x,y\rangle^-$ are the local maxima and minima of Path$\langle x,y\rangle$. Just as with Path$\langle x,y\rangle$, if X and Y are subsets of M then:

$$\begin{aligned}
\text{Path}\langle x,Y\rangle^- &:= \textstyle\bigcap_{y\in Y} \text{Path}\langle x,y\rangle^- \\
\text{Path}\langle X,y\rangle^- &:= \textstyle\bigcap_{x\in X} \text{Path}\langle x,y\rangle^- \\
\text{Path}\langle X,Y\rangle^- &:= \textstyle\bigcap_{\substack{x\in X\\ y\in Y}} \text{Path}\langle x,y\rangle^-
\end{aligned}$$

Note that Path$\langle x,y\rangle^-$ always has finite cardinality.

We are going to find a fixed point using (possibly transfinite) induction. Fix $b \in B$.

Base Case Pick $a_0 \in A$. We set $c_0 = a_0$ and let $D_0 = \{x \in A \ : \ c_0 \in \text{Path}\langle x,b\rangle\}$.

Successor Step Suppose we have $a_{\alpha-1}$, $c_{\alpha-1}$ and $D_{\alpha-1}$.

Pick $a_\alpha \in A \setminus D_{\alpha-1}$. Since $b \in \text{Path}\langle c_{\alpha-1}, b\rangle$ and $b \in \text{Path}\langle a_\alpha, b\rangle$,

$$\text{Path}\langle \{c_{\alpha-1}, a_\alpha\}, b\rangle \neq \emptyset$$

Let

$$C_\alpha := \{x \in \text{Path}\langle \{c_{\alpha-1}, a_\alpha\}, b\rangle \ : \ |\text{Path}\langle \{c_{\alpha-1}, a_\alpha\}, b\rangle^-| = |\text{Path}\langle x, b\rangle^-|\}$$

C_α is linearly ordered, and is bounded both above and below by elements of Path$\langle c_{\alpha-1}, b\rangle^- \cup \text{Path}\langle a_\alpha, b\rangle^-$. Since M is Rubin complete, C_α has both a maximal and a minimum element.

Let $c_\alpha \in C_\alpha$ be such that Path$\langle \{c_{\alpha-1}, a_\alpha\}, b\rangle$ = Path$\langle c_\alpha, b\rangle$.

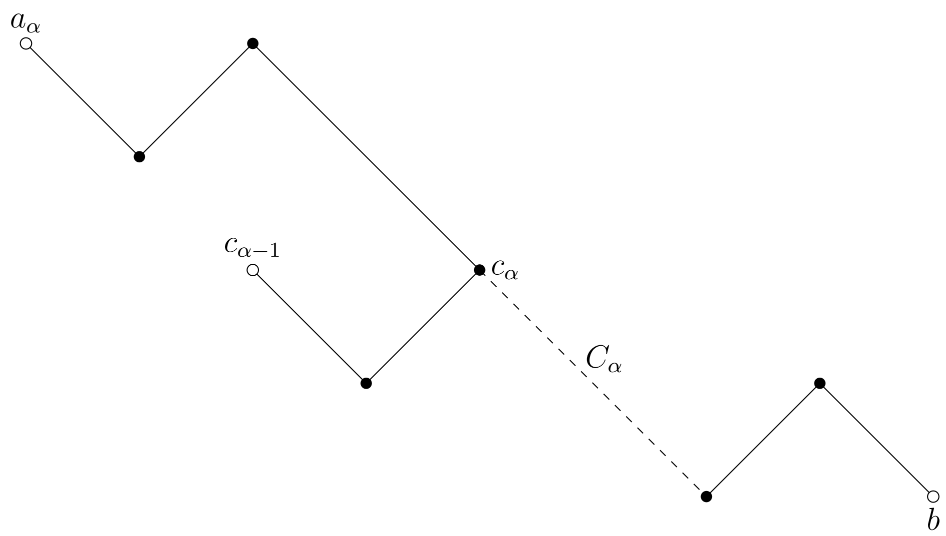

Figure 3.3: Finding c_α in Lemma 3.1.12

Since A is connected, Path$\langle c_{\alpha-1}, a_\alpha\rangle \subseteq A$, and since $c_\alpha \in$ Path$\langle c_{\alpha-1}, a_\alpha\rangle$, we have that $c_\alpha \in A$.

We define $D_\alpha = \{x \in A \ : \ c_\alpha \in \text{Path}\langle x, b\rangle\}$. If $D_\alpha = A$ then let $c = c_\alpha$ and stop.

Limit Step Let $n_\lambda = \min\{|\text{Path}\langle c_\alpha, b\rangle^-| \ : \ \alpha < \lambda\}$.

$$C_\lambda := \{x \in \text{Path}\langle c_\alpha, b\rangle \ : \ |\text{Path}\langle c_\alpha, b\rangle^-| = n_\lambda\}$$

C_α is linearly ordered, and is bounded both above and below by elements of $\bigcup_{\alpha<\lambda} \text{Path}\langle c_\alpha, b\rangle^-$, so has both a maximal and minimal element.

Let $c_\lambda \in C_\lambda$ be such that Path$\langle c_\lambda, b\rangle \subseteq$ Path$\langle \{c_\alpha, a_\alpha\}, b\rangle$. We define $D_\lambda = \{x \in A \ : \ c_\lambda \in \text{Path}\langle x, b\rangle\}$. If $D_\lambda = A$ then let $c = c_\lambda$ and stop.

We have found a c such that $c \in \text{Path}\langle A, b\rangle$. If we repeat this induction, fixing c and choosing b_α from B then we find a d such that

$$\text{Path}\langle c, d\rangle = \text{Path}\langle A, B\rangle$$

Let $\phi \in \text{Aut}(M)$.

$$\begin{aligned}\text{Path}\langle \phi(c), \phi(d)\rangle &= \phi(\text{Path}\langle c, d\rangle) \\ &= \phi(\text{Path}\langle A, B\rangle) \\ &= \text{Path}\langle \phi(A), \phi(B)\rangle \\ &= \text{Path}\langle A, B\rangle \\ &= \text{Path}\langle c, d\rangle\end{aligned}$$

Therefore both c and d are fixed by all automorphisms of M. $\square$

3.1.1 CFPO_n

Lemma 3.1.13 *If M is a connected CFPO_3 then M is treelike.*

Proof

A CFPO_3 can be split into three possibly empty sections, a tree which is above a linear order, which in turn is above a reverse ordering of a tree. If the tree section is empty the reverse tree cannot be empty, and vice versa.

By marking the reversed tree with a unary predicate and reversing its order we obtain a tree which has the same automorphism group as the CFPO_3. $\square$

Theorem 3.1.14 *If M is a connected CFPO_{2n+1} then M is treelike.*

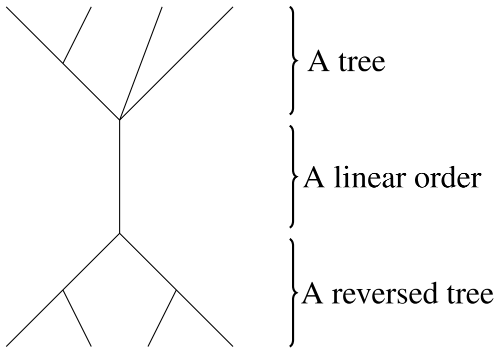

Figure 3.4: A typical CFPO_3

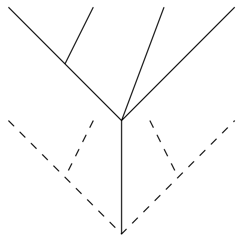

Figure 3.5: A Tree with the same Automorphism Group

Proof

Our strategy is to find a subset of M which is a CFPO_3 and is fixed setwise by Aut(M), and add cones to the tree corresponding to this CFPO_3 to obtain a tree with the same automorphism group as M.

We consider the $\phi(a_n)$ and $\phi(a_n^*)$, the images in M of the midpoints of Alt_{2n+1} and Alt^*_{2n+1} under all possible embeddings ϕ. Let C be the set of all such $\phi(a_n)$ and $\phi(a_n*)$. This is the candidate for the CFPO_3 we require for our strategy, but first we must show that it is indeed a CFPO_3, and that it is fixed setwise by Aut(M).

Suppose that C contains an antichain x_n, y_n. Since M is connected there must be a path between x_n and y_n. We also pick particular copies of either Alt_{2n+1} or Alt^*_{2n+1} that contain x_n and y_n, and label the points using x_i and y_i appropriately. X is the set $\{x_i\}$, while $Y = \{y_i\}$.

To show that the maximum length of a path though C is 3 we consider how the ends of Path$\langle x_n, y_n\rangle$ interact with X and Y.

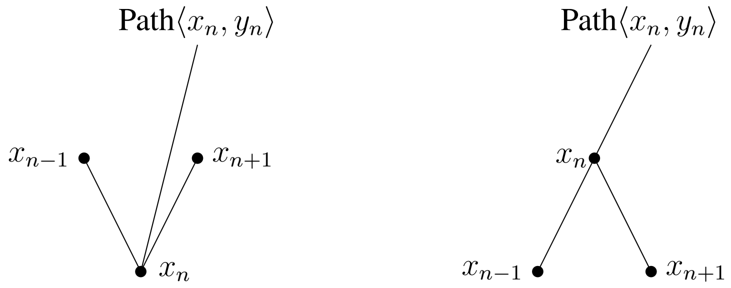

Figure 3.6: Interactions between X and Y

The cases where x_n is an upper point of Path$\langle x_n, y_n\rangle$ are reverse orderings of Cases 1 and 2, so will not be done explicitly. Also there is nothing special in our choice of X, so these arguments also apply to Y.

Case 1 In this case x_n is a lower point of both X and Path$\langle x_n, y_n\rangle$.

If $[x_n, x_{n+1}] \cap \text{Path}\langle x_n, y_n\rangle \neq \emptyset$ then $[x_n, x_{n-1}] \cap \text{Path}\langle x_n, y_n\rangle = \emptyset$, otherwise x_{n-1} and x_{n+1} would be related. So the union of at least one of $\{x_0, \ldots x_{n-1}\}$ or $\{x_{n+1}, \ldots x_{2n}\}$ with Path$\langle x_n, y_n\rangle$ is a copy of a finite section of Alt.

Case 2 In this case x_n is an upper point of X but a lower point of Path$\langle x_n, y_n\rangle$. As both x_{n-1} and x_{n+1} lie below x_n the two paths Path$\langle x_{n-1}, y_n\rangle$ and Path$\langle x_{n+1}, y_n\rangle$ both contain and have the same length as Path$\langle x_n, y_n\rangle$. We also know that x_{n-2} cannot be contained in Path$\langle x_{n-1}, y_n\rangle$, as this would require x_{n-2} and x_n to be related. Similarly x_{n+2} cannot be contained in Path$\langle x_{n+1}, y_n\rangle$. Thus we see that both $\{x_0, \ldots x_{n-2}\} \cup \text{Path}\langle x_{n-1}, y_n\rangle$ and $\{x_{2n}, \ldots x_{n+2}\} \cup \text{Path}\langle x_{n+1}, y_n\rangle$ are copies of a finite section of Alt.

Thus in both cases, at least one of $\{x_0, \dots x_{n-1}\}$ or $\{x_{n+1}, \dots x_{2n}\}$ with Path$\langle x_n, y_n\rangle$ is a copy of a finite section of Alt. M is a cycle free partial order so, assuming that the configurations of X, Y and Path$\langle x_n, y_n\rangle$ result in the shortest possible finite alternating chain,

$$P := \{x_0, \dots, x_{n-2}\} \cup \text{Path}\langle x_{n-1}, y_{n+1}\rangle \cup \{y_{n+2}, \dots y_{2n}\}$$

is a copy of a finite section of Alt. The length of P is

$$2n - 2 + \mid \text{Path}\langle x_{n-1}, y_{n+1}\rangle \mid$$

By assumption M is a CFPO_{2n+1}, so P has at most $2n+1$ elements, thus $\mid$ Path$\langle x_n, y_n\rangle \mid$ ≤ 3 and C is a CFPO_3.

To see that C is fixed setwise by automorphisms, simply note for any $x \in C$ and $\phi \in$ Aut(M), the image of the copy of Alt_{2n+1} that witnesses the fact that $x \in C$ will witness $\phi(x) \in C$.

We now have the CFPO_3 our strategy demands, so now we focus on how we may adjoin cones to it to obtain a tree with the same automorphism group as M.

For each $x \in C$, we define $B(x) := \{y \in M \ : \ \text{Path}\langle x, y\rangle \cap C = \{x\}\}$. If we introduce a predicate that fixes x to $B(x)$, then we are able to apply the construction in Definition 3.1.3 to $B(x)$ using x as the root to obtain $T(B(x))$. We also know that if there is an automorphism of M that maps x_0 to x_1 then $B(x_0) \cong B(x_1)$.

For each isomorphism type of $B(x)$, we add a colour predicate P_x to $\langle C, \leq\rangle$ such that $C \models P_x(y)$ if and only if $B(y) \cong B(x)$. We obtain $\langle C, \leq_M, \{P_x\}\rangle$, a CFPO_3 such that:

$$\text{Aut}(\langle C, \leq_M, \{P_x\}\rangle) \cong_P \{g \in \text{Aut}(C) \ : \ \exists h \in \text{Aut}(M)\, h|_C = g\}$$

Lemma 3.1.13 shows that there is a tree, which we call $T(C)$ such that

$$\text{Aut}(T(C)) \cong \{g \in \text{Aut}(C) \,:\, \exists h \in \text{Aut}(M)\, h|_C = g\}$$

We define T to be the structure whose domain is

$$T_C \cup \bigcup_{x \in C} T(B(x))$$

under the equivalence relation that identifies the root of $T_{B(x)}$ with the point of T_C that corresponds with x. We give T the transitive closure of the order inherited from T_C and all the $T_{B(x)}$. This structure is clearly a tree with the automorphism group of M.

Note that this method not only gives a tree T such that Aut$(M) \cong_A$ Aut(T), but also a tree T such that Aut$(T) \cong_P$ Aut(M). $\square$

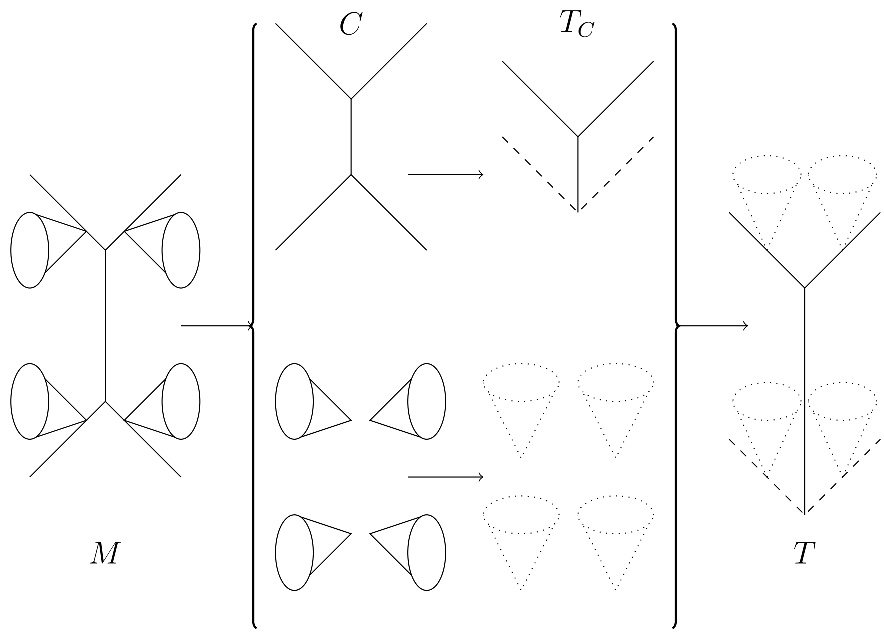

Figure 3.7: Turning a CFPO_{2n+1} into a Tree

Corollary 3.1.15 *If M is a connected CFPO_{2n} then M is treelike.*

Proof

Let $e \in M$ be an image of $a_0 \in \mathrm{Alt}_{2n}$ (if Alt_{2n} does not embed into M we may consider M^* instead). Below every point in $\mathrm{Or}(e)$ we adjoin a new point, coloured with a new unary predicate. This new structure is a CFPO_{2n+1} with the same automorphism group as M, so M shares its abstract automorphism group with a tree. $\square$

While we have found a tree T such that $\mathrm{Aut}(M) \cong_A \mathrm{Aut}(T)$, and thereby proved the corollary, we may do better than that. We can delete the points we added to M from T without introducing new automorphisms (as we added these points to every point in an orbit of M), getting a T^* such that $\mathrm{Aut}(M) \cong_P \mathrm{Aut}(T^*)$.

3.1.2 CFPO_ω

Definition 3.1.16 *If M is a* CFPO *then:*

1. *M is said to be a CFPO_ω if Alt_ω embeds but Alt does not; and*
2. *M is said to be a CFPO_∞ if Alt embeds.*

Theorem 3.1.17 *If M is a connected CFPO_ω then M is tree-like.*

Proof

This proof works in a similar fashion to the proofs of Theorem 3.1.11, Lemma 3.1.13 and Theorem 3.1.14; by altering the order on the CFPO we produce a tree, while maintaining the automorphism group. Let M be a Rubin-complete CFPO.

We say that $A \subseteq M$ is a maximal copy of either Alt_ω or Alt_ω^* if

- A is the image of Alt_ω (or Alt^*_ω respectively).

- There is no image of Alt_ω or Alt^*_ω that properly contains A.

Every copy of Alt_ω is contained in a maximal copy of either Alt_ω or Alt^*_ω. To see this, let $\{A_n \subseteq M \ : \ n \in \omega\}$ be such that each A_n is isomorphic to either Alt_ω or Alt^*_ω and if $n < m$ then $A_n \subsetneq A_m$. This means that

$$\bigcup_{n\in\mathbb{N}} (A_n \setminus A_0) \cong \mathrm{Alt}_\omega \text{ or } \mathrm{Alt}^*_\omega$$

and therefore

$$A_0 \cup \bigcup_{n\in\mathbb{N}} (A_n \setminus A_0) \cong \mathrm{Alt}$$

We now describe a procedure that transforms M into a tree while preserving its automorphism group. Again, we add a unary predicate U to remind us when we've changed direction.

1. Let M_0 be the following set:

   $$\{x \in M : x \text{ is the first element of a maximal copy of either } \mathrm{Alt}_\omega \text{ or } \mathrm{Alt}^*_\omega\}$$

   If $x \in M_0$ is witnessed by a maximal copy of Alt_ω then $x \in M_0$ cannot be witnessed by a maximal copy of Alt^*_ω. To see this, let $\{x, a_1, \ldots\}$ be a maximal copy of Alt_ω and let $\{x, b_1, \ldots\}$ be a maximal copy of Alt^*_ω.

   $b_1 > a_1$, but $b_2 || a_1$, as $b_2 || x$, so $\{b_3, b_2, b_1, a_1, \ldots\}$ is a copy of Alt_ω, contradicting the assumption that $\{x, a_1, \ldots\}$ was a maximal copy of Alt_ω.

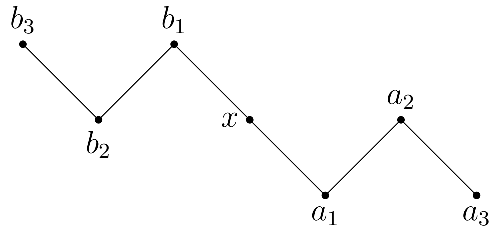

Figure 3.8: Witnessing $x \in M_0$

Let $\sim_C$ be the relation on M_0 given by

$$x \sim_C y \Leftrightarrow \left(\begin{array}{c} \{x, a_1, \ldots\} \text{ witnesses } x \in M_0 \\ \text{if and only if} \\ \{y, a_1, \ldots\} \text{ witnesses } y \in M_0. \end{array} \right)$$

That $\sim_C$ is an equivalence relation is readily apparent. We denote the $\sim_C$-equivalence classes as C_i^0.

Let $x \in M_0$, and let this be witnessed by $\{x, a_1, \ldots\}$, a copy of Alt_ω. For every $y \in [x]_{\sim_C}$, we know that $y > a_1$, and thus $[x]_{\sim_C} \cup a_1$ is a tree. Similarly, if $x \in M_0$ is witnessed by a copy of Alt_ω^* then $[x]_{\sim_C}$ is a reverse ordered tree.

Let $\{C_i^0\}$ be the set of $\sim_C$-equivalence classes of M_0.

2. Assume we have defined M_{n-1} and the C_i^{n-1}s. We define M_n to be:

$$\left\{ x \in M \setminus \bigcup_{i<n} M_i \; : \; \begin{array}{l} x \text{ is the first element of } A \text{which is a maximal} \\ \text{copy of either } \mathrm{Alt}_\omega \text{ or } \mathrm{Alt}_\omega^* \text{in } \left(M \setminus \bigcup_{i<n} M_i \right) \end{array} \right\}$$

   Again, M_n is a disjoint union of trees and reverse ordered trees, which we call C_i^n.

If C_i^n is a tree then $T(C_i^n) := \langle C_i^n, \leq, U \rangle$ where U is realised nowhere, and if C_i^n is a reverse ordered tree then $T(C_i^n) := \langle (C_i^n)^*, \leq, U \rangle$ where U is realised everywhere.

We define T_0 to be the disjoint union of $\{T(C_i^0)\}$ with no new relations added to the ordering. If we have already defined T_{n-1} then

$$T_n := T_{n-1} \cup \bigcup \{T(C_i^n)\}$$

We add to the order inherited from T_{n-1} and $T(C_i^n)$ pairs of the form (x, y) where

$$x \in T(C_i^n) \text{ for some } i$$

and y is in $T(C_j^{n-1})$, where C_j^{n-1} is a cone of x. We then take the transitive closure to obtain an ordering.

Put $T(M) := \bigcup_{n \in \mathbb{N}} T_n$. Since the M_n partition M and since at each stage we place trees above elements of trees, $T(M)$ is a tree.

If Aut(M) does not preserve the M_n then we would have a map that sends a maximal copy of Alt_ω or Alt_ω^* to a non-maximal copy. $T(M)$ realises U in monochromatic convex subsets. In the tree obtained by collapsing each of those subsets to a singleton, every maximal chain is isomorphic to ω^*, so $T(M)$ preserves the M_n set-wise too.

Since each $T(C_i^n)$ is monochromatic, and is order-isomorphic to either C_i^n or $(C_i^n)^*$, if Aut$(T(M)) \neq$ Aut(M) then we must either:

1. be unable send a $T(C_i^n)$ to a $T(C_j^n)$ where we can map C_i^n to a C_j^n; or

2. be able to send $T(C_i^n)$ to $T(C_j^n)$ where we cannot map C_i^n to C_j^n.

If $T(C_i^n) \cong T(C_j^n)$ but we cannot map one to the other using an automorphism of $T(M)$ then we must eventually attach $T(C_i^n)$ to something different to what we attach $T(C_j^n)$ to, but then C_i^n emanates from a point that is different to the point that C_i^n emanates from, and we cannot map C_i^n to C_i^n.

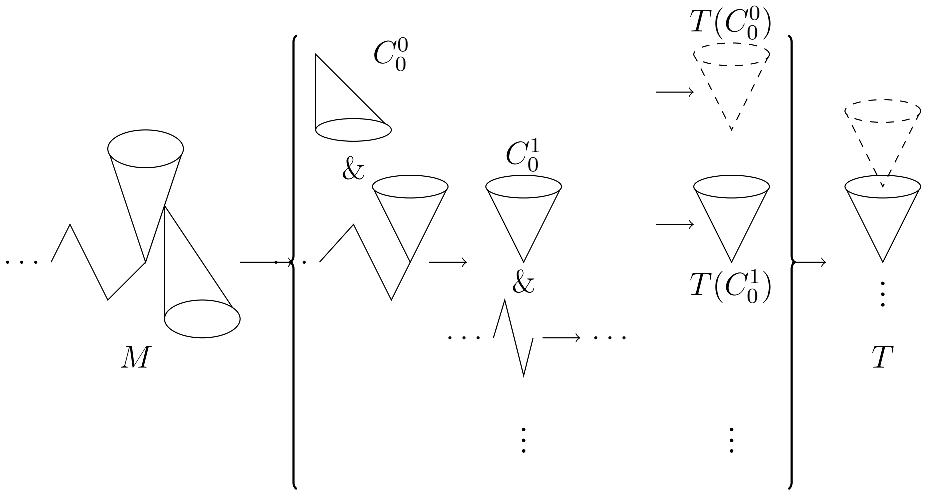

Figure 3.9: Turning a CFPO_ω into a Tree

If we do this argument in reverse we obtain point 2.

Therefore every Rubin-complete CFPO_ω is treelike. Let $\langle M, \leq_M\rangle$ be a not necessarily Rubin complete CFPO_ω, with Rubin completion $\langle M^R, \leq_M, I\rangle$. There is a tree $T(M)^R$ such that

$$\text{Aut}(\langle M^R, \leq_M, I\rangle) \cong_P \text{Aut}(\langle T(M^R), \leq_T, I, U\rangle)$$

We define $T(M) := \{x \in T(M^R) \ : \ T(M^R) \models \neg I(x)\}$. Then

$$\text{Aut}(\langle M, \leq_M\rangle) \cong_P \text{Aut}(\langle T(M), \leq_T, U\rangle)$$

□

3.1.3 Disconnected CFPOs

While this section has only proved results about connected CFPOs, they are readily extended to disconnected CFPOs.

Proposition 3.1.18 *Let M be a possibly disconnected CFPO with connected components A_i, where the i are indexed by I. If A_i is treelike for all $i \in I$ then M is treelike.*

Proof

For all $i \in I$, let $\langle T(A_i), \leq, U\rangle$ be the coloured tree such that $\mathrm{Aut}(\langle T(A_i), \leq_i, U_i\rangle) \cong_A \mathrm{Aut}(A_i)$.

$\mathcal{T} := \langle \{r\} \cup \bigcup(T(A_i)), \leq_T, U_T\rangle$ where

$$\begin{aligned}
\mathcal{T} \models (x \leq_T y) \quad &\Leftrightarrow \quad ((\exists i \in I\ (x \leq_i y)) \vee (x = r)) \\
\mathcal{T} \models U_T(x) \quad &\Leftrightarrow \quad \exists i \in I\ U_i(x)
\end{aligned}$$

$\mathrm{Aut}(M) \cong_A \mathrm{Aut}(\mathcal{T})$, as each of the cones of $r \in \mathcal{T}$ share an automorphism group with its corresponding A_i, and may only be mapped to one another by an automorphism of $\mathcal{T}$ if their corresponding A_i are isomorphic. $\square$

Remark 3.1.19 *If each of the $T(A_i)$ are obtained using Definition 3.1.3, then we may adapt the interpretation in Lemma 3.1.6 by changing ϕ_{Dom} to $x \neq r$ to obtain an interpretation of $\langle M, \leq_M\rangle$ in $\mathcal{T}$.*

3.2 The Infinite Dihedral Group

Definition 3.2.1 *D_∞, the **infinite dihedral group**, is the group with the following presentation $\langle \sigma, \tau \mid \sigma^2 = 1, \sigma\tau\sigma = \tau^{-1}\rangle$.*

How D_∞ occurs as a subgroup of an automorphism group of a CFPO characterises whether it is treelike or not. We will first examine how D_∞ can act on trees.

3.2.1 Dendromorphic Groups

Definition 3.2.2 *If T is a tree that contains points a and b then*

$$B(a;b) := \{t \in T \,:\, a < t \wedge b\}$$

$B(a;b)$ is the cone of a that contains b. If $a \not< b$ then $B(a;b) = \emptyset$. If B is a set such that $a \leq B$ then

$$B(a,B) := \bigcup_{b \in B} B(a;b)$$

Definition 3.2.3 *Given an abstract group G and a permutation group $(H, S, \mu(h,s))$ their wreath product, written as $G \wr_S H$, is the abstract group on domain*

$$H \times \{\eta : S \to G\}$$

We use $\eta(s)$ to denote the function $s \mapsto \eta(s)$, and so $\eta(s_0 s)$ is the function $s \mapsto \eta(s_0 s)$. The group operation of $G \wr_S H$ is given by

$$(h_0, \eta_0(x))(h_1, \eta_1(x)) = (h_0 h_1, \eta_0(\mu(h_1^{-1}, x))\eta_1(x))$$

When $G = \mathrm{Aut}(M)$ and $H = \mathrm{Aut}(N)$ their wreath product $G \wr H$ is the automorphism group of the structure obtained by replacing every element of N with a copy of G.

Remark 3.2.4 *$\mathbb{Z} \wr \mathbb{Z}_2$ is the automorphism group of the structure obtained by replacing the elements of a 2-element antichain by copies of $(\mathbb{Z}, \leq)$, while $\mathbb{Z}_2 \wr \mathbb{Z}$ is the automorphism group of the structure obtained by replacing the elements of $(\mathbb{Z}, \leq)$ with 2-element antichains (the lamplighter group).*

Remark 3.2.5 *If group G acts on set X, with $X_0 \subseteq X$, then $G_{\{X_0\}}$ is the set-wise stabiliser of X_0, while $G_{(X_0)}$ is the point-wise stabiliser. Similarly for automorphism*

Figure 3.10: Mnemonic for the Wreath Product

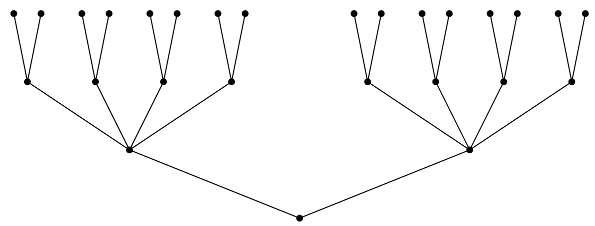

Figure 3.11: Example of a Regular Tree

groups, $\mathrm{Aut}_{\{X_0\}}(M)$ *is the set-wise stabiliser of* X_0 *in* M*, while* $\mathrm{Aut}_{(X_0)}(M)$ *denotes the point-wise stabiliser. If* $X_0 = \{x\}$ *then these two notions coincide and we use the pithier expression* G_x *or* $\mathrm{Aut}_{X_0}(x)$.

Definition 3.2.6 *A tree* T *is said to be regular if:*

1. *all the maximal chains are isomorphic to each other;*
2. *the maximal chains are isomorphic to a finite linear order or* $\mathbb{N}$*;*
3. *the ramification order of any non-maximal element of* T *is at least 2 but finite; and*
4. *if* $|T^{\leq x}| = |T^{\leq y}|$ *then the ramification order of* x *equals the ramification order of* y.

A tree T *is said to be fh-regular (finite height) if it is regular and the maximal chains are finite.*

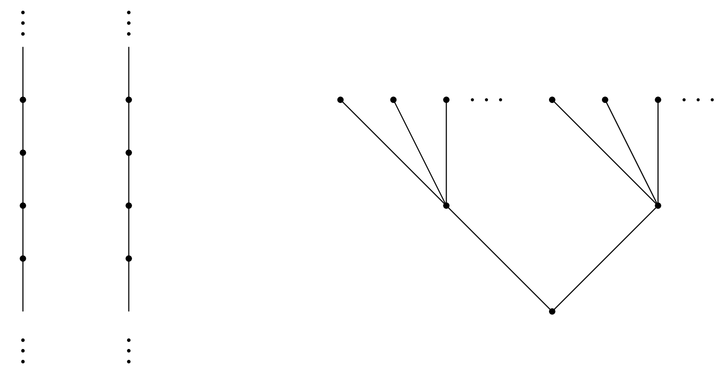

Figure 3.12: Example of Trees whose automorphism group is a dendromorphic group

Remark 3.2.7 *Let T be a finite tree.* Aut(T) *acts 1-transitively on the maximal elements of T if and only if T is fh-regular.*

Definition 3.2.8 *A group G is said to be a dendromorphic group if it is a Cartesian product of copies at least one of:*

1. $\mathbb{Z} \wr \mathbb{Z}_2$;
2. Sym(ω);
3. Sym$(\omega) \wr \mathbb{Z}_2$; *and*
4. *the automorphism group of a regular tree;*

Examples of the automorphism group of a regular tree include S_n, in particular $\mathbb{Z}_2$, and $(S_n \wr \mathbb{Z}_2)$.

Definition 3.2.9 *(Recall Definition 1.3.3) Let M be a CFPO, let $x \in M$ and let $G \subseteq$* Aut(M).

$$G(x) := \{y \in M \ : \ \exists g \in G \, g(x) = y\}$$

Theorem 3.2.10 *If T is a tree and there exists a $G \leq \mathrm{Aut}(T)$ such that $G \cong D_\infty$ then there exists an H such that $G \leq H \leq \mathrm{Aut}(T)$ and H is a dendromorphic group .*

Proof

Let T be a tree such that there is $G \leq \mathrm{Aut}(T)$ and $G \cong D_\infty$. We use the same presentation of D_∞ that we gave in Definition 3.2.1, so here σ and τ are automorphisms of T that generate G and satisfy the identities $\sigma^2 = 1$ and $\sigma\tau\sigma = \tau^{-1}$.

Let $t \in T$. How does σ constrain the structure of $G(t)$? If $t < \sigma(t)$ then $\sigma(t) < \sigma^2(t) = t$, which is a contradiction. Similarly $\sigma(t) < t$ also leads to a contradiction, so if $t \neq \sigma(t)$ then $t \parallel \sigma(t)$. Since $\sigma\tau \neq \tau\sigma$, we know that $\mathrm{supp}(\sigma) \cap \mathrm{supp}(\tau) \neq \emptyset$.

First suppose that $t \in T$ is such that $\{\phi|_{G(t)} \ : \ \phi \in G\} \not\cong D_\infty$. This means that there is some $n \in \mathbb{Z}$ and $i \in \{0, 1\}$ such that $\tau|^n_{G(t)}\sigma|^i_{G(t)} = id|_{G(t)}$.

1. If $\sigma|_{G(t)} = id|_{G(t)}$ then the identity $\sigma\tau = \tau^{-1}\sigma$ becomes $\tau = \tau^{-1}$ and we learn that $G(t) = \{t\}$ and $\mathrm{Aut}(G(t))$ is trivial.

2. If $\tau|^n_{G(t)} = id|_{G(t)}$ then $G(t)$ is a finite antichain and so $G(t)^+$ is a finite tree whose automorphism group acts transitively on its maximal elements, and by Remark 3.2.7 is fh-regular, so $\mathrm{Aut}(G(t))$ is the automorphism group of the fh-regular tree $G(t)^+$.

3. If $\sigma\tau^n|_{G(t)} = id|_{G(t)}$ then we can deduce that $\sigma|_{G(t)} = \tau^n|_{G(t)}$, and thus $\tau^{2n}|_{G(t)} = id|_{G(t)}$.

Now we suppose $t \in T$ is such that $\{\phi|_{G(t)} \ : \ \phi \in G\} \cong D_\infty$.

We now examine the possible action of τ on t. Since τ has infinite order, $\{\tau^n(t) \ : \ n \in \mathbb{Z}\}$ and $\{\tau^n\sigma(t) \ : \ n \in \mathbb{Z}\}$ are infinite. We now consider various cases to deduce the structure of $G(t)$.

Case 1: $t < \tau(t)$ or $t > \tau(t)$

Without loss of generality we assume that $t < \tau(t)$.

Since $t < \tau(t)$ we know that $\tau^m(t) < \tau^n(t)$ if and only if $m < n$, where $m, n \in \mathbb{Z}$. Suppose σ fixes one of these $\tau^m(t)$. Hence

$$\sigma\tau^m(t) = \tau^m(t)$$

but in D_∞ we know that $\tau^{-m}\sigma = \sigma\tau^m$, so

$$\begin{aligned} \tau^{-m}\sigma(t) &= \tau^m(t) \\ \sigma(t) &= \tau^{2m}(t) \end{aligned}$$

which means that σ maps t to $\tau^{2m}(t)$, which in this case is assumed to be greater than t, which we have already shown yields a contradiction, and thus σ does not fix any $\tau^n(t)$.

We suppose that there is an $n \in \mathbb{Z}$ such that $\tau^n(t) \leq t \wedge \sigma(t)$. We know that $\sigma\tau^n(t) \parallel \tau^n(t)$, which is the situation depicted in Figure 3.13.

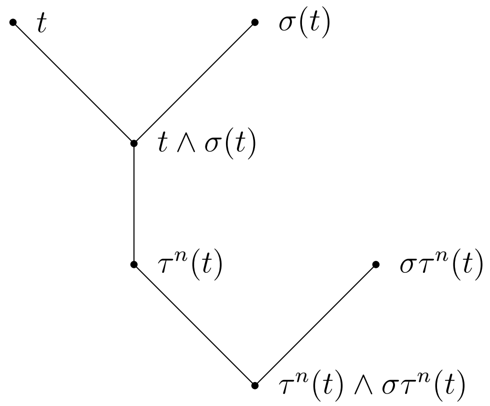

Figure 3.13: Deduced Structure of $G(t)$ if $(\tau^n(t) \leq t \wedge \sigma(t))$

However σ maps the pair $(t, \tau^n(t))$ to $(\sigma(t), \sigma\tau^n(t))$, so $\tau^n(t) < t$ implies that

$\sigma\tau^n(t) < \sigma(t)$, providing a contradiction.

So there is no n such that $\tau^n(t) \leq t \wedge \sigma(t)$ and then we are in the situation depicted in Figure 3.14.

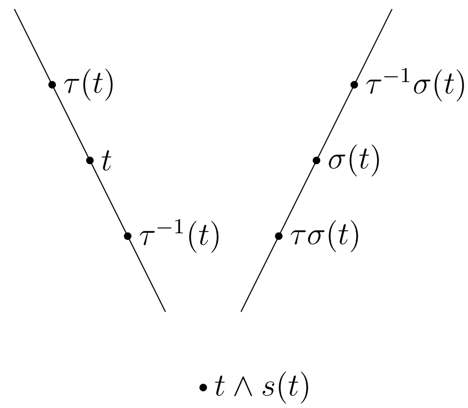

Figure 3.14: Deduced Structure of $G(t)$ if $(t \wedge \sigma(t) \leq \tau^i(t))$

The automorphism group of this structure is clearly $\mathbb{Z} \wr \mathbb{Z}_2$, and so

$$\mathrm{Aut}(G(t)) \cong \mathbb{Z} \wr \mathbb{Z}_2$$

Case 2: $t \parallel \tau(t)$ and $\tau^m(t) \wedge \tau^n(t) = \tau^{m'}(t) \wedge \tau^{n'}(t)$ for all $m \neq n, m' \neq n'$. We call denote common ramification point, $\tau^m(t) \wedge \tau^n(t)$ for $m \neq n$, by x. In other words, the $\tau^n(t)$ form an antichain, which ramifies from x.

If $x = \sigma(x)$ then the whole orbit of t is an infinite (as $G(t)$ is infinite) antichain above x, and thus Aut(T) is Sym(ω).

If $x \neq \sigma(x)$ then the whole orbit of t is two infinite (as both $\{\tau^n(t) \ : \ n \in \mathbb{Z}\}$ and $\{\tau^n\sigma(t) \ : \ n \in \mathbb{Z}\}$ are infinite) antichains , one ramifying from x, the other from $\sigma(x)$. In this case Aut$(T) \cong$ Sym$(\omega) \wr \mathbb{Z}_2$.

Case 3: $t \parallel \tau(t)$ and $\tau^m(t) \wedge \tau^n(t) \neq \tau^{m'}(t) \wedge \tau^{n'}(t)$ for some m, n, m', n'.

For $m \in \mathbb{N} \setminus \{0\}$ let $G_m := \{\sigma^i\tau^{mn} \ : \ i \in \{0, 1\} \ n \in \mathbb{Z}\}$. Note that $G_m \cong D_\infty$.

For brevity's sake, x_n will denote $\tau^{mn}(t) \wedge \tau^{m(n+1)}(t)$. Suppose that $x_i \neq x_{i+1}$ for all i. Note that $\tau^{mk}(x_n) = x_{n+k}$ because greatest lower bounds are preserved by automorphisms. For any $i \in \mathbb{Z}$ both x_i and x_{i+1} are below $\tau^{m(i+1)}(t)$, so $\{x_i : i \in \mathbb{Z}\}$ is linearly ordered and acted on by τ^m, showing that $\tau^m(x_i) < x_i$ or $\tau^m(x_i) > x_i$.

If $\{\phi|_{G_m(x_0)} : \phi \in G_m\} \ncong D_\infty$, then $G_m(x_0)$ is an antichain, but we have just established that $\tau(x_i) < x_i$ or $\tau(x_i) > x_i$, so $\{\phi|_{G_m(x_0)} : \phi \in G_m\} \cong D_\infty$, and we may now apply Case 1 to $G_m(x_0)$ and find that $\mathrm{Aut}(G_m(x_0)) \cong (\mathbb{Z} \wr \mathbb{Z}_2)$.

Since each $x_i \neq x_{i+1}$, we can deduce the structure depicted in Figure 3.15.

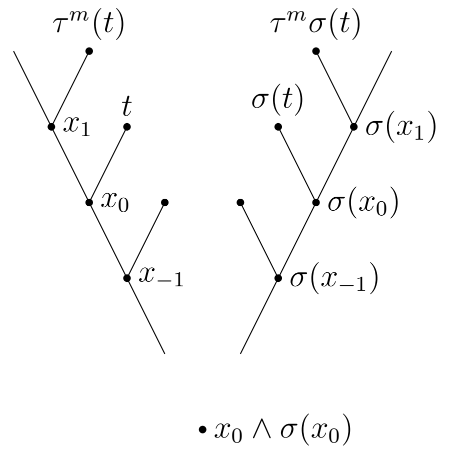

Figure 3.15: Deduced Structure needed for Case 3

Thus we see that $\mathrm{Aut}(G_m(t)) \cong (\mathbb{Z} \wr \mathbb{Z}_2)$. If we redefine $x_n := \tau^{mn+k}(t) \wedge \tau^{m(n+1)+k}(t)$ and repeat this argument, we see that $\mathrm{Aut}(G_m(\tau^k(t))) \cong (\mathbb{Z} \wr \mathbb{Z}_2)$

Let m_0 be the least element of the set

$$\{i = \mathrm{lcm}(n-m, n'-m') : \tau^m(t) \wedge \tau^n(t) \neq \tau^{m'}(t) \wedge \tau^{n'}(t)\}$$

Note that $\tau^{m_0 n}(t) \wedge \tau^{m_0(n+1)}(t) \neq \tau^{m_0(n+1)}(t) \wedge \tau^{m_0(n+2)}(t)$ for all n, so m_0 is in fact the least number such that $\mathrm{Aut}(G_{m_0}(t)) \cong (\mathbb{Z} \wr \mathbb{Z}_2)$.

$G(t)$ consists of $m_0 - 1$ copies of $G_{m_0}(t)$, which are preserved by σ, and τ acts cyclically on them, and indeed their least elements, which we call L. This gives us $\{\phi|_L : \phi \in G\} \ncong D_\infty$, and $\sigma|_L = \mathrm{id}|_L$, so L is trivial and $\mathrm{Aut}(G(t)) \cong (\mathbb{Z} \wr \mathbb{Z}_2)$

Therefore for all $t \in T$ the group $\mathrm{Aut}(G(t))$ is either trivial or :

1. $\mathbb{Z} \wr \mathbb{Z}_2$ (from Cases 1 and 3);
2. $\mathrm{Sym}(\omega)$ (from Case 2);
3. $\mathrm{Sym}(\omega) \wr \mathbb{Z}_2$ (from Case 2); or
4. the automorphism group of an fh-regular tree;

each of which is a dendromorphic group .

We pick one $t \in T$ such that $G(t) \neq \{t\}$, and let $s := \inf(G(t)^+)$. The next phase of this proof is to show that the additional automorphisms of $\mathrm{Aut}(G(t))$ extend to $B(s; G(t))$. We do this by addressing each of the possibilities in the above list individually.

Let $\lambda \in \mathrm{Aut}(G(t)) \setminus G$. We wish to extend λ to $B(s; G(t))$ and show that the group of the extensions of elements of $\mathrm{Aut}(G(t))$ is a dendromorphic group .

1. Suppose $\mathrm{Aut}(G(t)) \cong (\mathbb{Z} \wr \mathbb{Z}_2)$. Then λ is characterised by where it maps t and $\sigma(t)$. Let's suppose that $\lambda(t) = \tau^n(t)$ and $\lambda(\sigma(t)) = \tau^m\sigma(t)$. Then we define $\bar{\lambda}$ to be the following:
$$\bar{\lambda} : x \mapsto \begin{cases} \tau^n(x) & x \in B(s; t) \\ \tau^m(x) & x \in B(s; \sigma(t)) \end{cases}$$
If $\lambda(t) = \tau^n\sigma(t)$ and $\lambda(\sigma(t)) = \tau^m(t)$ then
$$\bar{\lambda} : x \mapsto \begin{cases} \tau^m\sigma(x) & x \in B(s; t) \\ \tau^n\sigma(x) & x \in B(s; \sigma(t)) \end{cases}$$

Thus we may extend λ to a unique element of Aut($(B(s;G(t)))$, so

$$\text{Aut}((B(s;G(t))) \cong (\mathbb{Z} \wr \mathbb{Z}_2)$$

2. Suppose Aut$(G(t)) \cong$ Sym(ω). If there is some $b \in G(t)$ such that $\sigma(b) = b$ and $\sigma|_{B(s;b)} \neq \mathrm{id}|_{B(s;b)}$ then there are two possible extensions of λ. If $x \in B(s,a)$ and $\tau^n(a) = \lambda(a) = \tau^m\sigma(a)$ then

$$\bar{\lambda}_0 : x \mapsto \tau^n(x)$$
$$\bar{\lambda}_1 : x \mapsto \tau^m\sigma(x)$$

Since each λ may be extended to two elements of Aut($(B(s;G(t)))$, we know that

$$\text{Aut}((B(s;G(t))) \cong (\mathbb{Z}_2 \times \text{Sym}(\omega))$$

Otherwise if $x \in B(s;a)$ and $\lambda(a) = \tau^n\sigma^i(t)$ then

$$\bar{\lambda} : x \mapsto \tau^n\sigma^i(x)$$

and we uniquely extend λ, showing

$$\text{Aut}((B(s;G(t))) \cong \text{Sym}(\omega)$$

3. Suppose Aut$(G(t)) \cong$ (Sym$(\omega) \wr \mathbb{Z}_2$). If $x \in B(s;a)$ and $\lambda(a) = \tau^n\sigma^i(t)$ then

$$\bar{\lambda} : x \mapsto \tau^n\sigma^i(x)$$

so we can uniquely extend λ, showing

$$\text{Aut}((B(s;G(t))) \cong (\text{Sym}(\omega) \wr \mathbb{Z}_2)$$

4. Suppose $G(t)^+$ is an fh-regular tree, and suppose that there is an $x \in B(s; G(t))$ such that $\{\phi|_{G(x)} : \phi \in G\} \cong D_\infty$. Clearly G preserves $B(s; G(t))$, so

$$G(x) \subseteq B(s; G(t))$$

Suppose that $x \in B(s; t)$. Then $\tau^n \sigma^i(x) \in B(s; \tau^n \sigma^i(t))$ for all $n \in \mathbb{Z}$ and $i \in \{0, 1\}$, therefore for all $y \in G(t)$

$$G(x) \cap B(s; y) \neq \emptyset$$

Rather than look at $\lambda \in \text{Aut}(G(t))$, we instead extend every $\mu \in \text{Aut}(G(x))$ to obtain a dendromorphic supergroup of G in $B(s, G(t))$.

Now we suppose that there is no $x \in B(s; G(t))$ such that $\{\phi|_{G(x)} : \phi \in G\} \cong D_\infty$. We will define by induction a family of sets that we will call X_k which will help us extend λ.

Let X_0 be the maximal subset of $B(s, G(t))$ such that for all $\phi, \psi \in G$

$$\phi|_{G(t)} = \psi|_{G(t)} \Rightarrow \phi|_{X_0} = \psi|_{X_0}$$

Let $x \in B(s; y)$ and let $\phi \in G$ be such that $\lambda(y) = \phi(y)$.

$$\bar{\lambda} : x \mapsto \phi(x)$$

Since all the possible ϕ agree, this map is a well-defined, unique extension of λ, so $\text{Aut}(X_0) \cong \text{Aut}(G(t)^+)$. If $X_0 = B(s; G(t))$ then we have extended λ to $B(s; G(t))$ and we are done.

Suppose that we have defined X_{k-1}, but $X_{k-1} \neq B(s; G(t))$. Let $x_k \in B(s, G(t)) \setminus$

X_{k-1}. Let X_k be the maximal subset of $B(s, G(x_1))$ such that for all $\phi, \psi \in G$

$$\phi|_{G(t)} = \psi|_{G(t)} \Rightarrow \phi|_{X_1} = \psi|_{X_1}$$

Again, $\mathrm{Aut}(X_k) \cong \mathrm{Aut}(G(x_k)^+)$ and if $X_k = B(s; G(t))$ then we have extended $\lambda \in \mathrm{Aut}(X_k)$ to $B(s; G(t))$ and we are done.

If $X_k \neq B(s; G(t))$ then we define $X := \bigcup_{k \in \mathbb{N}} X_k$. We know how to extend λ to X, so if we can show that:

(a) $X = B(s; G(t))$; and

(b) there is a regular tree F such that $\mathrm{Aut}(X) = \mathrm{Aut}(F)$;

then we will have shown that $\mathrm{Aut}(B(s, G(t)) \cong \mathrm{Aut}(F)$.

(a) For all k, the orbit $|G(x_k)| > |G(x_{k-1})|$, as there are $\phi, \psi \in G$ such that $\phi(x_{k-1}) = \psi(x_{k-1})$ but $\phi(x_k) \neq \psi(x_k)$, so the set $\{|G(x_k)| : k \in \mathbb{N}\}$ is unbounded.

If $y \in B(s; G(t)) \setminus X$ then for all k

$$\tau^{|G(x_k)|}(y) \neq y$$

so G acts as D_∞ on $G(y)$, and we have already seen how to extend λ to $B(s; G(t))$ in this case, so we may assume now that $X = B(s; G(t))$.

(b) Since X_k extends X_{k-1} and since s is the root of both $G(x_{k-1})^+$ and $G(x_k)^+$, we know that $G(x_k)^+$ is an extension of $G(x_{k-1})^+$. Therefore we consider the tree $F := \bigcup_{k \in \mathbb{N}} G(x_k)^+$.

Let $(s, y_1 \ldots)$ and $(s, z_1, \ldots)$ denote maximal chains of F. Since each $G(x_k)^+$ is an fh-regular tree, given any two maximal chains of F there is a partial automorphism from the initial k elements of the first to the initial k elements of the second. The union of all these partial automorphisms will be an automorphism of F, and thus Aut(F) acts transitively on every maximal chain, which is Condition 1 of Definition 3.2.6.

The initial section of every maximal chain of F finite, so every maximal chain is isomorphic to $\mathbb{N}$, Condition 2 of Definition 3.2.6.

If $y \in F$ then $y \in G(x_k)^+$ for some k, so the ramification order of any non-maximal element of F is at least 2 but finite, showing that F satisfies Condition 3 of Definition 3.2.6.

Finally, if $|F^{\leq y}| = |F^{\leq z}|$ then there is a k such that $y, z \in G(x_k)^+$ and $|(G(x_k)^+)^{\leq y}| = |(G(x_k)^+)^{\leq z}|$, so the fact that $G(x_k)^+$ is fh-regular implies that F satisfies Condition 3 of Definition 3.2.6, and is regular.

Therefore there is a regular tree F such that Aut$(B(s, G(t)) \cong$ Aut(F).

For any $t \in T$ let s_t be the root of $G(t)^+$. Consider the set

$$\mathcal{B} := \{B(s_t; G(t)) \ : \ |G(t)| \neq 1\} \cup \{\{t\} \ : \ |G(t)| \neq 1\}$$

Let H be the group of all automorphisms of T that fix every $B \in \mathcal{B}$ setwise.

$$H = \prod_{B \in \mathcal{B}} \text{Aut}(B)$$

Since the Cartesian product of dendromorphic groups is dendromorphic, H is also dendromorphic. We have already seen that G fixes every $B \in \mathcal{B}$ setwise, so $G \leq H$. $\square$

If you are familiar with automorphism groups as topological groups, you may have realised that in the proof of Theorem 3.2.10 we are essentially calculating the closure of the copy of D_∞. In Theorem 3.2.13 we will see that a CFPO is not treelike if and only if its automorphism group contains a closed copy of D_∞.

While describing this situation using the language of topological groups might have been more elegant, I prefer this approach as it makes it clear that these properties are recognisable from the abstract group.

3.2.2 D_∞ in CFPOs

Corollary 3.2.11 $\text{Aut}(\mathrm{Alt}) \ncong \text{Aut}(T)$ *for all trees* T.

Proof

$\text{Aut}(\mathrm{Alt}) \cong D_\infty$, so if $\text{Aut}(T) \cong \text{Aut}(\mathrm{Alt})$ then the whole automorphism group is a copy of D_∞, and so cannot be contained in a dendromorphic group . □

So we've established that D_∞ can occur as a subgroup of the automorphism group of a CFPO in a different way than it can as a subgroup of the automorphism group of a tree. The rest of this subsection is devoted to finding out how copies of D_∞ that aren't contained in a dendromorphic group can act on a CFPO.

Definition 3.2.12 *Let M be a CFPO. If $X \subseteq M$ then X^{cc}, the **connection closure** of X, is the following set*

$$\bigcup_{x,y \in X} \text{Path}\langle x, y\rangle$$

In particular, if $G \leq \text{Aut}(M)$ and $x \in M$ then this combines with the notation of Definition 1.3.3 to give:

$$G(x)^{cc} := \bigcup_{g,h \in G} \text{Path}\langle g(x), h(x)\rangle$$

Theorem 3.2.13 *Let M be a Rubin complete CFPO and let $G \leq \text{Aut}(M)$. If $G \cong D_\infty$ then either G is contained in a dendromorphic group or G acts on a copy of Alt in M, but not both.*

Proof

If M is a CFPO_n for some $n \in \mathbb{N}$ or a CFPO_ω then by Theorem 3.1.14, Corollary 3.1.15 and Theorem 3.1.15 there is a tree T such that Aut$(M) \cong$ Aut(T). Thus Theorem 3.2.10 shows that G is contained in a dendromorphic group and G cannot act on a copy of Alt, as M does not contain a copy of Alt. We now suppose that M is a connected CFPO_∞.

If G fixes $a \in M$ then $G \leq$ Aut$_a(M)$. By adding a colour predicate to M that only a realises, we find a CFPO with a fixed point whose automorphism group is Aut$_a(M)$. Since this CFPO has a fixed point it is treelike (Theorem 3.1.11), and Theorem 3.2.10 shows that there is a dendromorphic group X which is contained in Aut$_a(M)$ and contains G. Therefore if $M \setminus$ supp$(G) \neq \emptyset$ then G is contained in a dendromorphic group .

Now suppose that G has no fixed point and that $G(m)^{cc}$ is not a CFPO_∞ for any $m \in M$.

We can view the connected components of $M \setminus G(m)^{cc}$ as extended cones of elements of $G(m)^{cc}$. For all $a \in G(m)^{cc}$

$$C(a) := \{x \in M \setminus G(m)^{cc} \ : \ a \in \text{Path}\langle x, G(m)^{cc}\rangle\}$$

i.e. $C(a)$ is the union of all the extended cones of $M \setminus G(m)^{cc}$ that ramify from a. If $\phi \in$ Aut$(\langle G(m)^{cc}, \leq_M\rangle)$ does not extend to an automorphism of M then ϕ must map a to b but $C(a) \ncong C(b)$.

If for all CFPOs C such that $\exists a \in G(m)^{cc}\ C \cong C(a)$ we introduce a colour predicate P_C to $\langle G(m)^{cc}, \leq_M\rangle$ such that

$$\langle G(m)^{cc}, \leq_M\rangle \models P_C(a) \ \Leftrightarrow \ C(a) \cong C$$

Every automorphism of $\langle G(m)^{cc}, \leq_m, P_C\rangle$ is a restriction of an automorphism of M.

Each $G(m)^{cc}$ is G-invariant, as otherwise we would be able to map a path inside $G(m)^{cc}$ to one outside by an element of G, but this map must take the endpoints of this path with it, and these endpoints are elements of $G(m)^{cc}$.

We choose one $m \in M$. Since $G(m)^{cc}$ is not a CFPO$_\infty$, it is treelike. All of the extended cones that are contained in $M \setminus G(m)^{cc}$ are treelike if we fix the point in $G(m)^{cc}$ that they emanate from, so by replacing $G(m)^{cc}$ and the extended cones, we may find a tree T such that $G \leq \mathrm{Aut}(T) \leq \mathrm{Aut}(M)$, and so G is contained in a dendromorphic group .

So now suppose that $a \in M$ is such that $G(a)^{cc}$ is a CFPO$_\infty$. From such an a we define

$$b := \mathrm{Path}\langle a, \mathrm{Path}\langle \tau^{-1}(a), \tau(a)\rangle\rangle$$

(because $\mathrm{Path}\langle \tau^{-1}(b), b\rangle \cap \mathrm{Path}\langle b, \tau(b)\rangle = \{b\}$) and consider $G(b)^{-}$, the set of maximal and minimal points of $G(b)^{cc}$. If $\tau^{\mathbb{Z}}(b) = G(b)$ then $G(b)^{-}$ is a copy of Alt on which G acts.

If $\sigma(b) \notin \tau^{\mathbb{Z}}(b)$ then we consider G's action on

$$\mathrm{Path}\langle \tau^{\mathbb{Z}}(b), \tau^{\mathbb{Z}}(\sigma(b))\rangle$$

which, if non-empty, will be fixed pointwise by τ, and on which σ will have a fixed point, contradicting the assumption that G has no fixed points.

If $\mathrm{Path}\langle \tau^{\mathbb{Z}}(b), \tau^{\mathbb{Z}}(\sigma(b))\rangle$ is empty then we are in the situation depicted in Figure 3.17.

In Figure 3.17 c_0 is $\sigma(b)$ and $c_k := \tau^k(c_0)$, which forces $\sigma\tau^k(b)$ to be c_j for some j (whose relationship with k will be deduced shortly). Note that σ and τ satisfy the identity

$$\sigma\tau = \tau^{-1}\sigma$$

which implies the following equations:

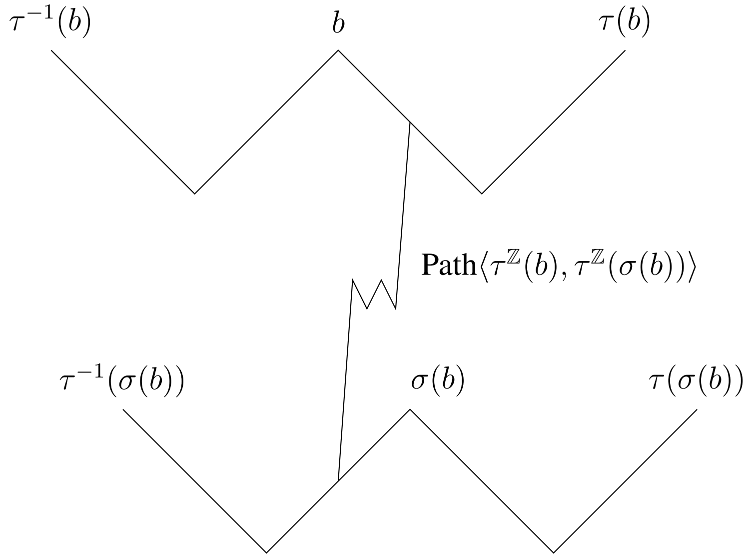

Figure 3.16: $\mathrm{Path}\langle\tau^{\mathbb{Z}}(b), \tau^{\mathbb{Z}}(\sigma(b))\rangle$

$$\begin{aligned} c_i &= \sigma\tau^j(b) \\ &= \tau^{-j}\sigma(b) \\ \tau^j(c_i) &= c_0 \\ c_{i+j} &= c_0 \end{aligned}$$

so $c_i = \sigma(\tau^{-i}(b))$. Let the d_i be the points fixed by σ on $\mathrm{Path}\langle\tau^i(b), c_i\rangle$ respectively. Then $\bigcup \mathrm{Path}\langle d_i, d_j\rangle$ is a copy of Alt which is acted on as desired.

Let $\mathcal{A}$ be the family of copies of Alt in M. We now show that if $\mathrm{Act}(A, \mathrm{Aut}(M))$ (the action of $\mathrm{Aut}(M)$ on A) is isomorphic to D_∞ for some $A \in \mathcal{A}$ then $\mathrm{Act}(A, \mathrm{Aut}(M))$

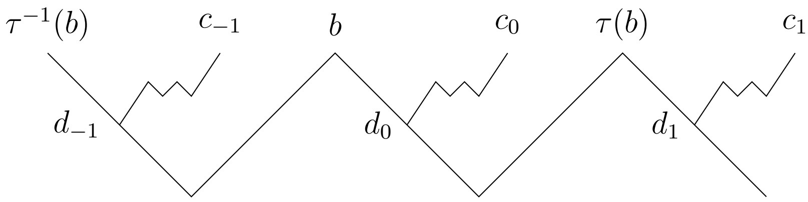

Figure 3.17: $\mathrm{Path}\langle\tau^i(b), c_i\rangle$

cannot be contained in a dendromorphic group , thus showing the exclusivity of the theorem.

If for some $A \in \mathcal{A}$ the action of Aut(M) is D_∞ then Act$(A, \text{Aut}(M)) \cong D_\infty$ and there is no dendromorphic group contained in Aut$_{\{A\}}(M)$ that contains Act$(A, \text{Aut}(M))$. Therefore if Act$(A, \text{Aut}(M))$ is contained in a dendromorphic group X, then

$$X \not\leq \text{Aut}_{\{A\}}(M)$$

In particular this implies that if $g \in X \setminus \text{Act}(A, \text{Aut}(M))$ then $g(A) \neq A$.

Let A^U be the set of upper points of A, enumerated by $\{\ldots, a_{-2}, a_0, a_2, \ldots\}$. Since $(A^U, \text{Act}(A, \text{Aut}(M)))$ is 1-transitive so is $(X(A^U), X)$, and so

$$(X(A^U), X) \cong (X(A^U), X_0)$$

where X_0 is one of the factors of X (i.e. Sym(ω), $\mathbb{Z} \wr \mathbb{Z}_2$ or $\prod S_n$). This X_0 cannot be Sym(ω) as then it would be possible to map the triple (a_{-2}, a_0, a_2) to (a_{-2}, a_2, a_0), but any map that does this has to change the length of Path$\langle a_{-2}, a_2 \rangle$, and so cannot be an isomorphism. This same argument prevents $X_0 \cong \prod S_n$.

Let σ be the infinite order generator of Act$(A, \text{Aut}(M))$ and τ be the finite order generator. Suppose $X_0 \cong \mathbb{Z} \wr \mathbb{Z}_2$, generated by α, β and γ, where α and β have infinite order and γ has finite order. Since Act$(A, \text{Aut}(M))$ contains an element of finite order, both supp(α) and supp(β) must have a non-empty intersection with A^U.

Since α, β and γ generate X and either preserve or switch supp(α) and supp(β), every member of Act$(A, \text{Aut}(M))$ either preserves or switches supp(α) and supp(β). So both supp$(\alpha) \cap A^U$ and supp$(\beta) \cap A^U$ cannot both be singletons, as only the identity will preserve supp$(\alpha) \cap A^U$ and supp$(\beta) \cap A^U$ and no member of Act$(A, \text{Act}(M))$ will swap them. Since supp$(\alpha) \cap A^U$ is not a singleton, the action on it determines the action on the whole of A^U, and so α and β cannot act independently. So X_0 cannot be isomorphic to $\mathbb{Z} \wr \mathbb{Z}_2$. □

Corollary 3.2.14 *Let M be a CFPO. If there is an $A \subseteq M$ and a $G \leq \mathrm{Aut}(M)$ such that:*

1. *A is a copy of Alt;*
2. *$G \cong D_\infty$; and*
3. *G acts on A.*

then M is not treelike.

Proof
If M is treelike then Theorem 3.2.10 shows that G is contained in a dendromorphic group, but Theorem 3.2.13 shows that this is impossible. □

3.3 CFPOs in Model Theory

The section promised all those pages ago in the introduction is finally here!

The theory of trees is known to have certain model theoretic properties. Parigot showed in 1982 that the theory of trees is NIP, and classified the stable ones [18], while Simon showed in 2011 that the theory of trees is inp-minimal [34]. The observations that have been made in this section give an easy method for extending these results to the theory of CFPOs.

3.3.1 NIP and Trees

Definition 3.3.1 *A formula $\phi(\bar{x}, \bar{y})$ is said to have the* ***independence property*** *(for a complete theory T) if in every model M of T there is, for each $n < \omega$, a family of tuples $\bar{b}_0, \bar{b}_1, \dots \bar{b}_{n-1}$ such that for every $I \subseteq \{0, 1, \dots n-1\}$ there is some tuple $\bar{a} \in M$ such that*

$$M \models \phi(\bar{a}, \bar{b}_i) \Leftrightarrow i \in I$$

T is said to be ***NIP*** *if no formula in T has the independence property.*

Note that if T is interpretable in S then if ϕ has the independence property for T then the interpretation of ϕ has the independence property for S. This means that if T is interpretable in S and S is NIP, then T is NIP.

The 'headline' result of [18] does not mention NIP.

Theorem 3.3.2 (Parigot, Theorem 2.6 of [18]) *A type over a tree never has more than $2^{\aleph_0}$ coheirs.*

'Coheirs' were defined by Poizat, appearing in [19] in 1981, the year before Parigot's paper was published. If you wish to read the proof of this theorem, but find Poizat's French too daunting, then I recommend the seminar notes of Casanovas [3], which are in English. I am not aware of any publicly available English translation or account of Parigot's paper.

Definition 3.3.3 (Poizat, [19]) *Let M, N be models such that $M \prec N$. Let $p(x) \subseteq q(x)$ where $q \in S_1(N)$ and $p \in S_1(M)$. We say that q is a coheir of p if q is finitely satisfiable in M.*

Theorem 3.3.4 (Poizat, [19]) *Let T be a theory.*

1. *If T has the NIP then for all M such that $T \models M$ and $|M| = \lambda \geq |T|$, for all $p \in S_1(M)$ there are at most 2^λ coheirs of p.*

2. *If T has the IP then for every $\lambda \geq |T|$ there is an M such that $T \models M$ and $|M| = \lambda \geq |T|$, and there is $p \in S_1(M)$ such that p has 2^{2^λ} coheirs.*

Parigot's results do not stop with trees, however. He extends to 'arborescent' structures, defined by Schmerl.

Definition 3.3.5 (Schmerl [28]) *Let $\mathcal{L} = \langle R_0, \ldots, R_{m-1}, U_0, \ldots, U_{n-1} \rangle$ be a finite language where each R_i is a binary predicate and each U_i is a unary predicate.*

Let $(x, y) \equiv (u, v)$ by the following quaternary formula:

$$x \neq y \wedge u \neq v \wedge \bigwedge_{i<m} ((R_i(x, y) \leftrightarrow R_i(u, v)) \wedge (R_i(y, x) \wedge R_i(v, u)))$$

Let M be an $\mathcal{L}$-structure. M is said to be ***arborescent*** *if for all finite $B \subseteq M$, if $|B| \geq 2$ then there are distinct $a, b \in B$ such that if $c \in B \setminus \{a, b\}$ then $(a, c) \equiv (b, c)$*

Finitely coloured trees are examples of arborescent structures.

Proposition 3.3.6 (Parigot, Corollary 2.8 of [18]) *All arborescent structures are NIP.*

3.3.2 inp-minimality and Trees

Definition 3.3.7 (Shelah, Definition 7.3 of [32]) *An independence pattern (an inp-pattern) of length κ is a sequence of pairs $(\phi^\alpha(x, y), k^\alpha)_{\alpha<\kappa}$ of formulas such that there exists an array $\langle a_i^\alpha \,:\, \alpha < \kappa\,, i < \lambda \rangle$ such that:*

- *Rows are k^α-inconsistent: for each $\alpha < \kappa$, the set $\{\phi^\alpha(x, a_i^\alpha) : i < \lambda\}$ is k^α-inconsistent,*
- *Paths are consistent: for all $\eta \in \lambda^\kappa$, the set $\{\phi^\alpha(x, a^\alpha_{\eta(\alpha)}) : \alpha < \kappa\}$ is consistent.*

Note that if M is interpretable in N then any independence pattern in M is also an independence pattern of N.

Definition 3.3.8 (Goodrick [13]) *A theory is inp-minimal if there is no inp-pattern of length two in a single free variable.*

Theorem 3.3.9 (Simon, Proposition 4.7 of [34]) *If $\langle T, \leq, C_i\rangle$ is a coloured tree then $\mathrm{Th}(\langle T, \leq, C_i\rangle)$ is inp-minimal.*

3.3.3 CFPOs

How can we apply these results to CFPOs?

Let M be a CFPO with connected components A_i, indexed by I. For each A_i, pick an $a_i \in A_i$ and introduce a new unary predicate A such that

$$M \models A(x) \Leftrightarrow \exists i \in I\ x = a_i$$

Since we are adding an additional symbol to the language $Th(\langle M, \leq_M\rangle)$ can be interpreted in $Th(\langle M, \leq_M, A\rangle)$ simply by forgetting A.

a_i is a fixed point of every $\langle A_i, \leq_M, A\rangle$ so we may invoke Remark 3.1.19 to note that $Th(\langle M, \leq_M, A\rangle$ is interpretable in $Th(\mathcal{T})$.

Therefore every CFPO is interpretable in an NIP, inp-minimal theory, and hence is NIP and inp-minimal.

This shows that if a property that is closed under taking an interpretation is possessed by the theory of coloured trees, then it is possessed by the CFPOs, but the interpretation here is of a special form. If we are allowed to fix points in a CFPO, we are essentially handling a tree, thus I expect any property of the coloured trees that allows reference to a set of parameters to also be possessed by the CFPOs.

Chapter 4

CFPOs with Transitivity Conditions

This chapter is concerned with the reconstruction of a certain class of CFPOs which fulfil some transitivity assumptions. These assumptions guarantee that the automorphism groups are rich enough to use the methods employed by Shelah in [31] and [30], and by Shelah and Truss in [33], for reconstructing the symmetric groups of cardinals and their quotients as permutation groups from them as abstract groups.

Throughout this chapter we will not assume that the CFPOs in question are Rubin complete, but we will assume that they are path complete.

4.1 Transitivity

Definition 4.1.1 *We say that CFPO M is **1-transitive** if for all $x, y \in M$ there exists a $\varphi \in \mathrm{Aut}(M)$ such that $\varphi(x) = y$.*

Definition 4.1.2 *We say that a CFPO M is **cone transitive** if it is 1-transitive and if C and D are cones emanating from point x in the same direction then there exists a $\varphi \in \mathrm{Aut}(M)$ such that $\varphi(C) = D$.*

Since every element may be sent to any other, if M is 1-transitive then M is monochromatic. M is cone transitive implies that M is one-transitive, so all cone transitive CFPOs are monochromatic.

Cone transitive CFPOs are the arena for an interpretation inspired by Shelah and Truss' work. Unfortunately, there are very few Rubin complete cone transitive CFPOs. However, these methods still work when we do not have Rubin completeness, so for this chapter we drop the assumption that M is Rubin complete.

We require one additional assumption before we begin our interpretation.

Definition 4.1.3 *The **upwards ramification order** of x in M, written as $Ro\uparrow(x)$, is the number of cones above x.*

*The **downwards ramification order** of x in M, written as $Ro\downarrow(x)$, is the number of cones below x.*

Proposition 4.1.4 *If M is 1-transitive then for all x and y*

$$Ro\uparrow(x) = Ro\uparrow(y) \text{ and } Ro\downarrow(x) = Ro\downarrow(y)$$

Proof

Any automorphism that maps x to y also maps the cones above x to the cones above y. The same is true for the cones below. □

Definition 4.1.5 *Let M be 1-transitive. The **upwards** (resp. **downwards**) **ramification order** of M, written as $ro\uparrow(M)$ (resp. $ro\downarrow(M)$), is equal to $Ro\uparrow(x)$ (resp. $Ro\downarrow(x)$) for some x.*

To get a sufficiently rich automorphism group we must also assume that both $ro\uparrow(M)$ and $ro\downarrow(M)$ are at least 5.

Definition 4.1.6 *Let K_{Cone} be the class of cone transitive CFPOs such that*

$$5 \leq ro\uparrow(M) \leq ro\downarrow(M)$$

for all $M \in K_{Cone}$.

We have made the additional assumption that $ro \uparrow (M) \leq ro \downarrow (M)$ because the reverse ordering of M always has the same automorphism group as M, but is not always isomorphic to M.

Definition 4.1.7 *Let $m, n \in \mathbb{N}$ be such that $5 \leq n \leq m$ and let L be a 1-transitive linear order. $X(n, m, L)$ is defined to be the CFPO where every point has upwards ramification order n, downwards ramification order m, and every maximal chain is isomorphic to L.*

We can use a back-and-forth argument to establish that if two CFPOs could possibly be represented by $X(n, m, L)$ then they are isomorphic, so $X(n, m, L)$ is well-defined.

Remark 4.1.8 $X(n, m, L) \in K_{Cone}$.

Before jumping into the interpretation, here are a few observations about the properties of the elements of K_{Cone}.

Definition 4.1.9 *Let L be a linear order and let $a \in L$. A neighbourhood of a is a convex subset of L that contains a. A discrete (resp. dense) neighbourhood is one where the convex set is discrete (resp. dense).*

Lemma 4.1.10 *Let $M \in K_{Cone}$ and let L be a maximal chain of M. If L has a discrete (resp. dense) neighbourhood then every maximal chain of M is discrete (resp. dense).*

Proof

Let $x \in M$ have a discrete neighbourhood in a copy of L that passes through it. This means that the cones above and below x that contain L have least and greatest elements respectively. Since M is cone transitive, all the cones that emanate from x have least or greatest elements. M is 1-transitive, so every element of x only has successors and predecessors, and M contains no dense chains. Therefore every maximal chain is discrete.

Similarly, if L has a dense neighbourhood, then there is a point which has no successors or predecessors, and so every maximal chain of M is dense. □

Proposition 4.1.11 *If $M \in K_{Cone}$ is Rubin complete then all the maximal chains of M are discrete and Dedekind complete.*

Proof

If M is Rubin complete then all of the cones above and below any point have least and greatest elements respectively. Hence all the maximal chains are discrete. Suppose that there is an L, a maximal chain of M which is not Dedekind complete. Let I be a cut of L. Since M is Rubin complete, the ideal $\{x \in M \ : \ x \leq_M m \text{ for some } m \in I\}$ has a maximal element, which we call a.

There is an $l \in L$ such that $\{x \in M \ : \ x \leq_M m \text{ for some } m \in I\} \leq l$, and so $a \leq l$. Since L is maximal, $a \in L$, giving a contradiction. □

Proposition 4.1.12 *Let $M \in K_{Cone}$. If M is Rubin complete then $M \cong X(n, m, \mathbb{Z})$ for some cardinal n, m such that $5 \leq n \leq m$.*

Proof

Let M be Rubin complete. By the definition of K_{Cone} there are cardinals n and m such that $ro\uparrow(M) = m$ and $ro\downarrow(M) = n$ and $5 \leq n \leq m$.

Proposition 4.1.11 shows that all the maximal chains are discrete. Let L be one of these discrete maximal chains of M. Since M is 1-transitive, there are maps taking a point to any of its successors and predecessors, and so any $x \in L$ is contained in a copy of $\mathbb{Z}$.

Thus $L \cong I \times \mathbb{Z}$ for some linear order I. This is only Dedekind complete if I is the one-element linear order.

Therefore every maximal chain of M is isomorphic to $\mathbb{Z}$, and hence M is of the form $X(n, m, \mathbb{Z})$. $\square$

This means that the structure of the cone-transitive Rubin-complete CFPOs is somewhat restricted, so I have developed the methods used in the following sections to work in path complete cone transitive CFPOs, which is a wider class.

4.2 Reconstructing Betweenness in K_{Cone}

All the CFPOs we are handling are from K_{Cone}, so are path complete, cone transitive, both $ro\uparrow(M)$ and $ro\downarrow(M)$ are greater than 4, and $ro\uparrow(M) \leq ro\downarrow(M)$.

We are now ready to give the interpretation of M inside Aut(M). The interpretation uses pairs of subgroups isomorphic to A_5, the alternating group on five elements, to represent the points of the CFPO. A_5 is chosen because it is the smallest non-abelian finite simple group.

Since we are now trying to find an interpretation of M in Aut(M), we will be seeing a lot of long formulas. I recommend tearing out the appendix so you don't have to keep flicking back and forth. It repeats the definitions and gives the intended meaning of the abbreviations, which may also illuminate matters.

Definition 4.2.1 *Let $\bar{f}$, $\bar{f}_0$, $\bar{f}_1$, $\bar{g}$, $\bar{g}_0$ and $\bar{g}_1$ be 60-tuples from* $\mathrm{Aut}(M)$.

1. *For all $\phi \in \mathrm{Aut}(M)$, if ϕ preserves X set-wise then $\phi|^X$, the **restriction** of ϕ to X, is the map obtained by taking the union of the standard restriction, which is a partial automorphism, and the restriction of the identity to $M \setminus X$. Symbolically:*

   $$\phi|^X := \phi|_X \cup id|_{M\setminus X}$$

   This is only a total automorphism in certain circumstances which crop up often in this chapter.

2. *(Definition 1.3.3) If $\bar{f} \in \mathrm{Aut}(M)$ and $x \in M$ then*

   $$\bar{f}(x) := \{y \in M \ : \ \exists f \in \bar{f}\ f(x) = y\}$$

3. *$A_5(\bar{f})$ is the formula that states "$\bar{f}$ satisfies the elementary diagram of A_5". This is the conjunction of formulas of the form $f_i f_j = f_k$ and $f_i f_j \neq f_k$.*

4. $\mathrm{Comm}(\bar{f}, \bar{g})$ *is the formula*

   $$\mathrm{Alt}_5(\bar{f}) \wedge \mathrm{Alt}_5(\bar{g}) \wedge \bigwedge_{\substack{f_i \in \bar{f} \\ g_j \in \bar{g}}} (f_i g_j = g_j f_i)$$

5. *if $\bar{f}$ and $\bar{g}$ satisfy Alt_5 and $\phi \in \mathrm{Aut}(M)$ is any automorphism then*

   $$\begin{array}{rclcrcl} \bar{f} * \bar{g} & := & (f_i g_i) & \quad & \bar{f}^{\phi} & := & (\phi f_i \phi^{-1}) \\ \phi * \bar{f} & := & (\phi f_i) & & \bar{f} * \phi & := & (f_i \phi) \end{array}$$

6. $\mathrm{Indec}(\bar{f})$ *is the formula*

$$\neg\exists\bar{g},\bar{h}(\bar{g}*\bar{h}=\bar{f}\wedge\mathrm{Comm}(\bar{g},\bar{h}))$$

7. $\mathrm{Disj}(\bar{f},\bar{g})$ *is the formula*

$$\mathrm{Indec}(\bar{f})\wedge\mathrm{Indec}(\bar{g})\wedge\mathrm{Comm}(\bar{f},\bar{g})$$

8. $[\text{supp}(\bar{f})\sqsubseteq\text{supp}(\bar{g})]$ *is the formula*

$$\begin{array}{ll}\mathrm{Indec}(\bar{f})\wedge\mathrm{Indec}(\bar{g})\wedge\neg\text{disj}(\bar{f},\bar{g}) & \wedge\\ \neg\exists\phi[\neg\text{disj}(\bar{f}^{\phi},\bar{f})\wedge\text{disj}(\bar{g}^{\phi},\bar{g})] & \wedge\\ \neg\exists\phi(\bar{f}^{\phi}=\bar{f}\wedge\bar{g}^{\phi}\neq\bar{g}) & \wedge\end{array}$$

9. $[\text{supp}(\bar{g})\sqsubset\text{supp}(\bar{f})]$ *is the formula*

$$[\text{supp}(\bar{g})\sqsubseteq\text{supp}(\bar{f})]\wedge\neg[\text{supp}(\bar{f})\sqsubseteq\text{supp}(\bar{g})]$$

10. $\mathrm{SamePD}(\bar{f},\bar{g})$ *(Same Point and Direction) is the formula*

$$\forall\bar{h}([\text{supp}(\bar{h})\sqsubset\text{supp}(\bar{f})]\leftrightarrow[\text{supp}(\bar{h})\sqsubset\text{supp}(\bar{g})])$$

11. $\mathrm{RepPoint}(\bar{f}_0,\bar{f}_1)$ *is the formula*

$$\text{disj}(\bar{f}_0,\bar{f}_1)\wedge\forall\bar{g}\exists\bar{h}\neg\text{disj}(\bar{g},\bar{h})\wedge\begin{pmatrix}\mathrm{SamePD}(\bar{f}_0,\bar{h}) & \vee\\ \mathrm{SamePD}(\bar{f}_1,\bar{h}) & \end{pmatrix}$$

12. $\mathrm{EqRepPoint}(\bar{f}_0, \bar{f}_1; \bar{g}_0, \bar{g}_1)$ *is the formula*

$$\mathrm{RepPoint}(\bar{f}_0, \bar{f}_1) \wedge \mathrm{RepPoint}(\bar{g}_0, \bar{g}_1) \wedge$$
$$(\mathrm{SamePD}(\bar{f}_0, \bar{g}_0) \wedge \mathrm{SamePD}(\bar{f}_1, \bar{g}_1)) \vee (\mathrm{SamePD}(\bar{f}_0, \bar{g}_1) \wedge \mathrm{SamePD}(\bar{f}_1, \bar{g}_0))$$

4.2.1 The Domain of the Interpretation

Lemma 4.2.2 *If* $\mathrm{Aut}(M) \models \mathrm{Alt}_5(\bar{g})$ *holds then* $\bar{g}$ *fixes at least one point.*

Proof

Every transitive action of Alt_5 is isomorphic to its action on a coset space $[\mathrm{Alt}_5 : H]$ for some $H \leq A_5$. The subgroups of A_5 can have orders 1, 2, 3, 4, 5, 6, 10, 12 and 60 and hence the possible values for $|\bar{g}(x)|$ are 60, 30, 20, 15, 12, 10, 6, 5 and 1.

Every element of $\bar{g}$ has finite order so for all x we know that $\bar{g}(x)$ is an antichain. Pick one x such that $|\bar{g}(x)| \neq 1$ (possible, as A_5 is not the identity), so there are g_i that act non-trivially.

Let

$$S := \bigcup_{x_i, x_j \in \bar{g}(x)} \mathrm{Path}\langle x_i, x_j \rangle^-$$

The definition of $\mathrm{Path}\langle x, y \rangle^-$ can be found in Lemma 3.1.12. Since each $\mathrm{Path}\langle x_i, x_j \rangle^-$ is finite and $\bar{g}(x)$ is finite, S is also finite, and therefore must be a CFPO_n for some n.

In Subsection 3.1.1 we showed that there was a tree T such that $\mathrm{Aut}(S) \cong_P \mathrm{Aut}(T)$. The root of T is fixed by every automorphism of S, and hence by every element of $\bar{g}$. □

Lemma 4.2.3 *If* $\mathrm{supp}(\bar{f})$ *and* $\mathrm{supp}(\bar{g})$ *are disjoint then* $\mathrm{Aut}(M) \models \mathrm{Comm}(\bar{f}, \bar{g})$.

Definition 4.2.4 *Let $\bar{f}$ be such that* Aut$(M) \models A_5(\bar{f})$ *and C_i are the connected components of* supp$(\bar{f})$. *We say that $E \subseteq M$ is an extended connected component of* supp$(\bar{f})$ *if:*

1. *E contains a union of C_i and at most one element from $M \setminus$ supp$(\bar{f})$, which we call e;*

2. *if $C \subseteq E$ then $\bar{f}(C) \subseteq E$;*

3. *if e exists then E contains at least two connected components, C_0 and C_1, and $\{e\} =$ Path$\langle C_0, C_1\rangle$; and*

4. *if D satisfies conditions 1-3 and $E \cap D \neq \emptyset$ then $E \subseteq D$.*

Lemma 4.2.5 *If X is an extended connected component of some* supp$(\bar{f})$ *then* Path$\langle X, M \setminus X\rangle$ *is a singleton.*

Proof

Condition 2 of Definition 4.2.4 shows that X is preserved setwise by $\bar{f}$, so by Lemma 3.1.12 there are x and y such that Path$\langle X, M \setminus X\rangle =$ Path$\langle x, y\rangle$. Both x and y are fixed by $\bar{f}$, so $x, y \in M \setminus X$.

Suppose one of the extended cones above x (Definition 1.3.19) intersects X and one of the cones below x intersects X. Let U be the upwards extended cone and let D be the downwards extended cone. $\bar{f}(U) \cap \bar{f}(D) = \emptyset$, as $\bar{f}$ fixes x, so $\bar{f}(U) \cap X$ satisfies conditions 1-3, and does not contain X giving a contradiction.

Therefore we may assume that X is contained in extended cones above x. Let y_0 and y_1 lie in different extended cones below x. The definition of extended cone guarantees that Path$\langle x, y_0\rangle \cap$ Path$\langle x, y_1\rangle = \{x\}$, so Path$\langle x, y\rangle = \{x\}$. □

Lemma 4.2.6 *Let $\bar{f}$ satisfy* Alt_5. *If we partition* supp$(\bar{f})$ *into two collections of extended connected components, which we will call X and Y, then $(f_i|^X)$ and $(f_i|^Y)$ satisfy* Alt_5.

Proof

First of all, we must show that this lemma makes sense, i.e. $\bar{f}$ preserves the extended connected components of supp$(\bar{f})$ set-wise and therefore $f_i|^X$ and $f_i|^Y$ are automorphisms.

Since the supports of $(f_i|^X)$ and $(f_i|^Y)$ are disjoint, $\mathrm{Comm}(f_i|^X), (f_i|^Y))$ holds. We consider the positive statements of the formula A_5 that $\bar{f}$ satisfies, which are of the form $f_i f_j = f_k$.

Since $f_i = f_i|^X f_i|^Y$ for all i we can deduce that

$$f_i|^X f_j|^X f_i|^Y f_j|^Y = f_k|^X f_k|^Y$$

and since $(f_\alpha|^X f_\alpha|^Y)|^X = f_\alpha|^X$ we conclude that $(f_i|^X)$ and $(f_i|^Y)$ satisfy all the positive statements of Alt_5. We now consider the negative statements, those of the form $f_i f_j \neq f_k$.

Repeating the argument for the positive statements allows us to deduce

$$f_i|^X f_j|^X f_i|^Y f_j|^Y \neq f_k|^X f_k|^Y$$

which only guarantees that at least one of $f_i|^X f_j|^X \neq f_k|^X$ or $f_i|^Y f_j|^Y \neq f_k|^Y$. Without loss of generality we assume that $f_i|^Y f_j|^Y \neq f_k|^Y$. In A_5 there is the positive statement $f_i f_j = f_l$ for some $f_l \neq f_k$, so if $f_i|^X f_j|^X = f_k|^X$, then $f_k|^X = f_l|^X$.

We define the homomorphism

$$\Phi : \begin{cases} \bar{f} & \rightarrow & (f_i|^X) \\ f_i & \mapsto & f_i|^X \end{cases}$$

$\Phi^{-1}(id)$ is a normal subgroup of $\bar{f}$. We have just found distinct f_k and f_l such that $f_k|^X = f_l|^X$, so since A_5 is simple, this means that $f_i|^X = id$ for all $f_i \in \bar{f}$, contradicting the fact that $X \cap \text{supp}(\bar{f}) \neq \emptyset$.

Therefore if $A_5(\bar{f})$ then $A_5((f_i|^X))$ and $A_5((f_i|^Y))$. $\Box$

Lemma 4.2.7 *If* $\bar{g} * \bar{h} = \bar{f}$ *and* $\text{Comm}(\bar{g}, \bar{h})$ *then* $\text{supp}(\bar{g}), \text{supp}(\bar{h}) \subseteq \text{supp}(\bar{f})$.

Proof

Suppose there is an x such that $g_j(x) \neq x$ for some j and $f_i(x) = x$ for all i. Therefore

$$\begin{array}{lrcl} \forall i & h_i g_i(x) & = & x \\ \forall i & g_i(x) & = & h_i^{-1}(x) \end{array}$$

There are g_j and g_k such that $g_j g_k(x) \neq g_k g_j(x)$ as A_5 is non-abelian, and if we substitute h_j^{-1} for g_j we find that

$$h_j^{-1} g_k(x) \neq g_k h_j^{-1}(x)$$

contradicting $M \models \text{Comm}(\bar{g}, \bar{h})$. $\Box$

Lemma 4.2.8 *Let X and Y be extended connected components of* $\text{supp}(\bar{f})$ *and* $\text{supp}(\bar{g})$. *If* $\text{Comm}(\bar{f}, \bar{g})$ *and* $|X \cap Y| \geq 1$ *then either $X \subseteq Y$ or $Y \subseteq X$.*

Proof

Let $\{x\} = \text{Path}\langle X, M \setminus X\rangle$ and $\{y\} = \text{Path}\langle Y, M \setminus Y\rangle$. These are singletons by Lemma 4.2.5. Suppose $X \nsubseteq Y$ and $Y \nsubseteq X$.

First suppose that $x = y$. This means that Path$\langle X, Y\rangle = \{x\}$, and that X and Y are entirely contained in the upwards and downwards extended cones of x, as illustrated in Figure 4.1.

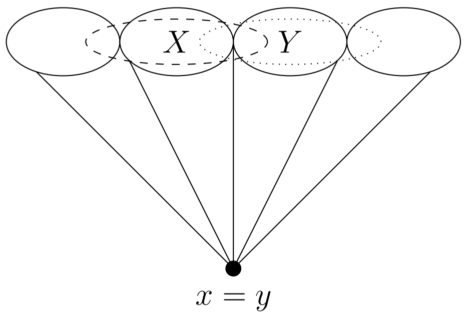

Figure 4.1: If $x = y$ in Lemma 4.2.8

Recall Definition 4.2.4, and note that $\bar{g}(X \cap Y)$ satisfies conditions 1 and 3 because both X and Y do, and by definition it satisfies condition 2. Therefore $Y \subseteq \bar{g}(X \cap Y)$. Thus if $\bar{g}(X \cap Y) \subseteq X$ then $Y \subseteq X$ and we are done. Similarly if $\bar{f}(X \cap Y) \subseteq Y$ then $X \subseteq Y$ and we are done.

We now suppose that there is a $z \in X \cap Y$ such that $\bar{f}(z) \nsubseteq Y$ and $\bar{g}(z) \nsubseteq X$. Let C_z be the extended cone of x that contains z. We consider the action of $\bar{f}$ and $\bar{g}$ on the set $\bar{f}(C_z) \cup \bar{g}(C_z)$.

Let $f_i \in \bar{f}$ map C_z into $X \setminus Y$ and let g_j map C_z into $Y \setminus X$. Then

$$f_i g_j(C_z) = g_j(C_z) \text{ and } g_j f_i(C_i) = f_i(C_z)$$

contradicting the assumption that Aut$(M) \models \mathrm{Comm}(\bar{f}, \bar{g})$. This is depicted in Figure 4.2.

Now suppose that $x \neq y$. Suppose $x \notin Y$ and $y \in X$, and let $z \in Y$. By definition, $y \in$ Path$\langle z, x\rangle$, and since x is an endpoint of that path, Path$\langle z, x\rangle \subseteq X$, and so $z \in X$. This is depicted in Figure 4.3.

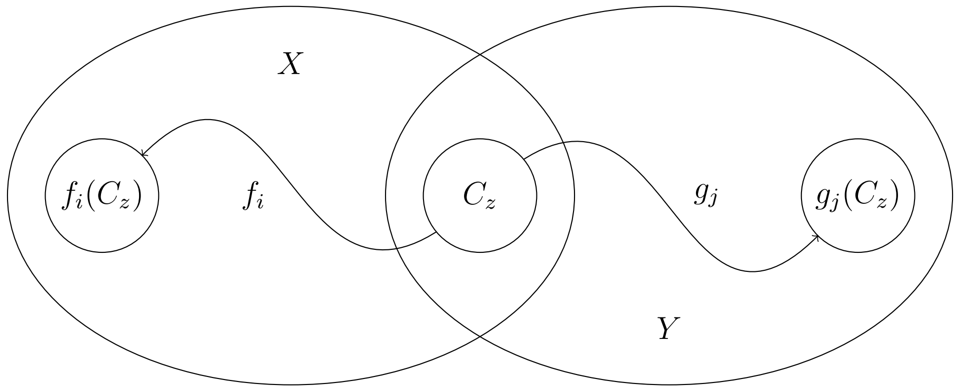

Figure 4.2: Images of C_z

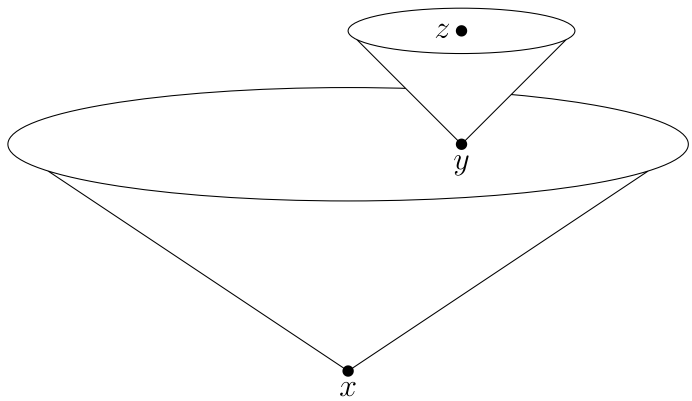

Figure 4.3: $x \notin Y$ and $y \in X$

If both $x \notin Y$ and $y \notin X$ then Path$\langle x, y\rangle \subseteq M \setminus (X \cup Y)$. This is depicted in Figure 4.4. Let $z \in Y$. By definition $y \in$ Path$\langle x, z\rangle$ and since Path$\langle x, y\rangle \nsubseteq X$, we know that $z \notin X$. Similarly, if $z \in X$ then $z \notin Y$, contradicting the assumption that $X \cap Y \neq \emptyset$.

We therefore suppose that $x \in Y$ and $y \in X$.

$x \in$ Path$\langle y, f_i(y)\rangle$ for any f_i, as otherwise X will not be an extended connected component.

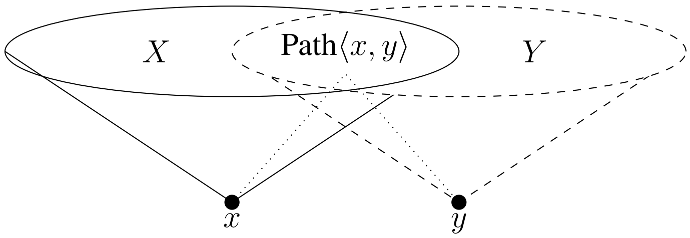

Figure 4.4: $x \notin Y$ and $y \notin X$

Path-betweenness is preserved by automorphisms, so

$$g_j(x) \in \text{Path}\langle g_j(y), g_j f_i(y)\rangle$$

and $\bar{f}$ and $\bar{g}$ commute, and y is fixed by $\bar{g}$, hence

$$g_j(x) \in \text{Path}\langle y, f_i(y)\rangle$$

By symmetry

$$y \in \text{Path}\langle x, g_j(x)\rangle$$

and

$$f_i(y) \in \text{Path}\langle x, g_j(x)\rangle$$

From these facts we can deduce the path-configuration of x, y, $g_j(x)$ and $f_i(y)$.

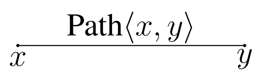

Figure 4.5: Path$\langle x, y\rangle$

Since $y \in \text{Path}\langle x, g_j(x)\rangle$ and $x \in \text{Path}\langle y, f_i(y)\rangle$ we may add to Figure 4.5 $f_i(y)$ and $g_j(x)$ to obtain Figure 4.6.

But we also know that $f_i(y) \in \text{Path}\langle x, g_j(x)\rangle$, so we deduce that $f_i(y) = x$. Similarly

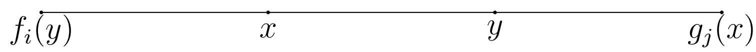

Figure 4.6: Path$\langle x, y\rangle$, $f_i(y)$ and $g_j(x)$

$g_j(x) \in$ Path$\langle y, f_i(y)\rangle$ shows that $g_j(x) = y$. This contradicts the fact that $\bar{f}$ fixes x and $\bar{g}$ fixes y, so we conclude that either $X \subseteq Y$ or $Y \subseteq X$.

$\square$

Lemma 4.2.9 *If* Aut$(M) \models \mathrm{Comm}(\bar{f}, \bar{g})$ *and* supp$(\bar{f}) \cap$ supp$(\bar{g}) \neq \emptyset$ *then* $\bar{f} * \bar{g}$ *has an orbit of length 20 in* supp$(\bar{f}) \cap$ supp$(\bar{g})$. *If* $\bar{f} * \bar{g}$ *has an orbit of length 20 then it also has a non-trivial orbit of some length other than 20.*

Proof

Lemma 3.5 of [33] is:

"Suppose that $\bar{f}, \bar{g}$ are subgroups of $\mathrm{Sym}(\mathcal{X})$ isomorphic to A_5 (in the specified listings) which centralize each other, and such that $\langle \bar{f}, \bar{g}\rangle$ is transitive on $\mathcal{X}$. Then $\bar{f} * \bar{g}$ has an orbit of length 20. Moreover, if $\bar{f} * \bar{g}$ has an orbit of length 20 then is also has an orbit of some other length greater than 1."

Let $\{A_i \ : \ i \in I\}$ be the ECC of supp$(\bar{f})$ and let $\{B_j \ : \ j \in J\}$ be the ECC of supp$(\bar{g})$. Lemma 4.2.8 shows that if $A_i \cap B_j \neq \emptyset$ then $A_i \subseteq B_j$ or $B_j \subseteq A_i$.

Pick one such A and B, and without loss of generality assume that $A \subseteq B$. Let X be a connected component of A.

$$\mathcal{X} := \langle \bar{f}, \bar{g}\rangle(X)$$

Each member of $\mathcal{X}$ is a translate of X.

We define $\phi_f \ : \ \bar{f} \ \rightarrow \ \mathrm{Sym}(\mathcal{X})$ as follows: $\phi_f(f_i) \ = \ (X \ \mapsto \ f_i(X))$. This is a homomorphism, and since A_5 is simple, so ϕ's kernel is trivial, and $\phi_f(\bar{f}) \ \cong \ A_5$.

Similarly, if we define $\phi_g : \bar{g} \to \mathrm{Sym}(\mathcal{X})$ as follows: $\phi_g(g_i) = (X \mapsto g_i(X))$. then $\phi_g(\bar{g}) \cong A_5$.

The 'specified listings' in Shelah and Truss' Lemma 3.5 refers to the fact that the formula $A_5(\bar{f})$ will be different depending on how we enumerate A_5. For example, we could insist that f_0 is the identity, and this would give a different formula to if we insisted that f_5 is the identity. Our formula A_5 is fixed so we need not worry about this assumption.

$\langle \phi_f(\bar{f}), \phi_g(\bar{g}) \rangle$ is transitive on $\mathcal{X}$ since $\mathcal{X}$ is an orbit of $\langle \bar{f}, \bar{g} \rangle$.

Therefore Lemma 3.5 of [33] is applicable to $\mathcal{X}$. $\square$

Lemma 4.2.10 *If* $\mathrm{Aut}(M) \models A_5(\bar{f})$ *then no orbit of* $\bar{f}$ *has length 60.*

Proof

Let $x \in M$ be such that $|\bar{f}(x)| = 60$, and let $\{x_0, \ldots x_{59}\}$ be an enumeration of $\bar{f}(x)$. Take $X(5, 60, \mathbb{Z})$, and pick an arbitrary $z \in X(5, 60, \mathbb{Z})$, and label the successors of z as $z_0, \ldots z_{59}$. For each $f_i \in \bar{f}$, let $g_i \in \mathrm{Aut}(X(5, 60, \mathbb{Z})$ be induced by the partial automorphism

$$z_n \mapsto z_m \text{ if } f_i(x_n) = x_m$$

$\mathrm{Aut}(X(5, 60, \mathbb{Z})) \models A_5(\bar{g})$ and $\bar{g}$ has an orbit of length 60. Let C_i be the extended cone of z that contains z_i.

For each $y \in \bar{g}(z)$ there is a unique $g_i \in \bar{g}$ such that $g_i(z) = y$, so we may label $\bar{g}(x)$ by elements of $\bar{g}$. In this way, we can view the action of $\bar{g}$ on $\bar{g}(x)$ as left multiplication.

We define $\bar{h}$ on each $g \in \bar{h}$ as follows:

$$h_i : g \mapsto g g_i^{-1}$$

This $\bar{h}$ commutes with $\bar{g}$, as

$$\begin{aligned} h_i g_j(g) &= h_i(g_j g) \\ &= (g_j g) g_i^{-1} \\ &= g_j(g g_i^{-1}) \\ &= g_j(h_i(g)) \\ &= g_j h_i(g) \end{aligned}$$

We may extend each $h_i \in \bar{h}$ to an automorphism of $X(5, 60, \mathbb{Z})$ as follows

$$y \mapsto \begin{cases} y & z \in X(5,60,\mathbb{Z}) \setminus \bigcup_{i<60} C_i \\ g_j(y) & y \in C_k \text{ and } g_j : C_k \mapsto h_i(C_k) \end{cases}$$

We now have a $\bar{h} \in$ Aut$(X(5, 60, \mathbb{Z}))$ such that Aut$(X(5, 60, \mathbb{Z})) \models \mathrm{Comm}(\bar{g}, \bar{h})$

Remember that $\mathrm{id} \in \bar{g}$ and consider $\bar{g} * \bar{h}$. For all $g_i h_i \in \bar{g} * \bar{h}$

$$\begin{aligned} g_i h_i(\mathrm{id}) &= g_i \mathrm{id} g_i^{-1} \\ &= \mathrm{id} \end{aligned}$$

Since $\mathrm{id}(x)$ was labelled as id, this means that $x \in$ supp$(\bar{g}) \cap$supp$(\bar{h})$, but $x \notin$ supp$(\bar{g} * \bar{h})$, contradicting Lemma 4.2.7.

□

Lemma 4.2.11 *If* Aut$(M) \models A_5(\bar{f})$ *and there is an* $x \in M$ *such that* $|\bar{f}(x)| = 30$ *then there are* $\bar{g}$ *and* $\bar{h}$ *such that* Aut$(M) \models \mathrm{Comm}(\bar{g}, \bar{h})$ *and* $\bar{f} = \bar{g} * \bar{h}$.

Proof

Let Aut$(M) \models A_5(\bar{f})$ be such that there is an $x \in M$ such that $|\bar{f}(x)| = 30$. Let X be the ECC of supp$(\bar{f})$ that contains x.

Let G and H be subgroups of A_5 such that $|G| = 12$ and $|H| = 10$. There is a transitive action of A_5 on $\{aG \ : \ a \in A_5\} \times \{bH \ : \ b \in A_5\}$ which is isomorphic (as permutation groups) to $\bar{f}$'s action on $\bar{f}(x)$.

We may therefore label each cone of X as (aG, bH).

We define $\bar{g}, \bar{h} \in \text{Aut}(M)$ as follows:

$$g_i : z \mapsto \begin{cases} f_i(z) & z \in M \setminus X \\ f_j(z) & f_j((aG, bH)) = (g_i aG, bH) \end{cases}$$

$$h_i : z \mapsto \begin{cases} z & z \in M \setminus X \\ f_j(z) & f_j((aG, bH)) = (aG, h_i bH) \end{cases}$$

If $\bar{g}$ and $\bar{h}$ are not well-defined then there is an f_i such that $f_i((aG, bH)) = (aG, bH)$ but there is a $z \in (aG, bH)$ such that $f_i(z) \neq z$. However $\bar{f}$ acts transitively on the (aG, bH), so $|\bar{f}(z)| = 60$. Lemma 4.2.10 shows that no such $\bar{f}$ exists.

If $z \in M \setminus X$ then $g_i h_i(z) = g_i(z) = f_i(z)$, so $(\bar{g} * \bar{h})|_{M \setminus X} = \bar{f}|_{M \setminus X}$. If $z \in (aG, bH)$ then $g_i h_i((aG, bH)) = (g_i(a)G, h_i(b)H) = f_i((aG, bH))$, therefore $g_i h_i(z) = f_i(z)$, and so $(\bar{g} * \bar{h})|_X = \bar{f}|_X$.

Together, we now have $(\bar{g} * \bar{h}) = \bar{f}$, so the lemma is proved. □

Proposition 4.2.12 $\text{Aut}(M) \models \text{Indec}(\bar{f})$ *if and only if* $\text{supp}(\bar{f})$ *has exactly one extended connected component and every orbit has less than 30 members.*

Proof

First we prove that if $\text{supp}(\bar{f})$ has exactly one extended connected component and every orbit has less than 30 members then $\text{Aut}(M) \models \text{Indec}(\bar{f})$ by contradiction. Let $\bar{g}$ and $\bar{h}$ witness the fact that $\bar{f}$ does not satisfy Indec, i.e. $\bar{f} = \bar{g} * \bar{h}$ and $\text{Aut}(M) \models \text{Comm}(\bar{g}, \bar{h})$.

If supp$(\bar{g}) \cap$ supp$(\bar{h}) = \emptyset$ then $\bar{f}$ fixes supp$(\bar{g})$ and supp$(\bar{h})$ setwise, and hence supp$(\bar{g})$ and supp$(\bar{h})$ lie in different ECCs of supp$(\bar{f})$.

If supp$(\bar{g}) \cap$ supp$(\bar{h}) \neq \emptyset$ then Lemma 4.2.9 shows that $\bar{g} * \bar{h}$ has an orbit of length at least 20. If $\bar{g} * \bar{h}$ has an orbit of length 20 then there is also another orbit of length other than 20. Since the length is other than 20, this other orbit cannot lie in the same ECC as the orbit of length 20.

Therefore if supp$(\bar{f})$ has exactly one extended connected component and every orbit has less than 30 members then Aut$(M) \models \mathrm{Indec}(\bar{f})$. We now turn our attention to the other direction, which we also do by contradiction.

Suppose supp$(\bar{f})$ has multiple extended connected components. We let X be one of these extended connected components and consider $\bar{f}|^X$ and $\bar{f}|^{M\setminus X}$. These two both satisfy Alt_5 (by Lemma 4.2.6) and their supports are disjoint, so they satisfy Comm. Finally $\bar{f}|^X * \bar{f}|^{M\setminus X} = \bar{f}$, showing that $\bar{f}|^X$ and $\bar{f}|^{M\setminus X}$ witness the fact that $\bar{f}$ does not satisfy Indec.

Lemma 4.2.10 shows that $\bar{f}$ cannot have an orbit of length 60. Lemma 4.2.11 shows that if $\bar{f}$ has an orbit of length 30 then Aut$(M) \models \neg\mathrm{Indec}(\bar{f})$. $\square$

Lemma 4.2.13 *If* Aut$(M) \models disj(\bar{f}, \bar{g})$ *then* supp$(\bar{f}) \cap$ supp$(\bar{g}) = \emptyset$.

Proof

Suppose supp$(\bar{f}) \cap$ supp$(\bar{g}) \neq \emptyset$. By Lemma 4.2.8 either

$$\mathrm{supp}(\bar{f}) \subseteq \mathrm{supp}(\bar{g}) \text{ or } \mathrm{supp}(\bar{f}) \subseteq \mathrm{supp}(\bar{g})$$

Now assume that supp$(\bar{f}) \subsetneqq$ supp$(\bar{g})$ and let $z \in$ supp$(\bar{f})$.

Path$\langle$supp$(\bar{f}), M \setminus$ supp$(\bar{f})\rangle$ is a singleton, as Aut$(M) \models \mathrm{Indec}(\bar{f})$. Let

$$\{x_f\} := \text{Path}\langle \text{supp}(\bar{f}), M \setminus \text{supp}(\bar{f})\rangle$$

$x_f \notin$ supp$(\bar{f})$, but since supp$(\bar{f}) \subsetneq$ supp$(\bar{g})$, we know that $x_f \in$ supp$(\bar{g})$.

Let $g_i \in \bar{g}$. By definition $x_f \in$ Path$\langle g_i^{-1}(x_f), z\rangle$. Since paths are preserved by automorphisms, this translates to

$$g_i(x_f) \in \text{Path}\langle x_f, g_i(z)\rangle$$

Thus if $g_i \neq \mathrm{id}$ then $g_i(z) \notin$ supp$(\bar{f})$, i.e. $f_j g_i(z) = g_i(z)$ for all j, but since $z \in$ supp$(\bar{f})$ there is a k such that $f_k(z) \neq z$. This is depicted in Figure 4.7.

$$\begin{aligned} g_i(z) &= f_k g_i(z) \\ &= g_i f_k(z) \\ &\neq g_i(z) \end{aligned}$$

This is a contradiction. Therefore if supp$(\bar{f}) \cap$ supp$(\bar{g}) \neq \emptyset$ then supp$(\bar{f}) =$ supp$(\bar{g})$.

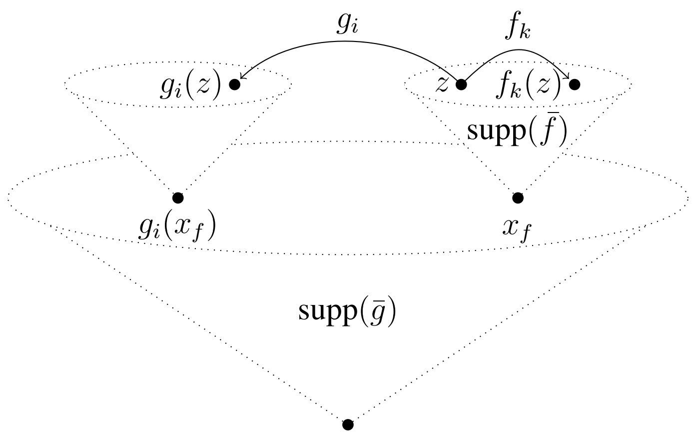

Figure 4.7: supp$(\bar{f}) \subsetneq$ supp$(\bar{g})$

Suppose that supp$(\bar{f}) =$ supp$(\bar{g})$. Again, Path$\langle$supp$(\bar{f}), M \setminus$ supp$(\bar{f})\rangle$ is a singleton, as

Aut$(M) \models \mathrm{Indec}(\bar{f})$ so we let

$$\{x_f\} := \text{Path}\langle \text{supp}(\bar{f}), M \setminus \text{supp}(\bar{f})\rangle$$

Both $\bar{f}$ and $\bar{g}$ must act transitively on the same antichain of immediate successors or predecessors of x_f, which $\bar{f} * \bar{g}$ must also act on. Since Aut$(M) \models \mathrm{Indec}(\bar{f})$ and Aut$(M) \models \mathrm{Indec}(\bar{g})$, Proposition 4.2.12 shows that this antichain must have less than 30 members, but Lemma 4.2.9 showed that $\bar{f} * \bar{g}$ must have an orbit of at least 20 members.

Lemma 4.2.9 also showed that if $\bar{f} * \bar{g}$ has an orbit of length 20 then there was another orbit. Therefore $\bar{f}$ acts transitively on a set with strictly more than 20 elements, and hence at least 30, which contradicts Proposition 4.2.12.

Therefore supp$(\bar{f}) \cap$ supp$(\bar{g}) = \emptyset$. $\square$

Lemma 4.2.14 *Recall that* $[\text{supp}(\bar{f}) \sqsubseteq \text{supp}(\bar{g})]$ *is the formula*

$$\begin{array}{ll} \mathrm{Indec}(\bar{f}) \wedge \mathrm{Indec}(\bar{g}) \wedge \neg\text{disj}(\bar{f}, \bar{g}) & \wedge \\ \forall\phi[\text{disj}(\bar{g}^{\phi}, \bar{g}) \rightarrow \text{disj}(\bar{f}^{\phi}, \bar{f})] & \wedge \\ \forall\phi(\bar{g}^{\phi} \neq \bar{g} \rightarrow \bar{f}^{\phi} = \bar{f}) & \wedge \end{array}$$

If $\bar{f}$ and $\bar{g}$ satisfy this formula then the support of $\bar{g}$ is contained in the support of $\bar{f}$.

Proof

The two sentences

$$\forall \bar{f}, \bar{g} \left(\begin{array}{c} (\forall\phi[\text{disj}(\bar{g}^{\phi}, \bar{g}) \rightarrow \text{disj}(\bar{f}^{\phi}, \bar{f})]) \\ \leftrightarrow \\ (\neg\exists\phi[\neg\text{disj}(\bar{f}^{\phi}, \bar{f}) \wedge \text{disj}(\bar{g}^{\phi}, \bar{g})]) \end{array} \right)$$

and

$$\forall \bar{f}, \bar{g} \left(\begin{array}{c} (\forall \phi(\bar{g}^{\phi} \neq \bar{g} \to \bar{f}^{\phi} = \bar{f})) \\ \leftrightarrow \\ (\neg \exists \phi(\bar{f}^{\phi} = \bar{f} \wedge \bar{g}^{\phi} \neq \bar{g})) \end{array} \right)$$

are tautologies, so the formula given here is equivalent to the one given in Definition 4.2.1.

Suppose that $\bar{f}$ and $\bar{g}$ are such that

$$\mathrm{Indec}(\bar{f}) \wedge \mathrm{Indec}(\bar{g}) \wedge \neg \text{disj}(\bar{f}, \bar{g})$$

This means that supp$(\bar{f})$ and supp$(\bar{g})$ each have exactly one ECC, which have a non-empty intersection. We define

$$x_f := \text{Path}\langle \text{supp}(\bar{f}), M \setminus \text{supp}(\bar{f})\rangle$$
$$x_g := \text{Path}\langle \text{supp}(\bar{g}), M \setminus \text{supp}(\bar{g})\rangle$$

If supp$(\bar{f}) =$ supp$(\bar{g})$ then supp$(\bar{f}^{\phi}) =$ supp$(\bar{g}^{\phi})$ for all $\phi \in$ Aut(M). Therefore for all $\phi \in$ Aut(M)

$$\text{Aut}(M) \models (\text{disj}(\bar{f}^{\phi}, \bar{f}) \leftrightarrow \text{disj}(\bar{g}^{\phi}, \bar{g}))$$

and

$$\text{Aut}(M) \models (\bar{g}^{\phi} \neq \bar{g} \leftrightarrow \bar{f}^{\phi} = \bar{f})$$

Thus Aut$(M) \models$ [supp$(\bar{f}) \sqsubseteq$ supp$(\bar{g})$].

We now suppose that supp$(\bar{f}) \neq$ supp$(\bar{g})$. In Case 1 we consider supp$(\bar{g}) \subsetneq$ supp$(\bar{f})$. In Case 3 we consider supp$(\bar{f}) \subsetneq$ supp$(\bar{g})$. If neither supp$(\bar{g}) \subsetneq$ supp$(\bar{f})$ nor supp$(\bar{f}) \subsetneq$ supp$(\bar{g})$ then we are either in Case 2, where $x_f \neq x_g$, or Case 4 where $x_f = x_g$.

In Case 3 we must prove that Aut$(M) \models$ [supp$(\bar{f}) \sqsubseteq$ supp$(\bar{g})$], while in Cases 1 and 2, we must show that the converse holds. Finally, in Case 4 we show that Aut$(M) \models$ [supp$(\bar{f}) \subseteq$ supp$(\bar{g})$] if and only if supp$(\bar{f}) =$ supp$(\bar{g})$.

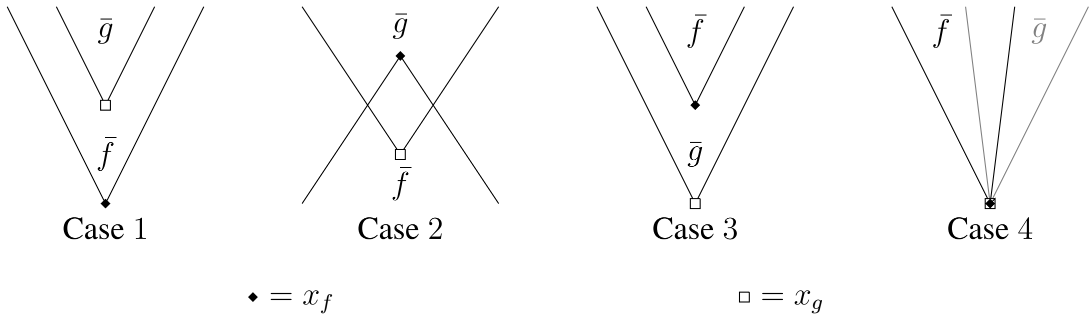

Figure 4.8: Cases of Lemma 4.2.14

Case 1 Since x_f is moved by $\bar{g}$ there is an x'_f such that x_f and x'_f lie in the same $\bar{g}$-orbit and $x_f \neq x'_f$. Let ϕ be an automorphism that switches x_f and x'_f, but fixes anything that it does not have to move. If $z \in \text{supp}(\bar{f})$ then $\phi(z) \notin \text{supp}(\bar{f})$ and so disj$(\bar{f}^\phi, \bar{f})$. Since Path$\langle x_f, x'_f\rangle \subseteq \text{supp}(\bar{g})$ we know that $\text{supp}(\bar{g}) = \text{supp}(\bar{g}^\phi)$ and therefore $\neg$disj$(\bar{g}^\phi, \bar{g})$

Thus ϕ witnesses the fact that $\bar{f}$ and $\bar{g}$ do not satisfy $[\text{supp}(\bar{g}) \sqsubseteq \text{supp}(\bar{f})]$.

Case 2 Let x'_f be such that $x_f \in \text{Path}\langle x_g, x'_f\rangle$ and $x_f \parallel x'_f$. Since $X(n, m, \mathbb{Z})$ is 1-transitive there is an automorphism ϕ such that $\phi(x_f) = x'_f$. We know that disj$(\bar{f}^\phi, \bar{f})$ as

$$\text{Path}\langle \text{supp}(\phi * \bar{f}), \text{supp}(\bar{f})\rangle = \text{Path}\langle f, f'\rangle$$

which cannot be empty, as $x_f \parallel x'_f$. Since $x_f \in \text{Path}\langle x_g, x'_f\rangle$ and $x_f \in \text{supp}(\bar{g})$ the support of $\bar{g}$ must contain x'_f. However x'_f is clearly contained in $\text{supp}(\phi * \bar{g})$, so $\neg$disj$(\phi * \bar{g}, \bar{g})$.

Thus ϕ witnesses the fact that $\bar{f}$ and $\bar{g}$ do not satisfy $[\text{supp}(\bar{g}) \sqsubseteq \text{supp}(\bar{f})]$.

Case 3 For a contradiction, assume that

$$\text{Aut}(M) \models \exists\phi[\text{disj}(\bar{f}^\phi, \bar{f}) \wedge \neg\text{disj}(\bar{g}^\phi, \bar{g})]$$

and let ϕ witness this. Since disj$(\bar{f}^\phi, \bar{f})$ holds, and supp$(\bar{g})$ is contained in supp$(\bar{f})$, we know that disj$(\bar{g}^\phi, \bar{g})$, giving a contradiction.

Now assume that

$$\text{Aut}(M) \models \exists\phi(\bar{f}^\phi = \bar{f} \wedge \bar{g}^\phi \neq \bar{g})$$

Let C_0, C_1 be two of the cones of x_f that are contained in the support of $\bar{f}$ and let $f_i \in \bar{f}$ map C_0 to C_1. Since $\bar{g}^\phi \neq \bar{g}$, there is an $x \in$ supp$(\bar{g})$ such that $\phi(x) \neq x$. We suppose without loss of generality that $x \in C_0$.

If $\phi(x) \notin C_1$ then f_i^ϕ will map x to $f_i\phi(x) \neq f_i(x)$ and so $\bar{f}^\phi \neq \bar{f}$. If $\phi(x) \in C_1$ then conjugation by ϕ will at least switch the roles C_0 and C_1, and so $\bar{f}^\phi \neq \bar{f}$.

Case 4 In this case, $x_f = x_g$. Let $C_0^f, \ldots$ be the cones of f that are contained in $\bar{f}$, and let $C_0^g, \ldots$ be the cones of x_g that are contained in $\bar{g}$. We may pick our indices such that $C_i^f \in$ supp$(\bar{f}) \cap$ supp$(\bar{g})$ if and only if $C_i^g \in$ supp$(\bar{f}) \cap$ supp$(\bar{g})$.

Assume that only one C_i^f is not in the intersection of the supports, and assume without loss of generality that this is C_0^f. Let $\phi \in$ Aut(M) be such that supp$(\phi) \subsetneq C_0^g$ and . Then Aut$(M) \models (\bar{f}^\phi = \bar{f} \wedge \bar{g}^\phi \neq \bar{g})$, showing that $\bar{f}$ and $\bar{g}$ do not satisfy [supp$(\bar{f}) \sqsubseteq$ supp$(\bar{g})$].

Now we assume that more that one C_i^f is not in the intersection of the supports, without loss of generality C_0^f and C_1^f. Let $\phi \in$ Aut(M) be such that ϕ swaps C_0^g and C_1^g and fixes everything else point-wise. Since ϕ fixes supp$(\bar{f})$ point-wise, Aut$(M) \models \bar{f}^\phi = \bar{f}$.

Now consider a elements of $\bar{g}$ which switches C_0^g and C_2^G. The corresponding elements of $\bar{g}^\phi$ will switch C_1^g and C_2^g, and so Aut$(M) \models \bar{g}^\phi \neq \bar{g}$. □

Corollary 4.2.15 *Recall that* [supp$(\bar{g}) \sqsubset$ supp$(\bar{f})$] *is the formula*

$$[\text{supp}(\bar{g}) \sqsubseteq \text{supp}(\bar{f})] \wedge \neg[\text{supp}(\bar{f}) \sqsubseteq \text{supp}(\bar{g})]$$

Aut$(M) \models [\text{supp}(\bar{g}) \sqsubset \text{supp}(\bar{f})]$ *if and only if* supp$(\bar{g})$ *is properly contained in* supp$(\bar{f})$.

Definition 4.2.16 *Let* Aut$(M) \models \mathrm{Indec}(\bar{f}) \wedge \mathrm{Indec}(\bar{g})$ *and let*

$$x_f := \text{Path}\langle \text{supp}(\bar{f}), M \setminus \text{supp}(\bar{f})\rangle$$
$$x_g := \text{Path}\langle \text{supp}(\bar{g}), M \setminus \text{supp}(\bar{g})\rangle$$

We say that $\bar{f}$ and $\bar{g}$ have the ***same direction****, or act in the same direction if*

$$\exists y \in \text{supp}(\bar{f})\ (x_f < y) \Leftrightarrow \exists z \in \text{supp}(\bar{g})\ (x_g < z)$$

We say that $\bar{f}$ and $\bar{g}$ have ***different directions****, or act in different directions if*

$$\exists y \in \text{supp}(\bar{f})\ x_f < y \Leftrightarrow \exists z \in \text{supp}(\bar{g})\ (x_g > z)$$

Lemma 4.2.17 *Recall that* $\mathrm{SamePD}(\bar{f}, \bar{g})$ *is the formula*

$$\forall \bar{h}([\text{supp}(\bar{h}) \sqsubset \text{supp}(\bar{f})] \leftrightarrow [\text{supp}(\bar{h}) \sqsubset \text{supp}(\bar{g})])$$

Let

$$\{x_f\} := \text{Path}\langle \text{supp}(\bar{f}), M \setminus \text{supp}(\bar{f})\rangle$$
$$\{x_g\} := \text{Path}\langle \text{supp}(\bar{g}), M \setminus \text{supp}(\bar{g})\rangle$$

If

$$\text{Aut}(M) \models \mathrm{SamePD}(\bar{f}, \bar{g})$$

then $f = g$ and $\bar{f}$ and $\bar{g}$ have the same direction.

Proof

Suppose Aut$(M) \models \mathrm{SamePD}(\bar{f}, \bar{g})$

We will first show that $x_f = x_g$ by contradiction. Suppose that $x_g \in$ supp$(\bar{f})$. If supp$(\bar{g}) \subset$ supp$(\bar{f})$ then $\bar{f}$ witnesses that $\bar{f}$ and $\bar{g}$ cannot satisfy $\mathrm{SamePD}(\bar{f}, \bar{g})$. If

$\text{supp}(\bar{g}) \not\subset \text{supp}(\bar{f})$ then the supports of $\bar{f}$ and $\bar{g}$ are as in the pictures in Case 2 of Figure 4.8.

Let $\bar{h}$ be a tuple such that:

1. $\text{Aut}(M) \models \text{Indec}(\bar{h})$;
2. $\{x_f\} = \text{Path}\langle \text{supp}(\bar{h}), M \setminus \text{supp}(\bar{h})\rangle$; and
3. $\bar{f}$ and $\bar{g}$ act in different directions.

Then $\text{supp}(\bar{h}) \subset \text{supp}(\bar{g})$ and $\text{supp}(\bar{h}) \cap \text{supp}(\bar{f}) = \emptyset$, giving a contradiction.

Now suppose that $x_g \notin \text{supp}(\bar{f})$ and $x_f \notin \text{supp}(\bar{g})$. We consider two situations, where the point of $\text{Path}\langle x_f, x_g\rangle$ next to x_f is in the same direction as $\bar{f}$ or in the other direction (depicted in Figure 4.9).

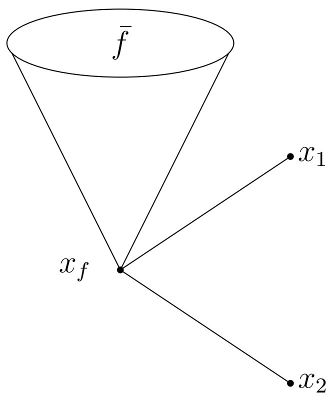

Figure 4.9: $\text{Path}\langle f, g\rangle$ and the Direction of $\bar{f}$

This picture depicts both situations. By "the point of $\text{Path}\langle x_f, x_g\rangle$ immediate to f is in the same direction as $\bar{f}$" we mean that $x_1 \in \text{Path}\langle f, g\rangle$, while $x_2 \in \text{Path}\langle f, g\rangle$ is the other situation we need to consider.

Suppose $x_1 \in \text{Path}\langle f, g\rangle$ and let ϕ be an automorphism of M which fixes f and switches x_1 with a member of $\text{supp}(\bar{f})$. Then $\phi * \bar{f}$ witnesses the fact that $\bar{f}$ and $\bar{g}$ cannot satisfy

$\mathrm{SamePD}(\bar{f}, \bar{g})$. If $x_2 \in$ Path$\langle f, g \rangle$ then any tuple that satisfies Indec, fixes f and moves x_2 will do as a witness.

We know that if Aut$(M) \models \mathrm{SamePD}(\bar{f}, \bar{g})$ then $x_f = x_g$. If $\bar{f}$ and $\bar{g}$ act in different directions then we may pick any point in supp$(\bar{g})$ and any tuple that fixes that point and moves x_f to find our counter-example.

It remains to show that if $\bar{f}$ and $\bar{g}$ fix the same point and have the same direction then they satisfy SamePD. Assume without loss of generality that $\bar{f}$ and $\bar{g}$ act on the successors of x_f. Let $\bar{h}$ be any tuple such that

$$[\text{supp}(\bar{f}) \sqsubset \text{supp}(\bar{h})]$$

This means that $\bar{h}$ moves x_f and all its successors, and therefore supp$(\bar{g})$ contains the support of $\bar{g}$, and so $\bar{h}$ satisfies [supp$(\bar{f}) \sqsubset$ supp$(\bar{h})$]. $\Box$

Lemma 4.2.18 *Recall that* $\mathrm{RepPoint}(\bar{f}_0, \bar{f}_1)$ *is the formula*

$$\text{disj}(\bar{f}_0, \bar{f}_1) \wedge \forall \bar{g} \exists \bar{h} (\neg \text{disj}(\bar{g}, \bar{h}) \wedge (\mathrm{SamePD}(\bar{f}_0, \bar{h}) \vee \mathrm{SamePD}(\bar{f}_1, \bar{h})))$$

Let

$$\{x_0\} := \text{Path}\langle \text{supp}(\bar{f}_0), M) \setminus \text{supp}(\bar{f}_0)\rangle$$
$$\{x_1\} := \text{Path}\langle \text{supp}(\bar{f}_1), M) \setminus \text{supp}(\bar{f}_1)\rangle$$

Then

$$\text{Aut}(M) \models \mathrm{RepPoint}(\bar{f}_0, \bar{f}_1)$$

if and only if $x_0 = x_1$ *and* $\bar{f}_0$ *and* $\bar{f}_1$ *act in different directions.*

Proof

First we will prove that if $\bar{f}_0$ and $\bar{f}_1$ are such that $x_0 = x_1$ and $\bar{f}_0$ and $\bar{f}_1$ act in different directions then

$$\text{Aut}(M) \models \mathrm{RepPoint}(\bar{f}_0, \bar{f}_1)$$

If $\neg$disj$(\bar{g}, \bar{f}_0)$ or $\neg$disj$(\bar{g}, \bar{f}_1)$ then we may take $\bar{h} = \bar{f}_0$ or $\bar{h} = \bar{f}_1$, so suppose that disj$(\bar{g}, \bar{f}_0)$ and disj$(\bar{g}, \bar{f}_1)$.

Let

$$\{x_g\} := \text{Path}\langle \text{supp}(\bar{g}), M) \setminus \text{supp}(\bar{g})\rangle$$

and let $\bar{h}$ be such that

$$(\mathrm{SamePD}(\bar{f}_0, \bar{h}) \vee \mathrm{SamePD}(\bar{f}_1, \bar{h}))$$

and Path$\langle x_0, x_g\rangle \subset$ supp$(\bar{h})$. Clearly this $\bar{h}$ is as required by the formula.

Now we must prove that if

$$\text{Aut}(M) \models \mathrm{RepPoint}(\bar{f}_0, \bar{f}_1)$$

then $\bar{f}_0$ and $\bar{f}_1$ are as desired. If $x_0 \neq x_1$ then there is some y such that none of the following hold

$$y \in \text{Path}\langle x_0, x_1\rangle \quad x_0 \in \text{Path}\langle y, x_1\rangle \quad x_1 \in \text{Path}\langle y, x_0\rangle$$

Let $\bar{g}$ be such that Path$\langle y, \{x_0, x_1\}\rangle \not\subset$ supp$(\bar{g})$ and

$$y = \text{Path}\langle \text{supp}(\bar{g}), M \setminus \text{supp}(\bar{g})\rangle$$

This $\bar{g}$ witnesses the fact that $\bar{f}_0$ and $\bar{f}_1$ do not satisfy $\mathrm{RepPoint}(\bar{f}_0, \bar{f}_1)$.

Now suppose that $x_0 = x_1$ but

$$\text{Aut}(M) \models \mathrm{SamePD}(\bar{f}_0, \bar{f}_1)$$

In this case any $\bar{g}$ whose support is disjoint from that of $\bar{f}_0$ and $\bar{f}_1$ and which fixes f_0 will be a witness. □

We now have our formula that defines the domain of interpretation, however there will be a lot of pairs that satisfy $\mathrm{RepPoint}$ but fix the same point.

Lemma 4.2.19 *Recall that* $\mathrm{EqRepPoint}(\bar{f}_0, \bar{f}_1; \bar{g}_0, \bar{g}_1)$ *is the formula*

$$\mathrm{RepPoint}(\bar{f}_0, \bar{f}_1) \wedge \mathrm{RepPoint}(\bar{g}_0, \bar{g}_1) \wedge$$
$$(\mathrm{SamePD}(\bar{f}_0, \bar{g}_0) \wedge \mathrm{SamePD}(\bar{f}_1, \bar{g}_1)) \vee (\mathrm{SamePD}(\bar{f}_0, \bar{g}_1) \wedge \mathrm{SamePD}(\bar{f}_1, \bar{g}_0))$$

Let

$$x_f := \text{Path}\langle \text{supp}(\bar{f}_0), M) \setminus \text{supp}(\bar{f}_0)\rangle$$
$$x_g := \text{Path}\langle \text{supp}(\bar{g}_0), M) \setminus \text{supp}(\bar{g}_0)\rangle$$

If

$$\text{Aut}(M) \models \mathrm{EqRepPoint}(\bar{f}_0, \bar{f}_1; \bar{g}_0, \bar{g}_1)$$

then $x_f = x_g$.

Proof

Clearly $x_f \neq x_g$ if and only if $\mathrm{SamePD}(\bar{f}_i, \bar{g}_j)$ holds for some choice of indices. □

4.2.2 Interpreting Betweenness

From now on we will adopt the convention that when a lower case letter, such as g, appears in one of our formulas, it is actually a pair $(\bar{g}_0, \bar{g}_1)$ that satisfies $\mathrm{RepPoint}$. We will refer to the point represented by g as x_g.

Definition 4.2.20 $\mathrm{Temp1PB}(g; h, k)$ *is the following formula:*

$$\exists l(\mathrm{EqRepPoint}(\bar{g_0}, \bar{g_1}; \bar{l_0}, \bar{l_1}) \wedge \left(\begin{array}{l} \neg\text{disj}(\bar{l_0}, \bar{h_0}) \wedge \neg\text{disj}(\bar{l_0}, \bar{h_1}) \wedge \\ \neg\text{disj}(\bar{l_1}, \bar{k_0}) \wedge \neg\text{disj}(\bar{l_1}, \bar{k_1}) \end{array} \right)$$

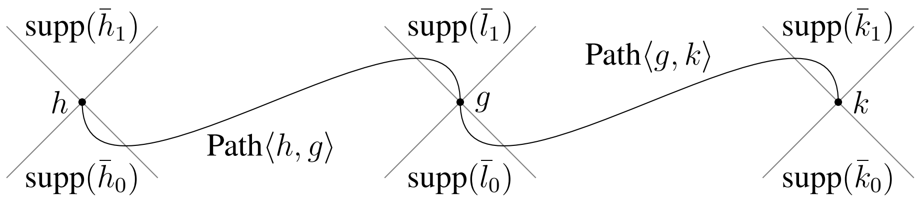

Figure 4.10: What is described by $\mathrm{Temp1PB}(g; h, k)$

$\mathrm{Temp2PB}(g; h, k)$ *is the formula*

$$\phi(g; h, k) \wedge \forall l\, \phi(l; h, k) \rightarrow \left[\begin{array}{ll} \mathrm{Temp1PB}(g; l, k) & \wedge \\ \mathrm{Temp1PB}(g; l, h) & \end{array} \right]$$

where ϕ is the formula that requires, using disj*, the configurations of the supports of $\bar{g}_0$, $\bar{g}_1$, $\bar{h}_0$, $\bar{h}_1$, $\bar{k}_0$ and $\bar{k}_1$ depicted in Figure 4.11, for all permutations of the indices and that each pair represents different points.*

$\mathrm{PathBetween}(g; h, k)$ *is the formula*

$$\mathrm{Temp1PB}(g; h, k) \vee \mathrm{Temp2PB}(g; h, k)$$

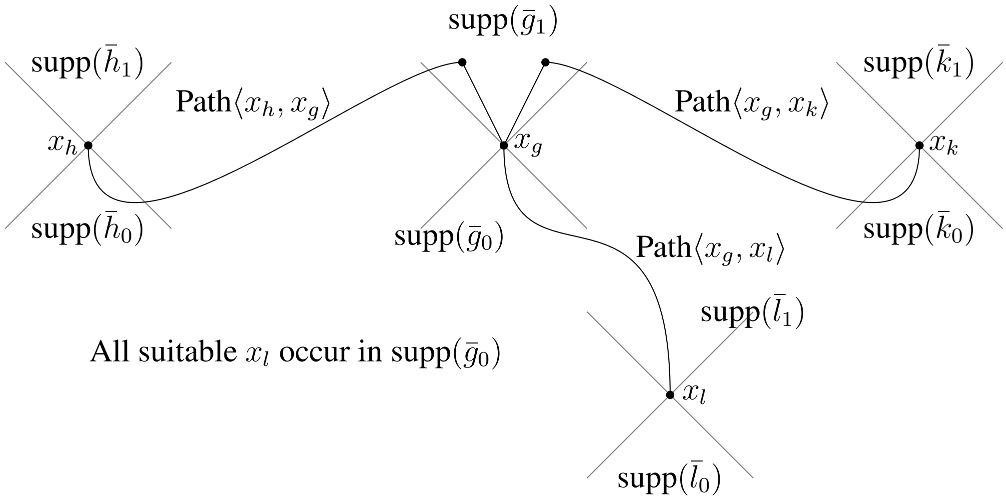

Figure 4.11: What is described by $\mathrm{Temp2PB}(g; h, k)$

Lemma 4.2.21 *The previously defined formulas express the following properties of the structure:*

1. $\mathrm{Temp1PB}(g; h, k)$ *holds if and only if* Path$\langle x_h, x_k\rangle$ *contains a chain of length at least three, of which x_g is one of the middle points.*
2. $\mathrm{Temp2PB}(g; h, k)$ *holds only if x_g is a local maximum or minimum of* Path$\langle x_h, x_k\rangle$.
3. $\mathrm{PathBetween}(x_g; x_h, x_k)$ *holds if and only if $x_g \in$* Path$\langle x_h, x_k\rangle$.

Proof

Without loss of generality, we suppose that the situation is the same as depicted in the diagrams above.

1. Since the formula $\mathrm{Temp1PB}$ insists that $x_h \in$ supp($\bar{g}_1$) and $x_k \in$ supp($\bar{g}_0$), and since any path between something in supp($\bar{g}_0$) and something in supp($\bar{g}_1$) must pass through x_g, we conclude that $x_g \in$ Path$\langle x_h, x_k\rangle$. Additionally, since $\bar{g}_0$ and $\bar{g}_1$ point in different directions there must be both an immediate successor and an

immediate predecessor of x_g lying on Path$\langle x_h, x_k\rangle$ thus showing that if $\mathrm{Temp1PB}$ holds then the properties it was intended to describe hold. The other direction is immediate.

2. Since the formula $\mathrm{Temp2PB}$ holds both x_h and x_k are in supp$(\bar{g}_1)$, if $x_g \in$ Path$\langle x_h, x_k\rangle$ then it is either a local maximum or a local minimum, as supp$(\bar{g}_0)$ is an extended connected component originating at x_g. If $x_g \notin$ Path$\langle x_h, x_k\rangle$ then

$$\{x_g\} \subsetneq \text{Path}\langle x_g, x_h\rangle \cap \text{Path}\langle x_g, x_k\rangle$$

Any

$$x_l \in [\text{Path}\langle x_g, x_h\rangle \cap \text{Path}\langle x_g, x_k\rangle] \setminus \{x_g\}$$

will prevent $\mathrm{Temp2PB}$ from holding. Again, the other direction is immediate.

3. If $x_g \in$ Path$\langle x_h, x_k\rangle$ then either x_g is a local maximum or minimum, or x_g lies on a chain of length at least 3, so $\mathrm{Temp1PB}$ and $\mathrm{Temp2PB}$ successfully cover every case.

□

Definition 4.2.22 $\mathrm{Related}(f, g)$ *is the formula*

$$\forall h(\mathrm{PathBetween}(h; f, g) \to \mathrm{Temp1PB}(h; f, g))$$

Lemma 4.2.23 $\mathrm{Related}(f, g)$ *holds if and only if* $x_f \leq x_g$ *or* $x_g < x_f$.

At this point we have recovered M up to order reversal. We may, if we wish, recover the full order using a variety of different methods, which I will detail later, but from here we can prove that the class is faithful by recovering the betweenness reduct of the CFPOs in question.

Definition 4.2.24 *$B(h; f, g)$ is the formula*

$$\mathrm{PathBetween}(h; f, g) \wedge \begin{pmatrix} \mathrm{Related}(f, g) & \wedge \\ \mathrm{Related}(f, h) & \wedge \\ \mathrm{Related}(g, h) & \end{pmatrix}$$

Lemma 4.2.25 *$B(h; f, g)$ if and only if x_h is between x_f and x_g.*

Theorem 4.2.26 *K_{Cone} is faithful.*

Proof

Let $\langle M, \leq\rangle, \langle N, \leq\rangle \in K_{Cone}$. Let Φ be the first-order interpretation comprising of:

- $\mathrm{RepPoint}(x)$ as the formula that defines the domain of interpretation;
- $\mathrm{EqRepPoint}(x, y)$ as the equivalence relation on the domain of interpretation;
- $B(z; x, y)$ as the betweenness relation.

We have established previously that for all M

$$\Phi(\text{Aut}(\langle M, \leq\rangle)) \cong \langle M, B\rangle$$

Therefore Aut$\langle M, \leq\rangle \cong$ Aut$\langle N, \leq\rangle$ if and only if $\langle M, B\rangle \cong \langle N, B\rangle$.

If $\langle M, B\rangle \cong \langle N, B\rangle$ then $\langle M, \leq\rangle \cong \langle N, \leq\rangle$ or $\langle M, \leq\rangle \cong \langle N, \leq^*\rangle$ (the reverse ordering). By assumption, this means that $\langle M, \leq\rangle \cong \langle N, \leq\rangle$, thus the class is faithful. $\square$

4.3 Reconstructing the Order

It is impossible to reconstruct the order of all members of K_{Cone} with a first-order interpretation. $X(5,5,\mathbb{Z})$, in many ways our best behaved member of K_{Cone}, is isomorphic to its own reverse ordering, and so the automorphism group has no idea which direction is 'up'.

In those circumstances, it will be necessary to make an artificial choice over which way is 'up'. When reconstructing linear orders in [26], McCleary and Rubin use a parameter pair for this purpose, obtaining a formula $\phi(x_1, x_2; y_1, y_2)$, which interprets

$$x_1 \leq x_2 \Leftrightarrow y_1 \leq y_2$$

This approach is also possible in this context, but not in a first order way.

Since all members of K_{Cone} embed the alternating chain, as the path between $\{x_1, x_2\}$ and $\{y_1, y_2\}$ grows, we require longer and longer formulas. We must use an $L_{\omega_1,\omega}$ formula to recover the order with this technique.

Another approach would be to exploit the fact that we have insisted that

$$ro\downarrow(M) \leq ro\uparrow(M)$$

Ramification order is definable when finite, so if $ro\downarrow(M) < \{ro\uparrow(M), \aleph_0\}$, then we can find a first order formula that depends on $ro\downarrow$ that interprets the order.

While first order, I find this far less satisfactory, as it gives lots of different formulas, each of which only work in limited circumstances. Even together they do not work everywhere. However, I will present both.

4.3.1 $ro\downarrow(M) < \{ro\uparrow(M), \aleph_0\}$

Definition 4.3.1 *Let* $K^n_{Cone} := \{M \in K_{Cone} \ : \ ro\downarrow(M) \leq n < ro\uparrow(M)\}$.

Definition 4.3.2 $x \lessdot_n y$ *is the following formula*

$$\mathrm{Related}(x,y) \wedge \exists x_0,\ldots,x_n \begin{pmatrix} \bigwedge_{0\leq i\leq n} & \mathrm{Related}(x,x_i) & \wedge \\ \bigwedge_{i\neq j} & \neg\mathrm{Related}(x_i,x_j) & \wedge \\ \bigwedge_{i\neq j} & \mathrm{PathBetween}(x;x_i,x_j) & \wedge \\ & \neg\mathrm{PathBetween}(x;y,x_0) & \end{pmatrix}$$

Theorem 4.3.3 *If* $M \in K^n_{Cone}$ *then* $\mathrm{Aut}(M) \models x \lessdot_n y$ *if and only if* $M \models x <_M y$.

Proof

By definition, $x \lessdot_n y \to \mathrm{Related}(x,y)$, so if Aut$(M) \models x \lessdot_n y$ then $M \models x \leq\geq_M y$.

Each of the x_i are related to x, but $\{x_i \ : \ i = 0,...,n\}$ forms an antichain. Suppose that none of the x_i's lie above x. Since $ro\downarrow(M) \leq n$ this means that at least two of the x_i's, say x_0 and x_1, are contained in the same downwards cone of x.

Therefore $x_0 \vee x_1 < x$, but the connecting set of the path from x_0 to x_1 must be

$$\{x_0, x_0 \vee x_1, x_1\}$$

which would imply that $x \notin$ Path$\langle x_0, x_1\rangle$, which contradicts the assumption that Aut$(M) \models x \lessdot_n y$. Thus at least one of the x_i's is above x.

Suppose, without loss of generality, that x_0 is above x. If any of the other x_i's lie below x_0 then they will be related to x_i, giving a contradiction. By the above argument, all of the x_i's lie in different cones.

On the other hand, any $n+1$ element antichain above x, where every element is contained in a different cone above x satisfies the all of the properties demanded of it, except

$$(\bigvee_{i \leq n} \neg \mathrm{PathBetween}(x; y, x_i))$$

If $x < y$ then we will be able to choose x_0 such that x_0 is contained in the same cone as y, so any such antichain will satisfy the formula.

If $y < x$ then any path from any of the x_i's to y will pass through x, and so the formula cannot be satisfied. □

4.3.2 Abandoning First Order Logic

Throughout this subsection, we assume that y_1 and y_2 satisfy $\mathrm{Related}$. All the formulas mentioned will use y_1 and y_2 as parameters. We will use y_1 and y_2 to indicate the direction of the order, so we suppose that $y_1 < y_2$.

Definition 4.3.4 $(x_1 <_0 x_2 \Leftrightarrow y_1 < y_2)$ *is the formula that insists that* x_1*,* x_2*,* y_1 *and* y_2 *are all related and using* $B(z; x, y)$ *insists that they lie in one of the configurations depicted below.*

Lemma 4.3.5 *If* $\mathrm{Aut}(M) \models (x_1 <_0 x_2 \Leftrightarrow y_1 < y_2)$ *then* $M \models x_1 <_M x_2$*.*

Proof
All possible cases are covered by the definition. □

Definition 4.3.6 $(x_1 <_1 x_2 \Leftrightarrow y_1 < y_2)$ *is the formula*

$$\neg(x_2 <_0 x_1 \Leftrightarrow y_1 < y_2) \wedge \neg(x_1 <_0 x_2 \Leftrightarrow y_1 < y_2) \wedge (\alpha_1 \vee \alpha_2 \vee \alpha_3 \vee \alpha_4)$$

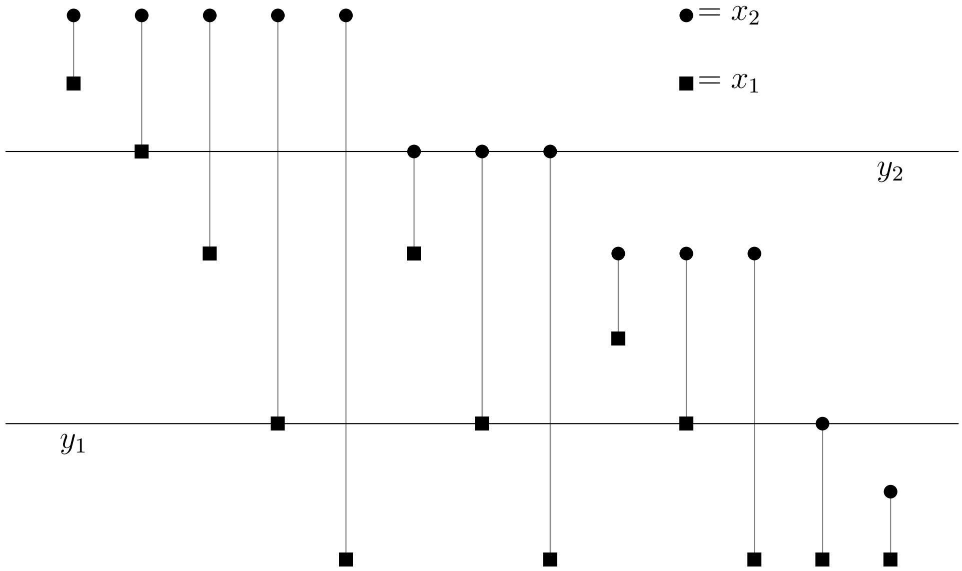

Figure 4.12: Defining $(x_1 <_0 x_2 \Leftrightarrow y_1 < y_2)$

where:

$$\begin{aligned}
\alpha_1 &:= B(y_2; y_1, x_2) \wedge \mathrm{Related}(x_1, x_2)\\
\alpha_2 &:= B(x_2; x_1, y_2)\\
\alpha_3 &:= B(y_1; x_1, y_2) \wedge \mathrm{Related}(x_1, x_2)\\
\alpha_4 &:= B(x_1; y_1, x_2)\\
\alpha_5 &:= \bigwedge_{i \neq j} \mathrm{Related}(x_i, y_j) \wedge \bigwedge_{i=j} \neg\mathrm{Related}(x_i, y_j) \wedge \mathrm{Related}(x_1, x_2)
\end{aligned}$$

Lemma 4.3.7 *If* Aut$(M) \models (x_1 <_1 x_2 \Leftrightarrow y_1 < y_2)$ *then* $M \models x_1 <_M x_2$.

Proof

Let $(x_0, x_1) \in M$ be such that Aut$(M) \models \neg(x_1 <_0 x_2 \Leftrightarrow y_1 < y_2)$. We will show that when Aut$(M) \models \alpha_i$ then $x_1 <_M x_2$ for each possible i.

First, assume that Aut$(M) \models \alpha_1$. Since Aut$(M) \models B(y_2; y_1, x_2)$ and we are supposing that $y_1 <_M y_2$, we know that $x_2 >_M y_2$. We also know that x_1 cannot be greater than

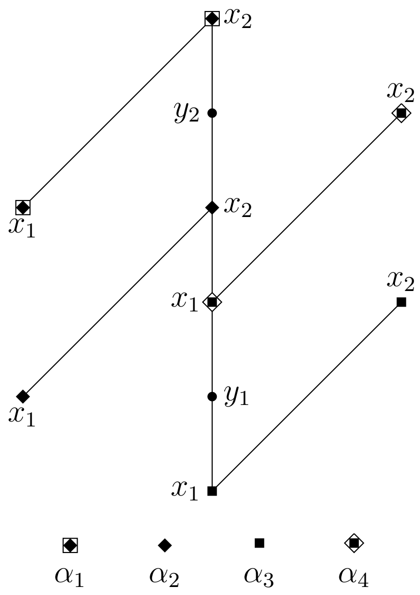

Figure 4.13: Defining $(x_1 <_1 x_2 \Leftrightarrow y_1 < y_2)$

x_2, as otherwise Aut$(M) \models (x_1 <_0 x_2 \Leftrightarrow y_1 < y_2)$. Since we have asserted that Aut$(M) \models \mathrm{Related}(x_1, x_2)$, this means that $x_1 <_M x_2$.

Now we assume that Aut$(M) \models \alpha_2$, so either $x_1 <_M x_2 <_M y_2$ or $y_2 <_M x_2 <_M x_1$, but the latter contradicts our assertion that not both of x_1 and x_2 are related to both y_1 and y_2.

Assume that Aut$(M) \models \alpha_3$, so $x_1 <_M y_1 <_M y_2$. If $x_2 <_M x_1$ then $x_2 <_M y_1, y_2$, contradicting Aut$(M) \models \neg(x_1 <_0 x_2 \Leftrightarrow y_1 < y_2)$.

Assume that Aut$(M) \models \alpha_2$, so either $x_2 <_M x_1 <_M y_1$ or $y_1 <_M x_1 <_M x_2$, but the former contradicts our assertion that not both of x_1 and x_2 are related to both y_1 and y_2.

Assume that Aut$(M) \models \alpha_5$, so Aut$(M) \models \mathrm{Related}(x_1, y_2) \wedge \neg\mathrm{Related}(x_1, y_1)$. This means that $x_1 <_M y_2$. If $x_2 <_M x_1$ then $x_2 <_M y_2$, but we have asserted that Aut$(M) \models \neg\mathrm{Related}(x_2, y_2)$. $\Box$

Definition 4.3.8 *Let $n \geq 2$. The formula $(x_1 <_n x_2 \Leftrightarrow y_1 < y_2)$ is defined to be the conjunction of the following four formulas:*

$$\forall z \left(\bigwedge_{i<n-1} \neg(x_1 <_i z \Leftrightarrow y_1 < y_2) \right) \wedge \forall z \left(\bigwedge_{i<n-1} \neg(z <_i x_2 \Leftrightarrow y_1 < y_2) \right)$$

to ensure that the order is yet to be resolved for either x_1 or x_2;

$$\begin{gathered} (\exists z\, ((x_1 <_{n-1} z \Leftrightarrow y_1 < y_2))) \wedge \forall z\, (\neg(z <_{n-1} x_2 \Leftrightarrow y_1 < y_2))) \\ \vee \\ (\exists z\, ((z <_{n-1} x_2 \Leftrightarrow y_1 < y_2))) \wedge \forall z\, (\neg(x_1 <_{n-1} z \Leftrightarrow y_1 < y_2))) \end{gathered}$$

to ensure that exactly one of x_1 and x_2 is related by $<_{n-1}$ to something;

$$\begin{gathered} \forall z\, (\neg(z <_{n-1} x_2 \Leftrightarrow y_1 < y_2)) \rightarrow \\ \exists w\, ((x_1 <_{n-1} w \Leftrightarrow y_1 < y_2) \wedge (x_1 <_1 x_2 \Leftrightarrow x_1 < w)) \end{gathered}$$

to describe what happens when x_1 is in the area where the order is defined, but x_2 is not, and;

$$\begin{gathered} \forall z\, (\neg(x_1 <_{n-1} z \Leftrightarrow y_1 < y_2)) \rightarrow \\ \exists w\, ((w <_{n-1} x_2 \Leftrightarrow y_1 < y_2) \wedge (x_1 <_1 x_2 \Leftrightarrow w < x_2)) \end{gathered}$$

to describe what happens when x_2 is in the area where the order is defined, but x_1 is not.

Proposition 4.3.9 *If* $\mathrm{Aut}(M) \models (x_1 <_n x_2 \Leftrightarrow y_1 < y_2)$ *then* $M \models x_1 <_M x_2$.

Proof

We proceed by induction, starting with the base case $n = 2$. Suppose that

$$\mathrm{Aut}(M) \models (x_1 <_2 x_2 \Leftrightarrow y_1 < y_2)$$

If

$$\mathrm{Aut}(M) \models \forall z\, (\neg(z <_{n-1} x_2 \Leftrightarrow y_1 < y_2))$$

then there is a $w \in M$ such that $\mathrm{Aut}(M) \models (x_1 <_1 w \Leftrightarrow y_1 < y_2)$ and so $x_1 <_M w$. Therefore $\mathrm{Aut}(M) \models (x_1 <_1 x_2 \Leftrightarrow x_1 < w)$ implies that $x_1 <_M x_2$.

If $\mathrm{Aut}(M) \models \forall z\, (\neg(x_1 <_1 z \Leftrightarrow y_1 < y_2))$ then there is a $w \in M$ such that

$$\mathrm{Aut}(M) \models (w <_1 x_2 \Leftrightarrow y_1 < y_2)$$

and so $w <_M x_2$. Therefore $\mathrm{Aut}(M) \models (x_1 <_1 x_2 \Leftrightarrow w < x_2)$ implies that $x_1 <_M x_2$.

We now examine the induction step. Suppose that if $\mathrm{Aut}(M) \models (x_1 <_{n-1} x_2 \Leftrightarrow y_1 < y_2)$ then $x_1 <_M x_2$ and also suppose that $\mathrm{Aut}(M) \models (x_1 <_n x_2 \Leftrightarrow y_1 < y_2)$.

If $\mathrm{Aut}(M) \models \forall z\, (\neg(z <_{n-1} x_2 \Leftrightarrow y_1 < y_2))$ then there is a $w \in M$ such that

$$\mathrm{Aut}(M) \models (x_1 <_{n-1} w \Leftrightarrow y_1 < y_2)$$

and so $x_1 <_M w$. Therefore $\mathrm{Aut}(M) \models (x_1 <_1 x_2 \Leftrightarrow x_1 < w)$ implies that $x_1 <_M x_2$.

If $\mathrm{Aut}(M) \models \forall z\, (\neg(x_1 <_{n-1} z \Leftrightarrow y_1 < y_2))$ then there is a $w \in M$ such that

$$\mathrm{Aut}(M) \models (w <_{n-1} x_2 \Leftrightarrow y_1 < y_2)$$

and so $w <_M x_2$. Therefore $\mathrm{Aut}(M) \models (x_1 <_1 x_2 \Leftrightarrow w < x_2)$ implies that $x_1 <_M x_2$. $\Box$

Definition 4.3.10 $(x_1 < x_2 \Leftrightarrow y_1 < y_2)$ *is defined to be the* $L_{\omega_1,\omega}$*-formula:*

$$\bigvee_{n<\omega} (x_1 <_n x_2 \Leftrightarrow y_1 < y_2)$$

Theorem 4.3.11 $\mathrm{Aut}(M) \models (x_1 < x_2 \Leftrightarrow y_1 < y_2)$ *if and only if* $M \models x_1 <_M x_2$.

Proof

Suppose $\mathrm{Aut}(M) \models (x_1 < x_2 \Leftrightarrow y_1 < y_2)$. In the first clause of their definition, we ensured that each of the formulas $(x_1 <_n x_2 \Leftrightarrow y_1 < y_2)$ are mutually exclusive. By Lemmas 4.3.5 and 4.3.7 and Proposition 4.3.9, so no matter which $\mathrm{Aut}(M)$ realises, we have ensured that $x_1 <_M x_2$.

Suppose $M \models x_1 <_M x_2$. We examine the length of Path$\langle\{x_1, x_2\}, \{y_1, y_2\}\rangle$, which we shall call j. If $j \leq 2$ then at least one of x_1 and x_2 is related to at least one of y_1 and y_2.

Suppose that both x_1 and x_2 are related to both y_1 and y_2. Since x_1 and x_2 must occur in one of the situations described by $(x_1 <_0 x_2 \Leftrightarrow y_1 < y_2)$.

Now suppose that not both of x_1 and x_2 are related to both y_1 and y_2, but at least one is. This situation is fully described by $(x_1 <_1 x_2 \Leftrightarrow y_1 < y_2)$.

If neither x_1 nor x_2 are related to either y_1 or y_2 then $\mathrm{Aut}(M)$ realises neither

$$(x_1 <_0 x_2 \Leftrightarrow y_1 < y_2) \quad \text{nor} \quad (x_1 <_1 x_2 \Leftrightarrow y_1 < y_2)$$

as both of those formulas contain instances of $B(z; x, y)$ that prevent this.

Now suppose that $j \geq 3$. We also assume that for all z_1 and z_2 such that $z_1 <_M z_2$, the length of Path$\langle\{z_1, z_2\}, \{y_1, y_2\}\rangle$ is i for $i < j$ if and only if

$$\mathrm{Aut}(M) \models (z_1 <_i z_2 \Leftrightarrow y_1 < y_2)$$

Suppose Path$\langle\{x_1, x_2\}, \{y_1, y_2\}\rangle = j$. We first wish to show that (x_0, x_1) satisfies the first clause of $(x_1 < x_2 \Leftrightarrow y_1 < y_2)$, vis.

$$\forall z \left(\bigwedge_{i<j-1} \neg(x_1 <_i z \Leftrightarrow y_1 < y_2) \right) \wedge \forall z \left(\bigwedge_{i<n-1} \neg(z <_i x_2 \Leftrightarrow y_1 < y_2) \right)$$

If there is a z such that $(x_1 <_i z \Leftrightarrow y_1 < y_2))$ for some $i < j-1$ then, by the induction hypothesis, the length of Path$\langle\{z, x_1\}, \{y_1, y_2\}\rangle$ is less than $j-1$. Since $x_1 <_M x_2$, this means that the length of Path$\langle\{x_1, x_2\}, \{y_1, y_2\}\rangle$ is less than j, contradicting our assumptions. If there is a z such that $(z <_i x_2 \Leftrightarrow y_1 < y_2))$ then we reach a similar contradiction.

Let us now examine the second clause:

$$\forall z\, (\neg(z <_{j-1} x_2 \Leftrightarrow y_1 < y_2)) \rightarrow$$
$$\exists w\, ((x_1 <_{j-1} w \Leftrightarrow y_1 < y_2) \wedge (x_1 <_1 x_2 \Leftrightarrow x_1 < w))$$

If there is a z such that Aut$(M) \models (z <_{j-1} x_2 \Leftrightarrow y_1 < y_2)$ then we are done, so suppose that there is no such z. Let $z_1, \ldots, z_j$ be the elements of the connecting set of Path$\langle\{x_1, x_2\}, \{y_1, y_2\}\rangle$, such that z_1 is related to x_1 and x_2.

If $z_2 <_M z_1 \leq_M x_2$ then Aut$(M) \models (z_2 <_{j-1} x_2 \Leftrightarrow y_1 < y_2)$, so we may assume that $x_1 \leq_M z_1 <_M z_2$.

Path$\langle\{z_1, z_2\}, \{y_1, y_2\}\rangle$ = Path$\langle z_2, \{y_1, y_2\}\rangle$ has length $j-1$, so Aut$(M) \models (x_1 <_{j-1} z_2 \Leftrightarrow y_1 < y_2)$. Additionally, we have deduced that $z_2 \parallel x_2$, and $x_1 \leq z_2, x_2$, so Aut$(M) \models (x_1 <_1 x_2 \Leftrightarrow x_1 < z_2)$, so (x_1, x_2) satisfies the second clause.

Recall that the third clause we must examine is:

$$\forall z\, (\neg(x_1 <_{n-1} z \Leftrightarrow y_1 < y_2)) \rightarrow$$
$$\exists w\, ((w <_{n-1} x_2 \Leftrightarrow y_1 < y_2) \wedge (x_1 <_1 x_2 \Leftrightarrow w < x_2))$$

If there is a z such that Aut$(M) \models (x_1 <_{j-1} z \Leftrightarrow y_1 < y_2)$ then we are done, so suppose that there is no such z. Again, let $z_1, \ldots, z_j$ be the elements of the connecting set of Path$\langle\{x_1, x_2\}, \{y_1, y_2\}\rangle$, such that z_1 is related to x_1 and x_2.

If $x_1 \leq_M z_1 <_M z_2$ then Aut$(M) \models (x_1 <_{j-1} z_2 \Leftrightarrow y_1 < y_2)$, so we may assume that $z_2 <_M z_1 \leq_M x_2$.

$\text{Path}\langle\{z_1, z_2\}, \{y_1, y_2\}\rangle = \text{Path}\langle z_2, \{y_1, y_2\}\rangle$ has length $j - 1$, so

$$\text{Aut}(M) \models (z_2 <_{j-1} x_2 \Leftrightarrow y_1 < y_2)$$

Additionally, we have deduced that $z_2 \parallel x_1$, and $x_2 \geq z_2, x_1$, so $\text{Aut}(M) \models (x_1 <_1 x_2 \Leftrightarrow z_2 < x_1)$, so (x_1, x_2) satisfies the third clause.

Now suppose that $\text{Aut}(M) \models (x_1 <_j x_2 \Leftrightarrow y_1 < y_2)$. Let $z_k, \ldots, z_j$ be the elements of the connecting set of $\text{Path}\langle\{x_1, x_2\}, \{y_1, y_2\}\rangle$, such that z_k is related to x_1 and x_2. Since

$$\text{Aut}(M) \models \forall z \left(\bigwedge_{i<j-1} \neg(x_1 <_i z \Leftrightarrow y_1 < y_2) \right) \wedge \forall z \left(\bigwedge_{i<n-1} \neg(z <_i x_2 \Leftrightarrow y_1 < y_2) \right)$$

the length of $\text{Path}\langle\{x_1, x_2\}, \{y_1, y_2\}\rangle$ has at least j elements, and thus $k \leq 1$. By the induction hypothesis, either

$$\text{Aut}(M) \models (z_1 <_{j-1} z_2 \Leftrightarrow y_1 < y_2) \quad \text{or} \quad \text{Aut}(M) \models (z_2 <_{j-1} z_1 \Leftrightarrow y_1 < y_2)$$

so if $k \neq 1$ then (x_1, x_2) cannot possibly satisfy the second and third coordinates. $\square$

Chapter 5

Decorated CFPOs and the Wreath Product

So far the classes of CFPOs we have reconstructed have been rather limited. The gap between those which have a singleton orbit and those with a single orbit is somewhat profound, and so this chapter seeks to redress this failing in a very direct way; we will combine treelike and members of K_{Cone} in such a way that the automorphism groups of the components are definable in the whole automorphism group, and so our previous reconstruction results will be applicable.

The first section will give the method of decoration and describe the resulting automorphism groups as wreath products of the automorphism groups of the components, while the second will define these components using second order logic. This is a desirable outcome, because if the components are definable, then we can perform our interpretations inside the definable sets rather than the whole group, reconstructing the component structures.

5.1 Decoration

We will first look at attaching trees above and between points of a member of K_{Cone}, and give conditions for when a general CFPO shares an automorphism group with such a CFPO.

Definition 5.1.1 *If M is a CFPO then we define M_{ap} to be the set*

$$\{(i,j) \in M^2 \ : \ i <_M j \wedge \forall k \in M \neg(i <_M k <_M j)\}$$

ap stands for 'adjacent pairs'.

Definition 5.1.2 *Let $\langle M, \leq_M\rangle$ be a CFPO and let $\langle S, \leq_S\rangle$ and $\langle T, \leq_T, L\rangle$ be trees, where L is a unary predicate that picks out a maximal chain of T. The structure* $\mathrm{Dec}(M, S, (T, L))$ *is the partial order with universe*

$$|M| \cup \biguplus_{i\in M} |S_i| \cup \biguplus_{(i,j)\in M_{ap}} |T_{(i,j)}|$$

where:

- *$S_i \cong S$ for every $i \in M$*
- *$T_{(i,j)} \cong T$ for every $(i,j) \in \mathcal{M}$. We use $L_{(i,j)}$ to denote the maximal chain of $T_{(i,j)}$ picked out by L.*

$\mathrm{Dec}(M, S, (T, L))$ *is ordered by $\leq_D$, which is the transitive closure of the following:*

$$\begin{array}{cc} x \leq_M y & \text{or} \\ x \leq_{S_i} y & \text{or} \\ y \in S_x & \text{or} \\ x \leq_{T_{(i,j)}} y & \text{or} \\ \exists z \in M\ L_{(x,z)}(y) & \text{or} \\ \exists z \in M\ L_{(z,y)}(x) & \end{array}$$

Informally, we attach a copy of S above every point of M, and glue a copy of T between every adjacent pair of M along L.

Note that if M_{ap} is empty, in other words if M is dense, then

$$\mathrm{Dec}(M, S, (T_0, L_0)) = \mathrm{Dec}(M, S, (T_1, L_1))$$

for all (T_0, L_0) and (T_1, L_1).

Example 5.1.3 *An illustration of the neighbourhood of an element of M in $\mathrm{Dec}(M, S, (T, L))$ is given in Figure 4.1. A more specific example of decorating is pictured in Figure 4.2. In this example, we do not need to specify an L, as B has exactly one maximal chain.*

Proposition 5.1.4 $\mathrm{Dec}(M, S, (T, L))$ *is a CFPO for any M, S and (T, L).*

Proof

Let a and b be such that there are two different paths between them, which we will call P_0 and P_1. If a and b are contained in the same copy of S or (T, L) then this contradicts our assumption that S and (T, L) are trees. If $a \in S_{m_a}$ then $\{m_a\} \subseteq P_0 \cap M \cap P_1$. If $a \in T_{(m_a, m'_a)}$ then either m_a or m'_a is in $P_0 \cap M$. Similarly either m_a or m'_a is in $P_1 \cap M$.

Thus the starting point of $P_0 \cap M$ is one of a, m_a or m'_a, while the ending point is one of b, m_b and m'_b. The same conclusion can be reached for $P_1 \cap M$. If $P_0 \cap M$ starts with m_a while $P_1 \cap M$ starts with m'_a then either P_0 or P_1 has to pass through the starting point of the other, which implies that one of the paths doubles back on itself, giving a contradiction. Since $P_0 \cap M$ and $P_1 \cap M$ have the same start and end points, the fact that M is a CFPO implies that they must be equal.

Then P_0 and P_1 'move through' M in the same way, and so must differ by their behaviour within the copies of S and (T, L). But both S and (T, L) are trees, so have unique paths and therefore $P_0 = P_1$. $\square$

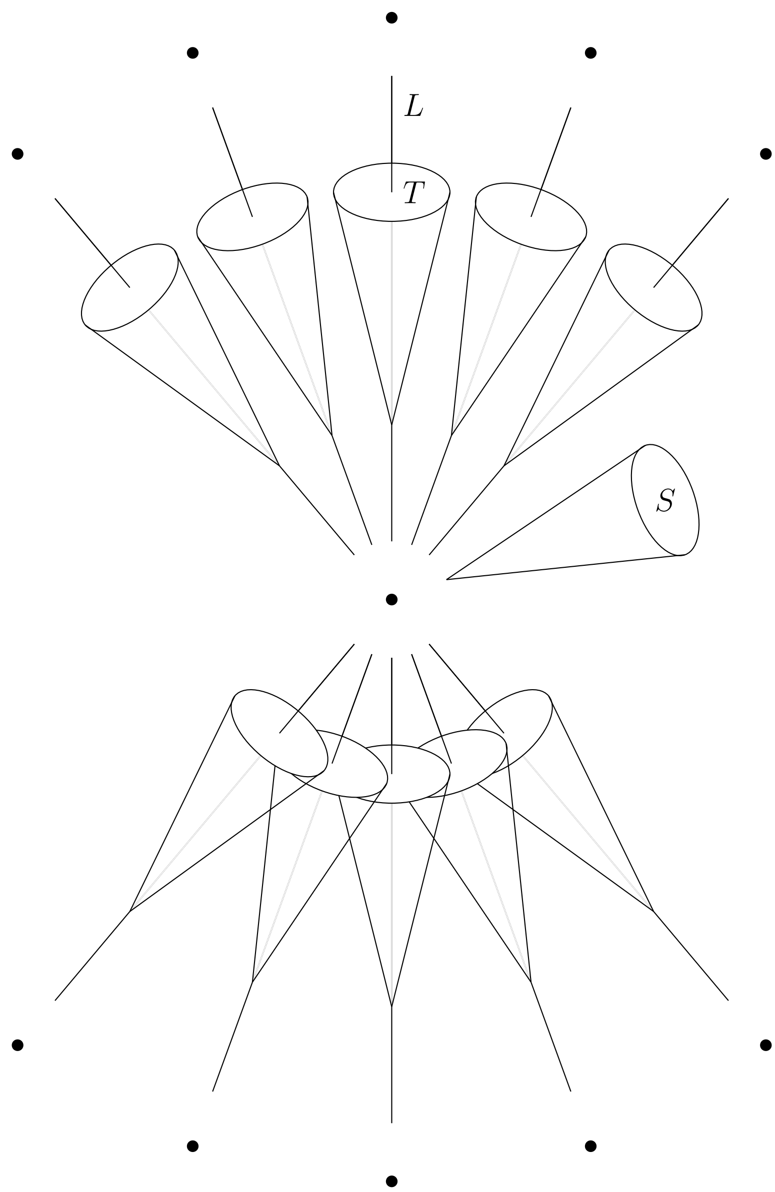

Figure 5.1: A vague illustration of Decoration

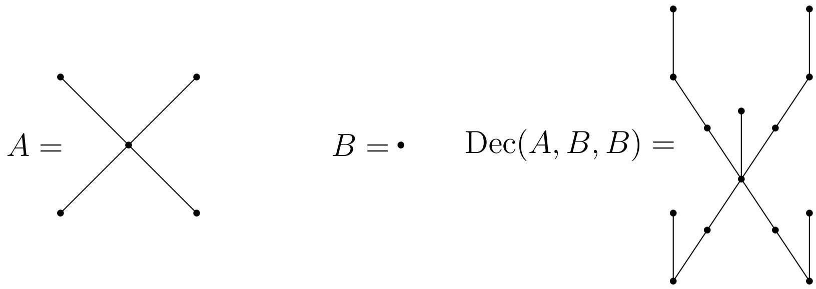

Figure 5.2: Example 5.1.3

Lemma 5.1.5 Aut$(\mathrm{Dec}(M, S, (T, L)))$ *preserves M setwise.*

Proof

Since $M \in K_{Cone}$, given any $a \in M$ there are $b_0, b_1 \in M$ such that $b_0||b_1$ and $a = b_0 \vee b_1$. Let $\phi \in$ Aut$(\mathrm{Dec}(M, S, (T, L))$. Since ϕ is an automorphism $\phi(a) = \phi(b_0) \vee \phi(b_1)$.

S and (T, L) are trees, so $\phi(b_0) \vee \phi(b_1)$ cannot be contained in a copy of S or (T, L), and so all automorphisms of $\mathrm{Dec}(M, S, (T, L))$ preserve M. $\square$

Theorem 5.1.6 *Let M be a CFPO, let A be a 1-orbit such that* Aut(M) *acts cone transitively on A, and for any $B \subset M$ let $\sim_B$ be the equivalence relation $x \sim y \Leftrightarrow$* Path$\langle x, y\rangle \cap B = \emptyset$*. We let $C \in (M \setminus A)/\sim_A$, and describe two conditions.*

1. *If* Path$\langle C, M \setminus C\rangle \neq \emptyset$ *then there is an $a_c \in A$ such that*

$$\text{Path}\langle C, M \setminus C\rangle = \{a_C\}$$

This says that if there is only one way to go from C to $M \setminus C$ then C is attached to a_c.

2. *If* Path$\langle C, M \setminus C\rangle = \emptyset$ *then:*

(a) *$(M \setminus C)/ \sim_C$ has exactly two elements which we call B_C and B'_C; and*

(b) *there is $(a_C, a'_C) \in A_{ap}$ such that*

$$\text{Path}\langle C, B_C\rangle = \{a_C\} \quad \text{and} \quad \text{Path}\langle C, B'_C\rangle = \{a'_C\}$$

This says that if there is more than one way to go from C to $M \setminus C$ then C lies between an adjacent pair of A.

If every $C \in (M \setminus A)/ \sim_A$ satisfy both 1. and 2. then there are trees S and (T, L) and a cone transitive CFPO X such that

$$\text{Aut}(M) \cong \text{Aut}(\text{Dec}(X, S, (T, L))$$

Proof

Suppose M has a 1-orbit A that satisfies the conditions of the theorem. We define X to be the substructure of M with domain A.

We define the following set:

$$\mathcal{C}_S := \{C \in (M \setminus A)/ \sim_A : \text{ Path}\langle C, M \setminus C\rangle \neq \emptyset\}$$

and let $C \in \mathcal{C}_S$. We wish to show that C, when acted on by $\text{Aut}_{\{C\}}(M)$, is treelike. If C does not embed Alt then C, even with its full automorphism group, is treelike (Definition 3.1.2), so we suppose that C does embed Alt, which we enumerate as $(\ldots c_{-1}, c_0, c_1, \ldots)$. There must be some i such that for all j

$$\text{Path}\langle a_C, c_i\rangle \subseteq \text{Path}\langle a_C, c_j\rangle \quad \text{or} \quad \text{Path}\langle a_C, c_{i+1}\rangle \subseteq \text{Path}\langle a_C, c_j\rangle$$

If $\phi \in \mathrm{Aut}(M)$ and $i \neq j$ then ϕ cannot map c_i to c_j, otherwise

$$\mathrm{Path}\langle a_C, a_i\rangle \cap \mathrm{Path}\langle \phi(a_C), a_j\rangle = \emptyset$$

which contradicts our assumption that $\mathrm{Path}\langle C, M \setminus C\rangle \neq \emptyset$. Thus $\mathrm{Aut}_{\{C\}}(M)$ cannot act as D_∞ on any copy of Alt that is contained in C, so C with the action of $\mathrm{Aut}_{\{C\}}(M)$ is treelike (Theorem 3.2.13), and we let $\langle S_C, \leq_C\rangle$ be a tree with the action of $\mathrm{Aut}(M)$.

Pick any $a \in A$ and let $\{C \in \mathcal{C}_S : \mathrm{Path}\langle C, M \setminus C\rangle = \{a\}\}$ be enumerated by $(C_i : i \in I)$. We define S to be the tree with domain

$$\{r\} \cup \bigcup_{i \in I} S_{C_i}$$

and order

$$x \leq_S y \text{ iff } \begin{cases} x = r & \text{or} \\ x \leq_{C_i} y & \end{cases}$$

S is independent of our choice of a because A is an orbit.

To find T, we define

$$\mathcal{C}_T := \{C \in (M \setminus A)/\sim_A :\ \mathrm{Path}\langle C, M \setminus C\rangle = \emptyset\}$$

Note that if $C, D \in \mathcal{C}_T$ are such that $a_C = a_D$ and $a'_C = a'_D$ then $C = D$, as there is a path from a_C to a'_C contained in both C and D.

Let $C \in \mathcal{C}_T$. Any automorphism of M that fixes C must also fix a_C and a'_C, and hence fixes $\mathrm{Path}\langle a_C, a'_C\rangle$ set-wise, so we may introduce a unary predicate L which is realised exactly on $\mathrm{Path}\langle a_C, a'_C\rangle$. We also use the symbol L to denote $\mathrm{Path}\langle a_C, a'_C\rangle$. Since a path cannot embed Alt, the set of points realising L is treelike, and indeed the resulting tree is a linear order, which we call L_T with ordering $\leq_L$.

Note that each of the elements of $(C \setminus L)/ \sim_L$ is also treelike, for the same reasons that the members of $\mathcal{C}_S$ are treelike. We enumerate the equivalence classes of $C \setminus L$ as $(D_j \,:\, j \in J)$, denote the tree which correspond to D_j by T_j, and for each j we partition L into

$$L'_j := \{l \in L \,:\, l \in \text{Path}\langle D_j, a_C\rangle\} \qquad \text{and}$$
$$L_j := \{l \in L \,:\, l \in \text{Path}\langle D_j, a'_C\rangle\} \setminus L'_j$$

Finally we are in a position to define our candidate for (T, L). The domain is

$$L_T \cup \bigcup_{j \in J} T_j$$

while the ordering is:

$$x \leq_T y \Leftrightarrow \begin{cases} x \leq_L y & \text{or} \\ x \leq_{T_j} y & \text{or} \\ y \in T_i \text{ and } x \in L_j & \end{cases}$$

and the predicate L is carried across from C. The (T, L) are independent of our choice of element from A_{ap} as Aut(M) acts cone transitively on A.

We now have candidates for X, S and (T, L).

Given $\phi \in$ Aut$(\mathrm{Dec}(X, S, (T, L))$ we seek to show how that ϕ can be viewed as an automorphism of M. Since ϕ preserves X setwise (Lemma 5.1.5), it preserves A.

Aut(M) acts cone transitively on A, so given any two $x, y \in A$ there is an automorphism of M that maps x to y, hence mapping $\{C \in \mathcal{C}_S \,:\, \text{Path}\langle C, M \setminus C\rangle = \{x\}\}$ to $\{C \in \mathcal{C}_S \,:\, \text{Path}\langle C, M \setminus C\rangle = \{y\}\}$. Therefore

$$\bigcup\{C \in \mathcal{C}_S \,:\, \text{Path}\langle C, M \setminus C\rangle = \{x\}\} \cong \bigcup\{C \in \mathcal{C}_S \,:\, \text{Path}\langle C, M \setminus C\rangle = \{y\}\}$$

By construction

$$\text{Aut}(S) \cong_A \text{Aut}(\bigcup\{C \in \mathcal{C}_S \ : \ \text{Path}\langle C, M \setminus C\rangle = \{x\}\})$$

So ϕ acts as an automorphism of $\bigcup \mathcal{C}_S$.

Aut(M) acts cone transitively on A, so given any two $(x_0, y_0), (x_1, y_1) \in A_{ap}$ there is an automorphism of M that maps (x_0, y_0) to (x_1, y_1). Each $C \in \mathcal{C}_T$ is uniquely determined by if a_C and a'_C therefore if $C_0, C_1 \in \mathcal{C}_T$ then $C_0 \cong C_1$. By construction, for all $C \in \mathcal{C}_T$

$$\text{Aut}(T) \cong_A \text{Aut}(C)$$

So ϕ acts as an automorphism of $\bigcup \mathcal{C}_T$.

Therefore every automorphism of $\text{Dec}(X, S, (T, L))$ is also an automorphism of M.

If ϕ is an automorphism of M then it preserves A, and thus X, and since every element of $\mathcal{C}_S$ and $\mathcal{C}_T$ is isomorphic to S or T respectively, it is also an automorphism of $\text{Dec}(X, S, (T, L))$. $\square$

Definition 5.1.7 *Given an abstract group G and a permutation group $(H, S, \mu(h, s))$ their wreath product, written as $G \wr_S H$, is the abstract group on domain*

$$H \times \{\eta : S \to G\}$$

We use $\eta(s)$ to denote the function $s \mapsto \eta(s)$, and so $\eta(s_0 s)$ is the function $s \mapsto \eta(s_0 s)$. The group operation of $G \wr_S H$ is given by

$$(h_0, \eta_0(x))(h_1, \eta_1(x)) = (h_0 h_1, \eta_0(\mu(h_1^{-1}, x))\eta_1(x))$$

Definition 5.1.8 *Let $X \in K_{Cone}$ and let $S, (T, L) \in K_{Rub}$. We introduce the notation*

$$W(X, S, (T, L)) := \mathrm{Aut}((T, L)) \wr_{X_{ap}} (\mathrm{Aut}(S) \wr_X \mathrm{Aut}(X))$$

where the action of $\mathrm{Aut}(S) \wr_X \mathrm{Aut}(X)$ *on X_{ap} is given by*

$$(\phi, \eta)(x, y) = (\phi(x), \phi(y))$$

When only one $W(X, S, (T, L))$ is being discussed, we may denote it as W for brevity.

Proposition 5.1.9 *Let X be a cone transitive CFPO and let S and (T, L) be trees.*

$$\mathrm{Aut}(\mathrm{Dec}(X, S, (T, L)) \cong W(X, S, (T, L))$$

Proof

Even through we regard $W(X, S, (T, L))$ as an abstract group, it has a natural action on $\mathrm{Dec}(X, S, (T, L))$, which we will call μ. We introduce the notation I^x_y for the identity map from S_x to S_y, and $I^{(x,y)}_{(w,z)}$ for the identity map from $T_{(x,y)}$ to $T_{(w,z)}$, and define μ as follows:

$$\mu((\phi, \eta, \zeta), x) = \begin{cases} (\phi(x)) & \text{if} \quad x \in X \\ I^{\alpha}_{\phi(\alpha)}(\eta(\alpha)(x)) & \text{if} \quad x \in S_\alpha \\ I^{(\alpha,\beta)}_{\phi(\alpha,\beta)}(\zeta((\alpha, \beta))(x)) & \text{if} \quad x \in T_{(\alpha,\beta)} \end{cases}$$

For any ϕ, η and ζ the function $x \mapsto \mu((\phi, \eta, \zeta), x)$ is an automorphism, as ϕ is an automorphism and for every α and β both

$$I^{\alpha}_{\phi(\alpha)}(\eta(\alpha)(x)) : S_\alpha \to S_{\phi(\alpha)} \quad \text{and} \quad I^{(\alpha,\beta)}_{\phi(\alpha,\beta)}(\zeta((\alpha, \beta))(x)) : T_{(\alpha,\beta)} \to T_{\phi(\alpha,\beta)}$$

are isomorphisms. Additionally, each (ϕ, η, ζ) results in a unique automorphism. To see this suppose for a contradiction that

$$\forall x\, \mu((\phi_0, \eta_0, \zeta_0), x) = \mu((\phi_1, \eta_1, \zeta_1), x)$$

Since this is for all x, it is true for all $x \in X$ in particular, and thus $\phi_0 = \phi_1$. We also have

$$\forall x \in S_\alpha\, \eta_0(\alpha)(x) = \eta_1(\alpha)(x) \quad \text{and} \quad \forall x \in T_{(\alpha,\beta)}\, \zeta_0((\alpha, \beta))(x) = \zeta_1((\alpha, \beta))(x)$$

and thus $\eta_0 = \eta_1$ and $\zeta_0 = \zeta_1$. Finally, if we are able to show that every automorphism of $\mathrm{Dec}(X, S, (T, L))$ can be represented in this way, we will have proved this proposition.

Let ψ be an automorphism of $\mathrm{Dec}(X, S, (T, L))$. We set $\phi := \psi|_X$ and we set the function components as follows:

$$\eta(\alpha) = \psi|_{S_\alpha} \quad \text{and} \quad \zeta((\alpha, \beta)) = \psi|_{T_{(\alpha,\beta)}}$$

Which gives an element of $W(X, S, (T, L))$ whose action on $\mathrm{Dec}(X, S, (T, L))$ via μ is the same as ψ. Thus the map

$$\begin{array}{rcl} W(X, S, (T, L)) & \to & \text{Aut}(\mathrm{Dec}(X, S, (T, L))) \\ (\phi, \eta, \zeta) & \mapsto & \mu((\phi, \eta, \zeta), x) \end{array}$$

is bijective and, since μ is a group action, an isomorphism. $\square$

5.2 Interpreting Inside a Wreath Product

When we interpreted $M \in K_{Cone}$ inside its automorphism group, we made use of the subgroups isomorphic to A_5. These subgroups still exist in the automorphism groups of the CFPOs we obtained through decoration, as Aut$(X) \leq W(X, S, (T, L))$.

If we can adapt the interpretation so that it ignores the decoration then we will be able to recover X. Subsection 5.2.1 works towards this by adding in a few clauses to the formulas of Chapter 4.

Subsection 5.2.2 gives second-order formulas that define subgroups of $W(X, S, (T, L))$ isomorphic to Aut(S) and Aut(T, L).

5.2.1 Reconstructing X

Lemma 5.2.1 *Recall that $A_5(\bar{f})$ is the formula that states that $\bar{f}$ satisfies the elementary diagram of A_5. If $W \models A_5(\bar{f})$ then $\bar{f}$ fixes an element of $X \subset \mathrm{Dec}(X, S, (T, L))$.*

Proof
The automorphisms of $\mathrm{Dec}(X, S, (T, L))$ preserve X (Lemma 5.1.5), so if $\bar{f}|_X \neq id$ then $\bar{f}$ has a fixed point in X by Lemma 4.2.2. If $\bar{f}|_X = id$ then $\bar{f}$ fixes X. □

Lemma 5.2.2 *Many of the formulas in Chapter 4 retain either their exact meaning, or something very similar, in $W(X, S, (T, L))$, which we call W.*

1. *If $W \models \mathrm{Indec}(\bar{f})$ then $\bigcup_{x,y \in \text{supp}(\bar{f})} \text{Path}\langle x, y\rangle \setminus \text{supp}(\bar{f})$ is a singleton, which we call f.*

2. *If $W \models \mathrm{Disj}(\bar{f}, \bar{g})$ then the support of $\bar{f}$ and $\bar{g}$ are disjoint.*

3. *If $W \models [\text{supp}(\bar{f}) \sqsubset \text{supp}(\bar{g})]$ then $\text{supp}(\bar{f}) \subset \text{supp}(\bar{g})$.*

4. *If $W \models \mathrm{SamePD}(\bar{f}, \bar{g})$ then either:*

   - *$f = g$,*

- *$f \in X$ and $g \in S_f$ or $g \in T_{(f,h)}$ for some h, or*
- *$g \in X$ and $f \in S_f$ or $f \in T_{(g,h)}$ for some h.*

Proof

Note that for all $\phi \in W$ if $x \in \text{supp}(\phi)$ then $S_x, T_{(x,y)} \subset \text{supp}(\phi)$ for all y.

1. Suppose $W \models \text{Indec}(\bar{f})$. If $\bar{f}|_X \neq id$ then the singleton we found in Lemma 4.2.2 (unique by Proposition 4.2.12) works in this context.

   Suppose that $\bar{f}|_X = id$. Since $\bar{f}$ is indecomposable, then either supp$(\bar{f}) \subseteq S_x$ or supp$(\bar{f}) \subseteq T_{(x,y)}$ for some $x \in X$ or $(x,y) \in X_{ap}$. We use x_f to denote the singleton Path$\langle$supp$(\bar{f}, \text{Dec}(X, S, (T, L)) \setminus$ supp$(\bar{f})\rangle$.

2. The proof of Lemma 4.2.13 does not require serious adaptation for this context.

3. If $W \models$ [supp$(\bar{f}) \sqsubset$ supp$(\bar{g})$] and at least one of x_f and x_g is in $W \setminus X$ then either supp$(\bar{f}) \subset$ supp$(\bar{g})$ or supp$(\bar{f}) \subset$ supp$(\bar{g})$, and the argument in the appropriate case of the proof of 4.2.15 suffices.

4. If both x_f and x_g are contained in X then Lemma 4.2.17 shows that $x_f = x_g$. If both x_f and x_g are in $W \setminus X$ then the proof of Lemma 4.2.17 shows that $x_f = x_g$.

   Suppose that $x_f \in X$ and $x_g \in W \setminus X$. If $x_g \in S_y$ or $x_g \in T_{(y,y')}$ for $y \neq x_f$ then the same witness that observes that $W \models \neg\text{SamePD}(\bar{f}, \bar{h})$ shows that $W \models \neg\text{SamePD}(\bar{f}, \bar{g})$, so $x_g \in S_f$ or $x_g \in T_{(x_f,y)}$ for some y.

   Similarly, if $x_g \in X$ and $x_f \in W \setminus X$ then $x_f \in S_f$ or $x_f \in T_{(x_g,y)}$ for some y.

□

Definition 5.2.3 *Let* $\mathrm{MeetsX}(\bar{f})$ *be the following formula*

$$\mathrm{Indec}(\bar{f}) \wedge \exists \bar{g} \left(\begin{array}{c} \neg \text{disj}(\bar{f}, \bar{g}) \wedge \neg \mathrm{SamePD}(\bar{f}, \bar{g})) \wedge \\ \neg [\text{supp}(\bar{f}) \sqsubset \text{supp}(\bar{g})] \wedge \neg [\text{supp}(\bar{g}) \sqsubset \text{supp}(\bar{f})] \end{array} \right)$$

Lemma 5.2.4 $W \models \mathrm{MeetsX}(\bar{f})$ *if and only if* $\text{supp}(\bar{f}) \cap X \neq \emptyset$.

Proof

First we suppose for a contradiction that both $W \models \mathrm{MeetsX}(\bar{f})$ and $\text{supp}(\bar{f}) \cap X = \emptyset$. Since $W \models \neg \mathrm{SamePD}(\bar{f}, \bar{g}))$, we know that $x_f \neq x_g$.

Since $\text{supp}(\bar{f}) \cap X = \emptyset$, the point x_f must be contained in one of the trees that we decorated X with, so $W \models \neg \text{disj}(\bar{f}, \bar{g})$ implies that $\text{supp}(\bar{f}) \subseteq \text{supp}(\bar{g})$ or $\text{supp}(\bar{g}) \subseteq \text{supp}(\bar{f})$, giving a contradiction.

Suppose $\text{supp}(\bar{f}) \cap X \neq \emptyset$. We can find a $\bar{g}$ to witness $W \models \mathrm{MeetsX}(\bar{f})$ by taking any tuple that fixes a point inside $\text{supp}(\bar{f})$ which moves x_f. □

Definition 5.2.5 *Let* $\mathrm{RepPointDec}(\bar{f}_0, \bar{f}_1)$ *be the formula*

$$\begin{array}{c} \text{disj}(\bar{f}_0, \bar{f}_1) \wedge \mathrm{MeetsX}(\bar{f}_0) \wedge \mathrm{MeetsX}(\bar{f}_1) \wedge \\ \forall \bar{g} \exists \bar{h} \left(\left(\begin{array}{c} \mathrm{MeetsX}(\bar{g}) \\ \mathrm{MeetsX}(\bar{h}) \end{array} \wedge \right) \rightarrow \neg \left(\text{disj}(\bar{g}, \bar{h}) \wedge \left(\begin{array}{c} \mathrm{SamePD}(\bar{f}_0, \bar{h}) \\ \mathrm{SamePD}(\bar{f}_1, \bar{h}) \end{array} \vee \right) \right) \right) \end{array}$$

Proposition 5.2.6 $W \models \mathrm{RepPointDec}(f_0, f_1)$ *if and only if* $x_{f_0} = x_{f_1} \in X$.

Proof

$\mathrm{RepPointDec}$ is only realised by tuples that satisfy MeetsX, so Lemma 4.2.17 shows that this proposition is true. □

Definition 5.2.7 *$\mathrm{EquivRepPointDec}$, $\mathrm{Temp1PBDec}$, $\mathrm{Temp2PBDec}$, $\mathrm{PathBetweenDec}$, $\mathrm{RelatedDec}$ and BDec are the formulas $\mathrm{EquivRepPoint}$, $\mathrm{Temp1PB}$, $\mathrm{Temp2PB}$, $\mathrm{PathBetween}$, $\mathrm{Related}$ and B with every instance of $\mathrm{RepPoint}$ replaced by $\mathrm{RepPointDec}$.*

Theorem 5.2.8 *$(\mathrm{RepPointDec}, \mathrm{EquivRepPointDec}, \mathrm{BDec})$ is a first order interpretation of (X, B) inside W.*

Proof

Since the other formulas in the interpretation only quantify over the points that realise $\mathrm{RepPointDec}$, the proofs of Subsection 4.2.2 apply directly. □

5.2.2 Reconstructing S and (T, L)

Now that we are able to refer to X inside W, we can exploit this fact to define subgroups isomorphic to Aut(S) and Aut(T, L) inside W. While the initial stages of the definitions are first order, I am unable to make the final jump without using second order logic.

Definition 5.2.9 *Let $\mathrm{FunctionPart}(\phi)$ be the formula*

$$\forall \bar{f}_0, \bar{f}_1, \bar{g}_0, \bar{g}_1 \left(\begin{array}{c} \left(\begin{array}{c} \mathrm{RepPointDec}(\bar{f}_0, \bar{f}_1) \\ \mathrm{RepPointDec}(\bar{g}_0, \bar{g}_1) \end{array} \wedge \right) \wedge \left(\begin{array}{c} (\bar{f}_0^{\phi} = \bar{g}_0 \wedge \bar{f}_1^{\phi} = \bar{g}_1) \\ (\bar{f}_1^{\phi} = \bar{g}_0 \wedge \bar{f}_0^{\phi} = \bar{g}_1) \end{array} \vee \right) \\ \rightarrow \mathrm{EquivRepPointDec}(\bar{f}_0, \bar{f}_1; \bar{g}_0, \bar{g}_1) \end{array} \right)$$

Lemma 5.2.10 *$W \models \mathrm{FunctionPart}(\phi)$ if and only if ϕ fixes X point-wise.*

Proof

ϕ fixes X point-wise if and only if $\psi(x) = \psi^{\phi}(x)$ for all $x \in X$. □

Proposition 5.2.11 $\mathrm{FunctionPart}(W) \cong \prod_{i \in X} \mathrm{Aut}(S_i) \times \prod_{(i,j) \in X_{ap}} \mathrm{Aut}(T_{(i,j)}, L_{(i,j)})$

Proof

$\phi \in \mathrm{FunctionPart}(W)$ if and only if ϕ fixes X point-wise, i.e. is of the form (id, η, ζ). □

Definition 5.2.12 $\mathrm{AboveWitness}(\phi; \bar{f}_0, \bar{f}_1)$ *is the formula*

$$\forall \bar{g}_0, \bar{g}_1 (\mathrm{EquivRepPointDec}(\bar{g}_0, \bar{g}_1; \bar{g}_0^{\phi}, \bar{g}_1^{\phi}) \rightarrow \mathrm{EquivRepPointDec}(\bar{f}_0, \bar{f}_1; \bar{g}_0, \bar{g}_1))$$

Definition 5.2.13 $\mathrm{BetweenWitness}(\phi; \bar{f}_0, \bar{f}_1, \bar{g}_0, \bar{g}_1)$ *is the formula*

$$\begin{array}{c} \mathrm{RelatedDec}(f, g) \wedge (\forall h \neg \mathrm{PathBetweenDec}(h; f, g)) \wedge \\ \forall \bar{h}_0, \bar{h}_1 \left(\begin{array}{l} \mathrm{EquivRepPointDec}(\bar{h}_0, \bar{h}_1; \bar{h}_0^{\phi}, \bar{h}_1^{\phi}) \rightarrow \\ \qquad \left(\begin{array}{l} \mathrm{EquivRepPointDec}(\bar{f}_0, \bar{f}_1; \bar{h}_0, \bar{h}_1) \\ \mathrm{EquivRepPointDec}(\bar{g}_0, \bar{g}_1; \bar{h}_0, \bar{h}_1) \end{array} \vee \right) \end{array} \right) \Bigg) \end{array}$$

Lemma 5.2.14 *If* $W \models \mathrm{AboveWitness}(\phi; \bar{f}_0, \bar{f}_1)$ *then for all* $g \in X$

$$\phi(g) = g \Leftrightarrow g = f$$

If $W \models \mathrm{BetweenWitness}(\phi; \bar{f}_0, \bar{f}_1)$ *then* f *is either a successor or predecessor of* g *and for all* $h \in X$

$$\phi(h) = h \Leftrightarrow h = f \text{ or } h = g$$

Proof

This is follows from the fact that if $(\bar{f}_0, \bar{f}_1)$ represents f then $(\bar{f}_0^{\phi}, \bar{f}_1^{\phi})$ represents $\phi(f)$. □

Finally we resort to second order logic to define subgroups of $\mathrm{FunctionPart}(W)$ isomorphic to $\mathrm{Aut}(S)$ and $\mathrm{Aut}(T, L)$.

Definition 5.2.15

1. $\mathrm{AboveTemp1}(A;f)$ *is the second order formula*

$$A \lneqq \mathrm{FunctionPart}(W) \wedge \forall\phi\Big(\ \mathrm{AboveWitness}(\phi;f)\ \ \rightarrow\ \ \phi A = A\ \Big)$$

$\mathrm{AboveTemp2}(A;f)$ *is the second order formula*

$$\begin{gathered}\mathrm{AboveTemp1}(A;f)\wedge\\ \forall B, C((\mathrm{AboveTemp1}(B;f)\wedge \mathrm{AboveTemp1}(C,f))\rightarrow BC\neq A)\end{gathered}$$

and $\mathrm{AboveTemp3}(A,f)$ *is the formula*

$$\begin{gathered}\mathrm{AboveTemp2}(A;f)\wedge\\ \forall B\neq A(\mathrm{AboveTemp2}(B;f)\rightarrow \neg\exists\phi(\phi(B)\leq A)\end{gathered}$$

2. $\mathrm{BetweenTemp1}(A;f,g)$ *is the second order formula*

$$A \lneqq \mathrm{FunctionPart}(W) \wedge \forall\phi\Big(\ \mathrm{BetweenWitness}(\phi;f,g)\ \ \rightarrow\ \ \phi A = A\ \Big)$$

and $\mathrm{BetweenTemp2}(A;f,g)$ *is the second order formula*

$$\begin{gathered}\mathrm{BetweenTemp1}(A;f)\wedge\\ \forall B, C((\mathrm{BetweenTemp1}(B;f,g)\wedge \mathrm{BetweenTemp1}(C,f))\rightarrow BC\neq A)\end{gathered}$$

3. $\mathrm{Between}(A,f,g)$ *is the second order formula*

$$\begin{gathered}\mathrm{BetweenTemp2}(A;f,g)\wedge\\ \forall B\neq A(\mathrm{BetweenTemp2}(B;f,g)\rightarrow \neg\exists\phi(\phi(B)\leq A))\end{gathered}$$

$\mathrm{Above}(A,f)$ *is the second order formula*

$$AboveTemp3(A,f)\wedge\forall B, g(\mathrm{Between}(B;f,g)\rightarrow\neg(A\subset B))$$

Theorem 5.2.16

1. *If* $M \models \mathrm{Above}(A; f)$ *then* $A \cong \mathrm{Aut}(S)$.

2. *If* $M \models \mathrm{Between}(A; f, g)$ *then* $A \cong \mathrm{Aut}(T, L)$.

Proof

Let π_x and $\pi_{(x,y)}$ be the projection functions from

$$\prod_{i \in X} \mathrm{Aut}(S_i) \times \prod_{(i,j) \in X_{ap}} \mathrm{Aut}(T_{(i,j)}, L_{(i,j)})$$

to $\mathrm{Aut}(S_x)$ and $\mathrm{Aut}(T_{(x,y)}, L_{(x,y)})$ respectively. Let B be such that

$$W \models \mathrm{AboveTemp1}(B, f)$$

Since for all ϕ such that $W \models \mathrm{AboveWitness}(\phi; f)$

$$\pi_x(B) = \pi_{\phi(x)}(B) \quad \text{and} \quad \pi_{(x,y)}(B) = \pi_{(\phi(x),\phi(y))}(B)$$

then for any $\phi \in W$ we may obtain by patching a ψ such that $\psi|_{S_f} = \phi|_{S_f}$ and

$$W \models \mathrm{AboveWitness}(\phi; f)$$

and so for any $a \in \mathrm{Aut}(S)$ there is a ψ such that $\pi_f(\psi) = a$, and since A is a subgroup preserved under composition with ψ, we know that $a \in \pi_f(B)$.

Variations on this argument show that for all x

$$\pi_x(B) = \mathrm{Aut}(S) \text{ or} \{id\} \quad \text{and} \quad \pi_{(x,y)}(B) = \mathrm{Aut}(T, L) \text{ or} \{id\}$$

Similarly, if $W \models \mathrm{BetweenTemp1}(B, f, g) \wedge \mathrm{BetweenWitness}(\phi; f, g)$ then

$$\pi_x(B) = \pi_{\phi(x)}(B) \quad \text{and} \quad \pi_{(x,y)}(B) = \pi_{(\phi(x),\phi(y))}(B)$$

and

$$\pi_x(B) = \mathrm{Aut}(S) \,\mathrm{or}\{id\} \quad \text{and} \quad \pi_{(x,y)}(B) = \mathrm{Aut}(T, L) \,\mathrm{or}\{id\}$$

If $W \models \mathrm{AboveTemp2}(A, f)$ then A cannot be realised as the composition of two subgroups that satisfy $\mathrm{AboveTemp1}$ and so if A is not the identity on S_x then A is the identity on all the $T_{(z_0,z_1)}$, and is not the identity on S_y if and only if

$$\exists \phi \in \mathrm{Aut}_f(W)\; \phi(x) = y$$

If A is not the identity on $T_{(z_0,z_1)}$ then A is the identity on all the S_x, and is not the identity on $T_{(y_0,y_1)}$ if and only if

$$\exists \phi \in \mathrm{Aut}_f(W)\; \phi((z_0, z_1)) = (y_0, y_1)$$

Similarly if $W \models \mathrm{BetweenTemp2}(A, f, g)$ then if A is not the identity on S_x then A is the identity on all the $T_{(z_0,z_1)}$, and is not the identity on S_y if and only if

$$\exists \phi \in \mathrm{Aut}_{(f,g)}(W)\; \phi(x) = y$$

If A is not the identity on $T_{(z_0,z_1)}$ then A is the identity on all the S_x, and is not the identity on $T_{(y_0,y_1)}$ if and only if

$$\exists \phi \in \mathrm{Aut}_{(f,g)}(W)\; \phi((z_0, z_1)) = (y_0, y_1)$$

If $W \models \text{AboveTemp3}(A, f)$ then given any $B \neq A$ that satisfies $\text{AboveTemp2}(A, f, g)$, we are unable to map B into A using members of W (other embeddings may exists, but not inside W). This means that either A does not act as the identity on S_f only, or A does not act as the identity on $\bigcup T_{(f,g)}$.

However, with $\text{Between}(A, f, g)$, the only family that does not permit B that satisfy BetweenTemp2 is the one that only acts non-trivially on $T_{(f,g)}$. Therefore

$$W \models \text{Between}(A, f, g) \Rightarrow A \cong \textsf{Aut}(T, L)$$

If $W \models \text{Above}(A, f)$ then A does not contain any subset that satisfies Between, so $A \cong \textsf{Aut}(S)$. □

5.3 Final Results

Definition 5.3.1 *Let Z be the one element partial order.*

$$K_{Dec} := \left\{ M \ : \begin{array}{c} \exists X \in K_{Cone} \cup \{\emptyset, Z\} \ \exists S, (T, L) \in K_{Rub} \cup \{\emptyset\} \\ M = \text{Dec}(X, S, (T, L)) \end{array} \right\}$$

Note that $\text{Dec}(Z, S, (T, L))$ equals S if S is non-empty, or Z if S is empty. $\text{Dec}(\emptyset, S, (T, L))$ is the empty partial order for any S and (T, L), and $\text{Dec}(X, \emptyset, \emptyset) = X$ for any $X \in K_{Cone}$.

We allow Z and $\emptyset$ as arguments in $\text{Dec}(X, S, (T, L))$ to ensure that $K_{Cone}, K_{Rub} \subseteq K_{Dec}$.

Theorem 5.3.2 *K_{Dec} is faithful.*

Proof

Let $\mathrm{Dec}(X_0, S_0, (T_0, L_0)), \mathrm{Dec}(X_1, S_1, (T_1, L_1)) \in K_{Dec}$ and assume that

$$\text{Aut}(\mathrm{Dec}(X_0, S_0, (T_0, L_0))) \cong \text{Aut}(\mathrm{Dec}(X_1, S_1, (T_1, L_1)))$$

Theorem 5.2.8 shows that $(X_0, B) \cong (X_1, B)$. For all $M \in K_{Cone}$

$$M^* \in K_{Cone} \Rightarrow M \cong M^*$$

Therefore $X_0 \cong X_1$.

Theorem 5.2.16 shows that $S_0 \cong S_1$ and $(T_0, L_0) \cong (T_1, L_1)$. $\Box$

Theorem 5.3.3 *Let M be a CFPO, let A be a 1-orbit such that* Aut(M) *acts cone transitively on A, and for any $B \subset M$ let $\sim_B$ be the equivalence relation $x \sim y \Leftrightarrow$* Path$\langle x, y\rangle \cap B = \emptyset$*. We let $C \in (M \setminus A)/\sim_A$, and describe two conditions.*

1. *If* Path$\langle C, M \setminus C\rangle \neq \emptyset$ *then there is an $a_c \in A$ such that*

$$\text{Path}\langle C, M \setminus C\rangle = \{a_C\}$$

   This says that if there is only one way to go from C to $M \setminus C$ then C is attached to a_c.

2. *If* Path$\langle C, M \setminus C\rangle = \emptyset$ *then:*
   (a) *$(M \setminus C)/\sim_C$ has exactly two elements which we call B_C and B'_C; and*
   (b) *there is $(a_C, a'_C) \in A_{ap}$ such that*

$$\text{Path}\langle C, B_C\rangle = \{a_C\} \quad \text{and} \quad \text{Path}\langle C, B'_C\rangle = \{a'_C\}$$

This says that if there is more than one way to go from C to $M \setminus C$ then C lies between an adjacent pair of A.

If every $C \in (M \setminus A)/ \sim_A$ satisfy both 1. and 2. then there is an $N \in K_{Dec}$ such that $\text{Aut}(M) \cong_A \text{Aut}(N)$.

This final theorem is a reformulation of Theorem 5.1.6. It describes the properties possessed by the members of K_{Dec} which aren't members of K_{Cone} or K_{Rub}. While restrictive, this is much wider than the K_{Cone}, and is as wide as this thesis can manage!

Chapter 6

Further Questions

6.1 Extensions

As proud as I am of the results contained in this thesis (whether justly or unjustly is up to you) there is a glaring deficiency: they do not reconstruct the full class of CFPOs, merely a well-behaved subclass. Chapter 3 does exactly what is asked, Chapter 5 gives the hand it's dealt a good try; the faults lies with Chapter 4.

The most immediate failing is the assumption that both $ro \uparrow (M)$ and $ro \downarrow (M)$ are at least than 5.

Question 6.1.1 *Is there an interpretation that works for cone transitive CFPOs where one of* $ro \uparrow (M)$ *and* $ro \downarrow (M)$ *is less than 5?*

The second transitivity condition of Chapter 4 is both strong and unnatural; simply assuming 1-transitivity is much weaker. In her Ph.D. thesis, Chicot gives a classification of the countable 1-transitive trees [4]. It is an impressive result; there are $2^{\aleph_0}$ many, and they are extremely wild. They may even have multiple non-isomorphic maximal branches, which are not even 1-transitive!

The maximal branches do have to be 'lower isomorphic', i.e. any two principal initial sections of any two maximal branches of a 1-transitive tree must be isomorphic. This suggests to me that the maximal chains of some 1-transitive CFPOs may be only 'interval isomorphic', meaning that any two intervals of the maximal chains are isomorphic.

It would be a wonderful thing to reconstruct the 1-transitive CFPOs. The frustrating thing is that this second condition is so necessary to the method that I don't believe there is a way to eliminate it. How can one use the subgroups isomorphic to A_5 without assuming that there are any?

Nonetheless, this presents a project:

Question 6.1.2 *Classify the (countable) 1-transitive CFPOs.*

Perhaps a method for reconstruction would present itself if they were better understood. But as I said, the classification of the 1-transitive trees was an impressive feat. I certainly do not have the energy for it at present. A more modest objective would be

Question 6.1.3 *Give an example of a 1-transitive CFPO where* $\mathrm{Aut}(M)$ *is unable to act as* A_5 *on the cones of a point, but* $ro\uparrow(M)$ *and* $ro\downarrow(M)$ *are greater than or equal to 5.*

Even if we had a reconstruction of the class of 1-transitive CFPOs, we would not be able to use decoration to reconstruct the whole class of CFPOs.

Example 6.1.4 $W(Alt, \mathbb{Z}, \emptyset)$ *is not the automorphism group of a tree, nor a 1-transitive CFPO, nor the automorphism group of the decoration of a 1-transitive CFPO by trees.*

Which informs the next question:

Question 6.1.5 *Is there a minimal class of CFPOs such that every automorphism group of a CFPO occurs as the automorphism group of a decoration of a member of the class by trees?*

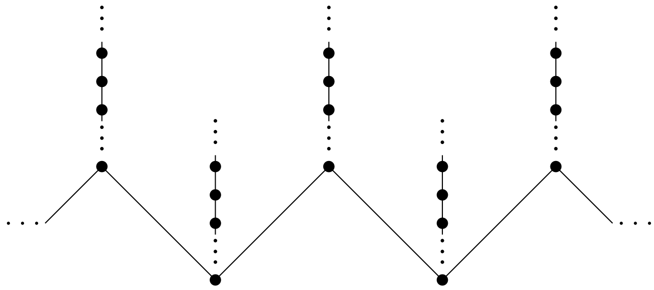

Figure 6.1: $Dec(Alt, \mathbb{Z}, \emptyset)$

While I find these questions interesting, really a method for reconstruction is available which sidesteps all these considerations.

Question 6.1.6 *Use locally moving groups to reconstruct the CFPOs.*

6.2 Ancillary Questions

In various variations, Lemma 2.2.12 is used frequently throughout this thesis. The CFPOs are not the widest class of partial orders where this is true, for example, instead of insisting that between any two points there is a unique path, we could insist that between any two points there are finitely many paths, and the same proof would work.

Definition 6.2.1 *Let $\mathcal{P}$ be the class of partial orders such that for all tuples $\bar{y}$ there is a supertuple $(x_0, \dots x_{n-1})$ such that*

$$\mathrm{qftp}(x_0, \dots, x_{n-1}) \cup \bigcup_{x_i \leq \geq x_j} \mathrm{tp}(x_i, x_j) \vdash \mathrm{tp}(x_0, \dots, x_{n-1})$$

Question 6.2.2 *Is $\mathcal{P}$ axiomatisable?*

This property is a useful tool for classifying according to various homogeneity and transitivity properties, so it seems natural to want to do the following:

Question 6.2.3 *Classify the $\aleph_0$-categorical members of $\mathcal{P}$.*

It might also be nice to extend this concept to other settings.

Question 6.2.4 *Develop analogous notions to $\mathcal{P}$ for other theories.*

In [18], as well as showing that all completions of the theory of trees are NIP, Parigot shows that the theory of a tree is stable if and only if every maximal branch has at most n elements for some $n \in \mathbb{N}$.

While I am almost certain that this is also for the CFPOs, there is perhaps scope for defining an infinite order even when the maximal branches are finite, for example in Alt the pairs (a_0, a_{2n}) have a natural order. While I would be shocked if this order is definable, I cannot see a way to prove that it is undefinable in all CFPOs.

Question 6.2.5 *Is a CFPO stable if and only if all its maximal branches have at most n elements for some fixed $n \in \mathbb{N}$.*

Bibliography

[1] G. Ahlbrandt and M. Ziegler. Quasi finitely axiomatisable totally categorical theories. *Annals of Pure and Applied Logic*, v.30:63–82, 1986.

[2] G. Campero-Arena and J. K. Truss. Countable, 1-transitive, coloured linear orderings i. *J. of Combinatorial Theory*, v.105 i.1:1–13, 2004.

[3] E. Casanovas. Nip formulas and theories. *http://www.ub.edu/modeltheory/documentos/nip.pdf*, 2010.

[4] K. M. Chicot. *Transitve Properties of Countable Trees*. PhD thesis, University of Leeds, 2004.

[5] P. Creed, J. K. Truss, and R. Warren. The structure of k-cs-transitive cycle-free partial orders with infinite chains. *Math. Proc. of the Cambridge Phil. Society*, 126:175–194, 1998.

[6] M. Droste. Structure of partially ordered sets with transitive automorphism groups. *Memoirs of the A.M.S.*, v.57, 1985.

[7] M. Droste. Partially ordered sets with transitive automorphism groups. *Proc. London Math. Soc.*, 54:517–543, 1987.

[8] M. Droste, W. C. Holland, and H. D. Macpherson. Automorphism groups of infinite semilinear orders (i). *Proceedings of the L.M.S.*, v.58:454–478, 1989.

[9] M. Droste, W. C. Holland, and H. D. Macpherson. Automorphism groups of infinite semilinear orders (ii). *Proceedings of the L.M.S.*, v.58:479–494, 1989.

[10] M. Droste, W. C. Holland, and H. D. Macpherson. Automorphism groups of infinite semilinear orders: Normal subgroups and commutators. *Cambridge Journal of Mathematics*, v.43 (4):721–737, 1991.

[11] M. Droste and H. D. Macpherson. On k-homogeneous posets and graphs. *J. of Combinatorial Theory*, 56:1–15, 1991.

[12] M. Giraudet and J. Truss. Recovering ordered structures from quotients of their automorphism groups. *J. of Symbolic Logic*, v.68 No. 4:1189–1198, 2003.

[13] J. Goodrick. A monotonicity theorem for dp-minimal densely ordered groups. *J. of Symbolic Logic*, v.75:221–238, 2010.

[14] R. Gray and J. K. Truss. Cycle-free partial orders and ends of graphs. *Math. Proc. of the Cambridge Phil. Society*, 146:535–550, 2009.

[15] W. Hodges. *Model Theory*. Cambridge University Press, 1993.

[16] A. C. Morel. A class of relation types isomorphic to the ordinals. *Mich. Math. J.*, 12:203–215, 1965.

[17] F. Mwesigye and J. K. Truss. Countable categorical coloured linear orders. *Mathematical Logic Quarterly*, v.56 No. 2:159–163, 2010.

[18] M. Parigot. Théories d'arbres. *J. of Symbolic Logic*, v.47(4):841–853, 1982.

[19] B. Poizat. Théories instables. *J. of Symbolic Logic*, v.46(3):513–530, 1981.

[20] J. G. Rosenstein. $\aleph_0$-categoricity of linear orderings. *Fund. Math*, v.64:1–5, 1969.

[21] J. G. Rosenstein. *Linear Orderings*. Academic Press, 1982.

[22] M. Rubin. On the reconstruction of boolean algebras from their automorphism groups. In R. Bonnet J. Monk, editor, *Handbook of Boolean Algebras*, volume v.2, pages 547–606. North-Holland, 1989.

[23] M. Rubin. *The Reconstruction of Trees from Their Automorphism Groups*. Contemporary mathematics (American Mathematical Society), 1991.

[24] M. Rubin. On the reconstruction of $\aleph_0$-categorical structures from their automorphism groups. *Proc. London Math. Soc.*, v.69(3):225–249, 1994.

[25] M. Rubin. Locally moving groups and reconstruction problems. In W. C. Holland, editor, *Ordered Groups and Infinite Permutation Groups*, pages 121–157. Kluwer Academic Publishing, 1996.

[26] M. Rubin and S. McCleary. Locally moving groups and the reconstructon problem for chains and circles. *arXiv:math/0510122*.

[27] J. Schmerl. On $\aleph_0$-categoricity and the theory of trees. *Fundamenta Mathematicae*.

[28] J. Schmerl. Arborescent structures. ii: Interpretability. *Transactions of the A.M.S.*, 226(2):629–643, 1981.

[29] J. H. Schmerl. Countable homogeneous partially ordered sets. *Algebra Universalis*, 9, 1979.

[30] S. Shelah. Errata to: First order theory of permutation groups. *Israel J. Math.*, v.15:437–441, 1973.

[31] S. Shelah. First order theory of permutation groups. *Israel J. Math.*, v.14:149162, 1973.

[32] S. Shelah. *Classification Theory: and the Number of Non-Isomorphic Models*. North-Holland, 1978.

[33] S. Shelah and J. Truss. On distinguishing quotients of symmetric groups. *Annals of Pure and Applied Logic*, v.97:47–83, 1999.

[34] P. Simon. On dp-minimal ordered structures. *J. of Symbolic Logic*, v.76(2):448–460, 2011.

[35] J. K. Truss. On k-cs-transitive cycle-free partial orders with finite alternating chains. *Order*, 15:151–165, 1998.

[36] J. K. Truss. Elementary properties of cycle-free partial orders and their automorphism groups. *Order*, v.18:359–379, 2001.

[37] J. K. Truss. Countable homogeneous and partially homogeneous ordered structures. In *Algebra, Logic, Set Theory: Festschrieft für Ulrich Felgner zum 65. Geburtstag*. College Publications London, 2006.

[38] R. Warren. *The Structure of k-CS-transitive Cycle-free Parital Orders*. PhD thesis, University of Leeds, 1992.

[39] R. Warren. The structure of k-cs-transitive cycle-free partial orders. *Memoirs of the A.M.S.*, v.129, 1997.

Appendix A

Appendix of Formulas

This appendix contains a table of the formulas defined in Chapters 4 and 5, with the meaning that I held in mind when I coined the expression. None of the formulas have been written out in full, for good reason.

The first formula, $A_5(\bar{f})$ will for each triple $(f_i, f_j, f_k) \subseteq \bar{f}$ contain an expression either of the form

$$f_i \cdot f_j = f_k \qquad \text{or} \qquad \neg f_i \cdot f_j = f_k$$

$\bar{f}$ has 60 elements, and there are 216,000 such triples. When writing the formula $A_5(\bar{f})$, we start with one expression for one of the triples, and when we add the next we also add a conjunction symbol, a left bracket and a right bracket. Thus there are at least $5 + 215,999 \times 8 = 1,727,997$ symbols in the formula $A_5(\bar{f})$.

An extremely generous estimate for the number of symbols documents formatted as this thesis can display per line is 100, with at most 30 lines per side. The full expression of the formula $A_(\bar{f})$ would take at least 575 sides, which on its own is longer than the maximum page count prescribed by the University of Leeds' thesis regulations.

Comm contains two instances of A_5, and a third clause of similar length, so let us say that $\mathrm{Comm}(\bar{f}, \bar{g})$ takes 750 pages. Indec adds a fourth clause of similar length to A_5, so would take 1,000 pages. Disj contains two Indec's and one Comm, so takes 2,750 pages, and [supp($\bar{f}$) $\subset$ supp($\bar{g}$)] uses 3 Disj's and 2 Indec's, so takes 10,250 pages. Let's say 10,000 for simplicity.

SamePD doubles that to 20,000 pages, and RepPoint, the first formula actually in the interpretation, takes 45,000 pages, and would weigh 2 tons if printed on standard A4 paper. This does not include the ink.

So I have only used abbreviations.

1. $A_5(\bar{f})$ states that $\bar{f}$ satisfies the elementary diagram of A_5.

2. $\mathrm{Comm}(\bar{f}, \bar{g})$ insists that $\bar{f}$ and $\bar{g}$ commute.

$$\mathrm{Alt}_5(\bar{f}) \wedge \mathrm{Alt}_5(\bar{g}) \wedge \bigwedge_{\substack{f_i \in \bar{f} \\ g_j \in \bar{g}}} (f_i g_j = g_j f_i)$$

3. $\mathrm{Indec}(\bar{f})$ insists that $\bar{f}$ is indecomposable.

$$\neg \exists \bar{g}, \bar{h} (\bar{g} * \bar{h} = \bar{f} \wedge \mathrm{Comm}(\bar{g}, \bar{h}))$$

4. disj$(\bar{f}, \bar{g})$ insists that the supports of $\bar{f}$ and $\bar{g}$ are disjoint.

$$\mathrm{Indec}(\bar{f}) \wedge \mathrm{Indec}(\bar{g}) \wedge \mathrm{Comm}(\bar{f}, \bar{g})$$

5. [supp$(\bar{g}) \sqsubseteq$ supp$(\bar{f})$] insists that the support of $\bar{f}$ is contained in the support of $\bar{g}$.

$$\begin{array}{ll} \mathrm{Indec}(\bar{f}) \wedge \mathrm{Indec}(\bar{g}) \wedge \neg\text{disj}(\bar{f}, \bar{g}) & \wedge \\ \neg \exists \phi [\neg \text{disj}(\bar{f}^{\phi}, \bar{f}) \wedge \text{disj}(\bar{g}^{\phi}, \bar{g})] & \wedge \\ \neg \exists \phi (\bar{f}^{\phi} = \bar{f} \wedge \bar{g}^{\phi} \neq \bar{g}) & \wedge \end{array}$$

6. [supp$(\bar{g}) \sqsubset$ supp$(\bar{f})$] is the properly contained version of the above formula.

$$[\text{supp}(\bar{g}) \sqsubseteq \text{supp}(\bar{f})] \wedge \neg[\text{supp}(\bar{f}) \sqsubseteq \text{supp}(\bar{g})]$$

7. $\mathrm{SamePD}(\bar{f}, \bar{g})$ (Same Point and Direction) insists that $\bar{f}$ and $\bar{g}$ emanate from the same point in the same direction.

$$\forall \bar{h} ([\text{supp}(\bar{h}) \sqsubset \text{supp}(\bar{f})] \leftrightarrow [\text{supp}(\bar{h}) \sqsubset \text{supp}(\bar{g})])$$

8. $\mathrm{RepPoint}(\bar{f}_0, \bar{f}_1)$ is the formula defining the domain of interpretation (ϕ_{Dom} in Definition 1.3.4).

$$\mathrm{disj}(\bar{f}_0, \bar{f}_1) \wedge \forall \bar{g} \exists \bar{h} \neg \mathrm{disj}(\bar{g}, \bar{h}) \wedge \begin{pmatrix} \mathrm{SamePD}(\bar{f}_0, \bar{h}) & \vee \\ \mathrm{SamePD}(\bar{f}_1, \bar{h}) & \end{pmatrix}$$

9. $\mathrm{EqRepPoint}(\bar{f}_0, \bar{f}_1; \bar{g}_0, \bar{g}_1)$ is the formula defining the equivalence on the domain of interpretation (ϕ_{Eq}).

$$\mathrm{RepPoint}(\bar{f}_0, \bar{f}_1) \wedge \mathrm{RepPoint}(\bar{g}_0, \bar{g}_1) \wedge$$
$$(\mathrm{SamePD}(\bar{f}_0, \bar{g}_0) \wedge \mathrm{SamePD}(\bar{f}_1, \bar{g}_1)) \vee (\mathrm{SamePD}(\bar{f}_0, \bar{g}_1) \wedge \mathrm{SamePD}(\bar{f}_1, \bar{g}_0))$$

10. $\mathrm{Temp1PB}(g; h, k)$ is a temporary formula that expresses path-betweenness in some circumstances.

$$\exists l (\mathrm{EqRepPoint}(\bar{g}_0, \bar{g}_1; \bar{l}_0, \bar{l}_1) \wedge \begin{pmatrix} \neg \mathrm{disj}(\bar{l}_0, \bar{h}_0) \wedge \neg \mathrm{disj}(\bar{l}_0, \bar{h}_1) \wedge \\ \neg \mathrm{disj}(\bar{l}_1, \bar{k}_0) \wedge \neg \mathrm{disj}(\bar{l}_1, \bar{k}_1) \end{pmatrix}$$

11. $\mathrm{Temp2PB}(g; h, k)$ is a temporary formula that expresses path-betweenness in other circumstances.

$$\phi(g; h, k) \wedge \forall l \, \phi(l; h, k) \rightarrow \begin{bmatrix} \mathrm{Temp1PB}(g; l, k) & \wedge \\ \mathrm{Temp1PB}(g; l, h) & \end{bmatrix}$$

12. $\mathrm{PathBetween}(g; h, k)$ insists that g lies on the path between the points h and k.

$$\mathrm{Temp1PB}(g; h, k) \vee \mathrm{Temp2PB}(g; h, k)$$

13. $\mathrm{Related}(x, y)$ insists that x are related y.

$$\forall z(\mathrm{PathBetween}(z; x, y) \to \mathrm{Temp1PB}(z; x, y))$$

14. $B(z; x, y)$ insists that z is between x and y.

$$\mathrm{PathBetween}(z; x, y) \wedge \begin{pmatrix} \mathrm{Related}(x, y) & \wedge \\ \mathrm{Related}(x, z) & \wedge \\ \mathrm{Related}(y, z) & \end{pmatrix}$$

15. $x \lessdot_n y$ is the formula that defines $x < y$ in K^n_{Cone}.

$$\mathrm{Related}(x, y) \wedge \exists x_0, \ldots, x_n \begin{pmatrix} (\bigwedge_{i=0,\ldots,n} \mathrm{Related}(x, x_i) & \wedge \\ (\bigwedge_{i \neq j} \neg\mathrm{Related}(x_i, x_j)) & \wedge \\ (\bigwedge_{i \neq j} \mathrm{PathBetween}(x; x_i, x_j)) & \wedge \\ (\neg\mathrm{PathBetween}(x; y, x_0)) & \end{pmatrix}$$

16. $(x_1 <_n x_2 \Leftrightarrow y_1 < y_2)$ is a first order clause of the infinite disjunction in 17. Its definition is too long to give here.

17. $(x_1 < x_2 \Leftrightarrow y_1 < y_2)$ is the $L_{\omega_1,\omega}$ formula that recovers $x_1 < x_2$ if and only if $y_1 < y_2$.

$$\bigvee_{n<\omega} (x_1 <_n x_2 \Leftrightarrow y_1 < y_2)$$

18. $\mathrm{MeetsX}(\bar{f})$ is the formula that says the support of $\bar{f} \subset \mathrm{Aut}(W(X, S, (T, L)))$ contains an element of X.

$$\mathrm{Indec}(\bar{f}) \wedge \exists \bar{g} \begin{pmatrix} \neg\mathrm{disj}(\bar{f}, \bar{g}) \wedge \neg\mathrm{SamePD}(\bar{f}, \bar{g}))\wedge \\ \neg[\mathrm{supp}(\bar{f}) \sqsubset \mathrm{supp}(\bar{g})] \wedge \neg[\mathrm{supp}(\bar{g}) \sqsubset \mathrm{supp}(\bar{f})] \end{pmatrix}$$

19. $\mathrm{RepPointDec}(\bar{f}_0, \bar{f}_1)$ is a formula that adapts $\mathrm{RepPoint}$ so that it performs the same duty as $\mathrm{RepPoint}$ in a decorated CFPO.

$$\begin{gathered}\text{disj}(\bar{f}_0, \bar{f}_1) \wedge \mathrm{MeetsX}(\bar{f}_0) \wedge \mathrm{MeetsX}(\bar{f}_1) \wedge \\ \forall \bar{g} \exists \bar{h} \left(\left(\begin{array}{c} \mathrm{MeetsX}(\bar{g}) \\ \mathrm{MeetsX}(\bar{h}) \end{array} \wedge \right) \rightarrow \neg \left(\text{disj}(\bar{g}, \bar{h}) \wedge \left(\begin{array}{c} \mathrm{SamePD}(\bar{f}_0, \bar{h}) \\ \mathrm{SamePD}(\bar{f}_1, \bar{h}) \end{array} \vee \right) \right) \right)\end{gathered}$$

20. $\mathrm{EquivRepPointDec}$ is the formula $\mathrm{EquivRepPoint}$ with every instance of $\mathrm{RepPoint}$ replaced by $\mathrm{RepPointDec}$. Similarly:

$\mathrm{Temp1PBDec}$	comes from	$\mathrm{Temp1PB}$
$\mathrm{Temp2PBDec}$	comes from	$\mathrm{Temp2PB}$
$\mathrm{PathBetweenDec}$	comes from	$\mathrm{PathBetween}$
$\mathrm{RelatedDec}$	comes from	$\mathrm{Related}$
BDec	comes from	B

21. $\mathrm{FunctionPart}(\phi)$ is a formula that lets us recognise the elements of Aut$(W(X, S, (T, L)))$ that fix X point-wise.

$$\forall \bar{f}_0, \bar{f}_1, \bar{g}_0, \bar{g}_1 \left(\begin{array}{c} \left(\begin{array}{c} \mathrm{RepPointDec}(\bar{f}_0, \bar{f}_1) \\ \mathrm{RepPointDec}(\bar{g}_0, \bar{g}_1) \end{array} \wedge \right) \wedge \left(\begin{array}{c} (\bar{f}_0^{\phi} = \bar{g}_0 \wedge \bar{f}_1^{\phi} = \bar{g}_1) \\ (\bar{f}_1^{\phi} = \bar{g}_0 \wedge \bar{f}_0^{\phi} = \bar{g}_1) \end{array} \vee \right) \\ \rightarrow \mathrm{EquivRepPointDec}(\bar{f}_0, \bar{f}_1; \bar{g}_0, \bar{g}_1) \end{array} \right)$$

22. The formulas written with the prefixes *Above* and *Between* are second-order formulas that recover Aut(S) and Aut(T, L) from Aut$(W(X, S, (T, L)))$.