Decomposition of semigroups into semidirect and Zappa-Szép products

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Abstract

This thesis focuses on semidirect and Zappa-Szép products in the context of semigroups and monoids. We present a survey of direct, semidirect and Zappa-Szép products and discuss correspondence between external and internal versions of these products for semigroups and monoids.

Particular attention in this thesis is paid to a wide class of semigroups known as restriction semigroups. We consider Zappa-Szép product of a left restriction semigroup S with semilattice of projections E and determine algebraic properties of it.

We prove that analogues of Green's lemmas and Green's theorem hold for certain semigroups where Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}$ and $\mathcal{D}$ are replaced by $\widetilde{\mathcal{R}}_E, \widetilde{\mathcal{L}}_E, \widetilde{\mathcal{H}}_E$ and $\widetilde{\mathcal{D}}_E$. We show that if $\widetilde{\mathcal{H}}_E$ is a congruence on a certain semigroup S, then any right congruence on the submonoid $\widetilde{H}_E^e$ (the $\widetilde{\mathcal{H}}_E$-class of e), where $e \in E$, can be extended to a congruence on S. We introduce the idea of an *inverse skeleton* U of a semigroup S and examine some conditions under which we obtain skeletons from monoids . We focus on a result of Kunze [37] for the Bruck-Reilly extension $\mathrm{BR}(M, \theta)$ of a monoid M, showing that $\mathrm{BR}(M, \theta)$ is a Zappa-Szép product of $\mathbb{N}^0$ under addition and a semidirect product $M \rtimes \mathbb{N}^0$. We put Kunze's result in more general framework and give an analogous result for certain restriction monoids.

We consider the λ-*semidirect product* of two left restriction semigroups and prove that it is left restriction. In the two sided case using the notion of double action we prove that the λ-semidirect product of two restriction semigroups is restriction.

We introduce the notion of (A, T)-properness to prove the results analogous to McAlister's covering theorem and O'Carroll's embedding theorem for monoids and left restriction monoids under some conditions.

We extend the notion of the λ-semidirect product of two restriction semigroups S and T to develop λ-Zappa-Szép products and construct a category. In the special case where S is a semilattice and T is a monoid we order our category to become inductive and thus obtain a restriction semigroup via the use of the standard pseudo-product.

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Preface

In this thesis we study the decomposition of semigroups and monoids into semidirect and Zappa-Szép products. Zappa-Szép products perhaps first appeared in the work of Neumann for groups in 1935 [56]. The Zappa-Szép product is a natural generalisation of the semidirect product of groups, whereas, the semidirect product is a natural generalisation of the direct product of groups. In the direct product of groups both factors are required to be normal, whereas in semidirect products this strong condition is replaced by the weaker condition that only one factor is required to be normal. More generally, in Zappa-Szép products, neither of the factors is required to be normal. These products have been studied from different points of view and under different names since 1935. The concept of Zappa-Szép products for *semigroups* was developed by Kunze [37] who used the terminology *bilateral semidirect products* for Zappa-Szép products. We provide a detailed historical background of these products in Chapter 3.

We give the basic definitions and results related to regular and inverse semigroups in Chapter 1. We also provide a brief introduction to non-regular classes of semigroups such as abundant, weakly abundant, adequate, weakly adequate, ample and weakly ample semigroups and their one-sided versions.

In Chapter 2 we provide introductory material of restriction semigroups. Restriction semigroups are a wide and natural class of semigroups to consider. Restriction semigroups and their one-sided versions have been studied from various points of view under different names for more than 40 years. Left restriction semigroups are a class of unary semigroups (that is, semigroups equipped with an additional unary operation) that are precisely the unary semigroups isomorphic to unary subsemigroups of partial transformation semigroups $\mathcal{PT}_X$ where the unary operation $^+$ takes a transformation α to the domain of α. Restriction semigroups are the two-sided version possessing a second unary operation $\alpha \mapsto \alpha^*$. We shall provide the definition of restriction semigroups as a class of algebras defined by identities. We shall look at the natural partial order on (left) restriction semigroups and provide the results related to this partial order to be used later in the thesis. In subsequent sections of this chapter we recall definitions of categories, inductive groupoids and inductive categories. We discuss the famous Ehresmann-Schein-Nambooripad Theorem that explains the category theoretic connection between inverse semigroups and inductive groupoids. We survey some results of Lawson from [33], that establish a connection between restriction semigroups and inductive categories.

Chapter 3 is a survey on direct, semidirect and Zappa-Szép products. We present a detailed historical background to Zappa-Szép products. We then discuss uniquely factorisable semigroups and monoids. We know from [7, Lemma 3.4] that if a semigroup U is uniquely factorisable as $U = ST$, then there is an idempotent $e \in S \cap T$ which is a right identity for S and a left identity for T. We provide an alternative proof for this result.

Zappa-Szép products are a natural generalisation of semidirect products whereas semidirect products generalise direct products. Therefore in the next section of Chapter 3 we discuss direct products. We give definitions of the internal and external direct products of monoids and semigroups. We see that the internal direct products of monoids are isomorphic to the external direct products of monoids and vice versa. We notice that for semigroups there is not such a complete correspondence between internal and external direct products as exists for monoids. When we consider the internal direct product $U = ST$ of semigroup U with subsemigroups S and T, then we notice that there is an element $e \in S \cap T$ which becomes the identity of S and T and hence of U. Thus U is forced to be a monoid with submonoids S and T which by the previous result is isomorphic to an external direct product of monoids. Naturally our next step is to see what happens if we consider an external direct product of semigroups. We see that it is not immediate to obtain an internal direct product from an external direct product of semigroups, but we can use Preston's techniques from [64] for semidirect products to overcome this difficulty.

In the subsequent sections, we discuss equivalence between external and internal versions of semidirect and Zappa-Szép products of semigroups and monoids. There is a correspondence between external and internal semidirect and Zappa-Szép products of monoids, but this is not true for semigroups in general. We discuss the conditions under which we get an equivalence for semidirect and Zappa-Szép products of semigroups. We discuss some known results that describe the correspondence between external and internal semidirect and Zappa-Szép products of monoids and consider why such a correspondence for semigroups does not exist. Under some conditions, we prove that we can find an equivalence between the external and internal versions of these products for semigroups.

In Chapter 4, we discuss algebraic properties of the Zappa-Szép product $E \bowtie S$ of a left restriction semigroup S with a semilattice of projections E. We determine the set of idempotents of $E \bowtie S$. We know that $E \bowtie S$ itself is not left restriction but we prove that we can find a subset T of $E \bowtie S$ which is left restriction. We notice that if S is also right restriction, then T is not always right restriction because, in particular we cannot always find a right identity of an element in T. Our results can be immediately applied to inverse semigroups. We characterise generalised Green's relations for Zappa-Szép products of certain monoids and semigroups.

Our main work starts in Chapter 5. The contents of Chapter 5 have already appeared in a joint paper with Victoria Gould [30]. In this chapter, we study semigroups possessing E-regular elements, where an element a of a semigroup S is *E-regular* if a has an inverse a° such that $aa^\circ, a^\circ a$ lie in E. In Section 5.1, we prove that under certain circumstances $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ behave like $\mathcal{R}$ and $\mathcal{L}$; in general, however, they do not. We show that where S possesses 'enough' (in a precisely defined way) E-regular elements, analogues of Green's lemmas hold, where Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}$ and $\mathcal{D}$ are replaced by $\widetilde{\mathcal{R}}_E, \widetilde{\mathcal{L}}_E, \widetilde{\mathcal{H}}_E$ and $\widetilde{\mathcal{D}}_E$. With some extra conditions on our semigroup we also have an analogue of Green's theorem. Namely, we show that under these conditions, if $a \,\widetilde{\mathcal{H}}_E\, a^2$, then $\widetilde{H}_E^a$, the $\widetilde{\mathcal{H}}_E$-class of a, is a monoid with identity from E.

In Section 5.2 we show that if $\widetilde{\mathcal{H}}_E$ is a congruence on a certain semigroup S, then any right congruence on the submonoid $\widetilde{H}_E^e$, where $e \in E$, can be extended to a congruence on S. We also have a result for two sided congruences, with some further restrictions on S. We stress that for regular semigroups with $E = E(S)$ we have $\widetilde{\mathcal{K}}_E = \mathcal{K}^* = \mathcal{K}$, so our results can be immediately applied to maximal subgroups of regular semigroups.

In Section 5.3 we introduce the idea of an *inverse skeleton* U of a semigroup S. Here U is an inverse subsemigroup of E-regular elements, such that $E \subseteq U$ and U intersects every $\widetilde{\mathcal{H}}_E$-class exactly once (it follows that $E = E(U)$). We examine some conditions under which we obtain skeletons from monoids having a particular submonoid L of the $\widetilde{\mathcal{L}}_E$-class of the identity. A monoid with such a submonoid L is called *special*. Our most complete results are for restriction monoids in Section 5.4, where we focus on a result of Kunze [37] for the Bruck-Reilly extension $\mathrm{BR}(M, \theta)$ of a monoid M, showing that $\mathrm{BR}(M, \theta)$ is a Zappa-Szép product of $\mathbb{N}^0$ under addition and a semidirect product $M \rtimes \mathbb{N}^0$. Certainly $\mathrm{BR}(M, \theta)$ is special, with L isomorphic to $\mathbb{N}^0$. We put Kunze's result in more general framework and prove in particular that a special $\widetilde{\mathcal{D}}_E$-simple restriction monoid can be decomposed in an analogous way. Again, our results apply immediately to inverse monoids.

We study the λ-semidirect product of restriction semigroups in Chapter 6. It is well known that, in general, the semidirect product $S \rtimes T$ of two inverse semigroups S and T

need not be inverse. To overcome this, Billhardt introduced the concept of a λ-semidirect product $S \rtimes^{\lambda} T$ [4]. The λ-semidirect product of two inverse semigroups is again inverse. Further, Billhardt generalised this result to the left ample case, where one component is a semilattice [5] (but he indicated that it would work for any two left ample semigroups). Again in this special case, this was extended further to the λ-semidirect product of a semilattice and a left restriction semigroup by Branco, Gomes and Gould [6]. We complete the picture to show that if S and T are restriction semigroups, then so is $S \rtimes^{\lambda} T$ subject to certain conditions on the actions.

We first consider the λ-semidirect product $S \rtimes^{\lambda} T$ of two left restriction semigroups S and T and following Billhardt's constructions prove that it is again left restriction. In the two sided case, it is not straightforward that the λ-semidirect product $S \rtimes^{\lambda} T$ of two restriction semigroups S and T is also *right* restriction. It turns out that we need both a left and a right action of T on S (given that restriction semigroups do not possess a natural involution in the way that groups and inverse semigroups do). This notion of double action was introduced in [24] to determine the structure of the free ample monoids. We present two proofs that the λ-semidirect product of two restriction semigroups is also *right* restriction. First a direct proof and second, by using the correspondence between the category of restriction semigroups and that of inductive categories.

In Chapter 7 we present structure theorems for certain monoids and left restriction monoids analogous to the approach for proper inverse semigroups. The well known *McAlister's P-theorem* [52] for proper (E-unitary) inverse semigroups is important due to McAlister's other major result that every inverse semigroup has a *proper cover* [51]. The P-theorem determines the structure of all proper inverse semigroups and has many important consequences such as *O'Carroll's embedding theorem* [59]. In Section 7.1, we give basic definitions of *proper* and *proper covers* in the context of inverse semigroups. We then define a *P-semigroup* and state *McAlister's P- theorem* and its *consequences*. McAlister's work has been extended for left ample, weakly left ample and left restriction semigroups. We give results for left restriction semigroups analogous to McAlister's theory in Section 7.2.

In Section 7.3, we consider a monoid S such that $S = AT$ for submonoids A, T, where T acts on A (as a semigroup) satisfying a condition (LAC) analogous to the left ample condition and introduce the notion of *(A, T)-properness*. We notice that our notion of (A, T)-properness generalises the notion of a proper left restriction semigroup. We then show how to construct an (A, T)-proper monoid from a semidirect product in Section 7.4 and state our covering theorem for (A, T)-proper monoids. As a second step in this section we present the embedding theorem for (A, T)-proper monoids subject to certain conditions.

In the final section of Chapter 7, we consider a left restriction monoid S with submonoids A, T such that $S = AT$, where T acts on A as a monoid by (2,1)-endomorphisms. We consider the λ-semidirect product of A by T to prove the results analogous to those in Section 7.4 for left restriction monoids. We adopted the notion of (A, T)-properness for left restriction monoids from Section 7.4 and under certain conditions prove the covering and embedding theorems for left restriction monoids.

In Chapter 8, we consider the λ-Zappa-Szép products of two restriction semigroups. The Zappa-Szép product of two restriction semigroups is not restriction in general. This is even the case for the Zappa-Szép product of two inverse semigroups. If $Z = S \bowtie T$ is a Zappa-Szép product of two inverse semigroups S and T, then it is known that by picking a certain subset $B_{\bowtie}(Z)$ of $Z = S \bowtie T$, $B_{\bowtie}(Z)$ is a groupoid under the restriction of the product in Z. In the special case where $S = E$ a semilattice and $T = G$ a group, this groupoid can be ordered to become inductive thus yielding an inverse semigroup via the use of the standard pseudo-product. These constructions are due to Gilbert and Wazzan [25]. We refer to this as the λ-Zappa-Szép product of S and T.

We extend the concept of the λ-Zappa-Szép product of inverse semigroups to the λ-Zappa-Szép product of restriction semigroups. In Section 8.1, we give results for the λ-Zappa-Szép product of inverse semigroups from [77]. We then present a comparison between the inverse and restriction cases in Section 8.2 and discuss the technical difficulties arising in the restriction case. We consider the Zappa-Szép product $Z = S \bowtie T$ of two restriction semigroups S and T and define two new actions of S and T on each other. We emphasise here that these new actions do not satisfy Zappa-Szép axioms. We then pick a certain set of pairs $V_{\bowtie}(Z)$ of Z and observe that if S and T are inverse semigroups regarded as restriction semigroups in the usual way, then $B_{\bowtie}(Z) = V_{\bowtie}(Z)$. In Section 8.3, we give $V_{\bowtie}(Z)$ the structure of a category. To obtain a restriction semigroup, we consider the Zappa-Szép product of a semilattice and a monoid. In this special case we order our category to become inductive and use the equivalence of categories between restriction semigroups and inductive categories to obtain a restriction semigroup.

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Author's Declaration

Chapter 1 and Chapter 2 mainly consist of definitions and results by other authors. Chapter 1 presents fundamental information about regular, inverse, abundant, weakly E-abundant, weakly E-adequate and ample semigroups, mostly from [34], [35], [17] and [18]. Introductory material on restriction semigroups in Chapter 2 is given from [28]. The remainder of Chapter 2 is from [44], [45] and [33] and consists of a survey of the ideas surrounding the Ehresmann-Schein-Nambooripad theorem.

Chapter 3 is a survey of direct, semidirect and Zappa-Szép products. Most of the material on direct and semidirect products is folklore, however we give some results about uniquely factorisable semigroups and semidirect products from [7] and [64]. Some of the contents on Zappa-Szép products in Chapter 3 are from [79], [37] and [77]. However we also make our contribution to provide a complete picture on the correspondence between internal and external direct, semidirect and Zappa-Szép products of semigroups and monoids.

Chapter 4 is the first chapter of new work in this thesis. The results are my own, making some small use of existing approaches.

The contents in Chapter 5 have already appeared in a joint paper with Victoria Gould [30]. The examples given are folklore, but they can be found in [11].

Chapters 6 and 7 will form a joint paper with Victoria Gould. The background material is from [4], [5], [45], [6], [77], [34], [51], [52] and [59] which is stated and referenced accordingly.

In Chapter 8 we extend the concept of the λ-Zappa-Szép products from [25] and [77] for inverse semigroups to restriction semigroups. In the earlier part of the chapter results are stated from [77]. All other work is my own.

Chapter 1

Regular, inverse, (weakly) abundant, (weakly) adequate semigroups and associated classes

In this chapter we provide definitions of Green's and generalised Green's relations. We give a brief introduction to regular and inverse semigroups. We explain how generalised Green's relations may be used to define the classes of non-regular semigroups given in the title to this chapter.

1.1 Green's relations, regular and inverse semigroups

Green's relations are five equivalence relations that characterise the elements of a semigroup in terms of the principal ideals they generate.

Let S be a semigroup and $a \in S$. Given that S need not be a monoid, it is not always true that

$$a \in aS = \{as : s \in S\}.$$

To overcome this problem, we need to consider the notion of *adjoining an identity*.

Let 1 be a symbol *not* in S. We extend the binary operation in S to $S \cup \{1\}$ by defining

$$1s = s = s1 \text{ for all } s \in S \text{ and } 11 = 1.$$

Then $S \cup \{1\}$ becomes a monoid with identity 1. We write

$$S^1 = \begin{cases} S & \text{If } S \text{ has an identity element} \\ S \cup \{1\} & \text{otherwise.} \end{cases} \tag{1.1}$$

We say that S^1 is the monoid obtained from S by *adjoining an identity if necessary.* We also put $S^{(1)} = S \cup \{1\}$ and say that $S^{(1)}$ is the monoid obtained from S by *adjoining an identity.*

We see that

$$aS^1 = aS \cup \{a\}$$

so that $a \in aS^1$. Note that aS^1 is a subset of S and so will not contain an adjoined identity. Precisely aS^1 is the smallest right ideal of S containing a, known as the *principal right ideal of S generated by a.* Dually $S^1a = Sa \cup \{a\}$ is a subset of S and will not contain an adjoined identity. Then S^1a is the smallest left ideal of S containing a, known as the *principal left ideal generated by a.* Also

$$S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}$$

is a subset of S which will not contain an adjoined identity and is the smallest two sided ideal of S containing a, known as the *principal ideal generated by a.*

The relations $\leq_{\mathcal{R}}$, $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{J}}$ depend on the principal ideals mentioned above and are defined on a semigroup S as follows: for any $a, b \in S$

- $a \leq_{\mathcal{R}} b$ if and only if $aS^1 \subseteq bS^1$;
- $a \leq_{\mathcal{L}} b$ if and only if $S^1a \subseteq S^1b$;
- $a \leq_{\mathcal{J}} b$ if and only if $S^1aS^1 \subseteq S^1bS^1$.

It is easy to check that $\leq_{\mathcal{R}}$, $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{J}}$ are pre-orders (quasi-orders) on S, that is, relations that are reflexive and transitive. We denote the associated equivalence relations by $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{J}$, respectively. Thus for any $a, b \in S$

- $a \,\mathcal{R}\, b$ if and only if $aS^1 = bS^1$;
- $a \,\mathcal{L}\, b$ if and only if $S^1a = S^1b$;
- $a \,\mathcal{J}\, b$ if and only if $S^1aS^1 = S^1bS^1$.

The next proposition gives an alternative characterisation of the relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{J}$.

Proposition 1.1.1. [34] *Let S be a semigroup and $a, b \in S$. Then*

(i) $a \mathcal{R} b$ if and only if there exists $x, y \in S^1$ such that $ax = b$, $by = a$;

(ii) $a \mathcal{L} b$ if and only if there exists $u, v \in S^1$ such that $ua = b$, $vb = a$;

(iii) $a \mathcal{J} b$ if and only if there exists $x, y, u, v \in S$ such that $xay = b$, $ubv = a$.

We note that $\leq_{\mathcal{R}}$ is left compatible, as if $a, b, c \in S$ and $a \leq_{\mathcal{R}} b$, then $aS^1 \subseteq bS^1$, and so $caS^1 \subseteq cbS^1$, that is, $ca \leq_{\mathcal{R}} cb$. Dually, $\leq_{\mathcal{L}}$ is right compatible. The following two lemmas give important properties of $\mathcal{R}$ and $\mathcal{L}$, the first follows from the observation concerning the compatibility properties of $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$.

Lemma 1.1.2. [35] *Let S be a semigroup. The relation $\mathcal{R}$ is a left congruence and $\mathcal{L}$ is a right congruence.*

Lemma 1.1.3. [35] *Let S be a semigroup. The relations $\mathcal{R}$ and $\mathcal{L}$ commute.*

The relations $\mathcal{H}$ and $\mathcal{D}$ are derived from $\mathcal{R}$ and $\mathcal{L}$ by letting $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$ be the intersection and $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$ be the join of $\mathcal{R}$ and $\mathcal{L}$. These equivalence relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, $\mathcal{H}$ and $\mathcal{D}$ are called Green's relations [31]. As these are equivalence relations on S, each of them yields a partition of S. The $\mathcal{R}$-class of a is denoted by R_a and similarly for other relations. In view of Lemma 1.1.3, we have

$$\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}.$$

It is convenient to visualise a $\mathcal{D}$-class as an 'egg box' in which each row represents an $\mathcal{R}$-class, each column represents an $\mathcal{L}$-class and each cell represents an $\mathcal{H}$-class.

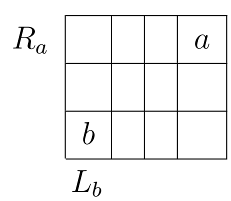

Figure 1.1: $\mathcal{D}$-class

The structure of $\mathcal{D}$-classes helps to determine the properties of a semigroup. The following important lemma, known as Green's Lemma, tells us that every $\mathcal{R}$-class and every $\mathcal{L}$-class in a $\mathcal{D}$-class has the same size. We first explain the maps restricted to particular domains used in this lemma, known as *right translations* and *left translations*.

Let S be a semigroup and $s \in S$. A *right translation* of S is a map $\rho_s : S \to S$ defined by

$$x\rho_s = xs \text{ for any } x \in S.$$

Dually, a *left translation* is a map $\lambda_t : S \to S$ such that

$$y\lambda_t = ty \text{ for any } y \in S.$$

Green's Lemma. [34] *Let S be a semigroup and $a, b \in S$. Suppose that $a \,\mathcal{D}\, b$.*

(1) If $a \,\mathcal{R}\, b$, let $s, s' \in S$ be such that

$$as = b, \ bs' = a.$$

Then the right translations $\rho_s : L_a \to L_b$ and $\rho_{s'} : L_b \to L_a$ are mutually inverse $\mathcal{R}$-class preserving bijections.

(2) If $a \,\mathcal{L}\, b$, let $t, t' \in S$ be such that

$$ta = b, \ t'b = a.$$

Then the left translations $\lambda_t : R_a \to R_b$ and $\lambda_{t'} : R_b \to R_a$ are mutually inverse $\mathcal{L}$-class preserving bijections.

A consequence of Green's Lemma is that if a, b are any $\mathcal{D}$-equivalent elements in a semigroup S, then $|H_a| = |H_b|$.

The following theorem, known as *Green's Theorem*, is of considerable use in Chapter 5.

Green's Theorem. [34] *Let S be a semigroup. Then the $\mathcal{H}$-class of an element $a \in S$ is a subgroup of S if and only if $a \,\mathcal{H}\, a^2$.*

Definition 1.1.4. An element a of a semigroup S is called *idempotent* if $a = a^2$.

Given a semigroup S, we denote by $E(S)$ its set of idempotents and we often use E to denote a subset of $E(S)$.

A consequence of Green's Theorem is that if e is an idempotent in S, then H_e is a subgroup of S and no $\mathcal{H}$-class in S contains more than one idempotent.

Definition 1.1.5. Let S be a semigroup. An element a of S is called *regular* if there exists x in S such that $a = axa$.

A semigroup S is *regular* if all of its elements are regular.

We denote the set of all regular elements in a semigroup S by $\mathrm{Reg}(S)$.

If a is a regular element of S with $a = axa$, and if $b \in R_a$, then there exist $u, v \in S^1$ such that $b = au$, $a = bv$. Then

$$b = au = axau = axb = b(vx)b$$

and so b is regular. The dual argument applies to any element in L_a and hence we have the following proposition.

Proposition 1.1.6. [34] *If a is a regular element of a semigroup S, then every element of D_a is regular.*

Thus every element in a $\mathcal{D}$-class D is regular or no element of D is regular. We say that a $\mathcal{D}$-class is *regular* if all of its elements are regular and *non-regular* otherwise.

If a is a regular element of S with $a = axa$, then clearly ax and xa are idempotents and

$$ax \,\mathcal{R}\, a \,\mathcal{L}\, xa.$$

We conclude that in a regular $\mathcal{D}$-class each $\mathcal{R}$-class and each $\mathcal{L}$-class contains an idempotent.

Definition 1.1.7. An element $a' \in S$ is said to be an inverse of $a \in S$ if

$$a = aa'a \text{ and } a' = a'aa'.$$

We denote the set of inverses of an element $a \in S$ by $\mathrm{V}(a)$. We notice that an element with an inverse is necessarily regular, as a regular element necessarily has an inverse for if a is regular element with $a = axa$, then by defining $a' = xax$, we see that

$$aa'a = axaxa = axa = a$$

and

$$a'aa' = (xax)a(xax) = x(axa)xax = xaxax = xax = a',$$

and hence a' is an inverse of a.

The following theorem is very useful to understand the structure of regular $\mathcal{D}$-classes.

Theorem 1.1.8. [34] *Let a be an element of a regular $\mathcal{D}$-class D in a semigroup S.*

(1) If a' is an inverse of a, then $a' \in D$ and the two $\mathcal{H}$-classes $R_a \cap L_{a'}$, $L_a \cap R_{a'}$ contain respectively the idempotents aa' and $a'a$.

(2) If $b \in D$ is such that $R_a \cap L_b$ and $L_a \cap R_b$ contain idempotents e, f, respectively, then H_b contains an inverse a' of a such that $aa' = e$ and $a'a = f$.

(3) No $\mathcal{H}$-class contains more than one inverse of a.

Theorem 1.1.8 and its following easy consequence will be of considerable use in Chapter 5.

Proposition 1.1.9. [34] *Let e, f be idempotents in a semigroup S. Then $e\,\mathcal{D}\,f$ if and only if there exists an element $a \in S$ and an inverse a' of a such that $aa' = e$ and $a'a = f$.*

Definition 1.1.10. A semigroup S is said to be *inverse* if each element $a \in S$ has exactly one inverse.

We will denote the unique inverse of a in S by a^{-1}.

An inverse semigroup is clearly regular, but the converse is not necessarily true.

Definition 1.1.11. A semigroup S is called a *semilattice* if all elements of S are idempotents and $ab = ba$ for all $a, b \in S$.

We now explain the concept of a *lower semilattice* and see that the notion of 'lower semilattice' and 'semilattice' are equivalent and interchangeable.

Definition 1.1.12. Let $(X, \leq)$ be a partially ordered set and let Y be a non-empty subset of X. An element $c \in Y$ is said to be a *lower bound* of Y if

$$c \leq y \text{ for all } y \in Y.$$

If the set of all lower bounds of Y is non-empty and has a maximum element l, then we say that l is the *greatest lower bound*, or *meet* of Y. The element l is unique if it exists and we write

$$l = \wedge\{y : y \in Y\}.$$

Definition 1.1.13. A *lower semilattice* is a partially ordered set $(X, \leq)$ such that $a \wedge b$ exists for all $a, b \in X$.

In a lower semilattice $(X, \leq)$ we have that

$$a \leq b \text{ if and only if } a \wedge b = a \text{ for all } a, b \in X.$$

We see in the following proposition that the notion of 'lower semilattice' coincides with 'semilattice'.

Proposition 1.1.14. [34, Proposition 1.3.2] *Let $(E, \leq)$ be a lower semilattice. Then $(E, \wedge)$ is a semilattice and*

$$a \leq b \text{ if and only if } a \wedge b = a \text{ for all } a, b \in E.$$

Conversely, suppose that $(E, \cdot)$ is a semilattice. Then the relation $\leq$ on E defined by

$$a \leq b \text{ if and only if } ab = a$$

is a partial order on E, with respect to which $(E, \leq)$ is a lower semilattice and the meet of $a, b \in E$ is their product ab.

The following theorem provides equivalent definitions for inverse semigroups.

Theorem 1.1.15. [35] *Let S be a semigroup with set of idempotents $E(S)$. Then the following are equivalent:*

(1) S is an inverse semigroup;
(2) S is regular and $E(S)$ is a semilattice;
(3) every $\mathcal{R}$-class and every $\mathcal{L}$-class contains exactly one idempotent;
(4) every element of S has a unique inverse.

The following proposition provides the elementary properties of inverse semigroups.

Proposition 1.1.16. *Let S be an inverse semigroup with semilattice of idempotents $E(S)$. Then*

(i) $(a^{-1})^{-1} = a$ for all $a \in S$;
(ii) $e^{-1} = e$ for all $e \in E(S)$;
(iii) $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in S$;
(iv) $aea^{-1} \in E(S)$, $a^{-1}ea \in E(S)$ for all $a \in S$ and $e \in E(S)$.

If S is an inverse semigroup with semilattice of idempotents $E(S)$, then the *natural partial order* $\leq$ on S is defined by the rule for $a, b \in S$

$$a \leq b \text{ if and only if } a = eb \text{ for some } e \in E(S).$$

This partial order on any inverse semigroup is compatible with the operations of multiplication and inverse. Several alternative characterisations of this partial order can be found in [34]. We give one of them in the following proposition.

Proposition 1.1.17. *Let S be an inverse semigroup with natural partial order $\leq$ and let $a, b \in S$. Then $a \leq b$ if and only if $a = bf$ for some $f \in E(S)$.*

If S is an inverse semigroup with semilattice of idempotents $E(S)$, then the restriction of $\leq$ to $E(S)$ is the usual partial order on $E(S)$. Thus for $e, f \in E(S)$

$$e \leq f \text{ if and only if } ef = e.$$

Definition 1.1.18. Let S be an inverse semigroup with semilattice of idempotents $E(S)$. The relation σ on S is defined by for all $s, t \in S$:

$$s\,\sigma\,t \Leftrightarrow es = et \text{ for some } e \in E(S).$$

The following result can be found in [45].

Theorem 1.1.19. *Let S be an inverse semigroup. Then*

(1) σ is a congruence on S;

(2) S/σ is a group;

(3) if ρ is any congruence on S such that S/ρ is a group, then $\sigma \subseteq \rho$.

The congruence σ is called the *minimum group congruence*.

From now onwards we assume that the reader is familiar with inverse semigroups and knows the early development made by Preston [61, 62, 63].

Regular $\mathcal{D}$-classes are particularly well understood, given that the left and right translations afforded by Green's lemmas result in Green's theorem. For non-regular $\mathcal{D}$-classes, indeed for non-regular semigroups, an approach using Green's relations is not always the most appropriate. As an alternative, one can make use of the extensions $\mathcal{K}^*$ of Green's relations $\mathcal{K}$, where $K \in \{R, L, H, D\}$ or the yet wider relations $\widetilde{\mathcal{K}}_E$, where E is a set of idempotents. We define these relations and semigroups built using these relations in the next section.

1.2 Abundant and weakly E-abundant semigroups

This section is concerned with classes of semigroups built using the relations $\mathcal{R}^*$, $\mathcal{L}^*$, $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$. Such semigroups do not have to be regular.

Definition 1.2.1. Let S be a semigroup. We define the relations $\leq_{\mathcal{R}^*}$ and $\leq_{\mathcal{L}^*}$ on S by the rule that for any $a, b \in S$

$$a \leq_{\mathcal{R}^*} b \text{ if and only if } a \leq_{\mathcal{R}} b \text{ in some oversemigroup of } S$$

and

$$a \leq_{\mathcal{L}^*} b \text{ if and only if } a \leq_{\mathcal{L}} b \text{ in some oversemigroup of } S,$$

where $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$ are defined in Section 1.1.

The following lemma gives an alternative characterisation of relations $\leq_{\mathcal{R}^*}$ and $\leq_{\mathcal{L}^*}$.

Lemma 1.2.2. [18] *Let S be a semigroup. Then the following statements are equivalent:*
(i) $(a, b) \in \leq_{\mathcal{R}^*}$ $((a, b) \in \leq_{\mathcal{L}^*})$;
(ii) for all $x, y \in S^1$, $xb = yb$ implies $xa = ya$ ($bx = by$ implies $ax = ay$).

It follows from Lemma 1.2.2 that the relations $\leq_{\mathcal{R}^*}$ and $\leq_{\mathcal{L}^*}$ are pre-orders. Also $\leq_{\mathcal{R}^*}$ is left compatible because $\leq_{\mathcal{R}}$ is left compatible and $\leq_{\mathcal{L}^*}$ is right compatible because $\leq_{\mathcal{L}}$ is right compatible. We denote the associated equivalence relations by $\mathcal{R}^*$ and $\mathcal{L}^*$ respectively. Therefore for any $a, b \in S$, $a \,\mathcal{R}^*\, b$ if and only if $a \,\mathcal{R}\, b$ in some oversemigroup of S. The relation $\mathcal{L}^*$ is defined dually. Clearly $\mathcal{R}^*$ is a left congruence and $\mathcal{L}^*$ is a right congruence. The following corollary gives an equivalent formulation of relations $\mathcal{R}^*$ and $\mathcal{L}^*$.

Corollary 1.2.3. [18] *Let S be a semigroup. Then the following statements are equivalent:*
(i) $(a, b) \in \mathcal{R}^*$ $((a, b) \in \mathcal{L}^*)$;
(ii) for all $x, y \in S^1$, $xa = ya$ if and only if $xb = yb$ ($ax = ay$ if and only if $bx = by$).

We note that an idempotent e of S acts as a left identity for its $\mathcal{R}^*$-class and a right identity for its $\mathcal{L}^*$-class. Also note that $\mathcal{R} \subseteq \mathcal{R}^*$ and $\mathcal{L} \subseteq \mathcal{L}^*$. A useful observation is that unlike Green's relations, the relations $\mathcal{R}^*$ and $\mathcal{L}^*$ need not commute.

Lemma 1.2.4. *Let S be a regular semigroup. Then $\mathcal{R} = \mathcal{R}^*$ and $\mathcal{L} = \mathcal{L}^*$.*

Proof. We have to show that $\mathcal{R}^* \subseteq \mathcal{R}$ and $\mathcal{L}^* \subseteq \mathcal{L}$. For this let $a, b \in S$ and $a \,\mathcal{R}^*\, b$. Then

$$xa = ya \text{ if and only if } xb = yb \text{ for all } x, y \in S^1.$$

As S is regular, we have $a = asa$ and $b = btb$ for some $s, t \in S$. Therefore

$$1a = asa \Rightarrow b = asb \text{ and } b = btb \Rightarrow a = bta.$$

Hence $a\,\mathcal{R}\,b$ and thus $\mathcal{R} = \mathcal{R}^*$. Dually, $\mathcal{L} = \mathcal{L}^*$. □

Following the conventions for Green's relations, the intersection of $\mathcal{R}^*$ and $\mathcal{L}^*$ is denoted by $\mathcal{H}^*$ and their join by $\mathcal{D}^*$. We denote the $\mathcal{R}^*$-class of an element a by R_a^* and similarly for other relations.

Definition 1.2.5. Let S be a semigroup with set of idempotents $E(S)$. Then S is called *left abundant* if each $\mathcal{R}^*$-class contains at least one element of $E(S)$.

Dually S is *right abundant* if each $\mathcal{L}^*$-class contains at least one element of $E(S)$.

Regular semigroups are abundant as in this case $\mathcal{R}^* = \mathcal{R}$ and $\mathcal{L}^* = \mathcal{L}$. The relations $\mathcal{R}^*$, $\mathcal{L}^*$, $\mathcal{H}^*$ and $\mathcal{D}^*$ play an important role in the theory of abundant semigroups which is to some extent analogous to that of Green's relations in the theory of regular semigroups.

If S is an abundant semigroup and $a \in S$, then we denote idempotents in the $\mathcal{R}^*$-class and $\mathcal{L}^*$-class by a^+ and a^*, respectively. Note that it is not necessary for a^+ and a^* to be unique.

A left (right) abundant semigroup S is said to be *left adequate (right adequate)* if $E(S)$ is a semilattice [17]. We say that a semigroup S is *adequate* if it is both left and right adequate. If S is an adequate semigroup, then for any $a \in S$, a^+ and a^* are unique. As if $a^\circ\,\mathcal{R}^*\,a^+$, where $a^\circ, a^+ \in E(S)$, then as idempotents are left identities for their $\mathcal{R}^*$-classes,

$$a^+ = a^\circ a^+ = a^+ a^\circ = a^\circ.$$

Definition 1.2.6. A left adequate semigroup is said to be *left ample* if if for all $a \in S$ and $e \in E(S)$, the left ample condition holds, that is:

$$(ae)^+ a = ae.$$

Dually, we have *right ample* semigroups where S is a right adequate semigroup and for all $a \in S$ and $e \in E(S)$:

$$a(ea)^* = ea.$$

We say that a semigroup S is *ample* if it is both left and right ample.

In particular, an inverse semigroup is ample, where $a^+ = aa^{-1}$ and $a^* = a^{-1}a$.

Definition 1.2.7. Let S be a semigroup and let E be a non-empty subset of $E(S)$ which we call a *distinguished set of idempotents*. The relations $\leq_{\widetilde{\mathcal{R}}_E}$ and $\leq_{\widetilde{\mathcal{L}}_E}$ on S are defined by the rule that for all $a, b \in S$ we have $a \leq_{\widetilde{\mathcal{R}}_E} b$ if and only if

$$\{e \in E : eb = b\} \subseteq \{e \in E : ea = a\}$$

and $a \leq_{\widetilde{\mathcal{L}}_E} b$ if and only if

$$\{e \in E : be = b\} \subseteq \{e \in E : ae = a\}.$$

It is clear that $\leq_{\widetilde{\mathcal{R}}_E}$ and $\leq_{\widetilde{\mathcal{L}}_E}$ are pre-orders on S. The associated equivalence relations are denoted by $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$, respectively. Thus for any $a, b \in S$ we have $a \,\widetilde{\mathcal{R}}_E\, b$ if and only if a and b have same set of left identities in E and $a \,\widetilde{\mathcal{L}}_E\, b$ if and only if a and b have same set of right identities in E.

Note that any $e \in E$ is a left (right) identity for its $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class). Because, if $a \in S$ and $a \,\widetilde{\mathcal{R}}_E\, e$, then since $e \in E$,

$$ee = e \Rightarrow ea = a.$$

In fact we can say more about it as we see in the following lemma.

Lemma 1.2.8. *[44] Let S be a semigroup and $E \subseteq E(S)$. If $a \in S$ and $e \in E$, then $a \,\widetilde{\mathcal{R}}_E\, e$ ($a \,\widetilde{\mathcal{L}}_E\, e$) if and only if $ea = a$ ($ae = a$) and for all $f \in E$,*

$$fa = a \Rightarrow fe = e \ \ (af = a \Rightarrow ef = e).$$

It is easy to see that $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$ and $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_E$. If S is regular and $E = E(S)$, then the foregoing inclusions are replaced by equalities. More generally, if $e, f \in E$ then $e \,\widetilde{\mathcal{R}}_E\, f$ if and only if $e \,\mathcal{R}\, f$ and $e \,\widetilde{\mathcal{L}}_E\, f$ if and only if $e \,\mathcal{L}\, f$. In general, however, the inclusions are strict.

We note that E does not necessarily have to be the whole of $E(S)$ but at times we consider the case when $E = E(S)$. If $E = E(S)$, then we use $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$ instead of $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$, respectively. Certainly, $\widetilde{\mathcal{R}} \subseteq \widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}} \subseteq \widetilde{\mathcal{L}}_E$ for any $E \subseteq E(S)$.

Following the usual practice for the relations $\mathcal{R}^*$ and $\mathcal{L}^*$, we denote idempotents in the $\widetilde{\mathcal{R}}_E$-class and $\widetilde{\mathcal{L}}_E$-class of an element a by a^+ and a^*, respectively. Note that a^+ and a^* are not unique unless E is a semilattice. We note that for any distinguished idempotents a^+ and a^* in $\widetilde{\mathcal{R}}_E$-class and $\widetilde{\mathcal{L}}_E$-class of a respectively, we have that

$$a^+a = a \ \text{ and } \ aa^* = a.$$

Unlike $\mathcal{R}$ and $\mathcal{R}^*$, the relation $\widetilde{\mathcal{R}}_E$ need not be a left congruence; of course the dual remark is also true. We say that S *satisfies the Congruence Condition* (C) *with respect to* E (or, more simply, S *satisfies* (C)) if $\widetilde{\mathcal{R}}_E$ is a left congruence and $\widetilde{\mathcal{L}}_E$ is a right congruence. A second observation is that, as is the case with $\mathcal{R}^*$ and $\mathcal{L}^*$, the relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ need not commute. We denote by $\widetilde{\mathcal{H}}_E$ and $\widetilde{\mathcal{D}}_E$ the intersection and join of $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ respectively. Note that from the previous remark, it is not usually the case that $\widetilde{\mathcal{D}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$.

Deviating slightly from standard terminology, we will denote the $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class, $\widetilde{\mathcal{H}}_E$-class, $\widetilde{\mathcal{D}}_E$-class) of any $a \in S$ by $\widetilde{R}^a_E$ ($\widetilde{L}^a_E$, $\widetilde{H}^a_E$, $\widetilde{D}^a_E$).

The next remark is folklore, but worth stating as a lemma.

Lemma 1.2.9. *If S satisfies (C), then $\widetilde{H}^e_E$ is a monoid with identity e, for any $e \in E$.*

Proof. Let $x, y \in \widetilde{H}^e_E$. Then as $\widetilde{\mathcal{L}}_E$ is a right congruence, we have

$$x \,\widetilde{\mathcal{L}}_E\, e \Rightarrow xy \,\widetilde{\mathcal{L}}_E\, ey = y \,\widetilde{\mathcal{L}}_E\, e$$

so that $xy \,\widetilde{\mathcal{L}}_E\, e$. Also as $\widetilde{\mathcal{R}}_E$ is a left congruence, we have

$$y \,\widetilde{\mathcal{R}}_E\, e \Rightarrow xy \,\widetilde{\mathcal{R}}_E\, xe = x \,\widetilde{\mathcal{R}}_E\, e.$$

Thus $xy \in \widetilde{H}^e_E$. It is clear that e is an identity for all elements in $\widetilde{H}^e_E$. Hence $\widetilde{H}^e_E$ is a monoid with identity e. $\square$

Definition 1.2.10. Let S be a semigroup with distinguished set of idempotents E. Then S is called *weakly E-abundant* if every $\widetilde{\mathcal{R}}_E$- and every $\widetilde{\mathcal{L}}_E$-class of S contains an element of E.

Clearly a regular semigroup S is weakly $E(S)$-abundant; on the other hand, any monoid is weakly $\{1\}$-abundant. A less extreme example is $M_n(R)$, the monoid of $n \times n$ matrices over a principal ideal domain, under matrix multiplication [19]. In such a monoid we have $\widetilde{\mathcal{R}}_E = \mathcal{R}^*$ and $\widetilde{\mathcal{L}}_E = \mathcal{L}^*$, where $E = E\big(M_n(R)\big)$, and further, every $\mathcal{H}^*$-class contains a regular element.

If the distinguished set of idempotents E in a weakly E-abundant semigroup S is the whole set of idempotents $E(S)$, then S is called *weakly abundant semigroup.*

Definition 1.2.11. A weakly E-abundant semigroup is said to be *weakly E-adequate* if E is a semilattice.

Lemma 1.2.12. *Let S be a weakly E-adequate semigroup. Then there is a unique idempotent in the $\widetilde{\mathcal{R}}_E$- and $\widetilde{\mathcal{L}}_E$-class of $a \in S$.*

Proof. For $e, f \in E$, let $a \widetilde{\mathcal{R}}_E e$ and $a \widetilde{\mathcal{R}}_E f$. Then $e \widetilde{\mathcal{R}}_E f$. Now as $ee = e$ and $ff = f$, therefore $ef = f$ and $fe = e$ and so

$$e = fe = ef = f.$$

Hence there is a unique idempotent in the $\widetilde{\mathcal{R}}_E$-class of a. Dually, there is a unique idempotent in the $\widetilde{\mathcal{L}}_E$-class of a. □

If the distinguished set of idempotents E is the whole set of idempotents in a weakly E-adequate semigroup S, then S is called a *weakly adequate semigroup*.

Definition 1.2.13. A weakly adequate semigroup S is said to be *weakly ample* if S satisfies C and for all $a \in S$ and $e \in E(S)$,

$$ae = (ae)^+a \text{ and } ea = a(ea)^*.$$

Of course one can give one-sided version of above definition as well, that is we say that S is *weakly left ample* if $\widetilde{\mathcal{R}}$ is a left congruence and left ample condition holds. Dually we have the notion of *weakly right ample* semigroups.

Chapter 2

Restriction semigroups and categories

In this chapter we shall provide basic definitions and results for restriction semigroups. We then recall some definitions of categories, inductive groupoids and inductive categories. We explain the category theoretic connection between inverse semigroups and inductive groupoids and then between restriction semigroups and inductive categories.

2.1 Restriction semigroups

Restriction semigroups and their one sided versions have been studied from various points of view and under different names since the 1960s. They were formerly called *weakly E-ample semigroups*, to emphasize that the class naturally extends the class of *ample semigroups*. The terminology *weakly E-ample* was first used in [21]. In the abstract definition of 'ample' the relations $\mathcal{R}^*$ and $\mathcal{L}^*$ were replaced by the generalised relation $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$, where E is special set of idempotents known as the *semilattice of projections* or, the *distinguished semilattice*. The terminology *restriction semigroup* has been adopted due to the connection between semigroup theory and category theory. Whereas (left, right) ample semigroups form quasi-varieties, that is, class of algebras of a certain type defined by quasi-identities, (left, right) restriction semigroups form varieties of algebras, where a *variety* is a class of algebras of a certain type defined by identities. It is important to note that varieties are closed under taking subalgebras, homomorphic images and direct products but quasi-varieties are not necessarily closed under homomorphic images. For detailed studies of the basic properties of these structures and a historical overview, the reader is referred to [24] and [28].

Definition 2.1.1. A semigroup S is called a *unary semigroup* if S is equipped with an additional unary operation.

If a semigroup S is equipped with two unary operations, then we say that S is a *bi-unary semigroup*.

An inverse semigroup S is a unary semigroup $(S, \cdot, ^{-1})$, where $^{-1}$ represents the the inverse unary operation on S.

Definition 2.1.2. A *left restriction semigroup* S is a unary semigroup $(S, \cdot, ^{+})$, where $(S, \cdot)$ is a semigroup and $^{+}$ is a unary operation such that the following identities hold:

$$a^+a = a,\ a^+b^+ = b^+a^+,\ (a^+b)^+ = a^+b^+ \text{ and } ab^+ = (ab)^+a.$$

Putting $E = \{a^+ : a \in S\}$, it is easy to see that E is a semilattice, since for $a^+, b^+ \in E$, we have that $a^+b^+ = (a^+b)^+ \in E$, $a^+b^+ = b^+a^+$ and $a^+a^+ = (a^+a)^+ = a^+$. We also notice that $(a^+)^+ = a^+$ as

$$(a^+)^+ = (a^+a^+)^+ = (a^+a)^+ = a^+.$$

These idempotents are called *projections* of S and we call E the *semilattice of projections (formerly distinguished semilattice)* of S.

Dually, a *right restriction* semigroup S is a unary semigroup $(S, \cdot, ^{*})$, where in this case the unary operation is denoted by * and the identities that define a right restriction semi group are:

$$aa^* = a,\ a^*b^* = b^*a^*,\ (ab^*)^* = a^*b^*,\ a^*b = b(ab)^*$$

and putting $E = \{a^* : a \in S\}$, it is easy to check that E is semilattice of projections of S. Thus left and right restriction semigroups form varieties of unary semigroups, that is, semigroups equipped with additional unary operations $^{+}$ and *, respectively.

Definition 2.1.3. A *restriction semigroup* is a bi-unary semigroup S which is both left restriction and right restriction and which also satisfies the linking identities

$$(a^+)^* = a^+ \text{ and } (a^*)^+ = a^*.$$

We remark that an inverse semigroup is restriction, where we define $a^+ = aa^{-1}$ and $a^* = a^{-1}a$. Also we will see that ample semigroups are restriction.

If a restriction semigroup S has an identity element 1, then it is easy to see that $1^+ = 1^* = 1$. Such a restriction semigroup is naturally called a *restriction monoid* and we call it *reduced restriction monoid*.

A restriction semigroup satisfies (C) (with respect to E) and is such that the $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class) of an element a contains a unique element of E, namely a^+ (a^*). The following lemma provides an alternative characterisation for left (right) restriction semigroups.

Lemma 2.1.4. *Let $S(S, \cdot, ^+)$ be a unary semigroup. Then S is left restriction with semilattice of projections E if and only if*

(i) E is a semilattice;

(ii) every $\widetilde{\mathcal{R}}_E$-class contains an idempotent of E;

(iii) the relation $\widetilde{\mathcal{R}}_E$ is a left congruence and

(iv) the left ample condition holds, that is, for all $a \in S$ and $e \in E$,

$$ae = (ae)^+a.$$

The dual result holds for right restriction semigroups and the relation $\widetilde{\mathcal{L}}_E$.

Clearly left (right) restriction semigroups are weakly left (right) E-adequate, so it follows from Lemma 1.2.12, that a^+ (a^*) is the unique idempotent of E in the $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class) of a.

We note the following simple condition for an element $a \in S$ to be $\widetilde{\mathcal{R}}_E$-related to an idempotent $e \in E$:

$$a \, \widetilde{\mathcal{R}}_E \, e \Leftrightarrow ea = a \text{ and for all } f \in E, \; fa = a \Rightarrow fe = e.$$

Thus it follows that a^+ is the smallest left identity of a and we call it the *minimum left identity* of a. Dually a^* is the *minimum right identity* in the $\widetilde{\mathcal{L}}_E$-class of a.

If S and T are left (right) restriction semigroups, then we write E_S and E_T to denote the semilattice of projections of S and T, respectively.

Definition 2.1.5. Let S and T be left restriction semigroups. A (2,1)-*morphism* is a map $\theta : S \to T$ which preserves both the binary operation and unary operation$^+$.

For $e \in E_S$, we have $e\theta \in E_T$ as

$$e\theta = e^+\theta = (e\theta)^+.$$

If S is a left restriction semigroup with semilattice of projections E, then a natural partial order on S is defined by the rule for $a, b \in S$

$$a \leq b \text{ if and only if } a = eb$$

for some $e \in E$. Equivalently

$$a \leq b \text{ if and only if } a = a^+b.$$

To see the equivalence we start with $a = eb$, where $a, b \in S$ and $e \in E$. Then

$$a^+ = (eb)^+ = (eb^+)^+ = eb^+,$$

so that

$$a = eb = eb^+b = a^+b.$$

The converse is clear.

It is easy to check that $\leq$ is reflexive and transitive. To see that it is anti-symmetric, let $a, b \in S$ and $a \leq b \leq a$. Then

$$a = a^+b \text{ and } b = b^+a,$$

so that

$$b = b^+a = b^+a^+b = a^+b^+b = a^+b = a.$$

Hence $\leq$ is anti-symmetric. Clearly, $\leq$ is right compatible and restricts to the usual order on E. To see that it is left compatible let $a, b \in S$ and $a \leq b$. Then $a = a^+b$, so that for $c \in S$,

$$ca = ca^+b = (ca)^+cb \text{ using the left ample condition.}$$

Thus $ca \leq cb$ and hence $\leq$ is compatible.

Lemma 2.1.6. *If S is a restriction semigroup, then for any $a, b \in S$*

$$a = eb \text{ for some } e \in E_S \Leftrightarrow a = a^+b \Leftrightarrow a = ba^* \Leftrightarrow a = bf \text{ for some } f \in E_S.$$

Proof. We have seen above that

$$a = eb \Leftrightarrow a = a^+b.$$

Dually, it is easy to check that

$$a = bf \Leftrightarrow a = ba^*.$$

Now we show that $a = a^+b$ if and only if $a = ba^*$. For this suppose that $a = a^+b$. Then

$$\begin{aligned} a &= a^+b \\ &= b(a^+b)^* \quad \text{using the right ample condition} \\ &= ba^*. \end{aligned}$$

Conversely, if $a = ba^*$, then

$$a = ba^* = (ba^*)^+b = a^+b.$$

□

Lemma 2.1.7. *Let S be a restriction semigroup. Let $a, b \in S$. Then $a \leq b$ implies that $a^+ \leq b^+$ and $a^* \leq b^*$.*

Proof. Let $a \leq b$, so that $a = a^+b$. Then

$$a^+ = (a^+b)^+ = (a^+b^+)^+ = a^+b^+$$

so that $a^+ \leq b^+$. Dually, $a^* \leq b^*$. □

As a^+ is the minimum left identity for a, the following lemma follows immediately from [17].

Lemma 2.1.8. *Let S be a left restriction semigroup with partial order $\leq$ and let $a, b \in S$. Then $(ab)^+ \leq a^+$.*

2.2 Categories

Categories are mathematical structures first introduced by Eilenberg and Maclane in 1945 [13]. Category theory has come to occupy a central position in contemporary mathematics and theoretical computer science and is also applied to mathematical physics. It is a theory that allows us to compare many mathematical structures.

Categories consist of two kind of entities: object and arrows, or morphisms, between these objects. We provide two approaches to categories, first the ‘big categories’- this is the most standard approach- and second ‘small’ categories which are considered to be a generalisation of monoids. The following definition of a category is from [36].

Definition 2.2.1. A category **C** consists of

- a class $\mathrm{Ob}(\mathbf{C})$ of objects;
- a class $\mathrm{Mor}(\mathbf{C})$ of morphisms (or arrows) between the objects;
- two assignments, $\mathbf{d}$ and $\mathbf{r}$, from $\mathrm{Mor}(\mathbf{C})$ to $\mathrm{Ob}(\mathbf{C})$. For $f \in \mathrm{Mor}(\mathbf{C})$ we indicate by

  $$f : A \to B$$

  that $\mathbf{d}(f) = A$ and $\mathbf{r}(f) = B$. Let $\mathrm{Mor}(A, B) = \mathrm{Mor}_{\mathbf{C}}(A, B)$ denote the set of all morphisms between $A, B \in \mathrm{Ob}(\mathbf{C})$;
- if $A, B, C, D \in \mathrm{Ob}(\mathbf{C})$, then there is a binary operation

  $$\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C),\ (f, g) \mapsto f \circ g,$$

  called *composition* of morphisms such that if $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$ and $h \in \mathrm{Mor}(C, D)$, then

  $$(f \circ g) \circ h = f \circ (g \circ h);$$
- for each $A \in \mathrm{Ob}(\mathbf{C})$, there exists a morphism $1_A \in \mathrm{Mor}(A, A)$ such that if $B \in \mathrm{Ob}(\mathbf{C})$ and $f \in \mathrm{Mor}(A, B)$, then

  $$1_A \circ f = f = f \circ 1_B.$$

A simple example of a category is the category $\mathcal{S}\mathbf{et}$ of sets. The objects of this category are sets and the morphisms are functions from one set to another. It is important to note that the objects of a category need not be sets (with or without additional structure) nor the morphisms functions between these sets. In particular, a category is called a *concrete* category if all objects are (structured) sets, morphisms from A to B are (structure preserving) mappings, composition of morphisms is the composition of mappings and the identities are the identity mappings.

Definition 2.2.2. A category $\mathbf{C}$ is called a *small category* if both $\mathrm{Ob}(\mathbf{C})$ and $\mathrm{Mor}(\mathbf{C})$ are sets.

We now give the definition of a functor. This is a structure preserving mapping between two categories, which allow us to compare categories.

Definition 2.2.3. Let $\mathbf{C}$ and $\mathbf{C}'$ be categories. A functor F from $\mathbf{C}$ to $\mathbf{C}'$ is a pair of maps

$$F : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{C}'),\ A \mapsto AF$$

and

$$F : \mathrm{Mor}(\mathbf{C}) \to \mathrm{Mor}(\mathbf{C}'),\ f \mapsto fF$$

which maps an element of $\mathrm{Mor}(G, H)$ to $\mathrm{Mor}(GF, HF)$

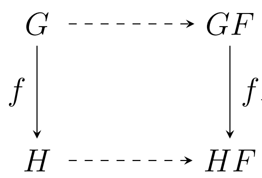

Figure 2.1: Functor: a pair of maps

such that
(i) if $\exists\, g \circ f$ in $\mathbf{C}$, then $(g \circ f)F = gF \circ fF$;
(ii) for $A \in \mathrm{Ob}(\mathbf{C})$, $1_A F = 1_{AF}$.

For any category $\mathbf{C}$, we denote the identity functor by $I_{\mathbf{C}}$. The identity functor assigns each object and morphism to itself.

Definition 2.2.4. Let $\mathbf{C}$ and $\mathbf{C}'$ be categories. Then $\mathbf{C}$ and $\mathbf{C}'$ are *isomorphic* if there exists functors

$$F : \mathbf{C} \to \mathbf{C}' \text{ and } G : \mathbf{C}' \to \mathbf{C}$$

such that $FG = 1_{\mathbf{C}'}$ and $GF = 1_{\mathbf{C}}$.

In this case we say that F is an *isomorphism* from $\mathbf{C}$ to $\mathbf{C}'$ (so G is an isomorphism from $\mathbf{C}'$ to $\mathbf{C}$) and call the pair $\mathrm{Hom}(F, G)$ an *isomorphism* between $\mathbf{C}$ and $\mathbf{C}'$. The functor G is called *inverse* of F (so F is called inverse of G).

We now give the second definition of a category which is convenient for our purposes from [45] and [33].

Let C be a set and let $\cdot$ be a partial binary operation on C. For $x, y \in C$, whenever we write '$\exists\, x \cdot y$', we mean that the product $x \cdot y$ is defined, so that when we will write '$\exists\, (x \cdot y) \cdot z$', it will be understood that we mean $\exists\, x \cdot y$ and $\exists\, (x \cdot y) \cdot z$. We say that an element $e \in C$ is idempotent if $\exists\, e \cdot e$ and $e \cdot e = e$. If e is an idempotent of C which satisfies:

$$\exists\, e \cdot x \Rightarrow e \cdot x = x \ \text{ and } \ \exists\, x \cdot e \Rightarrow x \cdot e = x,$$

then we say that e is an *identity* for C and we call this identity a *local identity* for C.

Definition 2.2.5. Let $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r})$, where $\cdot$ is a partial binary operation on C and $\mathbf{d}, \mathbf{r} : C \to C$ such that

(C1) $\exists\, x \cdot y$ if and only if $\mathbf{r}(x) = \mathbf{d}(y)$ and then

$$\mathbf{d}(x \cdot y) = \mathbf{d}(x) \text{ and } \mathbf{r}(x \cdot y) = \mathbf{r}(y);$$

(C2) $\exists\, x \cdot (y \cdot z)$ (so that from (C1), $\exists\, (x \cdot y) \cdot z$), then

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z;$$

(C3) $\exists\, \mathbf{d}(x) \cdot x$ and $\mathbf{d}(x) \cdot x = x$ and $\exists\, x \cdot \mathbf{r}(x)$ and $x \cdot \mathbf{r}(x) = x$.

Let $E = \{\mathbf{d}(x) : x \in C\}$. It follows from the axioms that $E = \{\mathbf{r}(x) : x \in C\}$ and $\mathbf{C}$ is a small category in standard sense with set of local identities E and set of objects identified with E. Further, $\mathbf{d}(x)$ is the domain of x and $\mathbf{r}(x)$ is the range of x.

Small categories can be seen as generalisations of monoids as if $\mathbf{C}$ has precisely one object, then all products are necessarily defined and thus $\mathbf{C}$ is essentially a monoid.

We make it clear that whenever we write $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r})$ for a category $\mathbf{C}$, we mean that $\mathbf{C}$ is a small category, described as in Definition 2.2.5.

We now define what we mean by an *ordering* on our category $\mathbf{C}$.

Definition 2.2.6. Let $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r})$ be a category with set of local identities E. Let $\leq$ be a partial order on C such that for all $e \in E$, $x, y \in C$:

(IC1) if $x \leq y$ then $\mathbf{r}(x) \leq \mathbf{r}(y)$ and $\mathbf{d}(x) \leq \mathbf{d}(y)$;

(IC2) if $x \leq y$ and $x' \leq y'$, $\exists\, x \cdot x'$ and $\exists\, y \cdot y'$, then $x \cdot x' \leq y \cdot y'$;

(IC3) if $e \leq \mathbf{d}(x)$ then $\exists$ unique ${}_e|x \in \mathbf{C}$ such that

$${}_e|x \leq x \quad \text{and} \quad \mathbf{d}({}_e|x) = e;$$

(IC4) if $e \leq \mathbf{r}(x)$ then $\exists$ unique $x|_e \in \mathbf{C}$ such that

$$x|_e \leq x \quad \text{and} \quad \mathbf{r}(x|_e) = e.$$

We then say that $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ is an *ordered category.*

The element ${}_e|x$ of Condition (IC3) is called the *restriction* of x to e and the element $x|_e$ of Condition (IC4) is called the *co-restriction* of x to e.

If $\mathbf{C}$ is an ordered category with set of local identities E, then for $e, f \in E$, we denote the greatest lower bound (meet) of e and f, where it exists, by $e \wedge f$.

Definition 2.2.7. Let $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ be an ordered category. Then $\mathbf{C}$ is an *inductive category* if (IC5) holds:

(IC5) $(E, \leq)$ is a meet semilattice.

Definition 2.2.8. A functor between two ordered categories is said to be *ordered* if it is order preserving. An ordered functor between two inductive categories is said to be *inductive* if it preserves the meet operation on the set of identities.

There is a category theoretic connection between inductive categories and restriction semigroups, but before explaining this we would like to explain the corresponding category theoretic connection between inverse semigroups and inductive groupoids as restriction semigroups generalise inverse semigroups in a natural way.

2.3 Inverse semigroups and inductive groupoids

Lie studied infinite continuous groups in the 1880s which are not groups at all but merely group-like, called *Lie pseudogroups* [48, 49]. Veblen and Whitehead generalised this structure to what they termed as *pseudogroups* to describe symmetries in differential geometry [76].

Definition 2.3.1. [76, p. 38] A *pseudogroup* Γ is a collection of partial homeomorphisms between open subsets of a topological space such that Γ is closed under composition and inverses, where $\alpha, \beta \in \Gamma$ are composed if $\operatorname{im}\alpha = \operatorname{dom}\beta$.

Inverse semigroups were only one solution to the problem of finding an abstract characterisation for pseudogroups. If in a pseudogroup, the composition

$$\operatorname{dom}\alpha\beta = [\operatorname{im}\alpha \cap \operatorname{dom}\beta]\alpha^{-1}$$

is given, then it is just an inverse semigroup of homeomorphisms between open sets of a topological space.

Ehresmann found a second solution to the problem of finding an abstract characterisation for pseudogroups starting from a different definition of the composition of partial bijections. Ehresmann noticed that a pseudogroup has the structure of an ordered groupoid with the obvious partial order of restriction of mappings, where an ordered groupoid is essentially an ordered category in which all arrows are invertible. As a second step Ehresmann imposed extra conditions on ordered groupoids and studied the so called *inductive groupoids*. Ehresmann's category theoretic work began in [14]. The reader is referred to [45] for more details on Ehresmann's work. We now recall the full definition of an inductive groupoid.

Definition 2.3.2. An *inductive groupoid* is an inductive category $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ in which the following conditions hold:

(IG1) for each $x \in C$, there is an $x^{-1} \in C$ such that $\exists\, x \cdot x^{-1}$ and $\exists\, x^{-1} \cdot x$, with

$$x \cdot x^{-1} = \mathbf{d}(x) \text{ and } x^{-1} \cdot x = \mathbf{r}(x).$$

(IG2) $x \leq y$ implies $x^{-1} \leq y^{-1}$ for all $x, y \in C$.

Thus all elements are invertible in an inductive groupoid. The next proposition justifies the introduction of inductive groupoids in the context of inverse semigroups.

Proposition 2.3.3. [45, Proposition 4.1.1] *Let S be an inverse semigroup. Then $(S, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ is an inductive groupoid, where $\mathbf{d}(a) = aa^{-1}$, $\mathbf{r}(a) = a^{-1}a$ and $\leq$ is the partial order on S. We refer to $\cdot$ as the* restricted product *on S:*

$$a \cdot b = ab \quad (\textit{the product in } S) \textit{ when } \mathbf{r}(a) = \mathbf{d}(b)$$

Thus an inductive groupoid can be constructed from an inverse semigroup. Conversely, to construct an inverse semigroup from an inductive groupoid, we need the notion of *pseudoproduct* which was defined by Ehresmann [14, 15] as follows:

Let G be an inductive groupoid and let $x, y \in G$. Then the pseudoproduct $x \otimes y$ of x and y is defined by:

$$x \otimes y = (x|_{\mathbf{d}(x)\wedge\mathbf{r}(y)})({}_{\mathbf{d}(x)\wedge\mathbf{r}(y)}|y).$$

Note that the pseudoproduct is everywhere defined in an inductive groupoid.

The connection between inverse semigroups and inductive groupoids was made explicit by Schein who proved that to any inverse semigroup there corresponds an inductive groupoid and vice versa [69, Theorem 3.4]. Later Nambooripad made his contributions to generalise Schein's results to the regular case [55] and put Schein's results in the context of an isomorphism between categories. These results were gathered together in a single theorem by Lawson in [45], who named it *The Ehresmann-Schein-Nambooripad theorem* to reflect the diverse origin of its various components. For the definition of *prehomomorphisms* the reader is referred to [45]. Inverse semigroups and semigroup homomorphisms form a category (of the first kind) as do inductive groupoids and inductive functors.

The Ehresmann-Schein-Nambooripad Theorem. *The category of inverse semigroups and prehomomorphisms is isomorphic to the category of inductive groupoids and ordered functors; and the category of inverse semigroups and homomorphisms is isomorphic to the category of inductive groupoids and inductive functors.*

2.4 Restriction semigroups and inductive categories

The connection between inverse semigroups and inductive groupoids raised natural questions to seek generalisations for classes such as ample semigroups which naturally generalise inverse semigroups. The Ehresmann-Schein-Nambooripad theorem received a succession of generalisations the first of these due to Armstrong [2]. She used the 'structure mappings' technique of Meakin [53] to study ample semigroups by considering mappings between $\mathcal{R}^*$- and $\mathcal{L}^*$-classes and extended the Ehresmann-Schein-Nambooripad theorem to ample semigroups and cancellative categories.

Lawson made two further generalisations of the Ehresmann-Schein-Nambooripad theorem. The first of these was to the case of weakly ample semigroups and inductive unipotent categories [43] and the second was to the case of restriction semigroups (weakly E-ample semigroups) and arbitrary inductive categories [44]. For a detailed survey, the reader is referred to [33] in which Hollings presented a complete historical overview of connection between semigroups and categories. Theorem 4.7 of [29] put all of these results into a general picture. We now list some results of Lawson [44] that were presented by Hollings [33] in terms of restriction semigroups.

The following theorem tells how to obtain inductive categories from restriction semigroups.

Theorem 2.4.1. [33, Theorem 7.2.4] *Let S be a restriction semigroup. Then $(S, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ is an inductive category with set of local identities E, where $\mathbf{d}(a) = a^+$ and $\mathbf{r}(a) = a^*$ and $\leq$ is the natural partial order on S. We refer to $\cdot$ the* restricted product *on S as follows:*

$$a \cdot b = ab \text{ (the product in } S\text{) when } \mathbf{r}(a) = \mathbf{d}(b).$$

The restriction ${}_e|a$ in this inductive category is just the product ea in S, because $ea \leq a$ and $(ea)^+ = (ea^+)^+ = e$. Similarly the co-restriction $a|_e = ae$ and $e \wedge f = ef$.

The pseudoproduct in an inductive category $\mathbf{C}$ defined by

$$a \otimes b = (a|_{\mathbf{d}(a)\wedge\mathbf{r}(b)})({}_{\mathbf{d}(a)\wedge\mathbf{r}(b)}|b)$$

is everywhere defined in $\mathbf{C}$ and coincides with the restricted product $\cdot$ in $\mathbf{C}$ (see [33]). Now we can get a restriction semigroup from an inductive category as we see in the following theorem.

Theorem 2.4.2. [33, Theorem 7.2.6] *If $(\mathbf{C}, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ is an inductive category, then $(\mathbf{C}, \otimes)$ is a restriction semigroup.*

Let S be a restriction semigroup and following the notation of [33], we will denote the inductive category associated to S by $\mathbf{C}(S)$. Also if C is an inductive category, then $\mathbf{S}(C)$ will denote the restriction semigroup associated to it. Now we can write the following result from [33, Theorem 7.2.7].

Theorem 2.4.3. *Let S be a restriction semigroup and C be an inductive category. Then*

$$\mathbf{S}(\mathbf{C}(S)) = S \text{ and } \mathbf{C}(\mathbf{S}(C)) = C.$$

Proposition 2.4.4. [33, Proposition 7.2.8] *Let S and T be restriction semigroups and $\varphi : S \to T$ be a (2,1,1)-morphism. Define $\mathbf{C}\varphi : \mathbf{C}(S) \to \mathbf{C}(T)$ to be the same function on the underlying sets. Then $\mathbf{C}\varphi$ is an inductive functor with respect to the restricted product in $\mathbf{C}(S)$ and $\mathbf{C}(T)$.*

Let $\psi : C \to D$ be an inductive functor of inductive categories C and D and define $\mathbf{S}\psi : \mathbf{S}(C) \to \mathbf{S}(D)$ to be the same function on the underlying sets. Then $\mathbf{S}\psi$ is a morphism with respect to the pseudoproduct in $\mathbf{S}(C)$ and $\mathbf{S}(D)$.

Further $\mathbf{S}(\mathbf{C}(\varphi)) = \varphi$ and $\mathbf{C}(\mathbf{S}(\psi) = \psi$.

Theorems 2.4.1, 2.4.2, 2.4.3 and Proposition 2.4.4 give the desired result which is generalisation of the Ehresmann-Schein-Nambooripad theorem for restriction semigroups.

Theorem 2.4.5. [33, Proposition 7.2.9] *The category of restriction semigroups and (2,1,1)-morphisms is isomorphic to the category of inductive categories and inductive functors.*

Chapter 3

Decomposition of semigroups and monoids

This chapter is a survey to direct, semidirect and Zappa-Szép products of semigroups and monoids. We shall provide historical background to Zappa-Szép products in the first section. In the second section we discuss uniquely factorisable semigroups. We discuss external and internal versions of direct, semidirect and Zappa-Szép products in the subsequent sections. We explain the correspondence between inernal and external versions of these products for semigroups and monoids.

The semidirect product of two inverse semigroups is not inverse in general. We provide results of Nico [58] and Preston [64] that tell us when the semidirect product of two inverse monoids (semigroups) is inverse. We consider the semidirect product of a right (left) restriction semigroup and a monoid and provide necessary and sufficient conditions for it to be right (left) restriction.

3.1 Historical background

The Zappa-Szép product is a natural generalisation of the semidirect product of groups, whereas, the semidirect product is a natural generalisation of the direct product of groups. In the direct product of groups both factors are required to be normal, whereas in semidirect products this strong condition is replaced by the weaker condition that only one factor is required to be normal. More generally, in Zappa-Szép products, neither of the factors is required to be normal.

The concept of Zappa-Szép products was perhaps first studied in 1935 for groups by

Neumann, who used the terminology *general decompositions* for these products [56]. The first systematic study of such products was done by Zappa in 1940 [79]. Further developments in the theory for groups were made by Casadio [9], Redei [65] and Szép [68, 71, 72, 73] from 1941 to the early 60s. Szép introduced the term *skew products* for Zappa-Szép products. The relations of Zappa-Szép products were introduced by Szép [71] to study the structural properties of groups. He also initiated the study of similar products in setting other than groups in [72] and [73]. The terminology Zappa-Szép product was suggested by Zappa.

Definition 3.1.1. Let U be a semigroup. We say that U is *factorisable (with factors S and T)* if there are subsemigroups S and T of U such that $U = ST$. If each element $u \in U$ can be uniquely written as a product of an element of S and an element of T, then we say that U is *uniquely factorisable.*

It is worth to mention here that this is also the definition of an *internal Zappa-Szép product* which we will discuss later in this chapter.

In Definition 3.1.1, if U is a group, then U is *factorisable* if in addition S and T are subgroups of U.

All possible factorisations of sporadic simple groups and all maximal factorisations of all finite simple groups and their automorphism groups have been completely determined. For a detailed history from the group theoretic point of view and for a complete list of references, the reader is referred to [77].

The concept of factorisable semigroups have been studied by Tolo in 1969 [74]. He characterised direct products of semigroups and found sufficient conditions for a semigroup to be factorisable, but the concept of Zappa-Szép product for semigroups was properly developed by Kunze in 1983 and he then continued his work on this subject in subsequent years [37, 38, 39, 40]. Kunze focused attention on transformation monoids and automata theory and termed Zappa-Szép products as *bilateral semidirect products.* He gave applications of Zappa-Szép product to translational hulls, Bruck-Reilly extensions and Rees matrix semigroups. He studied aperiodic transformation semigroups (X, S) and investigated a strong decomposition of its elements in terms of idempotents.

In 1997, the term *general products* for Zappa-Szép products was introduced by Lavers [41, 42] who found applications of these products in the theory of vine monoids and monoid presentations [41]. Lavers found conditions under which the Zappa-Szép product of two finitely presented monoids is itself finitely presented. Araújo [1] did a survey on finite representability of several semigroup constructions including Zappa-Szép products in 2002.

Coleman and Easdown [10] developed a decomposition theory for submonoids and subgroups of $(S, \circ)$, where S is a ring not necessarily with identity and the $\circ$ operation is defined by

$$a \circ b = a + b - ab.$$

They expressed these decompositions in terms of uniqely factorisable semidirect, reverse semidirect and Zappa-Szép products. Group presentations were obtained in terms of $(S, +)$ and $(S, \circ)$ by applying decomposition theory to ring of matrices over S when S is radical. A ring S is called radical if $S = \mathcal{J}(S)$, where $\mathcal{J}(S)$ is known as the Jacobson radical and is defined to be the largest ideal of S consisting of quasi-invertible elements of S, where an element $s \in S$ is said to be quasi-invertible if there is an element t such that

$$s \circ t = t \circ s = 0.$$

Brin recorded the behaviour of Zappa-Szép products in connection with properties that are needed to work with categories, monoids and group of fractions of monoids in 2005 [7]. He extended the family of Thompson's groups by constructing a braided version of Thompson's group that surjects onto Thompson's group using a Zappa-Szép product of the monoid of binary forests and the braid group on infinitely many strands.

In 2008, Zappa-Szép products of a free monoid and a group from the view of self similar group actions are studied by Lawson [46] who completely determined their structure by describing a correspondence between a class of left cancellative monoids and self similar group actions.

Gilbert and Wazzan studied Zappa-Szép products from the point view of regular and inverse semigroups. They generalised some results of Lawson and found necessary and sufficient conditions for these products to be regular and inverse [77]. Further, they generalised the concept of a λ-semidirect product of inverse semigroups to Zappa-Szép products by introducing the notion of a λ-Zappa-Szép product of a semilattice and a group [25]. By giving it the structure of an inductive groupoid, they proved that it is an inverse semigroup.

The concept of congruence pairs for a Zappa-Szép product $P = E \bowtie G$ of a semilattice E with an identity and a group G is introduced by Xiao and Li [78]. They prove that every congruence on P can be described by a congruence pair and the congruence lattice of P is isomorphic to a lattice of all congruence pairs for P. Group congruences for this Zappa-Szép product are also characterised.

Recently, Brownlowe and at al [8] used Zappa-Szép products to identify the class of C^*-

algebras associated to quasi-lattice ordered groups and self similar actions. Following Li's constructions [47], they produced a full C^*-algebra and provided its new presentation via generators and relations. An example of C^*-algebras associated to a self similar actions of a semigroup is also described.

3.2 Uniquely factorisable semigroups

We have seen the definition of uniquely factorisable semigroups in Section 3.1. In this section we discuss a result for such semigroups which is useful for our purpose later in this chapter.

If a semigroup U is uniquely factorisable as $U = ST$, then from [7, Lemma 3.4] there is an idempotent $e \in S \cap T$ which is a right identity for S and a left identity for T. We provide an alternative proof of this result in the following.

Lemma 3.2.1. *Let U be a semigroup and suppose U is uniquely factorisable as $U = ST$, where S and T are subsemigroups of U. Then there exists an idempotent $e \in S \cap T$ which is a right identity for S and a left identity for T.*

Proof. Let $s \in S$. Then s can be written as

$$s = uv \text{ for } u \in S,\, v \in T.$$

For $t \in T$,

$$st = uvt \text{ for all } t \in T$$

so that

$$t = vt \text{ for all } t \in T.$$

Thus v is a left identity for T and in particular $v = v^2$.

Next as $v \in T$, so we can write v as

$$v = pq \text{ for } p \in S,\, q \in T.$$

For $w \in S$

$$wv = wpq \text{ for all } w \in S$$

so that

$$w = wp \text{ for all } w \in S$$

which implies p is a right identity for S and in particular, $p = p^2$, so that

$$v = pq = ppq = pv.$$

Thus for $p \in S$, there exits an element $l = l^2 \in T$ which is a left identity for T and we can write p as $p = pl$. Now

$$\begin{array}{rrcll} & v & = & pv & \\ \Rightarrow & vl & = & pvl & \\ \Rightarrow & l & = & pl & \text{because } v \text{ is a left identity for } T \\ \Rightarrow & l & = & p & \text{because } p = pl. \end{array}$$

Hence $l = p = e \in S \cap T$ which is a right identity for S and a left identity for T. □

As Zappa-Szép products generalise semidirect products and semidirect products are natural generalisations of direct products, we start discussing direct products in the next section.

3.3 Direct products of semigroups and monoids

In this section we give definitions of *direct products* of semigroups and monoids. We discuss the equivalence between *internal and external direct products* of monoids and semigroups in the subsequent subsections.

Definition 3.3.1. Let S and T be semigroups and let

$$S \times T = \{(s, t) : s \in S, t \in T\}.$$

Define a binary operation on $S \times T$ by the rule

$$(s, t)(u, v) = (su, tv).$$

Then $S \times T$ is a semigroup and is known as the *(external) direct product* of semigroups S and T.

If S and T are monoids, then $S \times T$ is a monoid with identity $(1_S, 1_T)$ and we say that $S \times T$ is the *external direct product of monoids* S and T.

We now explain what we mean by an *internal direct product* of semigroups and monoids.

Definition 3.3.2. Let U be a semigroup and S and T are subsemigroups of U. Then U is the *internal direct product* of S and T if

(1) U is uniquely factorisable with factors S and T;

(2) for all $s \in S$ and for all $t \in T$, $st = ts$.

If U is a monoid with submonoids S and T in the above definition, then we say that U is an *internal direct product of submonoids* S and T.

3.3.1 Internal and external direct products of monoids

We now see that there is an equivalence between internal and external direct product of monoids.

Proposition 3.3.3. *Let U be a monoid and S, T be submonoids of U. Suppose U is the internal direct product of S and T. Then U is isomorphic to the external direct product $S \times T$.*

Conversely, suppose that $U = S \times T$ is the external direct product of monoids S and T. Then there are submonoids S' and T' of U such that $S \cong S'$, $T \cong T'$ and $U = S'T'$ is the internal direct product of S' and T'.

Proof. Suppose U is the internal direct product of monoids S and T. Then U is uniquely factorisable and $st = ts$ for all $s \in S$ and all $t \in T$. Define a map

$$\varphi : S \times T \to U$$

by $(s,t)\varphi = st$. Then φ is well defined and one-one due to unique factorisation. Also φ is onto. To check that φ is a morphism, let $(s,t), (s',t') \in S \times T$. Then

$$\begin{aligned} ((s,t)(s',t'))\varphi &= (ss',tt')\varphi \\ &= ss'tt' \\ &= sts't' && \text{because } st = ts \text{ for all } s \in S \text{ and all } t \in T \\ &= (s,t)\varphi(s',t')\varphi. \end{aligned}$$

Thus φ is morphism and hence $U \cong S \rtimes T$.

Conversely suppose that $U = S \times T$ is the external direct product of monoids S and T. Let

$$S' = \{(s, 1_T) : s \in S\} \text{ and } T' = \{(1_S, t) : t \in T\}.$$

Clearly S' and T' are submonoids of U. It is easy to check that $S \cong S'$ and $T \cong T'$ under the maps

$$s \mapsto (s, 1_T) \text{ and } t \mapsto (1_S, t)$$

respectively. Also each element $(s,t) \in U$ can be uniquely written as

$$(s,t) = (s, 1_T)(1_S, t).$$

Finally we note that elements of S' and T' commute, as for $(s, 1_T) \in S'$ and $(1_S, t) \in T'$

$$(s, 1_T)(1_S, t) = (s,t) = (1_S, t)(s, 1_T).$$

Hence U is the internal direct product of S' and T'.

□

3.3.2 Internal and external direct products of semigroups

There is not such a complete equivalence between internal and external direct products of semigroups as we have seen for monoids. When we consider internal direct product of semigroups, then we do not get an external direct product of semigroups but what we obtain is an internal direct product of monoids as we see in the following lemma.

Lemma 3.3.4. *Let U be a semigroup and S and T be subsemigroups of U. Suppose U is the internal direct product of S and T. Then U is a monoid and S and T are submonoids of U.*

Proof. As U is uniquely factorisable, so by Lemma 3.2.1, there exists an element $e \in S \cap T$ such that

$$se = s \text{ for all } s \in S \text{ and } et = t \text{ for all } t \in T.$$

We note that for $s \in S$

$$es = se = s,$$

because elements of S and T commute, so that e is identity of S. Thus for all $s \in S$ and for all $t \in T$

$$est = st,$$

implies that e is left identity for U. Dually e is right identity for U and hence U is a monoid with submonoids S and T. Thus U is the internal direct product of S and T, which by Proposition 3.3.3 is isomorphic to the external direct product $S \times T$. □

From the above lemma, we observe that $S \cap T = \{e\}$ as if $w \in S \cap T$ is another element, then

$$w = ew = we$$

gives $w = e$ because of unique factorisations. Hence $S \cap T = \{e\}$.

But $S \cap T = \{e\}$ does not imply that elements of U commute as it not true even in the group case.

Thus an internal direct product of semigroups is just an internal direct product of monoids. Now we would like to see what happens if we consider an external direct product of semigroups. It is not obvious to get an internal direct product from an external direct product of semigroups because for semigroups there do not exist such mappings $s \mapsto (s, 1_T)$ and $t \mapsto (1_S, t)$ as they exists for monoids, but we can use Preston's techniques from [64] for semidirect products to get round this difficulty.

Lemma 3.3.5. *Suppose S and T are semigroups and $U = S \times T$ is the external direct product of S and T. Let $S^{(1)}$ and $T^{(1)}$ be the semigroups obtained from S and T, respectively by adjoining an identity and let $V = S^{(1)} \times T^{(1)}$. Then there are subsemigroups S' and T' of $V = S^{(1)} \times T^{(1)}$ with $S \cong S'$, $T \cong T'$ such that $U = S'T'$ and every element of U is uniquely factorisable as $u = st$ for $s \in S'$ and $t \in T'$.*

Proof. Let $S^{(1)}$ and $T^{(1)}$ be the semigroups obtained from S and T, respectively by adjoining an identity and suppose $V = S^{(1)} \times T^{(1)}$. Let

$$S'' = \{(s, 1); s \in S^{(1)}\} \text{ and } T'' = \{(1, t) : t \in T^{(1)}\}.$$

By Proposition 3.3.3, S'' and T'' are submonoids of $V = S^{(1)} \times T^{(1)}$ and $V = S''T''$ is the internal direct product of S'' and T'' where

$$\alpha : S^{(1)} \to S'' \text{ and } \beta : T^{(1)} \to T''$$

are isomorphisms. Next let

$$S' = \{(s, 1) : s \in S\} \text{ and } T' = \{(1, t) : t \in T\}.$$

Clearly S' and T' are subsemigroups of S'' and T'' and hence of $V = S^{(1)} \times T^{(1)}$. Then

$$\alpha|_S : S \to S' \text{ and } \beta|_T : T \to T'$$

are isomorphisms. Also each element $(s,t) \in U$ can be written as

$$(s,t) = (s,1)(1,t) \text{ where } (s,1) \in S' \text{ and } (1,t) \in T'.$$

Moreover elements of S' and T' commute. Hence $U = S'T'$ is a subsemigroup of $V = S^{(1)} \times T^{(1)}$. □

3.4 Semidirect products of semigroups and monoids

The term *semidirect product* for semigroups was first used by Neumann [57] to construct wreath products of semigroups. We first explain that what we mean by an action of a semigroup T on a set X.

Definition 3.4.1. Let T be a semigroup and X be a set. Then T *acts on* X *on the left* if there is a map

$$T \times X \to X, \, (t,x) \mapsto t \cdot x$$

such that for all $x \in X$ and for all $t_1, t_2 \in T$ we have

(S1) $t_1 t_2 \cdot x = t_1 \cdot (t_2 \cdot x)$.

If T is a monoid and acts on a set X, then for a monoid action we insist that for all $x \in X$

(S2) $1 \cdot x = x$.

Dually, T *acts on the right* of X if there is a map

$$X \times T \to X, \, (x,t) \mapsto x^t$$

such that for all $x \in X$ and for all $t_1, t_2 \in T$ we have

$$x^{t_1 t_2} = (x^{t_1})^{t_2}.$$

Definition 3.4.2. Let S and T be semigroups and suppose that T acts on S on the left satisfying (S1). We say that T acts on S by *endomorphisms* if for all $t \in T$ and $s_1, s_2 \in S$ we have

(S3) $t \cdot (s_1 s_2) = (t \cdot s_1)(t \cdot s_2)$.

This is equivalent to saying that there is a homomorphism

$$\theta : T \to \operatorname{End} S$$

where $\operatorname{End} S$ is monoid of endomorphisms of S. Diverting from usual practice, we multiply elements of $\operatorname{End} S$ from right to left, that is, for $\theta, \varphi \in \operatorname{End} S$, to compute $\theta\varphi$ we first do φ then θ. Therefore we write θ on the right of the argument and for each $t \in T$, we denote $(\theta(t))s$ by $t \cdot s$.

If S and T are monoids, then T acts on S by endomorphisms if $\theta : T \to \operatorname{End} S$ is a monoid morphism, that is,

$$(\theta(t))(1) = 1 \text{ for all } t \in T.$$

This translates to $t \cdot 1 = 1$ for all $t \in T$.

Definition 3.4.3. Suppose S and T are semigroups such that T acts on the left of S by endomorphisms. Define a binary operation on $S \times T$ by

$$(s,t)(s',t') = (s(t \cdot s'), tt').$$

We check that this binary operation is associative. For this let $(s_1,t_1),(s_2,t_2),(s_3,t_3) \in S \times T$. Then

$$\begin{aligned}
\big((s_1,t_1)(s_2,t_2)\big)(s_3,t_3) &= (s_1(t_1 \cdot s_2), t_1t_2)(s_3,t_3) \\
&= (s_1(t_1 \cdot s_2)(t_1t_2 \cdot s_3), (t_1t_2)t_3) \\
&= (s_1(t_1 \cdot s_2)(t_1 \cdot (t_2 \cdot s_3)), t_1(t_2t_3)) \quad \text{using (S1)} \\
&= (s_1(t_1 \cdot (s_2(t_2 \cdot s_3))), t_1(t_2t_3)) \quad \text{using (S3)} \\
&= (s_1,t_1)(s_2(t_2 \cdot s_3), t_2t_3) \\
&= (s_1,t_1)\big((s_2,t_2)(s_3,t_3)\big).
\end{aligned}$$

Hence associativity holds and thus $S \times T$ is a semigroup known as an *(external) semidirect product* of S by T and denoted by $S \rtimes T$.

Lemma 3.4.4. *Let S and T be semigroups and suppose that $S \rtimes T$ is the external semidirect product of S by T. If T acts on S trivially, then $S \rtimes T$ becomes the external direct product $S \times T$.*

Proof. Suppose T acts on S trivially. Then $t \cdot s = s$ for all $t \in T$ and for all $s \in S$. In this case for $(s,t),(s',t') \in S \rtimes T$,

$$(s,t)(s',t') = (s(t \cdot s'), tt') = (ss', tt')$$

and hence $S \rtimes T$ is the external direct product $S \times T$. □

Lemma 3.4.5. *If S and T are monoids and T acts on the left of S by endomorphisms satisfying (S1), (S2) and (S3), then $S \rtimes T$ is a monoid with identity $(1_S, 1_T)$.*

Proof. We know that $S \rtimes T$ is a semigroup. Now as $\theta(t)$ is a monoid morphism and $t \cdot 1 = 1$, so $(1_S, 1_T)$ becomes identity of $S \rtimes T$, because for $(s,t) \in S \rtimes T$

$$(s,t)(1_S, 1_T) = (s(t \cdot 1_S), t1_T) = (s1_S, t) = (s,t),$$

and

$$(1_S, 1_T)(s,t) = (1_S(1_T \cdot s), 1_T t) = (s,t).$$

Hence $S \rtimes T$ is a monoid with identity $(1_S, 1_T)$ and we call it the *external semidirect product* of monoids S and T. □

Dually, we have the notion of *reverse* semidirect product, when S acts on T on the right by endomorphisms. Multiplication in a reverse semidirect product is defined by

$$(s,t)(s',t') = (ss', t^{s'}t').$$

We denote the reverse semidirect product of S by T by $S \ltimes T$.

We now give the definition of an *internal semidirect product* from [64].

Definition 3.4.6. Let U be a semigroup and S be a subset of U. An element $u \in U$ is said to be *left permutable with* S if

$$uS \subseteq Su.$$

If T is a subset of U, then T is said to be *locally left permutable with* S if each element of T is left permutable with S.

Definition 3.4.7. Let U be a semigroup and S, T be subsemigroups of U. Then U is the *internal semidirect product* of S and T if U is uniquely factorisable with factors S and T and T is locally left permutable with S.

If U is a monoid and S, T are submonoids of U in the above definition, then we say that U is the *internal semidirect product* of monoids S and T.

We now see that if U is the internal semidirect product of semigroups S and T, then there is an action of T on the left of S and thus we can get an external semidirect product.

Lemma 3.4.8. *Let U be a semigroup and suppose that $U = ST$ is the internal semidirect product of subsemigroups S and T. Then T acts on S such that $ts = (t \cdot s)t$ for all $s \in S$ and $t \in T$ and $U' = S \rtimes T$ is the external semidirect product of S and T.*

Proof. As $U = ST$ is the internal semidirect product of subsemigroups S and T, so for each $s \in S$ and $t \in T$, there exists a unique element $t \cdot s$ of S such that

$$ts = (t \cdot s)t.$$

By associativity we have that $(tt')s = t(t's)$ for any $t, t' \in T$ and $s \in S$. Now

$$(tt')s = ((tt') \cdot s)tt'$$

and

$$t(t's) = t((t' \cdot s)t') = (t \cdot (t' \cdot s))tt'.$$

By uniqueness we have $tt' \cdot s = t \cdot (t' \cdot s)$ and so (S1) holds.

Next for $t \in T$ and $s, s' \in S$

$$t(ss') = (t \cdot (ss'))t$$

and

$$(ts)s' = (t \cdot s)ts' = (t \cdot s)(t \cdot s')t.$$

By associativity and uniqueness, we have that

$$t \cdot (ss') = (t \cdot s)(t \cdot s'),$$

and thus (S3) holds. Hence we can form an external semidirect product $U' = S \rtimes T$ of S and T. □

We see in the following lemma that if U is the internal semidirect product of subsemigroups S and T, then one of the factors is forced to be a monoid.

Lemma 3.4.9. *Let U be a semigroup and suppose that $U = ST$ is the internal semidirect product of subsemigroups S and T. Then there exists an element $e \in S \cap T$ which is a right identity for S and a left identity for T such that*

$$e \cdot s = es,\ e = t \cdot e \text{ for all } s \in S \text{ and } t \in T.$$

Moreover T is a monoid with identity e.

Proof. As factorisations are unique, so from Lemma 3.2.1 there exists an element $e \in S \cap T$

which is a right identity for S and a left identity for T. Now for $s \in S$

$$es = (e \cdot s)e.$$

But e is right identity for S, so that

$$es = e(se) = (es)e.$$

Thus $(es)e = (e \cdot s)e$ and by uniqueness $es = e \cdot s$.

Next for $t \in T$

$$te = (t \cdot e)t.$$

But e is a left identity for T, so that

$$te = (et)e = e(te).$$

Thus $e(te) = (t \cdot e)t$ and by uniqueness $e = t \cdot e$ and $te = t$. Hence

$$e \cdot s = es,\ e = t \cdot e \ \text{ for all } s \in S \text{ and } t \in T.$$

Also as $te = t$ for all $t \in T$, so e is a right identity for T and hence T is a monoid. □

We now explain the correspondence between external and internal semidirect products of monoids and semigroups in the following subsections.

3.4.1 Internal and external semidirect products of monoids

There is an equivalence between internal and external semidirect products of monoids as we see in the following theorem.

Theorem 3.4.10. *Let U be a monoid and let $U = ST$ be the internal semidirect product of submonoids S and T. Then there exists an external semidirect product $U' = S \rtimes T$ such that $U \cong U'$.*

Conversely, let $U = S \rtimes T$ be the external semidirect product of monoids S and T. Let

$$S' = \{(s, 1_T) : s \in S\} \text{ and } T' = \{(1_S, t) : t \in T\}.$$

Then S' and T' are submonoids of U such that T' is locally left permutable with S' and U is uniquely factorisable with factors S' and T'. Further $S \cong S'$ and $T \cong T'$.

Proof. Suppose $U = ST$ is the internal semidirect product of submonoids S and T. As factorisations are unique, so from Lemma 3.4.8, (S1) and (S3) hold. Also as U is a monoid and $1_U = 1 \in S \cap T$, so for $s \in S$, we see that

$$s1 = 1s = (1 \cdot s)1$$

and by uniqueness $1 \cdot s = s$. Thus (S2) holds. Hence we can form an external semidirect product $U' = S \rtimes T$ of S and T. Now define a map

$$\varphi : U \to U'$$

by $(st)\varphi = (s, t)$. By uniqueness of factorisation, it is easy to see that φ is well defined and one-one. Clearly φ is onto. To show that φ is a morphism, let $st, s't' \in U$. Then

$$\begin{aligned} (st)\varphi(s't')\varphi &= (s,t)(s',t') \\ &= (s(t \cdot s'), tt') \\ &= (s(t \cdot s')tt')\varphi \\ &= (s(ts')t')\varphi \\ &= ((st)(s't'))\varphi. \end{aligned}$$

Also $1\varphi = (1, 1)$. Hence $U \cong U'$.

Conversely, let $U = S \rtimes T$ be the external semidirect product of monoids S and T. As $t \cdot 1 = 1$ for all $t \in T$, it is easy to see that U contains submonoids

$$S' = \{(s, 1_T) : s \in S\} \text{ and } T' = \{(1_S, t) : t \in T\}.$$

We note that every element of U has a unique expression

$$(s, t) = (s, 1_T)(1_S, t).$$

Let $s' = (s, 1_T) \in S'$ and $t' = (1_S, t) \in T'$. Then

$$\begin{aligned} t's' &= (1_S, t)(s, 1_T) \\ &= (t \cdot s, t) \\ &= (t \cdot s, 1_T)(1_S, t) \\ &= (t' \cdot s')t'. \end{aligned}$$

Thus $t'S' \subseteq S't'$ and hence T' is locally left permutable with S'. Thus $S \rtimes T$ is the internal semidirect product of S' and T'. Now define maps $\alpha : S \to S'$ and $\beta : T \to T'$ by

$$s \mapsto (s, 1_T) \text{ and } t \mapsto (1_S, t),$$

respectively. Then it is easy to check that $S \cong S'$ and $T \cong T'$. □

3.4.2 Internal and external semidirect products of semigroups

There is not such a complete equivalence between internal and external semidirect products of semigroups. It is not difficult to get an external semidirect product of semigroups from an internal semidirect product of semigroups as we have seen in Lemma 3.4.8, but the converse is not obvious, because for semigroups there do not exist such mappings

$$s \mapsto (s, 1_T) \text{ and } t \mapsto (1_S, t)$$

as they exist for monoids. However we can characterise those external semidirect products of semigroups which are also internal semidirect products.

Theorem 3.4.11. *Let U be a semigroup and let $U = ST$ be the internal semidirect product of subsemigroups S and T. Then there exists an external semidirect product $U' = S \rtimes T$ such that $U \cong U'$. Moreover there exists an element $e \in S \cap T$ which is a right identity for S and an identity for T such that*

$$e \cdot s = es,\ e = t \cdot e \text{ for all } s \in S \text{ and } t \in T.$$

Conversely, suppose that $U = S \rtimes T$ is the external semidirect product of semigroups S and T such that S has a right identity e_S and T is a monoid with identity e_T and

$$e_T \cdot s = e_S s,\ e_S = t \cdot e_S \quad \text{for all } s \in S \text{ and } t \in T.$$

Let

$$S' = \{(s, e_T) : s \in S\} \text{ and } T' = \{(e_S, t) : t \in T\}.$$

Then S' and T' are subsemigroups of U, $S \cong S'$, $T \cong T'$ and $U = S'T'$ is the internal semidirect product of S' and T'.

Proof. Suppose $U = ST$ is the internal semidirect product of subsemigroups S and T. Then because T is locally left permutable with S and factorisations are unique, so associativity and uniqueness gives us (S1) and (S3). Therefore we can form an external semidirect product $U' = S \rtimes T$ as in Theorem 3.4.10, so that $U \cong U'$. Also from Lemma 3.4.9, there exits an element $e \in S \cap T$ which is a right identity for S and an identity for T such that

$$e \cdot s = es,\ e = t \cdot e \text{ for all } s \in S \text{ and } t \in T.$$

Conversely, let $U = S \rtimes T$ be the external semidirect product of semigroups S and T such that S has a right identity e_S and T has an identity e_T with

$$e_T \cdot s = e_S s,\ e_S = t \cdot e_S \quad \text{for all } s \in S \text{ and } t \in T.$$

Put

$$S' = \{(s, e_T) : s \in S\} \text{ and } T' = \{(e_S, t) : t \in T\}.$$

Define maps $\alpha : S \to S'$ and $\beta : T \to T'$ by

$$s\alpha = (s, e_T) \text{ and } t\beta = (e_S, t),$$

respectively. Then it is easy to see that $S \cong S'$ and $T \cong T'$. Also each element in $(s,t) \in U$ can be written as

$$(s,t) = (s, e_T)(e_S, t)$$

and this decomposition is unique. Next let $s' = (s, e_S) \in S'$ and $t' = (e_S, t) \in T'$. Then

$$\begin{aligned}
t's' &= (e_S, t)(s, e_T) \\
&= (e_S(t \cdot s), te_T) \\
&= ((t \cdot e_S)(t \cdot s), t) && \text{because } e_S = t \cdot e_S \text{ and } te_T = t \\
&= (t \cdot (e_S s), t) && \text{using (S3)} \\
&= (t \cdot (e_T \cdot s), t) && \text{because } e_T \cdot s = e_S s \\
&= (te_T \cdot s, t) && \text{using (S1)} \\
&= (t \cdot s, t) && \text{because } te_T = t \\
&= (t \cdot s, e_T)(e_S, t) \\
&\in S't'.
\end{aligned}$$

Thus $t'S' \subseteq S't'$ and so T' is locally left permutable with S'. Hence $U = S'T'$ is the internal semidirect product of S' and T'. $\square$

Preston provided a characterisation of external semidirect products of semigroups and proved that it is always possible to consider a semidirect product of semigroups as a subsemigroup of semidirect product of monoids [64]. His construction works as follows:

Let S and T be semigroups and $U = S \rtimes T$ be an external semidirect product with $\theta : T \to \operatorname{End} S$. Let $S^{(1)}$ and $T^{(1)}$ be semigroups obtained from S and T, respectively by adjoining an extra element 1, that acts as an identity element as explained in Section 1.1 of Chapter 1. Let $t \cdot 1 = 1$ and define $\theta^{(1)} : T^{(1)} \to \operatorname{End} S^{(1)}$ to be the extension of θ where $\theta(1) = 1_{S^1}$. Set $W = S^{(1)} \rtimes T^{(1)}$. Then $\alpha : s \mapsto (s, 1)$ and $\beta : t \mapsto (1, t)$ embed S and T, respectively in W. Observe that $S\alpha \cap T\beta = \emptyset$ because if $x = s\alpha = t\beta$, then $(s, 1) = (1, t)$, which is impossible. Therefore by identifying S with $S\alpha$ and T with $T\beta$, S and T become disjoint subsemigroups of W. Thus we have the following result:

Theorem 3.4.12. [64, Theorem 8] *Let S and T be semigroups and $U = S \rtimes T$ be an external semidirect product with $\theta : T \to \operatorname{End} S$. Then setting $W = S^{(1)} \rtimes T^{(1)}$ and θ^1 as above, S and T are subsemigroups of W and $U' = ST$ is a subsemigroup of W. Moreover, $U \cong U'$.*

3.5 Semidirect products of inverse and right (left) restriction semigroups

The semidirect product of two inverse semigroups is not inverse in general. Nico provided necessary and sufficient conditions for the semidirect product of two inverse monoids to be inverse [58]. Nico considered the reverse semidirect product, that is, when S acts on T on the right by endomorphisms, but we state the result for semidirect products.

Theorem 3.5.1. [58, Theorem 2.6] *Let S and T be inverse monoids and $U = S \rtimes T$ be a semidirect product of S and T. Then U is an inverse monoid if and only if*

(i) S and T are inverse monoids and

(ii) for every $e \in E(T)$, $e \cdot s = s$ for all $s \in S$.

Preston generalised Nico's result by replacing monoids with semigroups in [64, Theorem 6]. His theorem is stated in the same way as Nico's but with a different proof.

We now consider the semidirect product of a right restriction semigroup and a monoid and give necessary and sufficient conditions for this semidirect product to be right restriction.

Theorem 3.5.2. *Let S be a monoid and T be a right restriction semigroup. Suppose T acts on the left of S by endomorphisms such that $t \cdot 1 = 1$ for all $t \in T$. Then $A = S \rtimes T$ is right* *restriction with* $(s,t)^* = (1,t^*)$ *if and only if* $e \cdot s = s$ *for all* $s \in S$ *and all* $e \in E_T$*, where* E_T *is the semilattice of projections of* T*.*

Proof. Suppose that $e \cdot s = s$ for all $s \in S$ and all $e \in E_T$. We have to show that A is right restriction. We put $E_A = \{(1,e) : e \in E_T\}$. Since $e \cdot 1 = 1$ for all $e \in E_T$, clearly, E_A is a semilattice isomorphic to E_T.

For any $(s,t) \in A$, we define $(s,t)^* = (1,t^*)$. We now check that the four identities that define a right restriction semigroup hold. For this let $(s,t) \in A$. Then

$$(s,t)(s,t)^* = (s,t)(1,t^*) = (s(t \cdot 1), tt^*) = (s,t),$$

because $tt^* = t$ as T is right restriction. Next let $(s,t),(u,v) \in A$. Then

$$(s,t)^*(u,v)^* = (u,v)^*(s,t)^*$$

because E_A is a semilattice isomorphic to E_T. Also

$$\begin{aligned}
((s,t)(u,v)^*)^* &= ((s,t)(1,v^*))^* \\
&= (s(t \cdot 1), tv^*)^* \\
&= (s, tv^*)^* \\
&= (1,(tv^*)^*) \\
&= (1, t^*v^*) && \text{because } T \text{ is right restriction, so } (tv^*)^* = t^*v^* \\
&= (1,t^*)(1,v^*) && \text{because } E_A \text{ is isomorphic to } E_T \\
&= (s,t)^*(u,v)^*.
\end{aligned}$$

Finally, let $(s,t) \in A$ and $(1,e) \in E_A$. Then

$$\begin{aligned}
(s,t)((1,e)(s,t))^* &= (s,t)(1(e \cdot s), et)^* \\
&= (s,t)(1,(et)^*) \\
&= (s(t \cdot 1), t(et)^*) \\
&= (s, et) && \text{using } t \cdot 1 = 1 \text{ and the right ample condition} \\
&= (1,e)(s,t) && \text{because } e \cdot s = s.
\end{aligned}$$

Thus all four identities are satisfied for elements of A and hence $A = S \rtimes T$ is right restriction.

Conversely, suppose that $A = S \rtimes T$ is right restriction with $(s,t)^* = (1,t^*)$. Let $s \in S$, $e \in E_T$. We have to show that

$$e \cdot s = s.$$

Let $t \in T$. Then $(s,t) \in A$, $(1,e) \in E_A$ and as the right ample condition holds in A, we have

$$\begin{array}{rrcl} & (s,t)((1,e)(s,t))^* & = & (1,e)(s,t) \\ \Rightarrow & (s,t)(e \cdot s, et)^* & = & (1(e \cdot s), et) \\ \Rightarrow & (s,t)(1,(et)^*) & = & (e \cdot s, et) \\ \Rightarrow & (s(t \cdot 1), t(et)^*) & = & (e \cdot s, et) \\ \Rightarrow & (s, et) & = & (e \cdot s, et). \end{array}$$

Hence $e \cdot s = s$ as required. $\square$

The left-right dual of Theorem 3.5.2 also holds as we see in the following corollary.

Corollary 3.5.3. *Let S be a left restriction semigroup and T be a monoid. Suppose S acts on T by endomorphisms such that $1^s = 1$. Then $A = S \ltimes T$ is left restriction with $(s,t)^+ = (s^+, 1)$ if and only if $t^e = t$ for all $t \in T$ and $e \in E_S$ where E_S is the semilattice of projections of S.*

3.6 Zappa-Szép products

For the convenience of the reader we begin by recalling the basic definitions relating to Zappa-Szép products.

Definition 3.6.1. Let S and T be semigroups and suppose that we have maps

$$T \times S \to S,\ (t,s) \mapsto t \cdot s \text{ and } T \times S \to T,\ (t,s) \mapsto t^s$$

such that for all $s, s' \in S, t, t' \in T$:

$$\begin{array}{ll} \text{(ZS1) } tt' \cdot s = t \cdot (t' \cdot s); & \text{(ZS3) } (t^s)^{s'} = t^{ss'}; \\ \text{(ZS2) } t \cdot (ss') = (t \cdot s)(t^s \cdot s'); & \text{(ZS4) } (tt')^s = t^{t' \cdot s} t'^s. \end{array}$$

From (ZS1) and (ZS3), we see that T acts on S from the left and S acts on T from the right, respectively. Also note that (ZS2) and (ZS4) are replacing the conditions we saw in the semidirect product case that the actions are by homomorphisms. Define a binary operation on $S \times T$ by

$$(s,t)(s',t') = (s(t \cdot s'), t^{s'} t').$$

We check that this binary operation is associative. Let (s_1,t_1), (s_2,t_2), $(s_3,t_3) \in S\times T$. Then

$$\begin{aligned}
\big((s_1,t_1)(s_2,t_2)\big)(s_3,t_3) &= (s_1(t_1\cdot s_2), t_1^{s_2}t_2)(s_3,t_3) \\
&= \big(s_1(t_1\cdot s_2)(t_1^{s_2}t_2\cdot s_3), (t_1^{s_2}t_2)^{s_3}t_3\big) \\
&= \big(s_1(t_1\cdot s_2)(t_1^{s_2}\cdot(t_2\cdot s_3)), (t_1^{s_2})^{t_2\cdot s_3}t_2^{s_3}t_3\big) && \text{using (ZS1) and (ZS4)} \\
&= \big(s_1(t_1\cdot s_2(t_2\cdot s_3)), t_1^{s_2(t_2\cdot s_3)}t_2^{s_3}t_3\big) && \text{using (ZS2) and (ZS3)} \\
&= (s_1,t_1)(s_2(t_2\cdot s_3), t_2^{s_3}t_3) \\
&= (s_1,t_1)\big((s_2,t_2)(s_3,t_3)\big).
\end{aligned}$$

Thus associativity holds and hence $S\times T$ is a semigroup, referred to as the (external) *Zappa-Szép product of S and T* and denoted by $S\bowtie T$.

Note that if one of the above actions is trivial (that is, one semigroup acts by the identity map), then the second action is by morphisms, and we obtain the semidirect product $S\rtimes T$ (if S acts trivially) or $S\ltimes T$ (if T acts trivially).

If S and T are monoids, then we insist that the following four axioms also hold:

$$\begin{array}{ll}
\text{(ZS5) } t\cdot 1_S = 1_S; & \text{(ZS7) } 1_T\cdot s = s; \\
\text{(ZS6) } t^{1_S} = t; & \text{(ZS8) } 1_T^s = 1_T.
\end{array}$$

(ZS6) and (ZS7) are saying that the actions are monoid actions and (ZS5) and (ZS8) are telling us that the identities are fixed under the actions. Then $S\bowtie T$ becomes a monoid with identity $(1_S,1_T)$.

We now give definition of an *internal Zappa-Szép product* and then we explain the correspondence between external and internal Zappa-Szép products of semigroups and monoids in the following subsections.

Definition 3.6.2. Let U be a semigroup and S,T be subsemigroups of U. Then U is the *internal Zappa-Szép product* of S and T if U is uniquely factorisable with factors S and T, that is, $U=ST$ and every element $u\in U$ has a *unique* expression as

$$u = st \text{ where } s\in S \text{ and } t\in T.$$

If U is a monoid with submonoids S and T in the above definition, then we say that U is the *internal Zappa-Szép product* of submonoids S and T.

In the following lemma we see that we can get an external Zappa-Szép product from an internal Zappa-Szép product of semigroups.

Lemma 3.6.3. *Let U be a semigroup and suppose that $U = ST$ is the internal Zappa-Szép product of subsemigroups S and T. Then there is an action of T on the left of S and an action of S on the right of T such that (ZS1)-(ZS4) hold and $U \cong S \bowtie T$.*

Proof. As each element $u \in U$ has a unique expression of the form $u = st$ for $s \in S$ and $t \in T$, therefore for $s \in S$, $t \in T$, there exists unique $s' \in S$ and $t' \in T$ such that $ts = s't'$. Writing $s' = t \cdot s$ and $t' = t^s$, we have two functions

$$T \times S \to S,\ (t,s) \mapsto t \cdot s \text{ and } T \times S \to T,\ (t,s) \mapsto t^s.$$

Thus $ts = (t \cdot s)t^s$ and for all $s, s' \in S$, $t, t' \in T$ we have that

$$(st)(s't') = s(t \cdot s')t^{s'}t'.$$

The associativity of the semigroup U and the uniqueness property of the decomposition give the axioms (ZS1)-(ZS4) for these actions. By associativity we have that $t(ss') = (ts)s'$. Now

$$t(ss') = (t \cdot ss')t^{ss'}$$

and

$$(ts)s' = (t \cdot s)t^s s' = (t \cdot s)(t^s \cdot s')(t^s)^{s'}.$$

By uniqueness we have:

(ZS2) $t \cdot ss' = (t \cdot s)(t^s \cdot s')$;

(ZS3) $t^{ss'} = (t^s)^{s'}$.

Thus (ZS2) and (ZS3) hold. Dually (ZS1) and (ZS4) hold. Hence we can form an external Zappa-Szép product $S \bowtie T$ of S and T.

Now define a map $\alpha : U \to S \bowtie T$ by $(st)\alpha = (s,t)$. Clearly α is well defined, one one and onto. Also α is homomorphism as

$$\begin{aligned} ((st)(s't'))\alpha &= \left(s(t \cdot s')t^{s'}t'\right)\alpha \\ &= \left(s(t \cdot s'), t^{s'}t'\right) \\ &= (s,t)(s',t') \\ &= (st)\alpha(s't')\alpha. \end{aligned}$$

Hence $U \cong S \bowtie T$. $\square$

3.6.1 Correspondence between internal and external Zappa-Szép products of monoids

There is an equivalence between internal and external Zappa-Szép products of monoids due to Kunze [37] as we see in the following theorem.

Theorem 3.6.4. [37] *Let U be a monoid and S, T be submonoids of U. Suppose that $U = ST$ is the internal Zappa-Szép product of S and T. Then there is an action of T on the left of S and an action of S on the right of T such that (ZS1)-(ZS8) hold and $U \cong S \bowtie T$.*

Conversely, suppose that $U = S \bowtie T$ is the external Zappa-Szep product of S and T. Let

$$S' = \{(s, 1_T) : s \in S\} \text{ and } T' = \{(1_S, t) : t \in T\}.$$

Then S' and T' are submonoids of U isomorphic to S and T respectively and $U = S'T'$ is the internal Zappa-Szép product of S' and T'.

Proof. Suppose that $U = ST$ is the internal Zappa-Szép product of submonoids S and T. Then from Lemma 3.6.3 there is an action of T on the left of S and an action of S on the right of T such that (ZS1)-(ZS4) hold. Also as $U = ST$ is a monoid and $1_U \in S \cap T$, thus

$$1_U t = t = t1_U = (t \cdot 1_U)t^{1_U}$$

and

$$s1_U = s = 1_U s = (1_U \cdot s)1_U^s.$$

Therefore by uniqueness:

(ZS5) $t \cdot 1_U = 1_U$;

(ZS6) $t^{1_U} = t$;

(ZS7) $1_U \cdot s = s$;

(ZS8) $1_U^s = 1_U$.

Thus the external Zappa-Szép product $S \bowtie T$ is a monoid Zappa-Szép product. Now defining a map $\varphi : U \to S \bowtie T$ by $(st)\varphi = (s, t)$, it is easy to see that $U \cong S \bowtie T$.

Conversely, suppose $U = S \bowtie T$ is the external Zappa-Szep product of monoids S and T. Let

$$S' = \{(s, 1_T) : s \in S\} \text{ and } T' = \{(1_S, t) : t \in T\}.$$

It is then immediate that S' and T' are submonoids and $S \cong S'$ and $T \cong T'$ under the maps

$$s \mapsto (s, 1_T) \text{ and } t \mapsto (1_S, t).$$

Also each element $(s,t) \in U$ can be written as

$$(s,t) = (s, 1_T)(1_S, t)$$

and this decomposition is evidently unique. Hence $U = S'T'$ is the internal Zappa-Szép product of S' and T'. □

3.6.2 Correspondence between internal and external Zappa-Szép products of semigroups

We have seen an equivalence between internal and external Zappa-Szép product of monoids in Theorem 3.6.4. In general, there is no such correspondence between internal and external Zappa-Szép product of semigroups. It is not difficult to get an external Zappa-Szép product from an internal Zappa-Szép product of semigroups as we have seen in Lemma 3.6.3 but the converse is not obvious. However we can characterise those external Zappa-Szép products of semigroups which are also internal ones.

Lemma 3.6.5. [77, Lemma 2.1.3] *Let U be a semigroup and suppose U is uniquely factorisable as $U = ST$, where S and T are subsemigroups of U. Then there exists an idempotent $e \in S \cap T$ which is a right identity for S and a left identity for T such that for $s \in S$, $t \in T$*

$$es = e \cdot s,\ e = e^s \text{ and } t \cdot e = e,\ t^e = te.$$

Proof. As U is uniquely factorisable with factors S and T, so from Lemma 3.2.1, there is an idempotent $e \in S \cap T$ which is a right identity for S and a left identity for T. Next for $s \in S$ we have

$$es = (e \cdot s)e^s \text{ because of unique factorisations.}$$

But $s = se$, because e is right identity for S, so that

$$es = e(se) = (es)e$$

and as factorisations are unique, hence

$$es = e \cdot s,\ e^s = e.$$

Dually for $t \in T$

$$te = (t \cdot e)t^e.$$

But $t = et$, because e is a left identity for T. Thus

$$te = (et)e = e(te)$$

and as factorisations are unique, therefore

$$t \cdot e = e,\ t^e = te.$$

□

Theorem 3.6.6. *Let U be a semigroup and suppose that $U = ST$ is the internal Zappa-Szép product of subsemigroups S and T. Then there is an action of T on the left of S and an action of S on the right of T such that (ZS1)-(ZS4) hold and $U \cong S \bowtie T$. Further, there exists an idempotent $e \in S \cap T$ such that e is right identity for S and left identity for T with*

$$es = e \cdot s,\ e = e^s \text{ and } t \cdot e = e,\ t^e = te$$

for all $s \in S$ and $t \in T$.

Conversely suppose that $U = S \bowtie T$ is an external Zappa-Szép product of semigroups S and T such that S has a right identity e_S and T has a left identity e_T with

$$e_S s = e_T \cdot s,\ e_T = e_T^s \text{ and } t \cdot e_S = e_S,\ t^{e_S} = te_T$$

for all $s \in S$ and $t \in T$. Let

$$S' = \{(s, e_T) : s \in S\} \text{ and } T' = \{(e_S, t) : t \in T\}.$$

Then $S \cong S'$, $T \cong T'$ and $U = S'T'$ is the internal Zappa-Szép product of S' and T'.

Proof. If $U = ST$ is the internal Zappa-Szép product of subsemigroups S and T, then from Lemma 3.6.3 there is an action of T on the left of S and an action of S on the right of T such that (ZS1)-(ZS4) hold and $U \cong S \bowtie T$. Also as factorisation is unique, therefore from

Lemma 3.6.5 there exists an idempotent e which is a right identity for S and a left identity for T such that

$$es = e \cdot s,\ e = e^s \text{ and } t \cdot e = e,\ t^e = te$$

for all $s \in S$ and $t \in T$.

Conversely, let $U = S \bowtie T$ be an external Zappa-Szép product of S and T. Suppose there is a right identity e_S of S and a left identity e_T of T satisfying

$$e_S s = e_T \cdot s,\ e_T = e_T^s \text{ and } t \cdot e_S = e_S,\ t^{e_S} = te_T$$

for all $s \in S$ and $t \in T$. Put

$$S' = \{(s, e_T) : s \in S\} \text{ and } T' = \{(e_S, t) : t \in T\}.$$

Define a map $\alpha : S \to S'$ by

$$s\alpha = (s, e_T).$$

Clearly α is well defined, one-one and onto. To check that α is a homomorphism, let $s, s' \in S$. Then

$$\begin{aligned}
(ss')\alpha &= (ss', e_T) \\
&= (se_S s', e_T) && \text{because } e_S \text{ is right identity for } S \\
&= (s(e_T \cdot s'), e_T^{s'} e_T) && \text{because } e_T \cdot s' = e_S s' \text{ and } e_T^{s'} = e_T \text{ by our supposition} \\
&= (s, e_T)(s', e_T) \\
&= (s\alpha)(s'\alpha).
\end{aligned}$$

Hence $S \cong S'$. Similarly defining a map $\beta : T \to T'$ by $t\beta = (e_S, t)$ it is easy to check that $T \cong T'$. Now for $(s, e_T) \in S'$ and $(e_S, t) \in T'$, we see that

$$\begin{aligned}
(s, e_T)(e_S, t) &= (s(e_T \cdot e_S), e_T^{e_S} t) \\
&= (se_S, e_T t) && \text{because } e_T \cdot e_S = e_S \text{ and } e_T^{e_S} = e_T \\
&= (s, t) && \text{because } e_S \text{ is right identity of } S \text{ and} \\
& && e_T \text{ is left identity of } T.
\end{aligned}$$

Thus each element $(s, t) \in U$ can be written as

$$(s, t) = (s, e_T)(e_S, t)$$

and this composition is evidently unique. Hence $U = S'T'$ is the internal Zappa-Szép product of S' and T'. $\square$

We can also use Preston's technique to get an internal Zappa-Szép product from an external Zappa-Szép product $S \bowtie T$ of semigroups S and T by adjoining identities to S and T and extending the actions to $S^{(1)}$ and $T^{(1)}$.

Theorem 3.6.7. *Let S and T be semigroups and let $U = S \bowtie T$ be the external Zappa-Szép product of S and T. Let $S^{(1)}$ and $T^{(1)}$ be the semigroups obtained from S and T, respectively by adjoining an identity. Then, setting $t \cdot 1 = 1$ and $1^s = 1$ and defining*

$$(1, s) \mapsto 1 \cdot s = s, \ (t, 1) \mapsto t^1 = t$$

to be the identity maps, the extended actions satisfy (ZS1)-(ZS8). Thus we can form the Zappa-Szép product $S^{(1)} \bowtie T^{(1)}$ of $S^{(1)}$ and $T^{(1)}$. Moreover
$S \cong S' = \{(s, 1) : s \in S\}$ under the map $s \mapsto (s, 1)$ and $T \cong T' = \{(1, t) : t \in T\}$ under the map $t \mapsto (1, t)$ and $U = S'T'$ is a subsemigroup of $S^{(1)} \bowtie T^{(1)}$, and is uniquely factorisable with factors S' and T'.

Proof. We first check that $S^{(1)} \bowtie T^{(1)}$ satisfies all axioms of a Zappa-Szép product.

(ZS1) Let $t, t' \in T^{(1)}$ and $s \in S^{(1)}$. Then to check that $tt' \cdot s = t \cdot (t' \cdot s)$, we have eight cases as follows:

(i) $t = t' = s = 1$;
(ii) $t = 1$, $t' \neq 1$ and $s = 1$;
(iii) $t \neq 1$, $t' \neq 1$ and $s = 1$;
(iv) $t \neq 1$, $t' = 1$ and $s = 1$;
(v) $t = 1$, $t' = 1$ and $s \neq 1$;
(vi) $t = 1$, $t' \neq 1$ and $s \neq 1$;
(vii) $t \neq 1$, $t' = 1$ and $s \neq 1$;
(viii) $t \neq 1$, $t' \neq 1$ and $s \neq 1$;

(i)-(iv) are obvious because $s = 1$ in all four cases and $t \cdot 1 = 1$ for all $t \in T^{(1)}$. For (v) we see that

$$tt' \cdot s = 11 \cdot s = 1 \cdot s = s = 1 \cdot (1 \cdot s) = t \cdot (t' \cdot s).$$

For (vi),

$$tt' \cdot s = 1t' \cdot s = t' \cdot s = 1 \cdot (t' \cdot s) = t \cdot (t' \cdot s).$$

For (vii)

$$tt' \cdot s = t \cdot s = t \cdot (1 \cdot s) = t \cdot (t' \cdot s).$$

For (viii) because $t \neq 1$, $t' \neq 1$ and $s \neq 1$ and $S^{(1)}$ and $T^{(1)}$ are semigroups obtained from S and T, respectively by adjoining an identity, so $t, t' \in T$ and $s \in S$. Also as $S \bowtie T$ is the external Zappa-Szép product of S and T, we have

$$tt' \cdot s = t \cdot (t' \cdot s).$$

Hence $T^{(1)}$ acts on $S^{(1)}$ from the left and thus (ZS1) holds.

(ZS2) Let $t \in T^{(1)}$ and $s, s' \in S^{(1)}$. We need to check that $t \cdot ss' = (t \cdot s)(t^s \cdot s')$. Now again we have eight cases as follows:

(i) $t = s = s' = 1$;

(ii) $t = 1$, $s = 1$ and $s' \neq 1$;

(iii) $t = 1$, $s \neq 1$ and $s' = 1$;

(iv) $t = 1$, $s \neq 1$ and $s' \neq 1$;

(v) $t \neq 1$, $s = 1$ and $s' = 1$;

(vi) $t \neq 1$, $s = 1$ and $s' \neq 1$;

(vii) $t \neq 1$, $s \neq 1$ and $s' = 1$;

(viii) $t \neq 1$, $s \neq 1$ and $s' \neq 1$;

(i)-(iv) are clear because $t = 1$ in all four cases and $1 \cdot s = s$, $1^s = 1$ for all $s \in S$. For (v), we see that $t \cdot (ss') = t \cdot 11 = t \cdot 1 = 1 = (t \cdot 1)(t^1 \cdot 1) = (t \cdot s)(t^s \cdot s')$. For (vi)

$$t \cdot (ss') = t \cdot s' = (t \cdot 1)(t^1 \cdot s') = (t \cdot s)(t^s \cdot s').$$

For (vii)

$$t \cdot (ss') = t \cdot s = (t \cdot s)(t^s \cdot 1) = (t \cdot s)(t^s \cdot s').$$

For (viii) because $t \neq 1$, $s \neq 1$ and $s' \neq 1$ and $S^{(1)}$ and $T^{(1)}$ are semigroups obtained from S and T, respectively by adjoining an identity, so $t \in T$ and $s, s' \in S$. Also as $S \bowtie T$ is the external Zappa-Szép product of S and T, we have

$$t \cdot (ss') = (t \cdot s)(t^s \cdot s').$$

Hence (ZS2) holds.

(ZS3) is dual of (ZS1)and (ZS4) is dual of (ZS2).

Also (ZS5)-(ZS8) hold because of the way in which we have extended our actions of S on

T and T on S to actions of $S^{(1)}$ on $T^{(1)}$ and $T^{(1)}$ on $S^{(1)}$ respectively. Hence we may form the Zappa-Szép product $V = S^{(1)} \bowtie T^{(1)}$ of the monoids $S^{(1)}$ and $T^{(1)}$.

Next let

$$S'' = \{(s, 1) : s \in S^{(1)}\} \text{ and } T'' = \{(1, t) : t \in T^{(1)}\}.$$

From Theorem 3.6.4, S'', T'' are submonoids of V and $V = S''T''$ is the internal Zappa-Szép product of S'' and T''. Moreover

$$\alpha : S^{(1)} \to S'' \text{ and } \beta : T^{(1)} \to T''$$

are isomorphisms. Now let

$$S' = \{(s, 1) : s \in S\} \text{ and } T' = \{(1, t) : t \in T\}.$$

Then S' and T' are subsemigroups of S'' and T'', respectively and hence of V. Clearly

$$\alpha|_S : S \to S' \text{ and } \beta|_T : T \to T'$$

are isomorphisms.

Note that any element $(s, t) \in U$ can be written as

$$(s, t) = (s, 1)(1, t) \text{ for } (s, 1) \in S' \text{ and } (1, t) \in T'$$

and this decomposition is unique.

Thus $U = S'T'$ is uniquely factorisable with factors S' and T'. $\square$

Chapter 4

Some algebraic properties of Zappa-Szép products

In this chapter, we determine some algebraic properties of Zappa-Szép products. We consider a left restriction semigroup S with semilattice of projections E and define left and right actions of S on E and E on S, respectively, to form the Zappa-Szép product $E \bowtie S$. We further investigate the properties of $E \bowtie S$. We find a subset of $E \bowtie S$ which is left restriction. We consider Zappa-Szép products of monoids and semigroups and characterise generalised Green's relations $\mathcal{R}^*$, $\mathcal{L}^*$, $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ for these Zappa-Szép products.

4.1 Zappa-Szép product of a left restriction semigroup and a semilattice

In this section we consider a left restriction semigroup with semilattice of projections E. By defining a left action of S on E and a right action of E on S, we see that $E \bowtie S$ becomes a Zappa-Szép product. We determine the set of idempotents of $E \bowtie S$. We see that $E \bowtie S$ is not itself left restriction but it contains a subsemigroup which is left restriction.

Lemma 4.1.1. *Let S be a left restriction semigroup with semilattice of projections E. Define an action of S on E by $s \cdot e = (se)^+$ and an action of E on S by $s^e = se$. Then $Z = E \bowtie S$ is a Zappa-Szép product of S and E.*

Proof. We check that these two actions satisfy the axioms of a Zappa-Szép product.

(ZS1) For $s_1, s_2 \in S$ and $e \in E$

$$\begin{aligned} s_1 \cdot (s_2 \cdot e) &= s_1 \cdot (s_2 e)^+ \\ &= (s_1 (s_2 e)^+)^+ \\ &= (s_1 (s_2 e))^+ \\ &= ((s_1 s_2) e)^+ \\ &= s_1 s_2 \cdot e. \end{aligned}$$

Hence (ZS1) holds.

(ZS2) Let $s \in S$ and $e, f \in E$. Then

$$\begin{aligned} (s \cdot e)(s^e \cdot f) &= (se)^+ (se \cdot f) \\ &= (se)^+ ((se)f)^+ \\ &= ((se)f)^+ \qquad \text{using Lemma 2.1.8 that } (ab)^+ \leq a^+ \text{ for any } a, b \in S \\ &= (s(ef))^+ \\ &= s \cdot ef. \end{aligned}$$

Thus (ZS2) holds.

(ZS3) For $s \in S$ and $e, f \in E$, we have

$$(s^e)^f = (se)^f = (se)f = s(ef) = s^{ef}.$$

Hence (ZS3) holds.

(ZS4) For $s_1, s_2 \in S, e \in E$

$$\begin{aligned} s_1^{s_2 \cdot e} {s_2}^e &= s_1^{(s_2 e)^+} (s_2 e) \\ &= s_1 (s_2 e)^+ s_2 e \\ &= s_1 (s_2 e) \\ &= (s_1 s_2) e \\ &= (s_1 s_2)^e. \end{aligned}$$

Hence (ZS4) holds. Thus $Z = E \bowtie S$ is a Zappa-Szép product under the binary operation

$$(e, s)(f, t) = (e(sf)^+, sft).$$

□

If S is left restriction, then the Zappa-Szép product $E \bowtie S$ obtained as above is the *standard Zappa-Szép product* of E and S.

We now compute the set of idempotents of $Z = E \bowtie S$.

Lemma 4.1.2. *Let S be a left restriction semigroup with semilattice of projections E. Suppose that $Z = E \bowtie S$ is the standard Zappa-Szép product of S and E. Then*

$$E(Z) = \{(e, s) : e \leq s^+, s = ses\}.$$

Also $\overline{E} = \{(e, e) : e \in E\}$ is a semilattice isomorphic to E and if $E(S) = E$, then $\overline{E} = E(Z)$.

Proof. Let $(e, s) \in Z$. Then

$$\begin{array}{rcccc} (e,s) \in E(Z) & \Leftrightarrow & (e,s)^2 & = & (e,s) \\ & \Leftrightarrow & (e,s)(e,s) & = & (e,s) \\ & \Leftrightarrow & (e(se)^+, ses) & = & (e,s) \\ & \Leftrightarrow & e = e(se)^+ & \text{and} & s = ses. \end{array}$$

Now $s = ses \Rightarrow s \,\mathcal{R}\, se \,\widetilde{\mathcal{R}}_E\, (se)^+$, so that $s^+ = (se)^+$. Hence

$$\begin{array}{rcccc} (e,s) \in E(Z) & \Leftrightarrow & (es^+, ses) & = & (e,s) \\ & \Leftrightarrow & e \leq s^+ & \text{and} & ses = s. \end{array}$$

Also $\overline{E} \subseteq E(Z)$. It is easy to check that $\overline{E}$ is a semilattice isomorphic to E.

Now if $E(S) = E$, then $\overline{E} = E(Z)$, for if $(e, s) \in E(Z)$, then from $s = ses$ we obtain $se = sese = (se)^+ = s^+$. Also we have

$$s = ses = (se)es = s^+es = es$$

giving $s^+ \leq e$, so that as $e \leq s^+$ it follows that $s^+ = e$. Hence $s = ses = s^2 = s^+ = e$. □

We now record some of the properties of $Z = E \bowtie S$.

Definition 4.1.3. Let S and T be semigroups and let $A \subseteq S$. Let $\varphi : S \to T$ be a morphism. We say that φ is *A-separating* if for $a, b \in A$

$$a\varphi = b\varphi \Rightarrow a = b.$$

If $A = E(S)$, we say that φ is *idempotent separating*.

Proposition 4.1.4. *Let S be a left restriction semigroup with semilattice of projections E. Suppose that $Z = E \bowtie S$ is the standard Zappa-Szép product of S and E. Let $(e,s) \in Z$, then:*

(a) $\overline{E} = \{(e,e) : e \in E\}$ is a semilattice isomorphic to $E(S)$;

(b) there is a morphism $\alpha : Z \to S$ separating the idempotents of $\overline{E}$;

(c) $(g,g)(e,s) = (e,s)$ if and only if $ge = e$ and $es = s$;

(d) (e,s) has a left identity in $\overline{E}$ if and only if $es = s$;

in this case $(e,s)\,\widetilde{\mathcal{R}}_{\overline{E}}\,(e,e)$ if and only if $es = s$;

(e) $(e,s)(f,f) = (e,s)$ if and only if $e \leq s^+, s = sf$;

(f) for $(e,s) \in Z$, $(e,s)\,\widetilde{\mathcal{L}}_{\overline{E}}\,(f,f)$ where $(f,f) \in \overline{E}$ if and only if $e \leq s^+$ and $s\,\widetilde{\mathcal{L}}_E\,f$;

(g) $(g,g)\,\widetilde{\mathcal{R}}_{\overline{E}}\,(e,s)\,\widetilde{\mathcal{L}}_{\overline{E}}\,(f,f)$ for some $g,\ f \in E$ implies $(e,s) = (s^+,s)$.

Further, there is a canonical embedding of S into $Z = E \bowtie S$ under $s \mapsto (s^+, s)$.

Proof. (a) From Lemma 4.1.2, we know that $\overline{E}$ is a semilattice isomorphic to $E(S)$.

(b) Define $\alpha : Z \to S$ by

$$(e,s)\alpha = es.$$

Clearly α is a surjection. To check that α is a homomorphism, let $(e,s),(f,t) \in Z$. Then

$$\begin{aligned} ((e,s)(f,t))\alpha &= (e(sf)^+, sft)\alpha \\ &= e(sf)^+sft \\ &= e(sf)t \\ &= esft \\ &= (e,s)\alpha(f,t)\alpha. \end{aligned}$$

Thus α is a homomorphism. Also for any (e,e), $(f,f) \in \overline{E}$, we have

$$(e,e)\alpha = (f,f)\alpha \Leftrightarrow e = f,$$

and so α separates idempotents of $\overline{E}$.

(c) Let $(e,s) \in Z$ and $(g,g) \in \overline{E}$. Then

$$\begin{aligned} & (g,g)(e,s) &=& \quad (e,s) \\ \Leftrightarrow\ & (g(ge), ges) &=& \quad (e,s) \\ \Leftrightarrow\ & ge = e &\text{and}& \quad ges = s \\ \Leftrightarrow\ & ge = e &\text{and}& \quad es = s. \end{aligned}$$

(d) Suppose now $(e,s)\,\widetilde{\mathcal{R}}_{\overline{E}}\,(h,h)$. By (c) we have $es=s$.

Conversely, if $es=s$, we note that (e,e) is a left identity of (e,s) as

$$(e,e)(e,s)=(e,es)=(e,s).$$

Next suppose that $(g,g)\in\overline{E}$ exists with $(g,g)(e,s)=(e,s)$. Then $ge=e$ from (c), so that, as $E\cong\overline{E}$, we have $(g,g)(e,e)=(e,e)$. Hence $(e,e)\,\widetilde{\mathcal{R}}_{\overline{E}}\,(e,s)$.

(e) For $(e,s)\in Z$ and $(f,f)\in\overline{E}$,

$$\begin{array}{rcll} (e,s)(f,f)=(e,s) & \Leftrightarrow & (e(sf)^+,sff) \quad = & (e,s) \\ & \Leftrightarrow & e(sf)^+=e \quad \text{and} & sf=s \\ & \Leftrightarrow & e\leq s^+ \quad \text{and} & sf=s. \end{array}$$

(f) Let $(e,s)\,\widetilde{\mathcal{L}}_{\overline{E}}\,(f,f)$, then $(e,s)(f,f)=(e,s)$ gives $e\leq s^+$ and $sf=s$.

Now suppose that $sg=s$ for some $g\in E$, then

$$(e,s)(g,g)=(e(sg)^+,sg)=(e,s),$$

and thus $(f,f)(g,g)=(f,f)$ as $(e,s)\,\widetilde{\mathcal{L}}_{\overline{E}}\,(f,f)$. Since $\overline{E}\cong E$, we have $fg=f$, so that $s\,\widetilde{\mathcal{L}}_E\,f$.

Conversely if $e\leq s^+$ and $s\,\widetilde{\mathcal{L}}_E\,f$, then

$$sf=s\Rightarrow(e,s)(f,f)=(e,s)$$

and if $(e,s)(g,g)=(e,s)$, then $sg=s$ and so $fg=f$, giving

$$(f,f)(g,g)=(f,f).$$

Hence $(e,s)\,\widetilde{\mathcal{L}}_{\overline{E}}\,(f,f)$.

(g) It is straight forward to prove that $(g,g)\,\widetilde{\mathcal{R}}_{\overline{E}}\,(e,s)\,\widetilde{\mathcal{L}}_{\overline{E}}\,(f,f)$ for some $g,f\in E$ implies that $(e,s)=(s^+,s)$ using (c) and (e).

Now let

$$U=\{(s^+,s):s\in S\}.$$

To prove that U is a subsemigroup of Z, let $(s^+, s), (t^+, t) \in U$, then

$$\begin{aligned}(s^+, s)(t^+, t) &= (s^+(st^+)^+, st^+t)\\ &= ((s^+st)^+, st)\\ &= ((st)^+, st) \in U.\end{aligned}$$

Clearly $S \cong U$ under $s \mapsto (s^+, s)$, so that U is a left restriction subsemigroup of Z where

$$(s^+, s)^+ = ((s^+)^+, s^+) = (s^+, s^+) \in U.$$

□

The Zappa-Szép product $Z = E \bowtie S$ of a left restriction semigroup S and E is not left restriction in general. However we have shown in Proposition 4.1.4 that Z contains the left restriction subsemigroup U, which is isomorphic to S. We now find a left restriction subsemigroup T of Z containing U, which, in view of Proposition 4.1.4 (d), is clearly maximum.

Theorem 4.1.5. *Let S be a left restriction semigroup with semilattice of projections E. Suppose that $Z = E \bowtie S$ is the standard Zappa-Szép product of S and E. Let*

$$T = \{(e, s) : s^+ \leq e\} = \{(e, s) : es = s\}.$$

Then T is a left restriction subsemigroup of Z with $(e, s)^+ = (e, e)$.

Proof. We first check T is a subsemigroup of $Z = E \bowtie S$. For this let $(e, s), (f, t) \in T$. Then $es = s, ft = t$, so that

$$\begin{aligned}(e, s)(f, t) &= (e(sf)^+, sft)\\ &= ((esf)^+, st)\\ &= ((sf)^+, st).\end{aligned}$$

Now $(sf)^+st = sft = st$ gives $(e, s)(f, t) \in T$. Hence T is subsemigroup of Z. Clearly $\overline{E} \subseteq T$.

To show that T is left restriction, let $(e, s) \in T$. Then $(e, s)^+ \widetilde{\mathcal{R}}_{\overline{E}} (e, e)$ follows from Theorem 4.1.4 (c) and the fact that $\overline{E} \subseteq T$.

Note that for any (e, s), $(f, t) \in T$,

$$(e, s) \widetilde{\mathcal{R}}_{\overline{E}} (f, t) \Leftrightarrow e = f.$$

To prove that $\widetilde{\mathcal{R}}_{\overline{E}}$ is a left congruence, let (e, s), $(f, t) \in T$ be such that $(e, s) \widetilde{\mathcal{R}}_{\overline{E}} (f, t)$, that is $e = f$. Then for any $(g, u) \in T$,

$$((g, u)(e, s)) = (g(ue)^+, ues) \, \widetilde{\mathcal{R}}_{\overline{E}} \, (g(uf)^+, uft) = (g, u)(f, t).$$

Therefore $(g, u)(e, s) \, \widetilde{\mathcal{R}}_{\overline{E}} \, (g, u)(f, t)$ and so $\widetilde{\mathcal{R}}_{\overline{E}}$ is a left congruence.

Finally to show that the left ample condition holds, let $(e, s) \in T$, $(f, f) \in \overline{E}$. Then

$$\begin{aligned}
((e, s)(f, f))^+(e, s) &= (e(sf)^+, sff)^+(e, s) \\
&= ((sf)^+, sf)^+(e, s) \\
&= ((sf)^+, (sf)^+)(e, s) \\
&= ((sf)^+((sf)^+e)^+, (sf)^+es) \\
&= ((sf)^+e, (sf)^+s) \qquad \text{because } es = s \\
&= (e(sf)^+, sf) \\
&= (e, s)(f, f).
\end{aligned}$$

Hence T is left restriction. □

It is important to note that, if S is also right restriction, then

$$T = \{(e, s) : s^+ \leq e\} = \{(e, s) : es = s\}$$

is not always right restriction, because we cannot always find a right identity of $(e, s) \in T$.

Also in the inverse case, if $Z = E \bowtie S$ is the Zappa-Szép product of an inverse semigroup S and its semilattice E, then $T = \{(e, s) : es = s\}$ is not inverse. However, from Theorem 4.1.4 there is a subsemigroup of T which is inverse.

Corollary 4.1.6. *Let $Z = E(S) \bowtie S$ be the standard Zappa-Szép product of an inverse semigroup S and semilattice of idempotents $E(S)$ and let $T = \{(e, s) : es = s\}$ be a subsemigroup of Z. Then there is a canonical embedding of S into $Z = E \bowtie S$ under $s \mapsto (ss^{-1}, s)$.*

4.2 Generalised Green's relations on Zappa-Szép products

In this section we consider Zappa-Szép products of semigroups and monoids and record the behaviour of the relations $\mathcal{R}^*$, $\mathcal{L}^*$, $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$.

Lemma 4.2.1. *If $Z = S \bowtie T$ is a Zappa-Szép product of monoids S and T, then $(a, b)\,\mathcal{R}^*\,(c, d)$ if and only if for all $x,\ u \in S$ and for all $y,\ v \in T$*

$$\begin{aligned} & x(y \cdot a) = u(v \cdot a) \quad \textit{and} \quad y^a b = v^a b \\ \Leftrightarrow\ & x(y \cdot c) = u(v \cdot c) \quad \textit{and} \quad y^c d = v^c d. \end{aligned}$$

Proof. Let (a, b), $(c, d) \in Z$. Then $(a, b)\,\mathcal{R}^*\,(c, d)$ if and only if for all (x, y), $(u, v) \in Z$,

$$\begin{aligned} & (x, y)(a, b) = (u, v)(a, b) \\ \Leftrightarrow\ & (x, y)(c, d) = (u, v)(c, d) \end{aligned}$$

if and only if for all $x, u \in S$ and for all $y, v \in T$,

$$\begin{aligned} & x(y \cdot a) = u(v \cdot a) \quad \text{and} \quad y^a b = v^a b \\ \Leftrightarrow\ & x(y \cdot c) = u(v \cdot c) \quad \text{and} \quad y^c d = v^c d. \end{aligned}$$

□

Proposition 4.2.2. *Suppose $Z = S \bowtie T$ is a Zappa-Szép product of monoids S and T. Then the following hold:*

(1) if $b,\ d \in T$ and $b\,\mathcal{R}^*\,d$, then $(a, b)\,\mathcal{R}^*\,(a, d)$ in Z;

(2) if $a,\ c \in S$ and $a\,\mathcal{L}^*\,c$, then $(a, b)\,\mathcal{L}^*\,(c, b)$ in Z.

Proof. (1) Suppose that $b\,\mathcal{R}^*\,d$ for some $b,\ d \in T$. Let $x,\ u \in S$ and $y,\ v \in T$, then

$$\begin{aligned} & x(y \cdot a) = u(v \cdot a) \quad \text{and} \quad y^a b = v^a b \\ \Leftrightarrow\ & x(y \cdot a) = u(v \cdot a) \quad \text{and} \quad y^a d = v^a d \quad \text{because } b\,\mathcal{R}^*\,d. \end{aligned}$$

By Lemma 4.2.1 $(a, b)\,\mathcal{R}^*\,(a, d)$ in Z.

The proof of (2) is dual. □

Proposition 4.2.3. *Let S, T be monoids and $Z = S \bowtie T$ be the Zappa-Szép product of S and T. Then*

(1) $(a, b)\,\mathcal{R}^*\,(c, d)$ in Z implies $a\,\mathcal{R}^*\,c$ in S;

(2) $(a, b)\,\mathcal{L}^*\,(c, d)$ in Z implies $b\,\mathcal{L}^*\,d$ in T.

Proof. (1) Suppose $(a, b)\,\mathcal{R}^*\,(c, d)$ in Z. To show that $a\,\mathcal{R}^*\,c$ in S, let $x, u \in S$ be such that

$$xa = ua.$$

Then

$$x(1 \cdot a) = u(1 \cdot a) \text{ and } 1^a b = 1^a b,$$

so by Lemma 4.2.1, $x(1 \cdot c) = u(1 \cdot c)$, that is, $xc = uc$. Thus together with the opposite direction, we obtain $a \mathcal{R}^* c$ in S.

The proof for $\mathcal{L}^*$ is dual. □

Definition 4.2.4. Let S and T be semigroups and $\alpha : S \to T$ be a map from S to T. We define $\operatorname{Ker} \alpha$ to be the relation

$$\operatorname{Ker} \alpha = \{(s, t) \in S \times S : s\alpha = t\alpha\}.$$

For more complete results, we now consider the reverse semidirect product $S \ltimes T$ of S and T and use the Lemma 4.2.1 in the context when one of the actions is trivial, because in the Zappa-Szép product $S \bowtie T$ of monoids (semigroups) S and T if one of the actions is trivial (that is, one semigroup acts by the identity map), then the second action is by morphisms, and we obtain the semidirect product $S \rtimes T$ (if S acts trivially) or $S \ltimes T$ (if T acts trivially).

Lemma 4.2.5. *Suppose $Z = S \ltimes T$ is a semidirect product of semigroups S and T, where T is right cancellative. Also suppose that for any a, $c \in S$, $a \mathcal{R}^* c$ implies* $\operatorname{Ker} a = \operatorname{Ker} c$ *(where* $\operatorname{Ker} a$ *is the kernel of the map induced by the right action of a). Then $a \mathcal{R}^* c$ in S implies that $(a, b) \mathcal{R}^* (c, d)$ in Z.*

Proof. Suppose $a \mathcal{R}^* c$ in S and let $x, u \in S$ and $y, v \in T$. Then

$$\begin{aligned} & xa = ua && \text{and} && y^a b = v^a b \\ \Leftrightarrow\ & xa = ua && \text{and} && y^a = v^a && \text{because T is cancellative} \\ \Leftrightarrow\ & xc = uc && \text{and} && y^c = v^c && \text{because } a \mathcal{R}^* c \text{ and } \operatorname{Ker} a = \operatorname{Ker} c \\ \Leftrightarrow\ & xc = uc && \text{and} && y^c d = v^c d && \text{because } T \text{ is cancellative.} \end{aligned}$$

Thus Lemma 4.2.1 gives $(a, b) \mathcal{R}^* (c, d)$ in Z. □

Corollary 4.2.6. *Suppose $Z = S \ltimes T$ is a semidirect product of monoids S and T, where T is right cancellative. Also suppose that S acts on T injectively. Then $a \mathcal{R}^* c$ in S implies that $(a, b) \mathcal{R}^* (c, d)$ in Z.*

Definition 4.2.7. Let M be a monoid and X be a set. An action of M on X is called *faithful* if for $m, n \in M$ with $m \neq n$, there exists an $x \in X$ such that $m \cdot x \neq n \cdot x$.

Theorem 4.2.8. *Let $Z = S \bowtie T$ be a Zappa-Szép product of monoids S and T, where T is right cancellative. Suppose S acts faithfully on the right of T and for any $a, c \in S$, $a \,\mathcal{R}^*\, c$ implies* $\operatorname{Ker} a = \operatorname{Ker} c$. *Then $a \,\mathcal{R}^*\, c$ in S implies that $(a, b) \,\mathcal{R}^*\, (c, d)$ in Z.*

Proof. Suppose (a, b), $(c, d) \in Z$ and $a \,\mathcal{R}^*\, c$ in S. To show that $(a, b) \,\mathcal{R}^*\, (c, d)$ in Z, let $x, u \in S$ and $y, v \in T$ be such that

$$x(y \cdot a) = u(v \cdot a) \text{ and } y^a b = v^a b.$$

Then $y^a = v^a$, as T is right cancellative. Also as $\operatorname{Ker} a = \operatorname{Ker} c$, we have $y^c = v^c$ and thus

$$y^c d = v^c d.$$

Now for any $z \in T$,

$$(z^x y)^a = z^{x(y \cdot a)} y^a = z^{u(v \cdot a)} v^a = (z^u v)^a$$

and so as $\operatorname{Ker} a = \operatorname{Ker} c$,

$$(z^x y)^c = (z^u v)^c,$$

gives

$$z^{x(y \cdot c)} y^c = z^{u(v \cdot c)} v^c.$$

But $y^c = v^c$ and T is right cancellative, therefore

$$z^{x(y \cdot c)} = z^{u(v \cdot c)}.$$

As this is true for any $z \in T$, and S acts faithfully, we have

$$x(y \cdot c) = u(v \cdot c).$$

Together with the opposite argument, Lemma 4.2.1 gives $(a, b) \,\mathcal{R}^*\, (c, d)$ in Z. □

Theorem 4.2.9. *Let S, T be monoids and let $Z = S \bowtie T$ be the Zappa-Szép product of S and T. Let $E \subseteq E(S)$ and $F \subseteq E(T)$. Put*

$$\overline{E} = \{(e, 1) : e \in E\} \text{ and } \overline{F} = \{(1, f) : f \in F\}.$$

Then $\overline{E}$ and $\overline{F}$ are sets of idempotents in Z and

(1) $(a, b) \,\widetilde{\mathcal{R}}_{\overline{E}}\, (c, d)$ in Z if and only if $a \,\widetilde{\mathcal{R}}_E\, c$ in S;

(2) $(a,b)\,\widetilde{\mathcal{L}}_{\overline{F}}\,(c,d)$ *in* Z *if and only if* $b\,\widetilde{\mathcal{L}}_F\,d$ *in* T.

Proof. It is easy to check that $\overline{E}\cup\overline{F}\subseteq E(Z)$.

(1) Suppose $(a,b)\,\widetilde{\mathcal{R}}_{\overline{E}}\,(c,d)$ in Z. Let $e\in E$. Then

$$\begin{array}{rrcll} ea=a & \Leftrightarrow \quad e(1\cdot a) & = & a & \text{by action of } T \text{ on } S \\ & \Leftrightarrow \quad (e,1)(a,b) & = & (a,b) & \\ & \Leftrightarrow \quad (e,1)(c,d) & = & (c,d) & \text{because } (a,b)\,\widetilde{\mathcal{R}}_{\overline{E}}\,(c,d) \\ & \Leftrightarrow \quad e(1\cdot c) & = & c & \\ & \Leftrightarrow \quad ec & = & c. & \end{array}$$

Hence $a\,\widetilde{\mathcal{R}}_E\,c$ in S.

Conversely suppose that $a\,\widetilde{\mathcal{R}}_E\,c$ in S and let $(a,b)\in Z$. Then for $(e,1)\in\overline{E}$ we have

$$\begin{array}{rrcll} & (e,1)(a,b) & = & (a,b) & \\ \Leftrightarrow & (e(1\cdot a),1^a b) & = & (a,b) & \\ \Leftrightarrow & (ea,b) & = & (a,b) & \\ \Leftrightarrow & ea & = & a & \\ \Leftrightarrow & ec & = & c & \text{because } a\,\widetilde{\mathcal{R}}_E\,c \\ \Leftrightarrow & (ec,d) & = & (c,d) & \\ \Leftrightarrow & (e(1\cdot c),1^c d) & = & (c,d) & \\ \Leftrightarrow & (e,1)(c,d) & = & (c,d). & \end{array}$$

Hence $(a,b)\,\widetilde{\mathcal{R}}_{\overline{E}}\,(c,d)$.

The proof for $\widetilde{\mathcal{L}}_E$ is dual. □

Theorem 4.2.10. *Let* S *and* T *be semigroups and* $Z=S\bowtie T$ *be the Zappa-Szép product of* S *and* T. *Let* $E\subseteq E(S)$ *and* $F\subseteq E(T)$. *Suppose there exists a right identity* e *for* S *and a left identity* f *for* T *such that*

$$\begin{array}{ll} b\cdot e=e, \quad b^e=b & \text{for all } b\in T \\ f\cdot a=a, \quad f^a=f & \text{for all } a\in S. \end{array} \tag{RLI}$$

Put

$$\overline{E}=\{(h,f):h\in E\} \text{ and } \overline{F}=\{(e,g):g\in F\}.$$

Then $\overline{E}$ *and* $\overline{F}$ *are sets of idempotents in* Z *and*

(i) $(a,b)\,\widetilde{\mathcal{L}}_{\overline{F}}\,(c,d)\Rightarrow b\,\widetilde{\mathcal{L}}_F\,d$ *and* $(a,b)\,\widetilde{\mathcal{R}}_{\overline{E}}(c,d)\Rightarrow a\,\widetilde{\mathcal{R}}_E\,c$*;*

(ii) $(a,b)\,\mathcal{L}^*\,(c,d) \Rightarrow b\,\mathcal{L}^*\,d$ *and* $(a,b)\,\mathcal{R}^*\,(c,d) \Rightarrow a\,\mathcal{R}^*\,c$.

Proof. We first check that$\overline{E}$ and $\overline{F}$ are sets of idempotents in Z. For this let $(h,f) \in \overline{E}$. Then

$$\begin{aligned}(h,f)(h,f) &= (h(f\cdot h), f^h f)\\ &= (hh, ff) \qquad \text{using (RLI)}\\ &= (h,f).\end{aligned}$$

Thus $\overline{E} \subseteq E(Z)$ and similarly $\overline{F} \subseteq E(Z)$.

(i) Let $(a,b),(c,d) \in Z$ and suppose that $(a,b)\,\widetilde{\mathcal{L}}_{\overline{F}}\,(c,d)$. Let $g \in F$ be such that $bg = b$. Then $b^e g = b$ using (RLI). Now as $b \cdot e = e$ and $ae = a$, we have

$$\begin{aligned} & (a,b)(e,g) = (a,b)\\ \Rightarrow\ & (c,d)(e,g) = (c,d) \qquad \text{because } (a,b)\,\widetilde{\mathcal{L}}_{\overline{E}}\,(c,d)\\ \Rightarrow\ & c(d\cdot e) = c \quad \text{and} \quad d^e g = d,\end{aligned}$$

giving $dg = d$. Together with the opposite direction, we obtain $b\,\widetilde{\mathcal{L}}_F\,d$.

Dually $(a,b)\,\widetilde{\mathcal{R}}_{\overline{E}}\,(c,d)$ implies $a\,\widetilde{\mathcal{R}}_E\,c$.

(ii) Suppose that $(a,b)\,\mathcal{R}^*\,(c,d)$ and let $u,v \in S$ be such that $ua = va$. Then,

$$\begin{aligned} & u(f\cdot a) = v(f\cdot a) \qquad \text{using (RLI)}\\ \Rightarrow\ & (u,f)(a,b) = (v,f)(a,b)\\ \Rightarrow\ & (u,f)(c,d) = (v,f)(c,d) \quad \text{because } (a,b)\,\mathcal{R}^*\,(c,d)\\ \Rightarrow\ & u(f\cdot c) = v(f\cdot c)\\ \Rightarrow\ & uc = vc.\end{aligned}$$

Further,

$$\begin{aligned} & a = va\\ & = v(f\cdot a)\\ \Rightarrow\ & (a,b) = (v,f)(a,b)\\ \Rightarrow\ & (c,d) = (v,f)(c,d) \quad \text{because } (a,b)\,\mathcal{R}^*\,(c,d)\\ \Rightarrow\ & c = v(f\cdot c)\\ \Rightarrow\ & c = vc.\end{aligned}$$

Together with the dual, we obtain $a\,\mathcal{R}^*\,c$.

Dually $(a,b)\,\mathcal{L}^*\,(c,d)$ implies that $b\,\mathcal{L}^*\,d$. □

Chapter 5

Semigroups with inverse skeletons and Zappa-Szép products

The content of this chapter has already appeared in a joint paper with Victoria Gould [30].

Throughout this chapter, E denotes a set of idempotents of a semigroup S.

The aim of this chapter is to study semigroups possessing E-regular elements, where an element a of a semigroup S is *E-regular* if a has an inverse a° such that $aa^\circ, a^\circ a$ lie in E. In Section 5.1, we show that where S possesses 'enough' (in a precisely defined way) E-regular elements, analogues of Green's lemmas and even of Green's theorem hold, where Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}$ and $\mathcal{D}$ are replaced by $\widetilde{\mathcal{R}}_E, \widetilde{\mathcal{L}}_E, \widetilde{\mathcal{H}}_E$ and $\widetilde{\mathcal{D}}_E$. With some extra conditions on our semigroup we also have an analogue of Green's theorem. Namely, we show that under these conditions, if $a \,\widetilde{\mathcal{H}}_E\, a^2$, then $\widetilde{H}_E^a$, the $\widetilde{\mathcal{H}}_E$-class of a, is a monoid with identity from E.

In Section 5.2 we show that if $\widetilde{\mathcal{H}}_E$ is a congruence on a certain semigroup S, then any right congruence on the submonoid $\widetilde{H}_E^e$, where $e \in E$, can be extended to a congruence on S. We also have a result for two sided congruences, with some further restrictions on S. We stress that for regular semigroups with $E = E(S)$ we have $\widetilde{\mathcal{K}}_E = \mathcal{K}^* = \mathcal{K}$, so our results can be immediately applied to maximal subgroups of regular semigroups.

In Section 5.3 we introduce the idea of an *inverse skeleton* U of a semigroup S. Here U is an inverse subsemigroup of E-regular elements, such that $E \subseteq U$ and U intersects every $\widetilde{\mathcal{H}}_E$-class exactly once (it follows that $E = E(U)$). We examine some conditions under which we obtain skeletons from monoids having a particular submonoid L of the $\widetilde{\mathcal{L}}_E$-class of the identity. A monoid with such a submonoid L is called *special*. Our most complete results are for restriction monoids in Section 5.4, where we focus on a result of Kunze [37] for the

Bruck-Reilly extension $\mathrm{BR}(M,\theta)$ of a monoid M, showing that $\mathrm{BR}(M,\theta)$ is a Zappa-Szép product of $\mathbb{N}^0$ under addition and a semidirect product $M \rtimes \mathbb{N}^0$. Certainly $\mathrm{BR}(M,\theta)$ is special, with L isomorphic to $\mathbb{N}^0$. We put Kunze's result in more general framework and prove in particular that a special $\widetilde{\mathcal{D}}_E$-simple restriction monoid can be decomposed in an analogous way. Again, our results apply immediately to inverse monoids.

5.1 The relations $\widetilde{\mathcal{R}}_E, \widetilde{\mathcal{L}}_E$ and analogues of Green's lemmas

In this section, we will show that, under certain circumstances, $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ behave like $\mathcal{R}$ and $\mathcal{L}$. In general, however, they do not.

We remind the reader from Chapter 1 that a semigroup S satisfies the congruence condition (C) if $\widetilde{\mathcal{R}}_E$ is a left congruence and $\widetilde{\mathcal{L}}_E$ is a right congruence.

Lemma 5.1.1. *Let S be a semigroup satisfying (C). Then if $a,b \in S$ and $a \,\widetilde{\mathcal{R}}_E\, e \,\widetilde{\mathcal{L}}_E\, b$, for some $e \in E$, we have that $a \,\widetilde{\mathcal{L}}_E\, ba \,\widetilde{\mathcal{R}}_E\, b$.*

Proof. As $a \,\widetilde{\mathcal{R}}_E\, e$ and $\widetilde{\mathcal{R}}_E$ is left congruence, we have $ba \,\widetilde{\mathcal{R}}_E\, be = b$. Dually, $ba \,\widetilde{\mathcal{L}}_E\, a$.

e		a
b		ba

Figure 5.1: 'Egg box' picture depicting behaviour of $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ in S

□

Definition 5.1.2. An element $c \in S$ is E-*regular* if c has an inverse c° such that $cc^\circ, c^\circ c \in E$.

We emphasise that the notation c° will always be used with this meaning. Of course, if c is E-regular, then so is c°. Observe that if $c \in S$ is E-regular and $g,h \in E$ with $g \,\widetilde{\mathcal{R}}_E\, c \,\widetilde{\mathcal{L}}_E\, h$, then $cc^\circ \,\mathcal{R}\, c \,\widetilde{\mathcal{R}}_E\, g$ and $c^\circ c \,\mathcal{L}\, c \,\widetilde{\mathcal{L}}_E\, h$, so that by an earlier remark from Chapter 1 that for $e,f \in E$, $e \,\widetilde{\mathcal{R}}_E\, f$ if and only if $e \,\mathcal{R}\, f$, we have $cc^\circ \,\mathcal{R}\, g$ and $c^\circ c \,\mathcal{L}\, h$. It follows from standard results for regular elements that c has an inverse c' such that $cc' = g$ and $c'c = h$.

Lemma 5.1.3. *Let S be a semigroup and suppose that $h,k \in S$ are E-regular. Then $h \,\widetilde{\mathcal{R}}_E\, k$ ($h \,\widetilde{\mathcal{L}}_E\, k$, $h \,\widetilde{\mathcal{H}}_E\, k$) if and only if $h \,\mathcal{R}\, k$ ($h \,\mathcal{L}\, k$, $h \,\mathcal{H}\, k$).*

Proof. As h, k are E-regular, so $hh^\circ, kk^\circ \in E$. Suppose that $h \,\widetilde{\mathcal{R}}_E\, k$. Then

$$hh^\circ \,\widetilde{\mathcal{R}}_E\, h \,\widetilde{\mathcal{R}}_E\, k \,\widetilde{\mathcal{R}}_E\, kk^\circ$$

so that we have $hh^\circ \,\mathcal{R}\, kk^\circ$. Thus there exists $u, v \in S^1$ such that

$$hh^\circ u = kk^\circ \text{ and } kk^\circ v = hh^\circ.$$

Hence $h(h^\circ uk) = kk^\circ k = k$ and $k(k^\circ vh) = hh^\circ h = h$ gives $h \,\mathcal{R}\, k$.

The reverse statement is clear because $\mathcal{R} \subseteq \widetilde{\mathcal{R}}_E$ $(\mathcal{L} \subseteq \widetilde{\mathcal{L}}_E, \mathcal{H} \subseteq \widetilde{\mathcal{H}}_E)$. □

Next we show that analogues of Green's Lemmas hold with $\mathcal{R}$, $\mathcal{L}$ replaced by $\widetilde{\mathcal{R}}_E$, $\widetilde{\mathcal{L}}_E$ where there is a suitable E-regular element.

Lemma 5.1.4. *Suppose that $\widetilde{\mathcal{L}}_E$ is a right congruence and S has an E-regular element c such that $e = cc^\circ$ and $f = c^\circ c$. Then the right translations*

$$\rho_c : \widetilde{L}_E^e \to \widetilde{L}_E^f \quad \textit{and} \quad \rho_{c^\circ} : \widetilde{L}_E^f \to \widetilde{L}_E^e$$

are mutually inverse $\widetilde{\mathcal{R}}_E$-class preserving bijections.

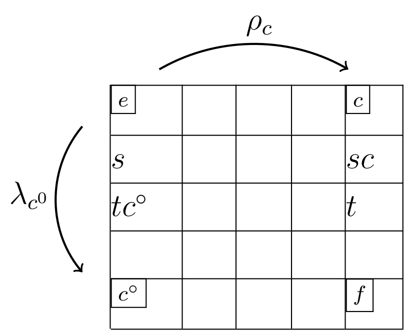

Figure 5.2: $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$-class preserving bijections

Proof. Notice that $e \,\mathcal{R}\, c \,\mathcal{L}\, f$. Let $s \in \widetilde{L}_E^e$. Since $\widetilde{\mathcal{L}}_E$ is a right congruence, $sc \,\widetilde{\mathcal{L}}_E\, ec = c$ so there is a map $\rho_c : \widetilde{L}_E^e \to \widetilde{L}_E^f$ defined by $s\rho_c = sc$. Now $s = se = scc^\circ \,\mathcal{R}\, sc$, so that certainly ρ_c is $\widetilde{\mathcal{R}}_E$-class preserving. Dually, $\rho_{c^\circ} : \widetilde{L}_E^f \to \widetilde{L}_E^e$ is $\widetilde{\mathcal{R}}_E$-class preserving.

For any $s \in \widetilde{L}_E^e$ and $t \in \widetilde{L}_E^f$ we have $s = se = s(cc^\circ) = s\rho_c\rho_{c^\circ}$ and similarly, $t = t\rho_{c^\circ}\rho_c$, so that ρ_c and ρ_{c° are mutually inverse on the specified domains. □

Note that we are not assuming that the $\widetilde{\mathcal{D}}_E$-class depicted above is an "egg-box", since as $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ need not commute, some of the cells may be empty.

For convenience we now state the dual of Lemma 5.1.4.

Lemma 5.1.5. *Suppose that $\widetilde{\mathcal{R}}_E$ is a left congruence and S has an E-regular element c such that $e = cc^\circ$ and $f = c^\circ c$. Then the left translations*

$$\lambda_{c^\circ} : \widetilde{R}_E^e \to \widetilde{R}_E^f \quad \textit{and} \quad \lambda_c : \widetilde{R}_E^f \to \widetilde{R}_E^e$$

are mutually inverse $\widetilde{\mathcal{L}}_E$-class preserving bijections.

Corollary 5.1.6. *Let S be a semigroup with (C). Let c be an E-regular element of S such that $e = cc^\circ$ and $f = c^\circ c$. Then $\widetilde{H}_E^e \cong \widetilde{H}_E^f$.*

Proof. By Lemmas 5.1.4 and 5.1.5, $\rho_c : \widetilde{H}_E^e \to \widetilde{H}_E^c$ and $\lambda_{c^\circ} : \widetilde{H}_E^c \to \widetilde{H}_E^f$ are bijections. Now For any $x, y \in \widetilde{H}_E^e$ we have

$$\begin{aligned} (xy)\rho_c\lambda_{c^\circ} &= c^\circ xyc \\ &= c^\circ xcc^\circ yc \qquad \text{as } cc^\circ = e \\ &= (x\rho_c\lambda_{c^\circ})(y\rho_c\lambda_{c^\circ}). \end{aligned}$$

Thus $\rho_c\lambda_{c^\circ}$ is an isomorphism and hence $\widetilde{H}_E^e \cong \widetilde{H}_E^f$. □

If we have enough E-regular elements, then we can say much more than in Corollary 5.1.6.

Lemma 5.1.7. *If every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element, then S is weakly E-abundant. Moreover if S has (C), then $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ (so that $\widetilde{\mathcal{D}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E$) and if $a, b \in S$ with $a \, \widetilde{\mathcal{D}}_E \, b$, then $|\widetilde{H}_E^a| = |\widetilde{H}_E^b|$.*

Proof. The first statement is clear. Suppose that $a, c \in S$ with $a \, \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E \, c$.

				$b^\circ b$
a		bb°		b
$cb^\circ a$				c

Figure 5.3: Picture showing that $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$

There exists an E-regular $b \in S$ such that $a \widetilde{\mathcal{R}}_E b \widetilde{\mathcal{L}}_E c$. Choose an inverse b° of b such that $bb^\circ, b^\circ b \in E$. Notice that $c \widetilde{\mathcal{L}}_E b^\circ b$ and $a \widetilde{\mathcal{R}}_E bb^\circ$. Using (C), $cb^\circ a \widetilde{\mathcal{R}}_E cb^\circ b = c$ and $cb^\circ a \widetilde{\mathcal{L}}_E bb^\circ a = a$. Then $a \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E c$. Together with the dual argument we have that

$$\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E.$$

In view of the remarks following Definition 5.1.2, the proof of the final statement follows easily from Lemmas 5.1.4 and 5.1.5. □

Green's theorem, a pivot of classical semigroup theory, states that if $k \in S$ and $k \mathcal{H} k^2$, then H_k is a group. We now consider semigroups with (C) such that the analogue of Green's theorem holds, by which we mean, if $k \widetilde{\mathcal{H}}_E k^2$, then $\widetilde{H}_E^k$ is a monoid with identity an element of E: in view of Lemma 1.2.9, this is equivalent to containing an element of E.

The set of idempotents $E(T)$ of any semigroup T may be endowed with the two pre-orders $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$, under which it has the structure of a *biordered set*; if T is regular, then $E(T)$ is a *regular biordered set*. Conversely, any biordered set is the biordered set of idempotents of a semigroup, which is regular if E is regular [55, 12]. Suppose now that S is our semigroup with $E \subseteq E(S)$; [55, Theorem 1.3] gives necessary and sufficient conditions such that E generates a regular subsemigroup $S' = \langle E \rangle$ of S such that $E(S') = E$. Clearly, if these conditions hold, and if $h \in S'$ with $h \widetilde{\mathcal{H}}_E h^2$ in S, then as $E \subseteq S'$ we have $h \widetilde{\mathcal{H}}_E h^2$ in S'. It follows that $h \mathcal{H} h^2$ in S' so that $h \mathcal{H} u$ in S' for some $u \in E(S') = E$. Certainly then $\widetilde{H}_E^h$ (in either S or S') contains u.

To obtain a more general result, we need to introduce the following concept.

Definition 5.1.8. We say that $E \subseteq E(S)$ is *closed under E-conjugation* if for any $e \in E$ and E-regular $c \in S$ (with $cc^\circ, c^\circ c \in E$), if $cec^\circ \in E(S)$, then $cec^\circ \in E$.

Notice that the above definition is symmetric, since $(c^\circ)^\circ = c$.

Lemma 5.1.9. *Let S be a restriction semigroup, let $c \in S$ be E-regular and let $e \in E$. Then cec° (and hence also $c^\circ ec$) lie in E.*

Proof. Let c, e be as above. Then

$$cec^\circ = (ce)^+ cc^\circ \in E$$

as E is a semilattice. □

The next lemma follows the pattern for regular semigroups, as stated in [32, Result 2]. However, we need a little care as E need not consist of all the idempotents of S.

Lemma 5.1.10. *The E-regular elements of S form a subsemigroup T with $E = E(T)$ if and only if ef is E-regular for any $e, f \in E$, and E is closed under E-conjugation.*

Proof. Let T denote the set of E-regular elements of S. The direct statement is clear.

Conversely, suppose that ef is E-regular for any $e, f \in E$, and E is closed under E-conjugation. Let $h, k \in T$ and choose inverses h°, k° of h and k respectively, such that $hh^\circ, f = h^\circ h, e = kk^\circ, k^\circ k \in E$. Let u be an inverse of fe such that $ufe, feu \in E$. It is easy to check that $k^\circ uh^\circ$ is an inverse of hk. We then have $(hk)(k^\circ uh^\circ) \in E(S)$ and

$$(hk)(k^\circ uh^\circ) = hf(kk^\circ)uh^\circ = h(feu)h^\circ,$$

so that $(hk)(k^\circ uh^\circ) \in E$ as $feu \in E$ and E is closed under E-conjugation. Similarly, $(k^\circ uh^\circ)hk \in E$. Thus $hk \in T$ as required. □

Corollary 5.1.11. *Suppose that ef is E-regular for any $e, f \in E$, and E is closed under E-conjugation. If $h \in S$ is E-regular and $h \,\widetilde{\mathcal{H}}_E\, h^2$, then $\widetilde{H}_E^h$ contains an idempotent of E; hence if S satisfies (C), then $\widetilde{H}_E^h$ is a monoid with identity from E.*

Proof. From Lemma 5.1.10 we have that the E-regular elements of S form a subsemigroup T with $E = E(T)$. Certainly $h, h^2 \in T$ with $h \,\widetilde{\mathcal{H}}_E\, h^2$ in T. Then $h \,\mathcal{H}\, h^2$ in T so that as $E = E(T)$ we have $\widetilde{H}_E^h$ (in either T or S) contains an idempotent of E. □

Whereas the previous result uses Green's theorem, the next does not, but has rather restrictive hypotheses.

Lemma 5.1.12. *Suppose that $E \subseteq E(S)$ is a band, every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element, $\widetilde{\mathcal{H}}_E$ is a congruence and S satisfies (C). Then for $k \in S$ with $k \,\widetilde{\mathcal{H}}_E\, k^2$, we have $E \cap \widetilde{H}_E^k \neq \emptyset$.*

Proof. Notice that as $\widetilde{\mathcal{H}}_E$ is a congruence and $k \,\widetilde{\mathcal{H}}_E\, k^2$, we have that $\widetilde{H}_E^k$ is a subsemigroup.

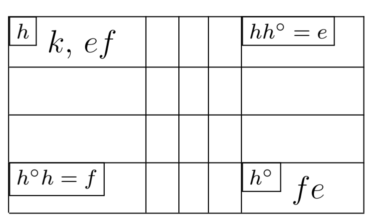

Figure 5.4: Examining the behaviour of $\widetilde{H}_E^k$

By hypothesis there exists an E-regular element $h \in \widetilde{H}_E^k$ such that $hh^\circ = e, h^\circ h = f \in E$. Then

$$h^\circ = h^\circ hh^\circ \,\widetilde{\mathcal{H}}_E\, h^\circ hhh^\circ = fe \in E.$$

By Lemma 5.1.1, $ef \in \widetilde{H}_E^k$ and $ef \in E$ as E is a band. Hence $E \cap \widetilde{H}_E^k \neq \emptyset$. □

5.2 Extending congruences

Let M be a subsemigroup of a semigroup S and let ρ be a congruence (respectively, right congruence) on M. We denote by $\widetilde{\rho}$ (respectively, $\bar{\rho}$) the congruence (respectively, right congruence) on S generated by ρ. We briefly review the circumstances under which

$$\rho = \widetilde{\rho} \cap (M \times M) \text{ or } \rho = \bar{\rho} \cap (M \times M),$$

where $M = \widetilde{H}_E^e$ for some $e \in E$, in the context of the conditions discussed in this chapter.

Definition 5.2.1. A subsemigroup M of a semigroup S has the *(right) congruence extension property* in S if for any (right) congruence ρ on M we have

$$\rho = \widetilde{\rho} \cap (M \times M) \text{ (respectively, } \rho = \bar{\rho} \cap (M \times M)).$$

Lemma 5.2.2. *Let S be a weakly E-abundant semigroup with (C). Suppose that $\widetilde{\mathcal{H}}_E$ is a congruence. Let $e \in E$. Then $M = \widetilde{H}_E^e$ has the right congruence extension property in S.*

Proof. Let ρ be a right congruence on M. Clearly $\rho \subseteq \bar{\rho} \cap (M \times M)$. Let $a \in M, b \in S$ and suppose $a \,\bar{\rho}\, b$. Then either $a = b$ (so that clearly $a \,\rho\, b$) or there exists a sequence

$$a = c_1t_1,\; d_1t_1 = c_2t_2,\; \cdots, d_nt_n = b$$

for some $n \in \mathbb{N}$, where $(c_i, d_i) \in \rho$, $t_i \in S$, $1 \leq i \leq n$ (see, for example, [36, Chapter 1]). As $a, c_1, d_1, \cdots, c_n, d_n \in M$, which has identity e, we have

$$a = c_1t_1',\; d_1t_1' = c_2t_2',\; \cdots, d_nt_n' = b \quad \text{where } t_i' = et_i.$$

Since $\widetilde{\mathcal{H}}_E$ is a congruence we have

$$a = c_1t_1' \,\widetilde{\mathcal{H}}_E\, et_1' = t_1' \,\widetilde{\mathcal{H}}_E\, d_1t_1' = c_2t_2' \,\widetilde{\mathcal{H}}_E\, et_2' = t_2' \,\widetilde{\mathcal{H}}_E \cdots \widetilde{\mathcal{H}}_E\, et_n' = t_n'.$$

We conclude that $t'_1, \cdots, t'_n \in M$ and so $b \in M$ and $a \, \rho \, b$. Hence M has the right congruence extension property. □

Note that what we have shown above is something a little stronger than claimed, namely that $\bar{\rho}$ saturates M, where ρ *saturates* M if

$$a \, \rho \, b \Rightarrow (a \in M \text{ if and only if } b \in M),$$

or equivalently, M is the union of ρ-classes.

Corollary 5.2.3. *Let S be a regular semigroup such that $\mathcal{H}$ is a congruence. Then for any $e \in E(S)$, the maximal subgroup H_e has the right congruence extension property.*

Note that it was shown in [23, Lemma 6.3] that every subgroup of a semigroup has the right congruence extension property and we are thankful to Maria Szendrei for making it explicit for us (in a private communication). Our result is in a sense stronger that we don't require $\widetilde{\mathcal{H}}_E$ to be a subgroup, but it is also weaker, because we need $\widetilde{\mathcal{H}}_E$ to be a congruence.

Let M be a subsemigroup of S and let ρ be a congruence on M. We say that ρ is *closed under E-conjugation* if for $u, v \in M$ with $u \, \rho \, v$ and for any E-regular $c \in S$ with $cuc^\circ, cvc^\circ \in M$, we have $cuc^\circ \, \rho \, cvc^\circ$; if $E = E(S)$, we simply say that ρ is *closed under conjugation.*

Proposition 5.2.4. *Let S be a semigroup with (C) such that every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element, $\widetilde{\mathcal{H}}_E$ is a congruence and if $k \, \widetilde{\mathcal{H}}_E \, k^2$, then $\widetilde{H}_E^k$ contains an idempotent of E. Let $e \in E$ and $M = \widetilde{H}_E^e$ and let ρ be a congruence on M. Then*

$$\rho = \widetilde{\rho} \cap (M \times M),$$

if and only if ρ is closed under E-conjugation.

Proof. It is clear that if $\rho = \widetilde{\rho} \cap (M \times M)$, then ρ is closed under E-conjugation.

Conversely, suppose that ρ is closed under E-conjugation. Let $a \in M, b \in S$ and suppose that

$$a = cpd, \; cqd = b,$$

where $(p, q) \in \rho$ and $c, d \in S^1$. As $p \, \widetilde{\mathcal{H}}_E^e \, q$ and $\widetilde{\mathcal{H}}_E$ is a congruence, we see that $b \in M$. It follows that

$$a = c'pd', \; c'qd' = b,$$

where $c' = ece$ and $d' = ede$. Then

$$a \leq_{\widetilde{\mathcal{R}}_E} c' \leq_{\widetilde{\mathcal{R}}_E} e \,\widetilde{\mathcal{R}}_E\, a,$$

so that $a \,\widetilde{\mathcal{R}}_E\, c'$. Dually, $a \,\widetilde{\mathcal{L}}_E\, d'$.

$\boxed{e}$ a	v°			c' u
u°				f
d' v u^*	g			$d'c'$ w

Figure 5.5: Congruence extension property

From the comments following Definition 5.1.2, there exist E-regular elements $u \in \widetilde{H}_E^{c'}$ and $v \in \widetilde{H}_E^{d'}$ such that $uu^\circ = e, u^\circ u = f \in E$ and $v^\circ v = e, vv^\circ = g \in E$. Now $vu \in \widetilde{R}_E^v \cap \widetilde{L}_E^u$ by Lemma 5.1.1 and $vu \,\widetilde{\mathcal{H}}_E\, d'c'$. Since

$$uv \,\widetilde{\mathcal{H}}_E\, c'd' = c'ed' \,\widetilde{\mathcal{H}}_E\, c'pd' = a \,\widetilde{\mathcal{H}}_E\, e$$

we have

$$vuvu \,\widetilde{\mathcal{H}}_E\, veu \,\widetilde{\mathcal{H}}_E\, vu.$$

By assumption, there exists an idempotent $w \in E \cap \widetilde{H}_E^{d'c'}$. Let $u^* \in \widetilde{H}_E^{d'}$ be an inverse of u such that $uu^* = e$ and $u^*u = w$. Then

$$a = c'wpwd' = (c'u^*)(upu^*)(ud') \text{ and } b = c'wqwd' = (c'u^*)(uqu^*)(ud').$$

Now $u^* \,\widetilde{\mathcal{H}}_E\, d'$ gives that $c'u^* \,\widetilde{\mathcal{H}}_E\, c'd' \,\widetilde{\mathcal{H}}_E\, e$, so $c'u^* \in M$ and similarly $u \,\widetilde{\mathcal{H}}_E\, c'$ gives that $ud' \,\widetilde{\mathcal{H}}_E\, c'd' \,\widetilde{\mathcal{H}}_E\, e$, so that $ud' \in M$. Further,

$$upu^* = e(upu^*)e \,\widetilde{\mathcal{H}}_E\, (c'u^*)(upu^*)(ud') = a \in M,$$

and similarly, $uqu^* \in M$. Since ρ is closed under E-conjugation it follows that $upu^* \,\rho\, uqu^*$ and so $a \,\rho\, b$.

Now consider $h \in M, k \in S$ with $h \,\widetilde{\rho}\, k$. Either $h = k$ (so that certainly $h \,\rho\, k$), or h is connected to k via a ρ-sequence

$$h = c_1 p_1 d_1, c_1 q_1 d_1 = c_2 p_2 d_2, \cdots, c_n q_n d_n = k,$$

for some $n \in \mathbb{N}$, where $(p_i, q_i) \in \rho$, $c_i, d_i \in S^1$, $1 \leq i \leq n$ (see, for example, [34, Chapter 1]). It follows from the above that $c_i q_i d_i \in M$ and $h \, \rho \, c_i q_i d_i$ for $1 \leq i \leq n$. Hence $h \, \rho \, k$ and

$$\rho = \widetilde{\rho} \cap (M \times M).$$

□

Corollary 5.2.5. *Let S be a regular semigroup such that $\mathcal{H}$ is a congruence. Let $G = H_e$ be the maximal subgroup with identity $e \in E(S)$. Then for any right congruence ρ on G we have $\rho = \widetilde{\rho} \cap (G \times G)$ if and only if ρ is closed under conjugation.*

Note that if E is a band, then from Lemma 5.1.12, the remaining hypotheses of Proposition 5.2.4 will guarantee that $\widetilde{H}_E^k$ contains an idempotent of E.

We remark here that if S is a completely semisimple semigroup, that is if $\mathcal{J} = \mathcal{D}$ and the principal factors are completely 0-simple, then ρ is closed under conjugation. In this case if $G = H_e$ is the maximal subgroup with identity $e \in E(S)$, then $\rho = \widetilde{\rho} \cap (G \times G)$.

In the following, M is a monoid with identity e.

Example 5.2.6. Let B be a band. With $E = \{e\} \times B$, the direct product $M \times B$ satisfies the hypotheses of Proposition 5.2.4.

The next three examples are essentially folklore, but they can all be found in [11].

Example 5.2.7. Let $S = \mathcal{B}^{\circ}(M, I)$ be a Brandt semigroup. That is,

$$S = (I \times M \times I) \cup \{0\}$$

with multiplication given by

$$(i, m, j)(j, n, k) = (i, mn, k),$$

all other products being 0. Then with

$$E = \{(i, 1, i) : i \in I\} \cup \{0\}$$

we have that for any $(i,m,j),(k,n,l) \in M$

$$(i,m,j)\,\widetilde{\mathcal{R}}_E\,(k,n,l) \text{ if and only if } i = k$$

and

$$(i,m,j)\,\widetilde{\mathcal{L}}_E\,(k,n,l) \text{ if and only if } j = l.$$

It follows that S is restriction with distinguished semilattice E, $\widetilde{\mathcal{H}}_E$ is a congruence on S and with

$$U = \{(i,e,j) : i,j \in I\} \cup \{0\}$$

we have that U is an inverse subsemigroup of E-regular elements, intersecting every $\widetilde{\mathcal{H}}_E$-class exactly once. In particular, S satisfies the hypotheses of Proposition 5.2.4.

Example 5.2.8. Let $S = \mathrm{BR}(M,\theta)$, where $\theta : M \to M$ is a monoid morphism. That is,

$$S = \mathbb{N}^0 \times M \times \mathbb{N}^0$$

and multiplication is given by

$$(m,a,n)(h,b,k) = (m-n+u, a\theta^{u-n}b\theta^{u-h}, k-h+u) \text{ where } u = \max\,(n,h).$$

With

$$E = \{(m,e,m) : m \in \mathbb{N}^0\}$$

we have that for any $(m,a,n),(h,b,k) \in S$,

$$(m,a,n)\,\widetilde{\mathcal{R}}_E\,(h,b,k) \text{ if and only if } m = h$$

and

$$(m,a,n)\,\widetilde{\mathcal{L}}_E\,(h,b,k) \text{ if and only if } n = k.$$

It is then easy to see that $\widetilde{\mathcal{H}}_E$ is a congruence on S and S is restriction. Moreover, with

$$U = \{(m,e,n) : m,n \in \mathbb{N}^0\}$$

we have that U is an inverse subsemigroup of E-regular elements of S intersecting every $\widetilde{\mathcal{H}}_E$-class exactly once. In particular, S satisfies the hypotheses of Proposition 5.2.4. Note that S is a monoid with identity $(0,e,0)$.

Note that the assumption in [11] that the image of θ is contained in $\widetilde{H}_E^1$, is not needed for the above.

Example 5.2.9. Let $S = \mathrm{BR}(M, \mathbb{Z}, \theta)$ be the extended Bruck-Reilly extension of a monoid M. The underlying set is

$$S = \mathbb{Z} \times M \times \mathbb{Z}$$

and the semigroup operation on S is defined as in Example 5.2.8. The semigroup S has the same properties as in that example, with the exception of being a monoid.

Example 5.2.10. Let $S = [Y; S_\alpha; \chi_{\alpha,\beta}]$ be a strong semilattice Y of monoids S_α, $\alpha \in Y$, with connecting morphims $\chi_{\alpha,\beta}$ for $\alpha \geqslant \beta$. Denoting the identity of S_α by e_α we have that S is restriction with

$$E = \{e_\alpha : \alpha \in Y\} \cong Y,$$

and the S_αs are the $\widetilde{\mathcal{H}}_E$-classes. Certainly then $\widetilde{\mathcal{H}}_E$ is a congruence on S and S satisfies the hypotheses of Proposition 5.2.4.

5.3 Semigroups with skeletons

We continue to examine semigroups with 'enough' E-regular elements, now moving towards decompositions of such semigroups. It is clear from Lemma 5.1.7 that if every $\widetilde{\mathcal{H}}_E$-class of a semigroup S with (C) contains an E-regular element, and $e \,\widetilde{\mathcal{D}}_E\, a$ where $e \in E$, then every element of $\widetilde{H}_E^a$ has a unique decomposition as upv, where u, v are fixed E-regular elements and $p \in \widetilde{H}_E^e$. For results leading further to structure theorems, we will concentrate in this section on the case where E is a semilattice.

Definition 5.3.1. Let $V \subseteq W$ be subsets of a semigroup S such that W is a union of $\widetilde{\mathcal{H}}_E$-classes. We say that V is an $\widetilde{\mathcal{H}}_E$-*transversal* of W if

$$|V \cap \widetilde{H}_E^a| = 1 \quad \text{for all } a \in W.$$

Lemma 5.3.2. *Let E be a semilattice and let $c \in S$ be E-regular. Then there is only one choice of c°. Moreover, if $d \in S$ is E-regular and $c \,\widetilde{\mathcal{H}}_E\, d$, then $c^\circ \,\widetilde{\mathcal{H}}_E\, d^\circ$.*

Proof. If c°, c' are both inverses of c with $cc^\circ, cc', c^\circ c, c'c \in E$, then we have

$$c \,\widetilde{\mathcal{L}}_E\, c^\circ c \,\widetilde{\mathcal{L}}_E\, c'c \text{ and } cc^\circ \,\widetilde{\mathcal{R}}_E\, c \,\widetilde{\mathcal{R}}_E\, cc'.$$

Since E is a semilattice, any $\widetilde{\mathcal{R}}_E$-class or $\widetilde{\mathcal{L}}_E$-class contains at most one idempotent of E, so that $c^\circ c = c'c = e$ and $cc^\circ = cc' = f$ say. Thus $c^\circ, c' \in R_e \cap L_f$ so that (as any $\mathcal{H}$-class contains at most one inverse of c) we have $c^\circ = c'$.

The proof of the second statement is similar. □

Clearly the above shows that if E is a semilattice and $c \in S$ is E-regular, then $(c^\circ)^\circ = c$. We recall that S is said to be *weakly E-adequate* if S is weakly E-abundant and E is a semilattice. In this case there is a unique idempotent in the $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class) of $a \in S$, which we denote by a^+ (a^*, respectively).

Note 5.3.3. *Let S be a weakly E-adequate semigroup and let $c \in S$ be E-regular. Then*

$$c \,\widetilde{\mathcal{R}}_E\, c^+ \,\widetilde{\mathcal{R}}_E\, cc^\circ,$$

so that we must have $c^+ = cc^\circ$ and similarly $c^* = c^\circ c$. Hence also $(c^\circ)^+ = c^\circ c$ and $(c^\circ)^* = cc^\circ$.

Proposition 5.3.4. *Let S be weakly E-adequate with $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$, and let $e \in E$. Suppose there is an $\widetilde{\mathcal{H}}_E$-transversal L of $\widetilde{L}^e_E$ such that every $c \in L$ is E-regular, and $e \in L$. Then:*

(1) $R = \{c^\circ : c \in L\}$ is an $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}^e_E$;

(2) $D = LR$ is an $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{D}^e_E$;

(3) if S has (C), then every element of $\widetilde{D}^e_E$ has a unique decompostion as cpd°, where $c, d \in L$ and $p \in \widetilde{H}^e_E$.

Proof. (1) Let $c \in L$. As E is a semilattice and $c \,\widetilde{\mathcal{L}}_E\, e$, we must have that $e = c^\circ c$ so that $e \,\widetilde{\mathcal{R}}_E\, c^\circ$. From Lemma 5.3.2, clearly R intersects any $\widetilde{\mathcal{H}}_E$-class at most once. On the other hand, let $a \in \widetilde{R}^e_E$. Then $a \,\widetilde{\mathcal{L}}_E\, f \in E$ and as $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$, we have that $f \,\widetilde{\mathcal{R}}_E\, c$ for some $c \in L$. It follows that $a \,\widetilde{\mathcal{H}}_E\, c^\circ$, so that R is an $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}^e_E$.

(2) It is clear from Lemma 5.1.1 that for any $c, d \in L$ we have $cd^\circ \in \widetilde{R}^c_E \cap \widetilde{L}^{d^\circ}_E$. Since $\widetilde{\mathcal{D}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$, it follows that D is an $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{D}^e_E$, as required.

(3) This follows from Lemmas 5.1.4 and 5.1.5.

□

We anticipate that Proposition 5.3.4 can be used to develop structure theorems for classes of weakly E-adequate semigroups analogous to those for inverse semigroups.

Definition 5.3.5. Let U be an inverse subsemigroup of S consisting of E-regular elements such that $E \subseteq U$. If U is an $\widetilde{\mathcal{H}}_E$-transversal of S, then U is an *inverse skeleton* of S.

Example 5.3.6. The semigroups of Examples 5.2.7, 5.2.8 and 5.2.10 all have inverse skeletons, with E being the skeleton in Example 5.2.10.

Lemma 5.3.7. *Let S be a semigroup containing an inverse skeleton U. Then $E = E(U)$ is a semilattice, S is weakly E-adequate and if in addition S has (C), we have $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$.*

Proof. We are given that $E \subseteq E(U)$. If $u \in E(U)$, then as u is E-regular, $u \, \mathcal{R} \, uu^\circ \in E$. We are given that $E(U)$ is a semilattice and so $u = uu^\circ \in E$. The remainder of the lemma is immediate from Lemma 5.1.7. $\square$

Naturally, we say that S is $\widetilde{\mathcal{D}}_E$*-simple* if it is a single $\widetilde{\mathcal{D}}_E$-class.

Theorem 5.3.8. *Let S be a $\widetilde{\mathcal{D}}_E$-simple weakly E-adequate monoid with $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$. Suppose there is a submonoid $\widetilde{\mathcal{H}}_E$-transversal L of $\widetilde{L}_E^1$ such that every $c \in L$ is E-regular and for all $c \in L$, $e \in E$ we have $cec^\circ, c^\circ ec \in E$. Let*

$$R = \{c^\circ : c \in L\}.$$

(1) R is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}_E^1$;

(2) $RL \subseteq \widetilde{R}_E^1 \cup \widetilde{L}_E^1$ if and only if E is a chain;

(3) if S is restriction then $U = \langle R \cup L \rangle$ is an inverse subsemigroup of S with $E(U) = E$;

(4) if S is restriction and $RL \subseteq R \cup L$, then $U = LR$ and U is an inverse skeleton for S.

Proof. From the condition that $cec^\circ, c^\circ ec \in E$ for all $c \in L$, and the fact that E is a semilattice, it is easy to see that for any $u, v \in R \cup L$ we have uv is E-regular with suitable inverse $v^\circ u^\circ$.

(1) From Proposition 5.3.4, we know that R is an $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}_E^1$. Let $c, d \in L$ so that $c^\circ, d^\circ \in R$. From the above, cd is E-regular with $(cd)^\circ = d^\circ c^\circ$. As $cd \in L$ we have $d^\circ c^\circ \in R$. Clearly, $1 = 1^\circ \in R$, so that R is a submonoid.

(2) Let $e, f \in E$ and let $c, d \in L$ be such that $cc^\circ = e, dd^\circ = f$. As above, $c^\circ d$ is E-regular with $(c^\circ d)^\circ = d^\circ c$. We have $c^\circ d \in \widetilde{R}_E^1$ if and only if $1 = c^\circ dd^\circ c$, which implies

(multiplying on the front by c and the back by c°) that $e = efe$ so that $e \leq f$. On the other hand, if $e \leq f$, then $c^\circ d \, \widetilde{\mathcal{R}}_E \, c^\circ ef = c^\circ e = c^\circ \, \widetilde{\mathcal{R}}_E \, 1$. Similarly, we see that $c^\circ d \in \widetilde{L}^1_E$ if and only if $f \leq e$. Statement (2) follows.

(3) Let $u = x_1 x_2 \dots x_k \in U$, where $x_i \in L \cup R$ for $1 \leq i \leq n$. We show by induction on k that u is E-regular with $u^\circ = x_k^\circ \dots x_1^\circ$. Clearly this is true for $k = 1$ and we commented above that this is true for $k = 2$.

Suppose now that $k \geqslant 3$ and the result is true for words in U of shorter length. Our inductive hypothesis gives that $x_1 \dots x_{k-1}$ is E-regular with inverse $x_{k-1}^\circ \dots x_1^\circ$. Then

$$\begin{aligned}(x_1 \cdots x_k)(x_k^\circ \cdots x_1^\circ)(x_1 \cdots x_k) &= (x_1 \cdots x_{k-1})(x_k x_k^\circ)[(x_{k-1}^\circ \cdots x_1^\circ)(x_1 \cdots x_{k-1})]x_k \\ &= (x_1 \cdots x_{k-1})[(x_{k-1}^\circ \cdots x_1^\circ)(x_1 \cdots x_{k-1})](x_k x_k^\circ)x_k \\ &= x_1 \cdots x_{k-1} x_k\end{aligned}$$

and

$$(x_1 \cdots x_k)(x_k^\circ \cdots x_1^\circ) = x_1(x_2 \cdots x_k x_k^\circ \cdots x_2^\circ)x_1^\circ \in E$$

by induction and hypothesis. Together with the dual argument, we obtain that $u = x_1 \cdots x_k$ is E-regular with $u^\circ = x_k^\circ \cdots x_1^\circ$.

Certainly $E \subseteq E(U)$. To show that U is inverse, we use the fact that S is restriction. Let $e \in E(U)$. Then

$$e^+ = ee^\circ = eee^\circ = ee^+$$

so that using the identity $xy^+ = (xy)^+x$ we have

$$e^+ = ee^+ = (ee)^+e = e^+e = e,$$

so that $E(U) = E$. Hence $E(U)$ is a semilattice and U is inverse.

(4) To see that $U = LR$, let $u \in U$. Since R and L are submonoids, we can write $u = l_1 r_1 l_2 r_2 \cdots l_m r_m$ where $l_1, \dots, l_m \in L$ and $r_1, \dots, r_m \in R$ and m is least with respect to such a decomposition of u. If $m \geqslant 2$, then either $r_1 l_2 \in R$ or $r_1 l_2 \in L$, so that as

$$u = l_1(r_1 l_2 r_2) \cdots l_m r_m = (l_1 r_1 l_2) r_2 \cdots l_m r_m$$

we have violated the minimality of m. Hence $m = 1$ and $U = LR$. From Proposition 5.3.4, U is an $\widetilde{\mathcal{H}}_E$-transversal of S, so that U is an inverse skeleton of S.

□

Example 5.3.9. Let $S = \mathrm{BR}(M, \theta)$ and put

$$L = \{(m, e, 0) : m \in \mathbb{N}^0\}.$$

We have that L is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{L}^1_E$ consisting of E-regular elements and $S \times S = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$. With

$$R = \{(0, e, m) : m \in \mathbb{N}^0\} = \{(m, e, 0)^\circ : m \in \mathbb{N}^0\}$$

we see that $RL \subseteq R \cup L$. Then U defined as in Theorem 5.3.8 coincides with U as given in Example 5.2.8.

5.4 $\widetilde{\mathcal{D}}_E$-simple monoids and Zappa-Szép products

We build on the results of previous sections to show how certain $\widetilde{\mathcal{D}}_E$-simple restriction monoids decompose as Zappa-Szép products of submonoids. In particular, we show how Kunze's [37] result for the Bruck-Reilly extension of a monoid may be put into a general framework.

Definition 5.4.1. Let S be a monoid. We say that S is *special* if there is a submonoid $\widetilde{\mathcal{H}}_E$-transversal L of $\widetilde{L}^1_E$ such that every $c \in L$ is E-regular.

Example 5.4.2. We have observed in Example 5.3.9 that $S = \mathrm{BR}(M, \theta)$ is special with

$$L = \{(m, e, 0) : n \in \mathbb{N}^0\}$$

being a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{L}^1_E$. Moreover, $\widetilde{\mathcal{H}}_E$ is a congruence on S.

Theorem 5.4.3. *Let S be a weakly E-adequate monoid with (C). Then S is $\widetilde{\mathcal{D}}_E$-simple with $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ and special if and only if S is the internal Zappa-Szép product of L and $\widetilde{R}^1_E$, where L is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{L}^1_E$.*

Proof. Suppose that S is the internal Zappa-Szép product of L and $\widetilde{R}^1_E$, where L is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{L}^1_E$.

Let $a, b \in S$ and write $a = lr$, $b = l'r'$ where $l, l' \in L$ and $r, r' \in \widetilde{R}^1_E$. Then $lr', l'r \in S$,

$$a = lr \,\widetilde{\mathcal{R}}_E\, lr' \,\widetilde{\mathcal{L}}_E\, l'r' = b$$

and

$$a = lr\,\widetilde{\mathcal{L}}_E\,l'r\,\widetilde{\mathcal{R}}_E\,l'r' = b.$$

Thus $\widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = S \times S$. Finally we need to show that L consists of E-regular elements. For this let $l \in L$ and write $l^+ = uv$ where $u \in L$ and $v \in \widetilde{R}^1_E$. Then $u\,\widetilde{\mathcal{R}}_E\,l$ so that $u = l$, since $|L \cap \widetilde{H}^a_E| = 1$ for all $a \in L$.

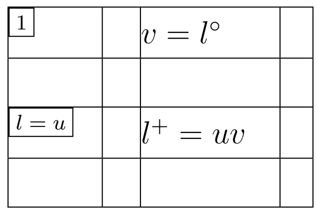

Figure 5.6: E-regular elements in $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{L}^1_E$

Therefore $l^+ = lv$ and $l = l\,1 = l^+l = l(vl)$ and $vl \in \widetilde{H}^1_E$ by Lemma 5.1.1. By uniqueness of factorisation, $vl = 1$. Thus $v = vlv$ and $lv, vl \in E$, so that l is E-regular as required. Thus S is special.

Conversely, suppose that $\widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = S \times S$ and S is special. Let $s \in S$. Then $1\,\widetilde{\mathcal{L}}_E\,l\,\widetilde{\mathcal{R}}_E s$ for some $l \in L$ and as l is E-regular we have $s = l^+s = ll^\circ s$. Now observe that $l^\circ s\,\widetilde{\mathcal{R}}_E\,l^\circ l = 1$ so that $l^\circ s \in \widetilde{R}^1_E$. To see that this factorisation is unique, suppose that $s = lr = kt$ where $l, k \in L$ and $r, t \in \widetilde{R}^1_E$. Now $\widetilde{\mathcal{R}}_E$ is a left congruence, so that $l\,\widetilde{\mathcal{R}}_E\,k$, giving $l = k$. As l is E-regular, we have $1 = l^\circ l$ and we deduce that $r = t$. Thus S is the internal Zappa-Szép product of L and $\widetilde{R}^1_E$. □

We now examine the actions in the situation where the hypotheses of Theorem 5.4.3 hold. For $r \in \widetilde{R}^1_E$ and $l \in L$ we have

$$rl = (rl)^+rl = dd^\circ rl$$

where $d \in L$. Observe now that $d^\circ rl\,\widetilde{\mathcal{R}}_E\,d^\circ(rl)^+ = d^\circ dd^\circ = d^\circ\widetilde{\mathcal{R}}_E\,1$. It follows that

$$r \cdot l = d \text{ and } r^l = d^\circ rl \text{ where } rl\,\widetilde{\mathcal{R}}_E\,d \in L.$$

We explain these actions with the help of a picture.

1		r	$r^l = d^\circ rl$
l			
$r \cdot l = d$			rl

Figure 5.7: Actions of $\widetilde{R}_E^1$ and L on each other

We can proceed further in Theorem 5.4.3 to decompose $\widetilde{R}_E^1$ as a Zappa-Szép product, under the additional hypothesis that for all $c \in L$ and $e \in E$ we have $cec^\circ, c^\circ ec \in E$. Recall from Theorem 5.3.8 that this guarantees that $R = \{c^\circ : c \in L\}$ is a submonoid $\widetilde{\mathcal{H}}_E$-transversal of $\widetilde{R}_E^1$.

Theorem 5.4.4. *Let S be a weakly E-adequate monoid with (C) such that S is $\widetilde{\mathcal{D}}_E$-simple with $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ and special. Suppose in addition that for all $c \in L$ and $e \in E$ we have $cec^\circ, c^\circ ec \in E$. Then $\widetilde{R}_E^1$ is the internal Zappa-Szép product of $\widetilde{H}_E^1$ and R.*

It follows that $\widetilde{R}_E^1 \cong \widetilde{H}_E^1 \bowtie R$. Further, if $\widetilde{\mathcal{H}}_E$ is a congruence on S, then the action of $\widetilde{H}_E^1$ on R is trivial and $\widetilde{R}_E^1 \cong \widetilde{H}_E^1 \rtimes R$.

Proof. Let $t \in \widetilde{R}_E^1$. For $r \in R$ with $r \,\widetilde{\mathcal{H}}_E\, t$, we have $rr^\circ = 1$ and $r^\circ r = f \in E$ and certainly $f \,\widetilde{\mathcal{L}}_E\, r$. From Lemma 5.1.4, $\rho_r : \widetilde{H}_E^1 \to \widetilde{H}_E^r$ is a bijection. Thus every element of $\widetilde{R}_E^1$ has a unique decomposition as hr for some $h \in \widetilde{H}_E^1$ and $r \in R$, that is, $\widetilde{R}_E^1 = \widetilde{H}_E^1 R$ is the internal Zappa-Szép product of $\widetilde{H}_E^1$ and R.

It follows that $\widetilde{R}_E^1 \cong \widetilde{H}_E^1 \bowtie R$. We now examine the mutual actions of $\widetilde{H}_E^1$ and R. Let $h \in \widetilde{H}_E^1, r \in R$ and let $t \in R$ be such that $rh \,\widetilde{\mathcal{H}}_E\, t$, so that $rh \,\widetilde{\mathcal{L}}_E\, f = t^\circ t$. Then $rh = (rh)f = (rh)(t^\circ t)$ and $rht^\circ \in \widetilde{H}_E^1$, again by Lemma 5.1.4. Hence

$$r \cdot h = rht^\circ \text{ and } r^h = t :$$

1 $\quad h \quad r \cdot h = rht^\circ$		r	$t = r^h$ $\quad rh$	
t°			$t^\circ t$	

Figure 5.8: Decomposition of $\widetilde{R}_E^1$ as a Zappa-Szép product of $\widetilde{H}_E^1$ and R

Finally, if $\widetilde{\mathcal{H}}_E$ is congruence, then $rh \,\widetilde{\mathcal{H}}_E\, r1 = r$, so that $t = r$ and $r^h = r$. $\square$

5.5 Some applications and examples

If S is such that every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element and S has (C), then we have noted in Lemma 5.1.7 that $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$. Moreover, if S is special and restriction, then we immediately see from Lemma 5.1.9 that for all $c \in L$ and $e \in E$ we have $cec^\circ, c^\circ ec \in E$. In particular, if S is an inverse monoid, then certainly with $E = E(S)$, S is restriction, every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element and $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ (since $\widetilde{\mathcal{K}}_E = \mathcal{K}$, for all relevant K). We thus immediately deduce from Theorems 5.4.3 and 5.4.4 the following: notice that we have reverted to the more usual notation of K_a for the $\mathcal{K}$-class of $a \in S$.

Theorem 5.5.1. *Let S be an inverse monoid. Then S is bisimple and special if and only if S is the internal Zappa-Szép product of L and R_1, where L is a submonoid $\mathcal{H}$-transversal of L_1. Moreover, in this case, R_1 is the internal Zappa-Szép product of H_1 and R where $R = \{r^{-1} : r \in L\}$, and is a semidirect product if $\mathcal{H}$ is a congruence.*

Now we deduce [37, Section 5.4].

Corollary 5.5.2. *Let $S = BR(M, \theta)$. Then with*

$$L = \{(n, e, 0) : n \in \mathbb{N}^0\} \text{ and } R = \{(0, e, n) : n \in \mathbb{N}^0\}$$

we have that

$$S \cong \mathbb{N}^0 \bowtie (M \rtimes \mathbb{N}^0).$$

Proof. We have observed that S is restriction and special with L and R as given. Moreover, $S \times S = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{H}}_E$ is a congruence. From Theorems 5.4.3 and 5.4.4 we have $S \cong L \bowtie (\widetilde{H}_E^1 \rtimes R)$ and then as $L \cong \mathbb{N}^0$, $\widetilde{H}_E^1 \cong M$ and $L \cong \mathbb{N}^0$, we deduce the result.

We now consider the relevant actions. For $(n, e, 0) \in L$ and $(0, a, m) \in \widetilde{R}_E^1$, with $k = \max(m, n)$ we have

$$(0, a, m)(n, e, 0) = (k - m, a\theta^{k-m}, k - n)$$

so that from the recipe in Theorem 5.4.3 we have

$$(0, a, m) \cdot (n, e, 0) = (k - m, e, 0) \text{ and } (0, a, m)^{(n,e,0)} = (0, a\theta^{k-m}, k - n).$$

Considering now the action of R on $\widetilde{H}_E^1$ we have

$$(0, e, m) \cdot (0, a, 0) = (0, e, m)(0, a, 0)(m, e, 0) = (0, a\theta^m, 0).$$

Using the natural isomorphisms $(n, e, 0) \mapsto n, (0, e, m) \mapsto m$ and $(0, a, 0) \mapsto a$ we have that $\mathbb{N}^0$ acts on S by

$$m \cdot a = a\theta^m$$

giving the semidirect product $S \rtimes \mathbb{N}^0$ and then $S \rtimes \mathbb{N}^0$ and $\mathbb{N}^0$ act on each other mutually by

$$(a, m) \cdot n = k - m \text{ and } (a, m)^n = (a\theta^{k-m}, k - n).$$

$\square$

Of course, the above can be applied to the bicyclic monoid (with M trivial) or to bisimple inverse ω-semigroups (with M a group).

We have set up the tools for further investigation. A structure theorem for $\widetilde{\mathcal{D}}_E$-simple restriction semigroups with ω-chain of idempotents using "weakly" Bruck-Reilly extensions is proved by S. Ma, X. Ren and Y. Yuan in [50]. It is proved that if S is a $\widetilde{\mathcal{D}}_E$-simple restriction semigroups with ω-chain of idempotents , then $S \cong \mathrm{WBR}(T, \theta)$, where $\mathrm{WBR}(T, \theta)$ is a "weakly" Bruck-Reilly extension of a monoid T where $T \cong \widetilde{H}_E^e$. Their result generalises the structure theorem of $*$-bisimple type A ω-semigroups given by U. Asibong-Ibe in 1985 [3].

Chapter 6

λ-Semidirect products of restriction semigroups

The semidirect product of two inverse semigroups is not inverse in general [60]. Billhardt showed how to modify the definition of a semidirect product of two inverse semigroups to obtain what he termed as a λ-*semidirect product* [4]. The λ-semidirect product of two inverse semigroups is again inverse. This result was generalised to the left ample case, again by Billhardt, where one component is a semilattice [5] (but he indicated that it would work for any two left ample semigroups). Again in this special case, this was extended further to the λ-semidirect product of a semilattice and a left restriction semigroup by Branco, Gomes and Gould [6]. Indeed [6] considers a yet more general λ-semidirect product, with one component a left regular band.

In this chapter following Billhardt's construction, we consider the λ-*semidirect product* of two left restriction semigroups and prove that it is left restriction.

In the two sided case, we show that if a restriction semigroup is acting *doubly* on another, then the λ-semidirect product is again restriction. We provide two proofs of this result. First a direct proof and second, by constructing an inductive category and then equipping it with the standard pseudo-product.

In the first section we provide background for λ-semidirect products and the work done so far regarding the λ-semidirect product of inverse, left ample and left restriction semigroups. We consider the λ-semidirect product of two arbitrary left restriction semigroups in the second section and prove that it is again left restriction. We then consider the two-sided case and by using the notion of double action prove that the λ-semidirect product of two restriction semigroups is again restriction. In the final section of this chapter, we consider the

λ-semidirect product of two restriction semigroups and give it the structure of an inductive category. By defining a pseudo-product on this inductive category, we obtain a restriction semigroup in a standard way.

6.1 λ-semidirect product of inverse, left ample and left restriction semigroups

We begin by defining the notion of the λ-semidirect product of two inverse semigroups.

Definition 6.1.1. [4] Let S and T be inverse semigroups and suppose that T acts on the left of S by endomorphisms. Put

$$S \rtimes^{\lambda} T = \{(a, t) \in S \times T : tt^{-1} \cdot a = a\}.$$

Define a binary operation on $S \rtimes^{\lambda} T$ by

$$(a, t)(b, u) = \Big(((tu)(tu)^{-1} \cdot a)(t \cdot b), tu\Big).$$

We say that $S \rtimes^{\lambda} T$ is the λ-*semidirect product* of S by T.

We will see later that this is a special case of the λ-semidirect product of left restriction semigroups. The following result is the consequence of Definition 6.1.1

Proposition 6.1.2. [4] *Let S and T be inverse semigroups such that T acts on the left of S by endomorphisms. Then $S \rtimes^{\lambda} T$ is an inverse semigroup. If in addition, S and T are monoids, then $S \rtimes^{\lambda} T$ is an inverse monoid with identity* $(1, 1)$.

The following proposition provides two special cases of this construction.

Proposition 6.1.3. [45] *Let S and T be inverse semigroups such that T acts on the left of S by endomorphisms.*

(1) If T is a group, then $S \rtimes^{\lambda} T = S \rtimes T$ is the semidirect product of S by T.

(2) If S and T are both groups, then $S \rtimes^{\lambda} T = S \rtimes T$ is a group.

In [77], Wazzan gave an alternative proof of Proposition 6.1.2, by using the correspondence in the Ehresmann-Schein-Nambooripad Theorem [Chapter 1] between inverse semigroups and inductive groupoids.

Theorem 6.1.4. [77] *Let S and T be inverse semigroups and let $P = S \rtimes T$ be the semidirect product of S by T. Then*

$$B(P) = \{(a,t) \in S \times T : tt^{-1} \cdot a = a\}$$

is an inductive groupoid under the restriction of binary operation in P where

$$\mathbf{d}(a,t) = (aa^{-1}, tt^{-1}), \mathbf{r}(a,t) = (t^{-1} \cdot a^{-1}a, t^{-1}t)$$

and with partial order on $B(P)$ defined by for $(a,t),(b,u) \in B(P)$,

$$(a,t) \leq (b,u) \text{ if and only if } a \leq tt^{-1} \cdot b \text{ and } t \leq u.$$

The set of local identities is then

$$E_{B(P)} = \{(e,f) \in P : e^2 = e, f^2 = f, f \cdot e = e\}$$

and for $(a,t) \in B(P)$,

$$(a,t)^{-1} = (t^{-1} \cdot a^{-1}, t^{-1}).$$

The restriction and co-restriction are defined for $(a,t) \in B(P)$, $(e,f) \in E_{B(P)}$ by

$${}_{(e,f)}|(a,t) = (e(f \cdot a), ft)$$

and

$$(a,t)|_{(e,f)} = ((tft^{-1} \cdot a)(tf \cdot e), tf).$$

Further, under the standard pseudo-product, $B(P) = S \rtimes^{\lambda} T$.

Billhardt later extended his construction, in the case where one component was a semilattice, to left ample semigroups [5]. This was extended further to the λ-semidirect product of a semilattice and a left restriction semigroup by Branco, Gomes and Gould [6].

Definition 6.1.5. Let T be a left restriction semigroup. Suppose T acts on a semigroup S by endomorphisms. Put

$$S \rtimes^{\lambda} T = \{(a,t) \in S \times T : t^{+} \cdot a = a\}.$$

Define a binary operation on $S \rtimes^{\lambda} T$ by

$$(a,t)(b,u) = (((tu)^{+} \cdot a)(t \cdot b), tu).$$

We say that $S \rtimes^{\lambda} T$ is the λ-*semidirect product* of S by T.

Lemma 6.1.6. *Let T be a left restriction semigroup such that T acts on a semigroup S on the left by endomorphisms. Then*

$$S \rtimes^{\lambda} T = \{(a,t) \in S \times T : \, t^{+} \cdot a = a\}$$

is a semigroup.

Proof. Let (a,t), $(b,u) \in S \rtimes^{\lambda} T$. We first show that $(a,t)(b,u) = ((tu)^{+} \cdot a)(t \cdot b), tu) \in S \rtimes^{\lambda} T$. We see that

$$\begin{aligned}
(tu)^{+} \cdot (((tu)^{+} \cdot a)(t \cdot b)) &= ((tu)^{+} \cdot ((tu)^{+} \cdot a))((tu)^{+} \cdot (t \cdot b)) \\
&= ((tu)^{+}(tu)^{+} \cdot a\big)((tu)^{+} t \cdot b) \\
&= ((tu)^{+} \cdot a)(tu^{+} \cdot b) \qquad \text{using the left ample condition} \\
&= ((tu)^{+} \cdot a)(t \cdot (u^{+} \cdot b)) \\
&= ((tu)^{+} \cdot a)(t \cdot b).
\end{aligned}$$

Thus multiplication is closed.

Now to check associativity, let (a,t), (b,u), $(c,v) \in S \rtimes^{\lambda} T$. Then

$$\begin{aligned}
\big((a,t)(b,u)\big)(c,v) &= \big(((tu)^{+} \cdot a)(t \cdot b), tu\big)(c,v) \\
&= \Big(\big((tuv)^{+} \cdot (((tu)^{+} \cdot a)(t \cdot b))\big)(tu \cdot c), (tu)v\Big) \\
&= \big(((tuv)^{+} \cdot ((tu)^{+} \cdot a)))((tuv)^{+} \cdot (t \cdot b))(tu \cdot c), t(uv)\big) \\
&= \big(((tuv)^{+}(tu)^{+} \cdot a)((t(uv)^{+})^{+} t \cdot b)(tu \cdot c), t(uv)\big) \\
&= \big(((tuv)^{+} \cdot a)(t(uv)^{+} \cdot b)(tu \cdot c), t(uv)\big) \\
&= \big(((t(uv))^{+} \cdot a)(t \cdot (((uv)^{+} \cdot b)(u \cdot c)), t(uv)\big) \\
&= (a,t)\big(((uv)^{+} \cdot b)(u \cdot c), uv\big) \\
&= (a,t)\big((b,u)(c,v)\big).
\end{aligned}$$

Thus the associative law holds and hence $S \rtimes^{\lambda} T$ is a semigroup. □

Proposition 6.1.7. [6] *Let E be a semilattice and T be a left restriction semigroup such that*

T acts on the left of E by endomorphisms. Then $E \rtimes^{\lambda} T$ is a left restriction semigroup with

$$(e,t)^+ = (e,t^+).$$

Then

$$E_{S\rtimes^{\lambda}T} = \{(e,t^+) \in E \times T : t^+ \cdot e = e\}$$

is the semilattice of projections of $S \rtimes^{\lambda} T$.

We extend the above in two ways. First, we consider the λ-semidirect product of arbitrary left restriction semigroups and prove that it is again left restriction. Using the notion of double actions taken from [24], we then introduce the λ-semidirect product of (two-sided) restriction semigroups. We give a direct argument that our construction yields a restriction semigroup. Then following Wazzan's technique we construct an inductive category and obtain the corresponding restriction semigroup.

6.2 λ-semidirect product of restriction semigroups

In this section we first consider the λ-semidirect product of two left restriction semigroups.

Theorem 6.2.1. *Let S and T be left restriction semigroups such that T acts on the left of S by endomorphisms of S regarded as a left restriction semigroup. Then*

$$S \rtimes^{\lambda} T = \{(a,t) \in S \times T :\ t^+ \cdot a = a\}$$

is left restriction under the binary operation defined by

$$(a,t)(b,u) = (((tu)^+ \cdot a)(t \cdot b), tu)$$

where $(a,t)^+ = (a^+, t^+)$ and

$$F = \{(a^+,t^+) : t^+ \cdot a^+ = a^+\}.$$

is the semilattice of projections of $S \rtimes^{\lambda} T$.

Proof. Note that by hypothesis,

$$t \cdot a^+ = (t \cdot a)^+ \quad \text{for any } t \in T \text{ and } a \in S.$$

Now if $(a,t) \in S \rtimes^{\lambda} T$, then

$$t^+ \cdot a^+ = (t^+ \cdot a)^+ = a^+,$$

so that $F \subseteq S \rtimes^{\lambda} T$. From Lemma 6.1.6, we know that the binary operation is associative. We now check the four identities that define a left restriction semigroup hold. For this let $(a,t) \in S \rtimes^{\lambda} T$. Then

$$\begin{aligned}
(a,t)^+(a,t) &= (a^+,t^+)(a,t) \\
&= \Big(((t^+t)^+ \cdot a^+)(t^+ \cdot a), t^+t\Big) \\
&= ((t^+ \cdot a^+)a, t) && \text{because } t^+ \cdot a = a \\
&= (a^+a, t) && \text{because } t^+ \cdot a^+ = a^+ \\
&= (a,t).
\end{aligned}$$

Next let $(a,t),(b,u) \in S \rtimes^{\lambda} T$. Then

$$\begin{aligned}
(a,t)^+(b,u)^+ &= (a^+,t^+)(b^+,u^+) \\
&= \Big((t^+u^+)^+ \cdot a^+)(t^+ \cdot b^+), t^+u^+\Big) \\
&= \Big((t^+u^+ \cdot a^+)(t^+ \cdot b^+), t^+u^+\Big) \\
&= \Big((u^+ \cdot (t^+ \cdot a^+))(t^+ \cdot b^+), t^+u^+\Big) \\
&= \Big((u^+ \cdot a^+)(t^+ \cdot b^+), t^+u^+\Big) && \text{because } t^+ \cdot a^+ = a^+ \\
&= (b,u)^+(a,t)^+,
\end{aligned}$$

the last step by symmetry.

Next we see that

$$\begin{aligned}
\Big((a,t)^+(b,u)\Big)^+ &= \Big((a^+,t^+)(b,u)\Big)^+ \\
&= \Big(((t^+u)^+ \cdot a^+)(t^+ \cdot b), t^+u\Big)^+ \\
&= \Big((t^+u^+ \cdot a^+)(t^+ \cdot b), t^+u\Big)^+ \\
&= \Big(((t^+u^+ \cdot a^+)(t^+ \cdot b))^+, (t^+u)^+\Big) \\
&= \Big((t^+u^+ \cdot a^+)(t^+ \cdot b^+), t^+u^+\Big) \\
&= (a^+,t^+)(b^+,u^+) \\
&= (a,t)^+(b,u)^+.
\end{aligned}$$

Finally we show that the left ample condition holds. For this we see that

$$\begin{aligned}
\Big((a,t)(b,u)\Big)^+(a,t) &= \Big(((tu)^+\cdot a)(t\cdot b), tu\Big)^+(a,t) \\
&= \Big((((tu)^+\cdot a)(t\cdot b))^+, (tu)^+\Big)(a,t) \\
&= \Big(\big((tu)^+t)^+\cdot(((tu)^+\cdot a)(t\cdot b))^+\big)((tu)^+\cdot a), (tu)^+t\Big) \\
&= \Big(\big((tu)^+\cdot(((tu)^+\cdot a)(t\cdot b))^+\big)((tu)^+\cdot a), tu^+\Big) && \text{as } (tu)^+t = tu^+ \text{ in } T \\
&= \Big((((tu)^+\cdot(((tu)^+\cdot a)(t\cdot b)))^+((tu)^+\cdot a), tu^+\Big) && \text{using } t\cdot a^+ = (t\cdot a)^+ \\
&= \Big((((tu)^+\cdot((tu)^+\cdot a))((tu)^+\cdot(t\cdot b)))^+((tu)^+\cdot a), tu^+\Big) \\
&= \Big((((tu)^+\cdot a)((tu)^+t\cdot b))^+((tu)^+\cdot a), tu^+\Big) \\
&= \Big((((tu)^+\cdot a)(tu^+\cdot b))^+((tu)^+\cdot a), tu^+\Big) && \text{as } (tu)^+t = tu^+ \text{ in } T \\
&= \Big(((tu)^+\cdot a)(tu^+\cdot b)^+, tu^+\Big) && \text{as } (rs)^+r = rs^+ \text{ in } S \\
&= \Big(((tu)^+\cdot a)(tu^+\cdot b^+), tu^+\Big) \\
&= (a,t)(b^+,u^+) && \text{because } u^+\cdot b^+ = b^+ \\
&= (a,t)(b,u)^+.
\end{aligned}$$

Thus all the four identities hold and hence $S \rtimes^\lambda T$ is left restriction with $(a,t)^+ = (a^+,t^+)$.

It is clear that

$$F = \{(a^+,t^+) : t^+\cdot a^+ = a^+\}$$

is the semilattice of projections of $S \rtimes^\lambda T$. □

Notice that in the above result,

$$\begin{aligned}
(a^+,t^+)(b^+,u^+) &= (((t^+u^+)^+\cdot a^+)(t^+\cdot b^+), t^+u^+) \\
&= ((t^+u^+\cdot a^+)(t^+\cdot(u^+\cdot b^+)), t^+u^+) && \text{because } b^+ = u^+\cdot b^+ \\
&= ((t^+u^+\cdot a^+)(t^+u^+\cdot b^+), t^+u^+) \\
&= (t^+u^+\cdot a^+b^+, t^+u^+),
\end{aligned}$$

so that $(a^+,t^+) \leq (b^+,u^+)$ if and only if

$$\begin{aligned}
& (a^+,t^+)(b^+,u^+) = (a^+,t^+) \\
\Leftrightarrow\ & (t^+u^+\cdot a^+b^+, t^+u^+) = (a^+,t^+) \\
\Leftrightarrow\ & t^+u^+\cdot a^+b^+ = a^+ \text{ and } t^+u^+ = t^+ \\
\Leftrightarrow\ & t^+\cdot a^+b^+ = a^+ \text{ and } t^+u^+ = t^+ \\
\Leftrightarrow\ & (t^+\cdot a^+)(t^+\cdot b^+) = a^+ \text{ and } t^+u^+ = t^+ \\
\Leftrightarrow\ & a^+(t^+\cdot b^+) = a^+ \text{ and } t^+u^+ = t^+ \\
\Leftrightarrow\ & a^+ \leq t^+\cdot b^+ \text{ and } t^+ \leq u^+.
\end{aligned}$$

In Theorem 6.2.1, we notice that F has an alternative description as

$$F = \{(f \cdot e, f) : f \in E_T, e \in E_S\},$$

for, if $(a^+, t^+) \in F$, we have $t^+ \cdot a^+ = a^+$. So

$$(a^+, t^+) = (t^+ \cdot a^+, t^+).$$

On the other hand, for $f = f^+ \in T$ and $e = e^+ \in S$, we have

$$f \cdot e = f^+ \cdot e^+ = (f^+ \cdot e)^+$$

and clearly $f^+ \cdot (f \cdot e) = f \cdot e$. Hence $(f \cdot e, f) \in F$.

6.2.1 Two sided case

We now consider the λ-semidirect product of two restriction semigroups. We have seen that the λ-semidirect product of two left restriction semigroups is again left restriction, but in general it is not easy to check that the λ-semidirect product of two restriction semigroups is also *right* restriction. We use the notion of double action introduced in [24] to determine the structure of the free ample monoids to obtain a right restriction semigroup from the λ-semidirect product of two right restriction semigroups,.

Definition 6.2.2. Let S and T be restriction semigroups. Then T acts *doubly* on S if T acts by restriction semigroup morphisms on the left and right of S satisfying the following compatibility condition:

$$\begin{aligned} (t \cdot s) \circ t = s \circ t^* &= t^* \cdot s \\ t \cdot (s \circ t) = s \circ t^+ &= t^+ \cdot s \end{aligned} \tag{CP1}$$

for all $s \in S$ and $t \in T$.

We notice that for all $s \in S$ and for all $f \in E_T$,

$$f \cdot s = s \circ f.$$

Theorem 6.2.3. *Let S and T be restriction semigroups. Suppose T acts doubly on S. Then*

$$S \rtimes^{\lambda} T = \{(a, t) \in S \times T : \ t^+ \cdot a = a\}$$

is a restriction semigroup where

$$(a,t)^+ = (a^+, t^+) \text{ and } (a,t)^* = (a^* \circ t, t^*);$$

further,

$$F = \{(a^+, t^+) : t^+ \cdot a^+ = a^+\}$$

is the semilattice of projections of $S \rtimes^\lambda T$.

Proof. From Theorem 6.2.1, we know that $S \rtimes^\lambda T$ is a left restriction semigroup with

$$(a,t)^+ = (a^+, t^+).$$

We now prove that $S \rtimes^\lambda T$ is also right restriction.

We first notice that for $(a,t) \in S \rtimes^\lambda T$, $(a^* \circ t, t^*) \in F$ as

$$\begin{aligned}
t^* \cdot (a^* \circ t) &= (a^* \circ t) \circ t^* \quad \text{because } f \cdot s = s \circ f \text{ for all } s \in S \text{ and } f \in E_T \\
&= a^* \circ tt^* \\
&= a^* \circ t.
\end{aligned}$$

Also we note that $((a,t)^+)^* = (a,t)^+$ and $((a,t)^*)^+ = (a,t)^*$, as

$$\begin{aligned}
((a,t)^+)^* &= (a^+, t^+)^* \\
&= ((a^+)^* \circ t^+, (t^+)^*) \\
&= (a^+ \circ t^+, t^+) && \text{because } S \text{ and } T \text{ are restriction} \\
&= (t^+ \cdot a^+, t^+) && \text{using (CP1)} \\
&= (a^+, t^+) && \text{because } t^+ \cdot a^+ = a^+ \\
&= (a,t)^+,
\end{aligned}$$

and

$$\begin{aligned}
((a,t)^*)^+ &= (a^* \circ t, t^*)^+ \\
&= ((a^* \circ t)^+, (t^*)^+) \\
&= (a^* \circ t, t^*) && \text{because } a^* \circ t \in E_S \text{ and } t^* \in E_T \\
&= (a,t)^*.
\end{aligned}$$

Next we check that the four identities that define a right restriction semigroup hold. Let

$(a,t) \in S \rtimes^{\lambda} T$. Then

$$\begin{aligned}
(a,t)(a,t)^* &= (a,t)(a^* \circ t, t^*) \\
&= \Big(((tt^*)^+ \cdot a)(t \cdot (a^* \circ t), tt^*\Big) \\
&= \Big((t^+ \cdot a)(t^+ \cdot a^*), t\Big) && \text{using (CP1)} \\
&= \Big(a(t^+ \cdot a)^*, t\Big) && \text{because action preserves }^* \\
&= (aa^*, t) \\
&= (a,t).
\end{aligned}$$

Next let $(a,t),(b,u) \in S \rtimes^{\lambda} T$. Then clearly $(a,t)^*(b,u)^* = (b,u)^*(a,t)^*$, because F is a semilattice.

Next we see that

$$\begin{aligned}
\Big((a,t)(b,u)^*\Big)^* &= \Big((a,t)(b^* \circ u, u^*)\Big)^* \\
&= \Big(((tu^*)^+ \cdot a)(t \cdot (b^* \circ u)), tu^*\Big)^* \\
&= \Big((((tu^*)^+ \cdot a)(t \cdot (b^* \circ u)))^* \circ tu^*, (tu^*)^*\Big) \\
&= \Big((((tu^*)^+ \cdot a)^*(t \cdot (b^* \circ u)))^* \circ tu^*, t^*u^*\Big) \\
&= \Big((((tu^*)^+ \cdot a^*)(t \cdot (u^* \cdot (b^* \circ u))))^* \circ tu^*, t^*u^*\Big) \\
&= \Big(((tu^* \cdot (a^* \circ tu^*))(tu^* \cdot (b^* \circ u)))^* \circ tu^*, t^*u^*\Big) && \text{using (CP1)} \\
&= \Big(((tu^* \cdot ((a^* \circ t) \circ u^*))(tu^* \cdot (b^* \circ u)))^* \circ tu^*, t^*u^*\Big) \\
&= \Big(((tu^* \cdot (u^* \cdot (a^* \circ t))(tu^* \cdot (b^* \circ u)))^* \circ tu^*, t^*u^*\Big) && \text{using (CP1)} \\
&= \Big(((tu^* \cdot (a^* \circ t))(tu^* \cdot (b^* \circ u)))^* \circ tu^*, t^*u^*\Big) \\
&= \Big(((tu^* \cdot ((a^* \circ t)(b^* \circ u)))^* \circ tu^*, t^*u^*\Big) \\
&= \Big(((tu^* \cdot ((a^* \circ t)(b^* \circ u))^*) \circ tu^*, t^*u^*\Big) && \text{because action preserves }^* \\
&= \Big(((tu^*)^* \cdot ((a^* \circ t)(b^* \circ u)), t^*u^*\Big) && \text{using (CP1)} \\
&= \Big((t^*u^* \cdot ((a^* \circ t)(b^* \circ u)), t^*u^*\Big) \\
&= \Big(((t^*u^* \cdot (a^* \circ t))(t^*u^* \cdot (b^* \circ u)), t^*u^*\Big) \\
&= \Big((t^*u^* \cdot (a^* \circ t))(t^* \cdot (u^* \cdot (b^* \circ u))), t^*u^*\Big) \\
&= \Big((t^*u^* \cdot (a^* \circ t))(t^* \cdot ((b^* \circ u) \circ u^*)), t^*u^*\Big) && \text{using (CP1)} \\
&= \Big((t^*u^*) \cdot (a^* \circ t))(t^* \cdot (b^* \circ uu^*)), t^*u^*\Big) \\
&= \Big((t^*u^* \cdot (a^* \circ t))(t^* \cdot (b^* \circ u)), t^*u^*\Big) \\
&= (a^* \circ t, t^*)(b^* \circ u, u^*) \\
&= (a,t)^*(b,u)^*.
\end{aligned}$$

Finally to check the right ample condition, let $(a,t),(b,u) \in S \rtimes^{\lambda} T$. Then

$$
\begin{aligned}
(a,t)\big((b,u)(a,t)\big)^* &= (a,t)\big(((ut)^+\cdot b)(u\cdot a), ut\big)^* \\
&= (a,t)\Big(\big(((ut)^+\cdot b)(u\cdot a)\big)^* \circ ut, (ut)^*\Big) \\
&= \Big(((t(ut)^*)^+\cdot a)\big(t\cdot\big((((ut)^+\cdot b)(u\cdot a))^*\circ ut\big)\big), t(ut)^*\Big) \\
&= \Big(((u^*t)^+\cdot a)\big(t\cdot\big(((((ut)^+\cdot b)(u\cdot a))^*\circ u)\circ t\big)\big), u^*t\Big) && \text{because } T \text{ is right restriction} \\
&= \Big((u^*t^+\cdot a)\big(t^+\cdot\big(((((ut)^+\cdot b)(u\cdot a))^*\circ u)\big)\big), u^*t\Big) && \text{using (CP1)} \\
&= \Big((u^*\cdot(t^+\cdot a))\big(((((ut)^+\cdot b)(u\cdot a))^*\circ u)\circ t^+\big), u^*t\Big) && \text{using (CP1)} \\
&= \Big((u^*\cdot a)\big((((ut)^+\cdot b)(u\cdot a))^*\circ ut^+\big), u^*t\Big) \\
&= \Big((u^*\cdot a)\big((((ut)^+\cdot b)^*(u\cdot a))^*\circ ut^+\big), u^*t\Big) && \text{because } S \text{ is right restriction} \\
&= \Big((u^*\cdot a)\big((((ut)^+\cdot b^*)(u\cdot a))^*\circ ut^+\big), u^*t\Big) && \text{because action preserves } {}^* \\
&= \Big((u^*\cdot a)\big(((ut\cdot(b^*\circ ut))(u\cdot a))^*\circ ut^+\big), u^*t\Big) && \text{using (CP1)} \\
&= \Big((u^*\cdot a)\big(((u\cdot(t\cdot((b^*\circ u)\circ t)))(u\cdot a))^*\circ ut^+\big), u^*t\Big) \\
&= \Big((u^*\cdot a)\big(((u\cdot(t^+\cdot(b^*\circ u)))(u\cdot a))^*\circ ut^+\big), u^*t\Big) && \text{using (CP1)} \\
&= \Big((u^*\cdot a)\big(((u\cdot(b^*\circ ut^+))(u\cdot a))^*\circ ut^+\big), u^*t\Big) \\
&= \Big((u^*\cdot a)\big(((u\cdot((b^*\circ ut^+)a))^*\circ ut^+\big), u^*t\Big) \\
&= \Big((u^*\cdot a)\big((u\cdot((b^*\circ ut^+)a)^*)\circ ut^+\big), u^*t\Big) \\
&= \Big((u^*\cdot a)\big(((u\cdot((b^*\circ ut^+)a)^*)\circ u)\circ t^+\big), u^*t\Big) \\
&= \Big((u^*\cdot a)\big((u^*\cdot((b^*\circ ut^+)a)^*)\circ t^+\big), u^*t\Big) && \text{using (CP1)} \\
&= \Big((u^*\cdot a)\big(t^+\cdot(u^*\cdot((b^*\circ ut^+)a)^*)\big), u^*t\Big) && \text{using (CP1)} \\
&= \Big((u^*\cdot a)\big(t^+u^*\cdot((b^*\circ ut^+)a)^*\big), u^*t\Big) \\
&= \Big((u^*\cdot a)\big(u^*t^+\cdot((b^*\circ ut^+)a)\big)^*, u^*t\Big) \\
&= \Big((u^*\cdot a)\big((u^*t^+\cdot(b^*\circ ut^+))(u^*t^+\cdot a)\big)^*, u^*t\Big) \\
&= \Big((u^*\cdot a)\big(((u^*t)^+\cdot(b^*\circ ut^+))(u^*\cdot(t^+\cdot a))\big)^*, u^*t\Big) \\
&= \Big((u^*\cdot a)\big(((u^*t^+\cdot(b^*\circ ut^+))(u^*\cdot a)\big)^*, u^*t\Big) && \text{because } t^+\cdot a = a \\
&= \Big((u^*\cdot a)\big(((u^*t^+\cdot((b^*\circ u)\circ t^+)))(u^*\cdot a)\big)^*, u^*t\Big) \\
&= \Big((u^*\cdot a)\big(((u^*t^+\cdot(t^+\cdot(b^*\circ u)))(u^*\cdot a)\big)^*, u^*t\Big) && \text{using (CP1)} \\
&= \Big((u^*\cdot a)\big(((u^*t^+\cdot(b^*\circ u))(u^*\cdot a)\big)^*, u^*t\Big) \\
&= \Big((u^*t^+\cdot(b^*\circ u))^*(u^*\cdot a), u^*t\Big) && \text{because } S \text{ is right restriction} \\
&= \Big(((u^*t)^+\cdot(b^*\circ u)^*)(u^*\cdot a), u^*t\Big) \\
&= \Big(((u^*t)^+\cdot(b^*\circ u))(u^*\cdot a), u^*t\Big) \\
&= (b^*\circ u, u^*)(a,t) \\
&= (b,u)^*(a,t).
\end{aligned}
$$

Thus all the four identities hold and hence $S \rtimes^{\lambda} T$ is restriction. $\square$

Now we check that (CP1) holds in the following example.

Example 6.2.4. Let T be a restriction semigroup and let $S = E_T = E$. Suppose T acts on the left and right of S by

$$t \cdot e = (te)^+$$

and

$$e \circ t = (et)^*$$

It is shown in [24] that T acts on E by morphisms. We check that (CP1) holds. For this let $t \in T$ and $e \in E$, then

$$(t \cdot e) \circ t = (te)^+ \circ t = ((te)^+ t)^* = (te)^*.$$

Also

$$e \circ t^* = (et^*)^* = (t^* e)^* = (te)^*$$

and

$$t^* \cdot e = (t^* e)^+ = ((te)^*)^+ = (te)^*.$$

Hence we see that

$$(t \cdot e) \circ t = e \circ t^* = t^* \cdot e.$$

Dually it is easy to check that

$$t \cdot (e \circ t) = e \circ t^+ = t^+ \cdot e$$

and hence (CP1) holds.

6.3 Inductive categories and restriction semigroups

In this section we provide another proof that $S \rtimes^{\lambda} T$ is a restriction semigroup. We consider the semidirect product of two restriction semigroups and from the underlying set of $S \rtimes^{\lambda} T$ we construct a category $\mathbf{C}$. We then order this category so that it becomes inductive which yields a restriction semigroup via the use of the standard pseudo-product.

We first prove a result that is needed later to show that $\mathbf{C}$ is an inductive category.

Lemma 6.3.1. *Let T be a semigroup acting on the left of a left restriction semigroup S by endomorphisms of S regarded as a left restriction semigroup. Then the action of T preserves $\leq$ on S (and hence on E_S).*

Proof. Let $a, b \in S$ and $t \in T$ and let $a \leq b$. Then $a = a^+b$, so that

$$\begin{array}{rrcll} & t \cdot a & = & t \cdot (a^+b) & \\ \Rightarrow & t \cdot a & = & (t \cdot a^+)(t \cdot b) & \text{because action is by morphisms} \\ & & = & (t \cdot a)^+(t \cdot b) & \text{because action preserves }^+ \\ \Rightarrow & t \cdot a & \leq & t \cdot b. & \end{array}$$

□

Theorem 6.3.2. *Let S and T be restriction semigroups. Suppose T acts doubly on S satisfying (CP1) and let $U = S \rtimes T$. Let*

$$C = S \rtimes^{\lambda} T = \{(a, t) \in S \times T : t^+ \cdot a = a\}.$$

Then $\mathbf{C} = (C, \bullet, \mathbf{d}, \mathbf{r})$ is a category under the restriction of the binary operation in U with set of local identities

$$E_{\mathbf{C}} = \{(f \cdot e, f) : f \in E_T, e \in E_S\}$$

where

$$\mathbf{d}(a, t) = (a^+, t^+) \text{ and } \mathbf{r}(a, t) = (a^* \circ t, t^*).$$

Proof. We refer to $\bullet$ as the *restricted product* in C:

$$(a, t) \bullet (b, u) = (a(t \cdot b), tu) \text{ if } \mathbf{r}(a, t) = \mathbf{d}(b, u).$$

We first check that $\bullet$ is a closed binary operation. For this let $(a, t), (b, u) \in C$ such that $(a, t) \bullet (b, u)$ is defined. Then

$$(a^* \circ t, t^*) = (b^+, u^+).$$

We see that

$$\begin{array}{rcll} (tu)^+ \cdot (a(t \cdot b)) & = & \big((tu)^+ \cdot a\big)\big((tu)^+ \cdot (t \cdot b)\big) & \\ & = & \big((tt^*)^+ \cdot a\big)\big((tu)^+ t \cdot b)\big) & \text{because } t^* = u^+ \\ & = & (t^+ \cdot a)(tu^+ \cdot b) & \text{because } T \text{ is restriction} \\ & = & a(t \cdot (u^+ \cdot b)) & \text{because } t^+ \cdot a = a \\ & = & a(t \cdot b) & \text{because } u^+ \cdot b = b. \end{array}$$

Hence $(a,t)\bullet(b,u)\in C$.

Next we note that C is closed under $\mathbf{d}$ and $\mathbf{r}$, for if $(a,t)\in C$, then

$$t^+\cdot a^+ = (t^+\cdot a)^+ = a^+,$$

so that $\mathbf{d}(a,t)=(a^+,t^+)\in C$;

further,

$$\begin{aligned} t^*\cdot(a^*\circ t) &= (a^*\circ t)\circ t^* \quad \text{because } e\cdot s = s\circ e \text{ for all } s\in S \text{ and } e\in E_T \\ &= a^*\circ tt^* \\ &= a^*\circ t, \end{aligned}$$

so that $\mathbf{r}(a,t)=(a^*\circ t, t^*)\in C$.

It is easy to check that for any $(a,t)\in C$,

$$\mathbf{d}(\mathbf{r}(a,t)) = \mathbf{r}(a,t) \text{ and } \mathbf{r}(\mathbf{d}(a,t)) = \mathbf{d}(a,t)$$

and

$$\operatorname{im}\mathbf{d} = \operatorname{im}\mathbf{r} = \{(f\cdot e, f) : f\in E_T, e\in E_S\}.$$

Now we show that C satisfies the axioms of a category.

(C1) Let $(a,t)\,,\,(b,u)\in C$ and suppose $\exists\ (a,t)\bullet(b,u)$. Then $\mathbf{r}(a,t)=\mathbf{d}(b,u)$ so that

$$(a^*\circ t, t^*) = (b^+, u^+).$$

Therefore

$$\begin{aligned} \mathbf{d}((a,t)\bullet(b,u)) &= \mathbf{d}(a(t\cdot b), tu) \\ &= \Big((a(t\cdot b))^+, (tu)^+\Big) \\ &= \Big((a(t\cdot b)^+)^+, (tu^+)^+\Big) \\ &= \Big((a(t\cdot b^+))^+, (tt^*)^+\Big) \qquad \text{because } t^* = u^+ \\ &= \Big((a(t\cdot(a^*\circ t)))^+, t^+\Big) \qquad \text{because } a^*\circ t = b^+ \\ &= \Big(((t^+\cdot a)(t^+\cdot a^*))^+, t^+\Big) \quad \text{using (CP1) and the fact } t^+\cdot a = a \\ &= \Big((t^+\cdot(aa^*))^+, t^+\Big) \\ &= (a^+, t^+) \\ &= \mathbf{d}(a,t) \end{aligned}$$

and thus $\mathbf{d}((a,t)\bullet(b,u)) = \mathbf{d}(a,t)$. Now

$$\begin{aligned}
\mathbf{r}((a,t)\bullet(b,u)) &= \mathbf{r}(a(t\cdot b), tu) \\
&= \Big((a(t\cdot b))^* \circ tu, (tu)^*\Big) \\
&= \Big((a^*(t\cdot b))^* \circ tu, (t^*u)^*\Big) \\
&= \Big(((t^+\cdot a)^*(t\cdot b))^* \circ tu, (t^*u)^*\Big) && \text{because } t^+\cdot a = a \text{ and } t^* = u^+ \\
&= \Big(\big((t^+\cdot a^*)(t\cdot b)\big)^* \circ tu, (u^+u)^*\Big) \\
&= \Big(\big((t\cdot(a^*\circ t))(t\cdot b)\big)^* \circ tu, u^*\Big) && \text{using (CP1)} \\
&= \Big(\big(t\cdot((a^*\circ t)b)\big)^* \circ tu, u^*\Big) \\
&= \Big((t\cdot(b^+b))^* \circ tu, u^*\Big) && \text{because } a^*\circ t = b^+ \\
&= \Big((t\cdot b)^* \circ tu, u^*\Big) \\
&= \Big((t\cdot b^*) \circ tu, u^*\Big) \\
&= \Big(((t\cdot b^*)\circ t) \circ u, u^*\Big) \\
&= \Big((t^*\cdot b^*) \circ u, u^*\Big) && \text{using (CP1)} \\
&= \Big((u^+\cdot b^*) \circ u, u^*\Big) && \text{because } t^* = u^+ \\
&= \Big((u^+\cdot b)^* \circ u, u^*\Big) \\
&= \Big(b^* \circ u, u^*\Big) && \text{because } u^+\cdot b = b \text{ as } (b,u)\in C \\
&= \mathbf{r}(b,u)
\end{aligned}$$

and hence $\mathbf{r}((a,t)\bullet(b,u)) = \mathbf{r}(b,u)$.

(C2) This directly follows from (C1) and the fact that multiplication in a semidirect product is associative.

(C3) Let $(a,t)\in C$. Then $\mathbf{r}(\mathbf{d}(a,t)) = \mathbf{d}(a,t)$. Thus $\exists\ \mathbf{d}(a,t)\bullet(a,t)$ and

$$\begin{aligned}
\mathbf{d}(a,t)\bullet(a,t) &= (a^+, t^+)\bullet(a,t) \\
&= (a^+(t^+\cdot a), t^+t) \\
&= (a^+a, t) \\
&= (a,t).
\end{aligned}$$

Also $\mathbf{d}(\mathbf{r}(a,t)) = \mathbf{r}(a,t)$. Thus $\exists\ (a,t) \bullet \mathbf{r}(a,t)$ and

$$\begin{array}{rcll} (a,t) \bullet \mathbf{r}(a,t) & = & (a,t) \bullet (a^* \circ t, t^*) & \\ & = & (a(t \cdot (a^* \circ t))), tt^*) & \\ & = & (a(t^+ \cdot a^*), t) & \text{using (CP1)} \\ & = & (a(t^+ \cdot a)^*, t) & \\ & = & (aa^*, t) & \\ & = & (a,t). & \end{array}$$

Hence $\mathbf{C} = (C, \bullet, \mathbf{d}, \mathbf{r})$ is a category. □

Now we order our category $\mathbf{C}$ to obtain an inductive category.

Theorem 6.3.3. *Let S and T be restriction semigroups. Suppose T acts doubly on S satisfying (CP1) and let $U = S \bowtie T$. Let*

$$C = S \bowtie^{\lambda} T = \{(a,t) \in S \times T :\ t^+ \cdot a = a\}.$$

Then $\mathbf{C} = (C, \bullet, \mathbf{d}, \mathbf{r}, \leq)$ is an inductive category under the restriction of the binary operation in U with set of local identities

$$E_{\mathbf{C}} = \{(f \cdot e, f) : f \in E_T, e \in E_S\}$$

where

$$\mathbf{d}(a,t) = (a^+, t^+) \ \text{ and } \ \mathbf{r}(a,t) = (a^* \circ t, t^*)$$

The partial order $\leq$ on $\mathbf{C}$ is defined by

$$(a,t) \leq (b,u) \quad \text{if and only if} \quad a \leq t^+ \cdot b\,,\ t \leq u.$$

For $(a,t) \in C$ and $(f \cdot e, f) \in E_{\mathbf{C}}$, the restriction is defined by

$$_{(f \cdot e, f)}|(a,t) = ((f \cdot e)(f \cdot a), ft) = (f \cdot ea, ft)$$

where $(f \cdot e, f) \leq \mathbf{d}(a,t)$ and co-restriction is defined as:

$$(a,t)|_{(f \cdot e, f)} = \Big(((tf)^+ \cdot a)(t \cdot (f \cdot e)), tf\Big)$$

where $(f \cdot e, f) \leq \mathbf{r}(a,t)$.

Proof. From Theorem 6.3.2, we know that **C** is a category. We now show that **C** is an inductive category.

We first check that $\leq$ is a partial order on **C**. It is clear that $\leq$ is reflexive.

Let $(a,t),(b,u) \in C$ and suppose that

$$(a,t) \leq (b,u) \text{ and } (b,u) \leq (a,t).$$

Then $a \leq t^+ \cdot b$, $t \leq u$ and $b \leq u^+ \cdot a$, $u \leq t$, so that

$$t \leq u\ ,\ u \leq t \Rightarrow t = u,$$

and $a \leq t^+ \cdot b$ implies

$$\begin{aligned} a &= a^+(t^+ \cdot b) \\ &= a^+(u^+ \cdot b) \quad \text{because } t = u \\ &= a^+ b \end{aligned}$$

and so $a \leq b$. By symmetry $b \leq a$ and so $a = b$. Thus $(a,t) = (b,u)$ and hence $\leq$ is antisymmetric.

Next let $(a,t),(b,u),(c,v) \in C$ be such that

$$(a,t) \leq (b,u) \quad \text{and} \quad (b,u) \leq (c,v).$$

Then $a \leq t^+ \cdot b$, $t \leq u$ and $b \leq u^+ \cdot c$, $u \leq v$. Now

$$t \leq u\,,\ u \leq v \Rightarrow t \leq v,$$

and $a \leq t^+ \cdot b$, $b \leq u^+ \cdot c$ implies

$$\begin{aligned} a &= a^+(t^+ \cdot b) \\ &= a^+\Big(t^+ \cdot (b^+(u^+ \cdot c))\Big) && \text{because } b \leq u^+ \cdot c, \text{ so } b = b^+(u^+ \cdot c) \\ &= a^+(t^+ \cdot b^+)(t^+ \cdot (u^+ \cdot c))) \\ &= a^+(t^+ \cdot b)^+(t^+u^+ \cdot c) \\ &= a^+(t^+ \cdot c) && \text{because } a^+ = a^+(t^+ \cdot b)^+ \text{ and } t^+ \leq u^+. \end{aligned}$$

Thus $a \leq t^+ \cdot c$ and

$$t \leq v \Rightarrow t^+ \leq v^+,$$

giving $(a,t) \leq (c,v)$. Hence $\leq$ is a partial order on **C**.

(IC1) Let $(a,t),(b,u) \in C$ and let $(a,t) \leq (b,u)$. Then

$$a \leq t^+ \cdot b \;\text{ and } t \leq u$$

To show that

$$\mathbf{d}(a,t) \leq \mathbf{d}(b,u),$$

we see that

$$\begin{array}{rcll} a & \leq & t^+ \cdot b & \\ \Rightarrow \; a^+ & \leq & (t^+ \cdot b)^+ & \text{using Lemma 2.1.7} \\ \Rightarrow \; a^+ & \leq & t^+ \cdot b^+. & \end{array}$$

Also $t \leq u$ implies that $t^+ \leq u^+$. Thus

$$(a^+, t^+) \leq (b^+, u^+).$$

Hence $\mathbf{d}(a,t) \leq \mathbf{d}(b,u)$. Next we show that

$$\mathbf{r}(a,t) \leq \mathbf{r}(b,u), \text{ that is, } (a^* \circ t, t^*) \leq (b^* \circ u, u^*),$$

we need to check that

$$a^* \circ t \leq t^* \cdot (b^* \circ u) \;\text{ and }\; t^* \leq u^*.$$

Now $t \leq u$ implies $t^* \leq u^*$ by Lemma 2.1.7. Also

$$\begin{array}{rcll} a & \leq & t^+ \cdot b & \\ \Rightarrow \; a^* \circ t & \leq & (t^+ \cdot b)^* \circ t & \text{using Lemma 6.3.1} \\ & = & (t^+ \cdot b^*) \circ t & \\ & = & (b^* \circ t^+) \circ t & \text{using (CP1)} \\ & = & b^* \circ t^+ t & \\ & = & b^* \circ t. & \end{array}$$

Hence

$$\begin{array}{rcll} (a^* \circ t)(t^* \cdot (b^* \circ u)) & = & (a^* \circ t)((b^* \circ u) \circ t^*) & \text{using (CP1)} \\ & = & (a^* \circ t)(b^* \circ ut^*) & \\ & = & (a^* \circ t)(b^* \circ t) & \text{because } t \leq u \\ & = & a^* \circ t. & \end{array}$$

Thus

$$(a^* \circ t, t^*) \leq (b^* \circ u, u^*),$$

and hence $\mathbf{r}(a,t) \leq \mathbf{r}(b,u)$.

(IC2) Let $(a,t),(b,u),(c,v),(d,p) \in \mathbf{C}$ be such that

$$(a,t) \leq (b,u),\ (c,v) \leq (d,p)$$

so that

$$a \leq t^+ \cdot b\ ,\ t \leq u \quad \text{and} \quad c \leq v^+ \cdot d\ ,\ v \leq p.$$

Suppose also $\exists\ (a,t) \bullet (c,v)$ and $\exists\ (b,u) \bullet (d,p)$. Then

$$\mathbf{r}(a,t) = \mathbf{d}(c,v) \quad \text{and} \quad \mathbf{r}(b,u) = \mathbf{d}(d,p),$$

so that

$$(a^* \circ t, t^*) = (c^+, v^+) \quad \text{and} \quad (b^* \circ u, u^*) = (d^+, p^+).$$

To show that

$$(a,t) \bullet (c,v) \leq (b,u) \bullet (d,p),$$

we need to check that

$$\Big(a(t \cdot c), tv\Big) \leq \Big(b(u \cdot d), up\Big).$$

Now $t \leq u$, $v \leq p$ implies that $tv \leq up$. Also

$$\begin{array}{rcll}
a(t \cdot c) & = & a^+(t^+ \cdot b)\Big(t \cdot (c^+(v^+ \cdot d))\Big) & \text{because } a \leq t^+ \cdot b \text{ and } c \leq v^+ \cdot d \\
& = & a^+(t^+ \cdot b)(t \cdot c^+)(t \cdot (v^+ \cdot d)) & \\
& = & a^+(t^+ \cdot b)(t \cdot c)^+(tv^+ \cdot d) & \\
& = & a^+\Big((t^+ \cdot b)(t \cdot c)^+\Big)^+(t^+ \cdot b)((tv)^+ t \cdot d) & \text{using left ample condition} \\
& = & \Big(a^+(t^+ \cdot b)(t \cdot c)\Big)^+((tt^*)^+ \cdot b)((tv)^+ t^+ u \cdot d) & \text{because } t \leq u \\
& = & (a(t \cdot c))^+((tv^+)^+ \cdot b)((tv)^+ \cdot (u \cdot d)) & \text{because } a \leq t^+ \cdot b \text{ and } t^* = v^+ \\
& = & (a(t \cdot c))^+((tv)^+ \cdot b)((tv)^+ \cdot (u \cdot d)) & \\
& = & (a(t \cdot c))^+((tv)^+ \cdot (b(u \cdot d))) & \\
\Rightarrow \quad a(t \cdot c) & \leq & (tv)^+ \cdot b(u \cdot d). &
\end{array}$$

Hence $(a(t \cdot c), tv) \leq (b(u \cdot d), up)$ giving

$$(a,t) \bullet (c,v) \leq (b,u) \bullet (d,p).$$

(IC3) Let $(f \cdot e, f) \in E_{\mathbf{C}}$ and $(a,t) \in C$ be such that

$$(f \cdot e, f) \leq \mathbf{d}(a,t) = (a^+, t^+)$$

so that

$$f \cdot e \leq f \cdot a^+ \quad \text{and} \quad f \leq t^+.$$

We show that $(f \cdot (ea), ft)$ is the unique element of C such that

$$(f \cdot (ea), ft) \leq (a,t) \text{ and } \mathbf{d}(f \cdot (ea), ft) = (f \cdot e, f).$$

We first check that $(f \cdot (ea), ft) \in C$. For this we see that

$$\begin{aligned}
(ft)^+ \cdot (f \cdot (ea)) &= ft^+ f \cdot (ea) \\
&= f \cdot (ea) \qquad \text{because } f = ft^+ \text{ as } f \leq t^+
\end{aligned}$$

Thus $(f \cdot (ea), ft) \in C$. Now

$$f \leq t^+ \Rightarrow ft \leq t^+ t \Rightarrow ft \leq t.$$

Also

$$\begin{aligned}
f \cdot (ea) &= (f \cdot e)(f \cdot a) \\
&\leq (f \cdot a^+)(f \cdot a) \\
&= f \cdot a \\
&= ft^+ \cdot a \\
&= (ft)^+ \cdot a.
\end{aligned}$$

Thus $(f \cdot (ea), ft) \leq (a,t)$. Also

$$\begin{aligned}
\mathbf{d}(f \cdot (ea), ft) &= \big((f \cdot (ea))^+, (ft)^+\big) \\
&= \big(f \cdot (ea)^+, ft^+\big) \\
&= (f \cdot (ea^+), f) \qquad \text{because } f = ft^+ \text{ as } f \leq t^+ \\
&= ((f \cdot e)(f \cdot a^+), f) \\
&= (f \cdot e, f) \qquad \text{because } f \cdot e \leq f \cdot a^+.
\end{aligned}$$

Next suppose that (r,s) is another element such that

$$(r,s) \leq (a,t) \quad \text{and} \quad \mathbf{d}(r,s) = (f \cdot e, f).$$

Then

$$\begin{array}{rrcll} & r & \leq & s^+ \cdot a & \\ \Rightarrow & r & \leq & f \cdot a & \text{because } s^+ = f \\ \Rightarrow & r & = & r^+(f \cdot a) & \\ \Rightarrow & r & = & (f \cdot e)(f \cdot a) & \text{as } r^+ = f \cdot e \\ \Rightarrow & r & = & f \cdot (ea). & \end{array}$$

Also

$$s \leq t \Rightarrow s = s^+ t = ft.$$

Hence $(f \cdot (ea), ft)$ is the unique element such that

$$(f \cdot (ea), ft) \leq (a, t) \text{ and } \mathbf{d}(f \cdot ea, ft) = (f \cdot e, f).$$

Hence $(f \cdot (ea), ft) = {}_{(f \cdot e, f)}|(a, t)$.

(IC4) Let $(a, t) \in C$, $(f \cdot e, f) \in E_{\mathbf{C}}$ be such that

$$(f \cdot e, f) \leq \mathbf{r}(a, t) = (a^* \circ t, t^*),$$

so that

$$f \cdot e \leq f \cdot (a^* \circ t) \text{ and } f \leq t^*.$$

We will prove that $\big(((tf)^+ \cdot a)(t \cdot (f \cdot e)), tf\big)$ is the unique element such that

$$\big(((tf)^+ \cdot a)(t \cdot (f \cdot e)), tf\big) \leq (a, t) \quad \text{and } \mathbf{r}\big(((tf)^+ \cdot a)(t \cdot (f \cdot e)), tf\big) = (f \cdot e, f).$$

We first check that $\big(((tf)^+ \cdot a)(t \cdot (f \cdot e)), tf\big) \in C$. For this we see that

$$\begin{aligned} (tf)^+ \cdot ((tf)^+ \cdot a)(t \cdot (f \cdot e)) &= \big((tf)^+ \cdot ((tf)^+ \cdot a)\big)\big((tf)^+ \cdot (tf \cdot e)\big) \\ &= ((tf)^+(tf)^+ \cdot a)((tf)^+(tf) \cdot e) \\ &= ((tf)^+ \cdot a)(tf \cdot e) \\ &= ((tf)^+ \cdot a)(t \cdot (f \cdot e)). \end{aligned}$$

Hence $\big(((tf)^+ \cdot a)(t \cdot (f \cdot e)), tf\big) \in C$. Now

$$f \leq t^* \Rightarrow tf \leq tt^* \Rightarrow tf \leq t.$$

Also since $t \cdot (f \cdot e) = tf \cdot e \in E_S$, therefore

$$((tf)^+ \cdot a)(t \cdot (f \cdot e)) \leq (tf)^+ \cdot a.$$

Hence

$$\Big(((tf)^+ \cdot a)(t \cdot (f \cdot e)), tf\Big) \leq (a, t).$$

Also

$$\begin{aligned}
\mathbf{r}\Big(((tf)^+ \cdot a)(t \cdot (f \cdot e)), tf\Big) &= \Big(\Big(((tf)^+ \cdot a)(tf \cdot e)\Big)^* \circ tf, (tf)^*\Big) \\
&= \Big(\Big(((tf)^+ \cdot a)^*(tf \cdot e)\Big)^* \circ tf, t^*f\Big) \\
&= \Big(((tf)^+ \cdot a^*)(tf \cdot e) \circ tf, ft^*\Big) \\
&= \Big(\Big(tf \cdot (a^* \circ tf)\Big)(tf \cdot e) \circ tf, f\Big) && \text{using (CP1) and } f \leq t^* \\
&= \Big(\Big(tf \cdot ((a^* \circ t) \circ f)\Big)(tf \cdot e) \circ tf, f\Big) \\
&= \Big(\Big(tf \cdot (f \cdot (a^* \circ t))\Big)(tf \cdot e) \circ tf, f\Big) && \text{using (CP1)} \\
&= \Big(((tf \cdot (a^* \circ t))(tf \cdot e)\Big) \circ tf, f\Big) \\
&= \Big((tf \cdot ((a^* \circ t)e)) \circ tf, f\Big) \\
&= \Big((tf)^* \cdot ((a^* \circ t)e), f\Big) && \text{using (CP1)} \\
&= \Big((t^*f)^* \cdot ((a^* \circ t)e), f\Big) \\
&= \Big(f \cdot ((a^* \circ t)e), f\Big) && \text{because } f \leq t^* \\
&= \Big((f \cdot e)(f \cdot (a^* \circ t)), f\Big) \\
&= (f \cdot e, f) && \text{because } f \cdot e \leq f \cdot (a^* \circ t).
\end{aligned}$$

Thus $\mathbf{r}\Big(((tf)^+ \cdot a)(tf \cdot e), tf\Big) = (f \cdot e, f)$. Next suppose that (m, n) is another element such that

$$(m, n) \leq (a, t) \quad \text{and} \quad \mathbf{r}(m, n) = (f \cdot e, f),$$

so that

$$m \leq n^+ \cdot a\,,\ n \leq t\,,\ m^* \circ n = f \cdot e \ \text{ and } n^* = f.$$

Then $n \leq t$ implies

$$n = tn^* = tf.$$

Also $m \leq n^+ \cdot a$ implies

$$\begin{array}{rcll} m & = & (n^+ \cdot a)m^* & \\ & = & ((tf)^+ \cdot a)m^* & \text{because } n = tf \\ & = & ((tf)^+ \cdot a)(n^+ \cdot m^*) & \text{because } m^* = (n^+ \cdot m)^* = n^+ \cdot m^* \\ & = & ((tf)^+ \cdot a)(n \cdot (m^* \circ n)) & \text{using (CP1)} \\ & = & ((tf)^+ \cdot a)(tf \cdot (f \cdot e)) & \text{because } n = tf \text{ and } m^* \circ n = f \cdot e \\ & = & ((tf)^+ \cdot a)(tf \cdot e). & \end{array}$$

Hence $\Big(((tf)^+ \cdot a)(t \cdot (f \cdot e)), tf\Big)$ is the unique element such that

$$\Big(((tf)^+ \cdot a)(t \cdot (f \cdot e)), tf\Big) \leq (a, t) \quad \text{and } \mathbf{r}\Big(((tf)^+ \cdot a)(t \cdot (f \cdot e)), tf\Big) = (f \cdot e, f)$$

and so we write $\Big(((tf)^+ \cdot a)(t \cdot (f \cdot e)), tf\Big) = (a, t)|_{(f \cdot e, f)}$.

(IC5) To show that $(E_{\mathbf{C}}, \leq)$ is a meet semilattice, let $(f \cdot e, f), (h \cdot g, h) \in E_{\mathbf{C}}$. We show that $(f \cdot e, f) \wedge (h \cdot g, h)$ exists and

$$(f \cdot e, f) \wedge (h \cdot g, h) = (fh \cdot (eg), fh).$$

Clearly $(fh \cdot (eg), fh) \in E_{\mathbf{C}}$ and $fh \leq f, h$. Also as $eg \leq e$, so

$$\begin{array}{rcll} fh \cdot (eg) & \leq & fh \cdot e & \text{using Lemma 6.3.1} \\ & = & fhf \cdot e & \\ & = & fh \cdot (f \cdot e). & \end{array}$$

Thus

$$(fh \cdot (eg), fh) \leq (f \cdot e, f).$$

Similarly

$$(fh \cdot (eg), fh) \leq (h \cdot g, h).$$

Next let $(m \cdot l, m) \in E_{\mathbf{C}}$ be such that

$$(m \cdot l, m) \leq (f \cdot e, f) \quad \text{and} \quad (m \cdot l, m) \leq (h \cdot g, h).$$

This implies that

$$m \cdot l \leq m \cdot (f \cdot e)\,,\, m \leq f \quad \text{and} \quad m \cdot l \leq m \cdot (h \cdot g)\,,\, m \leq h.$$

Now $m \leq f$, $m \leq h$ implies that $m \leq fh$ and

$$\begin{array}{rcll} m \cdot l & \leq & (m \cdot (f \cdot e))(m \cdot (h \cdot g)) & \\ & = & (mh \cdot (f \cdot e))(mf \cdot (h \cdot g)) & \text{because } m \leq f \text{ and } m \leq h \\ & = & (m \cdot (hf \cdot e))(m \cdot (fh \cdot g)) & \\ & = & m \cdot ((fh \cdot e)(fh \cdot g) & \\ & = & m \cdot (fh \cdot (eg))) & \\ \Rightarrow \quad m \cdot l & \leq & m \cdot (fh \cdot (eg)), & \end{array}$$

so that $(m \cdot l, m) \leq (fh \cdot (eg), fh)$. Hence $(f \cdot e, f) \wedge (h \cdot g, h)$ exists and

$$(f \cdot e, f) \wedge (h \cdot g, h) = (fh \cdot (eg), fh).$$

Thus $E_{\mathbf{C}} = \{(f \cdot e, f) : f \in E_T, e \in E_S\}$ is a meet semilattice.

Hence $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ is an inductive category. □

We now consider the pseudo-product on $\mathbf{C}$ and show that it coincides with the λ-semidirect product.

Theorem 6.3.4. *Let* $\mathbf{C} = (C, \bullet, \mathbf{d}, \mathbf{r}, \leq)$ *be the inductive category as defined in Theorem 6.3.3. Let* $(a,t), (b,u) \in \mathbf{C}$ *and define* $\otimes$ *by the rule*

$$(a,t) \otimes (b,u) = ((a,t)|_{\mathbf{r}(a,t) \wedge \mathbf{d}(b,u)})({}_{\mathbf{r}(a,t) \wedge \mathbf{d}(b,u)}|(b,u)).$$

Then as a restriction semigroup, $(\mathbf{C}, \otimes) = S \rtimes^{\lambda} T$.

Proof. We first compute $\mathbf{r}(a,t) \wedge \mathbf{d}(b,u)$. We have

$$\begin{array}{rcl} \mathbf{r}(a,t) \wedge \mathbf{d}(b,u) & = & (a^* \circ t, t^*) \wedge (b^+, u^+) \\ & = & \Big((t^* u^+ \cdot a^* \circ t)(t^* \cdot b^+), t^* u^+\Big) \\ & = & \Big((u^+ \cdot (t^* \cdot (a^* \circ t)))(t^* \cdot b^+), t^* u^+\Big) \\ & = & \Big((u^+ \cdot (a^* \circ t))(t^* \cdot b^+), t^* u^+\Big). \end{array}$$

Now

$$\begin{array}{rcl} ((a,t)|_{\mathbf{r}(a,t) \wedge \mathbf{d}(b,u)}) & = & ((a,t)|_{((u^+ \cdot (a^* \circ t))(t^* \cdot b^+), t^* u^+)}) \\ & = & \Big(\Big((t(t^* u^+))^+ \cdot a\Big)\Big(t \cdot ((u^+ \cdot (a^* \circ t))(t^* \cdot b^+)\Big), tt^* u^+\Big) \\ & = & \Big(\Big((tu)^+ \cdot a\Big)\Big(t \cdot (u^+ \cdot (a^* \circ t))\Big)\Big(t \cdot (t^* \cdot b^+)\Big), tu^+\Big) \\ & = & \Big(((tu)^+ \cdot a)(tu^+ \cdot (a^* \circ t))(t \cdot b^+), tu^+\Big). \end{array}$$

Also

$$
\begin{aligned}
(_{\mathbf{r}(a,t)\wedge\mathbf{d}(b,u)}|(b,u)) &= (_{((u^+\cdot(a^*\circ t))(t^*\cdot b^+),t^*u^+)}|(b,u)) \\
&= \big((u^+\cdot(a^*\circ t))(t^*\cdot b^+)(t^*u^+\cdot b), t^*u^+u\big) \\
&= \big((u^+\cdot(a^*\circ t))(t^*\cdot b^+)(t^*\cdot b), t^*u\big) \qquad \text{because } u^+\cdot b = b \\
&= \big((u^+\cdot(a^*\circ t))(t^*\cdot b), t^*u\big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
(a,t)\otimes(b,u) &= ((a,t)|_{\mathbf{r}(a,t)\wedge\mathbf{d}(b,u)})(_{\mathbf{r}(a,t)\wedge\mathbf{d}(b,u)}|(b,u)) \\
&= \big(((tu)^+\cdot a)(tu^+\cdot(a^*\circ t))(t\cdot b^+), tu^+\big)\bullet\big((u^+\cdot(a^*\circ t))(t^*\cdot b), t^*u\big) \\
&= \big(((tu)^+\cdot a)(tu^+\cdot(a^*\circ t))(t\cdot b^+)\big(tu^+\cdot((u^+\cdot(a^*\circ t))(t^*\cdot b)\big), tu^+t^*u\big) \\
&= \big(((tu)^+\cdot a)(tu^+\cdot(a^*\circ t))(t\cdot b^+)(tu^+\cdot(u^+\cdot(a^*\circ t)))(tu^+\cdot(t^*\cdot b)), tu\big) \\
&= \big(((tu)^+\cdot a)(tu^+\cdot(a^*\circ t))((t\cdot b^+)(tu^+\cdot(a^*\circ t))(tu^+\cdot b), tu\big) \\
&= \big(((tu)^+\cdot a)(tu^+\cdot(a^*\circ t))(tu^+\cdot(a^*\circ t))(t\cdot b^+)(t\cdot b), tu\big) \quad \text{as } u^+\cdot b = b \\
&= \big(((tu)^+\cdot a)(tu^+\cdot(a^*\circ t)))(t\cdot(b^+b)), tu\big) \\
&= \big(((tu)^+\cdot a)((tu)^+t\cdot(a^*\circ t)))(t\cdot b), tu\big) \\
&= \big(((tu)^+\cdot a)((tu)^+\cdot(t\cdot(a^*\circ t))))(t\cdot b), tu\big) \\
&= \big(((tu)^+\cdot a)((tu)^+\cdot(t^+\cdot a^*))(t\cdot b), tu\big) \qquad \text{using (CP1)} \\
&= \big(((tu)^+\cdot a)((tu)^+\cdot a^*)(t\cdot b), tu\big) \\
&= \big(((tu)^+\cdot(aa^*))(t\cdot b), tu\big) \\
&= \big(((tu)^+\cdot a)(t\cdot b), tu\big).
\end{aligned}
$$

Hence $\otimes$ coincides with the λ-semidirect product. Also note that $^+$ and * correspond as well. □

Chapter 7

Monoids and left restriction semigroups decomposed into products: an analogue of the approach for E-unitary inverse semigroups

A remarkable structure theorem for proper (E-unitary) inverse semigroups, now known as the 'P-theorem' was given by McAlister [52], which is important due to McAlister's other major result that every inverse semigroup has a proper cover [51]. The P-theorem determines the structure of all proper inverse semigroups and has many important consequences such as O'Carroll's embedding theorem [59], which states that any proper inverse semigroup can be embedded into a semidirect product of a semilattice by a group.

In this chapter, we aim to generalise O'Carroll's embedding theorem [59] for certain monoids and left restriction semigroups. We consider a monoid S such that $S = AT$ for submonoids A, T, where T acts on A (as a semigroup) satisfying a condition analogous to the left ample condition. We then introduce the notion of (A, T)-properness for S and prove covering and embedding theorems analogous to those of McAlister.

To prove the corresponding results for left (right) restriction semigroups is not so easy. The semidirect product of two left (right) restriction semigroups need not be left (right) restriction in general as we explained in Chapter 6. We therefore consider λ-semidirect products of left restriction monoids to obtain covering and embedding theorems analogous to the monoid case. We consider a left restriction monoid S with submonoids A, T and use the notion of (A, T)-properness defined for the monoid case to prove that a left restriction monoid has an (A,T)-proper cover and that every (A,T)-proper left restriction monoid can be embedded in the λ-semidirect product of two left restriction monoids.

7.1 Inverse semigroups: coverings, P-theorems and consequences

We start this section by giving the definitions of *proper* and *proper covers* in the context of inverse semigroups. We first give the definition of an E-unitary semigroup.

Definition 7.1.1. Let S be a semigroup and let $E = E(S)$ be the set of idempotents of S. Then S is *E-unitary* if for all $e \in E$ and for all $s \in S$

$$es \in E \Rightarrow s \in E.$$

This definition is two sided due to the following proposition, which is true for regular semigroups (see [34]), but we prove it for an arbitrary semigroup.

Proposition 7.1.2. *Let S be a semigroup with set of idempotents $E = E(S)$. Then S is E-unitary if and only if for $s \in S$ and $e \in E$,*

$$se \in E \Rightarrow s \in E.$$

Proof. Suppose that S is E-unitary. Let $s \in S$, $e \in E$ and suppose that $se \in E$. Then

$$(ese)^2 = (ese)(ese) = e(se)(se) = ese$$

and so $ese \in E$. Next we see that $(es)^2 \in E$ as

$$\begin{aligned} (es)^2(es)^2 &= (eses)(eses) \\ &= e(se)^3 s \\ &= eses \qquad \text{because } se \in E \\ &= (es)^2, \end{aligned}$$

and hence $(es)^4 = (es)^2 \in E$. Thus $ese \in E$ and $(ese)s \in E$ implies $s \in E$ because S is E-unitary.

The converse argument is dual. $\square$

Now we give the definition of a *proper inverse semigroup*.

Definition 7.1.3. Let S be an inverse semigroup with semilattice of idempotents E. Let σ be the minimum group congruence on S. Then S is *proper* if and only if $\mathcal{R} \cap \sigma = \imath$.

We remind the reader from Chapter 1 that if S is inverse, the least group congruence σ is given by the rule for all $s, t \in S$:

$$s \, \sigma \, t \Leftrightarrow es = et \text{ for some } e \in E(S).$$

Proposition 7.1.4. [34]. *Let S be an inverse semigroup and let σ be the least group congruence on S. Then the following are equivalent:*

(i) S is proper;

(ii) $\mathcal{L} \cap \sigma = \imath$;

(iii) S is E-unitary.

Definition 7.1.5. Let S be an inverse semigroup. A *proper cover* of S is a proper inverse semigroup U together with an onto, idempotent separating morphism $\psi : U \to S$.

We comment in the above definition that $\psi|_{E(U)} : E(U) \to E(S)$ is an isomorphism.

McAlister's Covering Theorem. [51] *Every inverse semigroup S has a proper cover.*

To define a *P-semigroup* and state *McAlister's P-theorem* and its *consequences*, we first have a look at some ideas needed for our purpose.

Definition 7.1.6. Let G be a group and $(\mathcal{X}, \leq)$ be a partially ordered set. Then G acts on $\mathcal{X}$ by *order automorphisms* if G acts on $\mathcal{X}$ and for $A, B \in \mathcal{X}$, $g \in G$

$$A \leq B \Leftrightarrow g \cdot A \leq g \cdot B.$$

Definition 7.1.7. Let $(\mathcal{X}, \leq)$ be a partially ordered set and $\mathcal{Y}$ be a subset of $\mathcal{X}$. Then $\mathcal{Y}$ is an *order ideal* of $\mathcal{X}$ if for all $A, B \in \mathcal{X}$,

$$A \in \mathcal{Y} \text{ and } B \leq A \Rightarrow B \in \mathcal{Y}.$$

To give the definition of a *P-semigroup*, we first need to define a *McAlister triple*.

Definition 7.1.8. Let G be a group acting on a partially ordered set $(\mathcal{X}, \leq)$ by order automorphisms. Let $\mathcal{Y}$ be a subset of $\mathcal{X}$ and suppose that the following conditions hold:

(P1) $\mathcal{Y}$ is a semilattice with respect to $\leq$;
(P2) $G\mathcal{Y} = \mathcal{X}$; that is, for every $X \in \mathcal{X}$ there exists $g \in G$ and $Y \in \mathcal{Y}$ such that $g \cdot Y = X$;
(P3) $\mathcal{Y}$ is an order ideal of $\mathcal{X}$;
(P4) for all $g \in G$, $g \cdot \mathcal{Y} \cap \mathcal{Y} \neq \emptyset$.
Then $(G, \mathcal{X}, \mathcal{Y})$ is called a *McAlister triple*.

We now give the definition of a *P-semigroup*.

Definition 7.1.9. Let $(G, \mathcal{X}, \mathcal{Y})$ be a McAlister triple and let

$$P(G, \mathcal{X}, \mathcal{Y}) = \{(A, g) \in \mathcal{Y} \times G : g^{-1} \cdot A \in \mathcal{Y}\}.$$

Define a binary operation on $P(G, \mathcal{X}, \mathcal{Y})$ by the rule

$$(A, g)(B, h) = (A \wedge g \cdot B, gh)$$

for $(A, g), (B, h) \in P(G, \mathcal{X}, \mathcal{Y})$.

Lemma 7.1.10. *If $(G, \mathcal{X}, \mathcal{Y})$ is a McAlister triple, then $P(G, \mathcal{X}, \mathcal{Y})$ is a semigroup.*

We call the semigroup $P(G, \mathcal{X}, \mathcal{Y})$ in Lemma 7.1.10 a *P-semigroup*.

We now state *McAlister's P-theorem* which determines the structure of all proper inverse semigroups.

McAlister's P-Theorem. [52] *Let P be a P-semigroup. Then P is a proper inverse semigroup. Conversely, any proper inverse semigroup is isomorphic to a P-semigroup.*

An important consequence of McAlister's P-theorem is O'Carroll's embedding theorem which tell us that any proper inverse semigroup can be embedded into a much simpler structure than a P-semigroup.

O'Carroll's Embedding Theorem. [59] *Let S be an inverse semigroup. Then S is proper if and only if S can be embedded into the semidirect product of a semilattice by a group.*

The McAlister's P-theorem for proper inverse semigroups prompted work for larger classes of semigroups, see for example [22, 70, 75]. McAlister's work has been extended in several directions, we mention here two of these, one to keep the condition of S being regular but weaken the condition that $E(S)$ forms a semilattice. The other direction is to drop the regularity of S but retain commutativity of idempotents.

7.2 Left restriction semigroups: coverings, P-theorems and consequences

A theory analogous to that for inverse semigroups has been developed for left ample, weakly left ample and left restriction semigroups.

Definition 7.2.1. Let S be a left restriction semigroup with semilattice of projections E. The relation σ_E on S is defined by

$$\sigma_E = \{(a,b) \in S \times S : \exists e \in E \text{ such that } ea = eb\}.$$

This relation is the least congruence on S identifying all the elements of E [28]. It is explained in [28] that σ_E can be regarded either as a semigroup congruence or a congruence in the augmented signature. When $E = E(S)$, we just write σ for σ_E. If S is inverse, then $\sigma = \sigma_E$ is the least group congruence on S. If S is left ample, then it is known that $\sigma = \sigma_E$ is the least right cancellative monoid congruence on S [16] and if S is a weakly left ample monoid, then σ is the least unipotent congruence on S [27].

Dually, if S is right restriction, then $a \, \sigma_E \, b$ if and only if $af = bf$ for some $f \in E$. Thus σ_E is again the least congruence identifying all the elements of E. Hence if S is restriction, then either characterisation of σ_E will suffice.

Definition 7.2.2. A left (right) restriction semigroup is *proper* if

$$\widetilde{\mathcal{R}}_E \cap \sigma_E = \imath \ (\widetilde{\mathcal{L}}_E \cap \sigma_E = \imath).$$

A restriction semigroup is *proper* if it is proper as both a left and as a right restriction semigroup.

If a restriction semigroup S_{res} is obtained from an inverse semigroup S by putting $a^+ = aa^{-1}$, $a^* = a^{-1}a$, then we have observed that σ_E is the least group congruence on S, $\widetilde{\mathcal{R}}_E = \mathcal{R}$ and so S_{res} is proper if and only if S is E-unitary. Thus the relation σ_E generalises the least group congruence on an inverse semigroup and the notion of a proper restriction semigroup generalises that of an E-unitary inverse semigroup. Moreover, the role played among inverse semigroups by groups is taken over among restriction semigroups by monoids, because groups are inverse semigroups with one idempotent and monoids are reduced restriction semigroups, that is, restriction semigroups in which $a^+ = 1 = a^*$ for all elements a. For inverse semigroups, we know that each inverse semigroup has a proper cover and proper inverse semigroups are just the inverse subsemigroups of the semidirect products of semilattices by groups [51, 52, 34]. There is an analogous theory for restriction semigroups where groups are replaced by monoids. This theory was developed in [20] for left ample semigroups.

Definition 7.2.3. Let S be a left restriction semigroup with semilattice of projections E. A *cover* of S is a left restriction semigroup U together with an onto morphism

$$\psi : U \to S,$$

which separates idempotents of E.

Corresponding to McAlister's covering theorem, a covering theorem has been given for left restriction semigroups in [24, Lemma 6.6] and [6, Theorem 6.4].

Theorem 7.2.4. [6] *Every left restriction semigroup has a proper cover.*

A structure theorem analogous to McAlister's P-theorem has been given for left restriction semigroups in terms of monoids (regarded as left restriction semigroups) acting on semilattices [6].

Definition 7.2.5. Let M be a monoid acting by morphisms on the left of a semilattice $\mathcal{X}$. Let $\mathcal{Y}$ be a subsemilattice of $\mathcal{X}$ and suppose that $\mathcal{Y}$ has an upper bound $\epsilon \in \mathcal{X}$ such that the following hold:

(a) for all $m \in M$, there exists $e \in \mathcal{Y}$ such that $e \leq m \cdot \epsilon$;

(b) for all $e, f \in \mathcal{Y}$ and all $m \in M$,

$$e \leq m \cdot \epsilon \Rightarrow e \wedge m \cdot f \in \mathcal{Y}.$$

Then the triple $(M, \mathcal{X}, \mathcal{Y})$ is called a *strong left $\mathcal{M}$-triple.*

Given a strong left $\mathcal{M}$-triple $(M, \mathcal{X}, \mathcal{Y})$, define

$$\mathcal{M} = \mathcal{M}(M, \mathcal{X}, \mathcal{Y}) = \{(e, m) \in \mathcal{Y} \times M : e \leq m \cdot \epsilon\}$$

with binary operation defined by

$$(e, m)(f, n) = (e \wedge m \cdot f, mn)$$

for $(e, m), (f, n) \in \mathcal{M}$. Then $\mathcal{M}(M, \mathcal{X}, \mathcal{Y})$ is a semigroup and is called a *strong $\mathcal{M}$-semigroup.*

Lemma 7.2.6. [6, Lemma 7.1] *Let* $(M, \mathcal{X}, \mathcal{Y})$ *be a strong left* $\mathcal{M}$*-triple. Then* $\mathcal{M}(M, \mathcal{X}, \mathcal{Y})$ *is a proper left restriction semigroup with*

$$(e, m)^+ = (e, 1),\ E = \{(e, 1) : e \in \mathcal{Y}\} \cong \mathcal{Y} \text{ and } \mathcal{M}(M, \mathcal{X}, \mathcal{Y})/\sigma_E \cong M.$$

Moreover, if $\epsilon \in \mathcal{Y}$*, then* $\mathcal{M}(M, \mathcal{X}, \mathcal{Y})$ *is a monoid with identity* $(\epsilon, 1)$.

Theorem 7.2.7. [6, Theorem 7.2] *A semigroup (monoid)* S *is proper left restriction if and only if it is isomorphic to a strong* $\mathcal{M}$*-semigroup (monoid) of the form* $\mathcal{M}(S/\sigma_E, \mathcal{X}, \mathcal{Y})$ *where* $\mathcal{Y} \cong E_S$.

The above results are specialised to (weakly) left ample semigroups because a left restriction semigroup with $E = E(S)$ is weakly left ample and if $\widetilde{\mathcal{R}}_{E(S)} = \mathcal{R}^*$, then S is left ample. In Lemma 7.2.6 M is unipotent for weakly left ample semigroups (monoids) and is right cancellative for left ample semigroups (monoids). In Theorem 7.2.7 S/σ_E is unipotent if S is weakly left ample and right cancellative if S is left ample.

We notice that the above results present a theory analogous to McAlister's theory and O'Carroll's work for inverse semigroups, but the order in which things were done in the inverse case is different. Namely, in inverse semigroups we start with a partially ordered set as, an inverse semigroup is proper if and only if it is isomorphic to a P-semigroup $P(G, \mathcal{X}, \mathcal{Y})$, where G is a group, $\mathcal{X}$ is a partially ordered set containing a semilattice $\mathcal{Y}$ as a sub-partially ordered set, subject to certain conditions. We also note that if S is proper inverse, then $S \cong P(S/\sigma, \mathcal{X}, E(S))$, where $E(S)$ is the set of idempotents of S and σ is the least congruence identifying idempotents of $E(S)$. In the left restriction case we start with a semilattice and we notice that, though we take $\mathcal{X}$ to be a semilattice, we lose the condition $G\mathcal{X} = \mathcal{Y}$, which appears in McAlister's result.

7.3 (A, T)-proper monoids

In this section we consider semidirect products of monoids A and T and introduce the notion of (A, T)-proper monoids. We aim to find an (A, T)-proper monoid U and an onto morphism from U to S, where S is the internal product of A and T.

Let A and T be monoids with identities $1_A, 1_T$ (we drop the subscripts where convenient) and suppose that $A \rtimes T$ is the (external) semidirect product of A by T where T is acting on the left of A by semigroup endomorphisms, that is, we do not insist that $t \cdot 1 = 1$. Let

$$A' = \{(a, 1) : a \in A\} \text{ and } T' = \{(t \cdot 1, t) : t \in T\}.$$

Clearly, A' is a submonoid of $A \rtimes T$ with identity $(1_A, 1_T)$ and $A \cong A'$. To check that T' is a submonoid of $A \rtimes T$, let $(t \cdot 1, t), (u \cdot 1, u) \in T'$. Then

$$\begin{aligned}
(t \cdot 1, t)(u \cdot 1, u) &= \big((t \cdot 1)(t \cdot (u \cdot 1)), tu\big) \\
&= \big(t \cdot (1(u \cdot 1)), tu\big) \\
&= (t \cdot (u \cdot 1), tu) \\
&= (tu \cdot 1, tu) \in T'.
\end{aligned}$$

Clearly $(1_A, 1_T)$ is an identity of T' and hence T' is a submonoid of $A \rtimes T$ which clearly is isomorphic to T.

Example 7.3.1. Let S be a left restriction semigroup and let $E = E_S$. Then S acts on E by semigroup morphisms by $s \cdot e = (se)^+$. We check that this is an action by morphisms. For this let $s, t \in S$ and $e \in E$. Then

$$st \cdot e = (ste)^+ = (s(te)^+)^+ = s \cdot (te)^+ = s \cdot (t \cdot e).$$

Also for $s \in S$ and $e, f \in E$

$$s \cdot (ef) = (s(ef))^+ = (sef)^+ = ((se)^+ sf)^+ = (se)^+(sf)^+ = (s \cdot e)(s \cdot f),$$

and hence S is acting on E by semigroup morphisms.

Note that if S (and hence E) is a monoid, then this is not a monoid action unless S is reduced.

We cannot immediately call upon the results of Chapter 3 to write $A \rtimes T$ as an internal semidirect product, since T does not necessarily act on A by monoid morphisms. We therefore show that $U = A'T'$ is a monoid subsemigroup of $A \rtimes T$.

Lemma 7.3.2. *Let $S = A \rtimes T$ be the semidirect product of monoids A and T, where T acts on A by semigroup endomorphisms, and let*

$$U = \{(a, t) \in A \times T : a(t \cdot 1) = a\}.$$

Then U is a subsemigroup of S. Further, U is a monoid with identity $(1_A, 1_T)$ containing the submonoids A' and T', $U = A'T'$ and each element $u \in U$ can be uniquely written as a product of an element of A and an element of T.

Proof. First we observe that $t \cdot 1$ is an idempotent, as

$$t \cdot 1 = t \cdot (11) = (t \cdot 1)(t \cdot 1).$$

Next let $(a,t), (b,u) \in U$. Then

$$a(t \cdot 1) = a \text{ and } b(u \cdot 1) = b.$$

We see that

$$\begin{aligned} a(t \cdot b)(tu \cdot 1) &= a(t \cdot b)(t \cdot (u \cdot 1)) \\ &= a(t \cdot (b(u \cdot 1))) \\ &= a(t \cdot b) \qquad\qquad \text{because } b(u \cdot 1) = b. \end{aligned}$$

Hence $(a,t)(b,u) \in U$. Also

$$(1,1)(a,t) = (a,t) \text{ and } (a,t)(1,1) = (a(t \cdot 1), t) = (a,t).$$

Thus U is a monoid. Clearly A', T' are submonoids of U as $t \cdot 1$ is idempotent for all $t \in T$ and each element $(a,t) \in U$ can be written uniquely as

$$(a,t) = (a,1)(t \cdot 1, t) \text{ where } (a,1) \in A' \text{ and } (t \cdot 1, t) \in T'.$$

Hence $U = A'T'$. □

Example 7.3.3. If S is a left restriction monoid, $A = E_S$, $T = S$ with action as in Example 7.3.1, then

$$U = \{(e,s) : e = e(s \cdot 1)\} = \{(e,s) : e \leq s^+\}.$$

Lemma 7.3.4. *Let $S = A \rtimes T$ be a semidirect product of monoids A and T where T acts on A by semigroup endomorphisms. Then T' acts on A' by $(t \cdot 1, t) \odot (a, 1) = (t \cdot a, 1)$ such that the action satisfies*

$$(t \cdot 1, t)(a, 1) = \big((t \cdot 1, t) \odot (a, 1)\big)(t \cdot 1, t).$$

Proof. We first check that the action of T' on A' is a monoid action. For this let

$(t\cdot 1,t),(u\cdot 1,u)\in T'$ and $(a,1)\in A'$. Then

$$\begin{aligned}(t\cdot 1,t)\odot\Big((u\cdot 1,u)\odot(a,1)\Big) &= (t\cdot 1,t)\odot(u\cdot a,1)\\ &= (t\cdot(u\cdot a),1)\\ &= (tu\cdot a,1)\\ &= (tu\cdot 1,tu)\odot(a,1)\\ &= \Big((t\cdot 1,t)(u\cdot 1,u)\Big)\odot(a,1).\end{aligned}$$

Also

$$(1_A,1_T)\odot(a,1)=(1_T\cdot a,1_T)=(a,1).$$

Hence action of T' on A' is a monoid action. Also it is easy to check that the action of T' on A' is by homomorphisms. Next we see that

$$\begin{aligned}(t\cdot 1,t)(a,1) &= \Big((t\cdot 1)(t\cdot a),t\Big)\\ &= (t\cdot a,t)\\ &= (t\cdot(a1),t)\\ &= ((t\cdot a)(t\cdot 1),t)\\ &= (t\cdot a,1)(t\cdot 1,t)\\ &= \Big((t\cdot 1,t)\odot(a,1)\Big)(t\cdot 1,t).\end{aligned}$$

Hence the action of T' on A' satisfies $(t\cdot 1,t)(a,1)=\Big((t\cdot 1,t)\odot(a,1)\Big)(t\cdot 1,t)$. $\square$

We now consider a monoid S with submonoids A,T such that $S=AT$ and we are supposing that T acts on A (as a semigroup) satisfying

$$ta=(t\cdot a)t \qquad \text{(LAC)}$$

Then for any $t\in T$ we can write $t=(t\cdot 1)t$, so that $at=a(t\cdot 1)t$. Since $t\cdot 1$ is idempotent for any $t\in T$, we have for any $s\in S$

$$s=at \quad \text{where } a=a(t\cdot 1)\ .$$

Example 7.3.5. If S is left restriction, then $S=ES$, and with action as in Example 7.3.1, for $s\in S$ and $e\in E$

$$(s\cdot e)s=(se)^+s=se.$$

Definition 7.3.6. Let S be a monoid with submonoids A,T such that $S=AT$ where T acts on A (as a semigroup) satisfying (LAC). Let $E = \{t \cdot 1 : t \in T\}$ so that $E \subseteq E(S)$. Define σ_A on T by

$$t \,\sigma_A\, u \Leftrightarrow et = fu \text{ for some } e, f \in \langle E \rangle.$$

It is clear that σ_A is reflexive, symmetric and right compatible. To check that it is also left compatible let $t \,\sigma_A\, u$. Then

$$et = fu \quad \text{where } e = (u_1 \cdot 1) \cdots (u_h \cdot 1) \text{ and } f = (v_1 \cdot 1) \cdots (v_k \cdot 1).$$

For $w \in T$,

$$\begin{array}{rrcl} & w(u_1 \cdot 1) \cdots (u_h \cdot 1)t & = & w(v_1 \cdot 1) \cdots (v_k \cdot 1)u \\ \Rightarrow & (w \cdot (u_1 \cdot 1))w(u_2 \cdot 1) \cdots (u_h \cdot 1)t & = & (w \cdot (v_1 \cdot 1))w(v_2 \cdot 1) \cdots (v_k \cdot 1)u \\ \Rightarrow & (wu_1 \cdot 1)(wu_2 \cdot 1)w \cdots (u_h \cdot 1)t & = & (wv_1 \cdot 1)(wv_2 \cdot 1)w \cdots (v_k \cdot 1)u \\ & \vdots & & \vdots \\ \Rightarrow & (wu_1 \cdot 1)(wu_2 \cdot 1) \cdots (wu_h \cdot 1)wt & = & (wv_1 \cdot 1)(wv_2 \cdot 1) \cdots (wv_k \cdot 1)wu. \end{array}$$

Hence σ_A is left compatible. Thus σ_A^t, where σ_A^t is the transitive closure of σ_A, is a congruence.

If idempotents in E commute, then $\sigma_A = \sigma_A^t$. For if $t, u, v \in T$ and $t \,\sigma_A\, u \,\sigma_A\, v$, then

$$et = fu \text{ and } gu = hv,$$

so that

$$get = gfu = fgu = fhv$$

and hence $t \,\sigma_A\, v$.

Example 7.3.7. If S is left restriction, $A = E$ and $T = S$, then σ_E (as σ_A) is the same σ_E as usual because E is a semilattice.

Now we give the definition of an (A, T)-proper monoid. We will see later in this chapter that this definition can be adopted for left restriction monoids with an extra condition.

Definition 7.3.8. Let $S = AT$, where S is a monoid with submonoids A, T such that T acts on A (as a semigroup) satisfying (LAC). Suppose the idempotents in $E = \{t \cdot 1 : t \in T\}$ commute. We say that S is (A, T)-*proper* if for $a, b \in A$, $t, u \in T$
where $a = a(t \cdot 1)$, $b = b(u \cdot 1)$,

$$at = bu \Leftrightarrow a = b \text{ and } t \,\sigma_A\, u.$$

Note that if S is (A,T)-proper, and $t\,\sigma_A\,u$, then putting $a=b=(t\cdot 1)(u\cdot 1)$ we deduce that

$$(t\cdot 1)(u\cdot 1)t=(t\cdot 1)(u\cdot 1)u,$$

so that $(u\cdot 1)t=(t\cdot 1)u$ because idempotents in E commute.

The following Lemma shows that our notion of (A,T)-properness generalises the notion of a proper left restriction semigroup.

Lemma 7.3.9. *Suppose S is a left restriction monoid with semilattice of projections E. Then S is proper if and only if S is (E,S)-proper.*

Proof. Suppose that S is (E,S)-proper and let $s,t\in S$ be such that $s\,(\widetilde{\mathcal{R}}_E\cap\sigma_E)\,t$. Then $s^+=t^+$ and $s\,\sigma_E\,t$. Now by the above remark, as S is (E,S)-proper,

$$s\,\sigma_E\,t\Rightarrow t^+s=s^+t,$$

so that $s=t$ as $s^+=t^+$ and hence S is proper.

Conversely we suppose that S is proper. Let $e,f\in E$ and $t,s\in S$ with $e\le s^+$ and $f\le t^+$. To prove that S is (E,S)-proper, we need to show that

$$es=ft\Leftrightarrow e=f \text{ and } s\,\sigma_E\,t.$$

First we suppose that $es=ft$. Clearly $s\,\sigma_E\,t$ by definition. Further,

$$e=es^+\,\widetilde{\mathcal{R}}_E\,es=ft\,\widetilde{\mathcal{R}}_E\,ft^+=f.$$

Thus $e=f$.

Conversely, let $e=f$ and $s\,\sigma_E\,t$. Then

$$t^+s\,\sigma_E\,s^+t,$$

and

$$t^+s\,\widetilde{\mathcal{R}}_E\,t^+s^+=s^+t^+\,\widetilde{\mathcal{R}}_E\,s^+t.$$

Therefore $t^+s\,(\widetilde{\mathcal{R}}_E \cap \sigma_E)\,s^+t$ giving $t^+s = s^+t$, because S is proper. Now

$$\begin{array}{rrcll} & t^+s & = & s^+t & \\ \Rightarrow & et^+s & = & es^+t & \\ \Rightarrow & ft^+s & = & es^+t & \text{because } e = f \\ \Rightarrow & fs & = & et & \text{because } e = es^+ \text{ and } f = ft^+ \\ \Rightarrow & es & = & ft & \text{because } e = f. \end{array}$$

Hence S is (E, S)-proper. □

7.4 Covering and embedding theorems for monoids

We now see in the following lemma how to construct an (A, T)-proper monoid.

Lemma 7.4.1. *Let $S = A \rtimes T$, where A and T are monoids such that T is acting on A (as a semigroup) and let $E = \{t \cdot 1 : t \in T\}$ be a set of commuting idempotents. Then T' acts on A' by $(t \cdot 1, t) \odot (a, 1) = (t \cdot a, 1)$ and satisfies (LAC). Let*

$$U = \{(a, t) \in A \times T : a(t \cdot 1) = a\},$$

so that $U = A'T'$ and U is a monoid subsemigroup of $A \rtimes T$. Moreover, $\sigma_{A'} = \imath_{T'}$ and U is (A', T')-proper.

Proof. From Lemma 7.3.2 and 7.3.4, we need only to show that $\sigma_{A'} = \imath_{T'}$ and U is (A', T')-proper.

Clearly $E' = \{(t \cdot 1, 1) : t \in T\}$ is a set of commuting idempotents, because idempotents in E commute. It is easy to check that $\sigma_{A'} = \imath_{T'}$ as for $(t \cdot 1, t), (u \cdot 1, u) \in T'$ if $(t \cdot 1, t)\,\sigma_{A'}\,(u \cdot 1, u)$, then

$$\begin{array}{rrcl} & (p_1 \cdot 1, 1) \cdots (p_m \cdot 1, 1)(t \cdot 1, t) & = & (q_1 \cdot 1, 1) \cdots (q_n \cdot 1, 1)(u \cdot 1, u) \\ \Rightarrow & ((p_1 \cdot 1) \cdots (p_m \cdot 1)(t \cdot 1), t) & = & ((q_1 \cdot 1) \cdots (q_n \cdot 1)(u \cdot 1), u) \\ \Rightarrow & (p_1 \cdot 1) \cdots (p_m \cdot 1)(t \cdot 1) & = & (q_1 \cdot 1) \cdots (q_n \cdot 1)(u \cdot 1) \text{ and } t = u \end{array}$$

and thus $(t \cdot 1, t) = (u \cdot 1, u)$.

Now to show that U is (A', T')-proper, let $(a, 1), (b, 1) \in A'$ and $(t \cdot 1, t), (u \cdot 1, u) \in T'$ such that

$$(a, 1) = (a, 1)\big((t \cdot 1, t) \odot (1_A, 1_T)\big) \text{ and } (b, 1) = (b, 1)\big((u \cdot 1, u) \odot (1_A, 1_T)\big),$$

that is,

$$(a,1) = (a,1)(t \cdot 1, 1) = (a(t \cdot 1), 1) \text{ and } (b,1) = (b,1)(u \cdot 1, 1) = (b(u \cdot 1), 1).$$

Suppose $(a,1)(t \cdot 1, t) = (b,1)(u \cdot 1, u)$. Then

$$\begin{array}{rrcll} & (a(t \cdot 1), t) & = & (b(u \cdot 1), u) & \\ \Rightarrow & a(t \cdot 1) & = & b(u \cdot 1) & \text{and } t = u \\ \Rightarrow & a & = & b & \text{and } t = u \\ \Rightarrow & (a,1) & = & (b,1) & \text{and } (t \cdot 1, t) = (u \cdot 1, u). \end{array}$$

Hence $(t \cdot 1, t)\, \sigma_{A'}\, (u \cdot 1, u)$. On the other hand if

$$(a,1) = (b,1) \text{ and } (t \cdot 1, t)\, \sigma_{A'}\, (u \cdot 1, u)$$

then clearly $(a,1)(t \cdot 1, t) = (b,1)(u \cdot 1, u)$ because $\sigma_{A'} = \imath_{T'}$. Hence U is (A', T')-proper. □

From Lemma 7.4.1, we see how to construct a monoid U which is (A', T')-proper. Now we can state our covering theorem.

Theorem 7.4.2. *Suppose S is a monoid. Let A, T be submonoids of S such that $S = AT$, where T acts on A (as a semigroup) satisfying (LAC) and $E = \{t \cdot 1 : t \in T\}$ is a set of commuting idempotents. Then there is an (A', T')-proper monoid U and an onto morphism $\theta : U \to S$ such that $\theta|_{A'} : A' \to A$ is an isomorphism.*

Proof. From Lemma 7.4.1, we know that $U = \{(a,t) \in A \times T : a(t \cdot 1) = a\}$ is (A', T')-proper. Define

$$\theta : U \to S$$

by $(a,t)\theta = at$. We need to check that θ is a morphism. Let $(a,t), (b,u) \in U$. Then

$$\begin{array}{rcl} ((a,t)(b,u))\theta & = & (a(t \cdot b), tu)\theta \\ & = & a(t \cdot b)tu \\ & = & atbu \\ & = & (a,t)\theta(b,u)\theta. \end{array}$$

Let $s \in S$. Then there exists $a \in A, t \in T$ such that $s = at = (a,t)\theta$. Hence θ is an onto morphism.

Finally, it is easy to check that $\theta|_{A'}$ defined by $(a,1)\theta|_{A'} = a$ is an isomorphism. □

If S is a left restriction monoid, $A = E_S = E$ and $T = S$, then from Examples 7.3.1, 7.3.3, 7.3.5 and 7.3.7 we know that S acts on E by semigroup morphisms where $s \cdot e = (se)^+$,

$$U = \{(e, s) : e = e(s \cdot 1)\} = \{(e, s) : e \leq s^+\},$$

for $s \in S$ and $e \in E$, $(s \cdot e)s = se$ and σ_E (as σ_A) is the same as that defined in Definition 7.2.1, since

$$\{s \cdot 1 : s \in S\} = \{s^+ : s \in S\}.$$

Also from Lemma 7.3.9, we know that S is proper if and only S is (E, S)-proper. Hence we deduce the following from [6]

Corollary 7.4.3. [6, Theorem 6.4] *Let S be a left restriction monoid with semilattice of projections E. Suppose that $S = ES$ where S acts on E by semigroup morphisms by $s \cdot e = (se)^+$. Then there is a proper left restriction monoid U and an onto morphism $\theta : U \to S$.*

We now show that for an (A, T)-proper monoid, there exists a semidirect product in which S embeds.

Lemma 7.4.4. *Suppose S is an (A, T)-proper monoid, where $S = AT$, A, T are submonoids of S, T acts on A (as a semigroup) satisfying (LAC) and $E = \{t \cdot 1 : t \in T\}$ is a set of commuting idempotents. Then for $e, f \in E$ and $t, u \in T$*

$$et = fu \Rightarrow e(t \cdot a) = f(u \cdot a) \text{ for all } a \in A.$$

Proof. Let $e, f \in E$ and $t, u \in T$. Then

$$\begin{array}{rrcll} & et & = & fu & \\ \Rightarrow & eta & = & fua & \\ \Rightarrow & e(t \cdot a)t & = & f(u \cdot a)u & \text{using (LAC)} \\ \Rightarrow & e(t \cdot a)(t \cdot 1)t & = & f(u \cdot a)(u \cdot 1)u & \text{because } t = (t \cdot 1)t \text{ and } u = (u \cdot 1)u \\ \Rightarrow & e(t \cdot a)(t \cdot 1) & = & f(u \cdot a)(u \cdot 1) & \text{because } S \text{ is } (A, T)\text{-proper} \\ \Rightarrow & e(t \cdot a) & = & f(u \cdot a). & \end{array}$$

□

Theorem 7.4.5. *Let $S = AT$ where S is a monoid with submonoids A, T such that T acts on A (as a semigroup) and let $E = \{t \cdot 1 : t \in T\}$. Suppose that S is (A, T)- proper and E*

is central in A, so that, idempotents in E commute . Then there exists a semidirect product $U = \mathcal{A} \rtimes T/\sigma_A$ where $\mathcal{A}$ contains a submonoid $A' \cong A$ and an embedding $\theta : S \to U$ such that $\theta|_A : A \to A' \times \{1\}$ is an isomorphism.

Proof. Let $I = \{H \subseteq A : H\langle E\rangle = H\}$. Then I is a subsemigroup of subsets of A. Let $\mathcal{A} = I^{T/\sigma_A} = \{f : T/\sigma_A \to I\}$ be the set of all maps from T/σ_A to I with pointwise multiplication, that is,

$$[t](fg) = ([t]f)([t]g)$$

for all $[t] \in T/\sigma_A$ and $f, g \in \mathcal{A}$, which is clearly associative. Define an action of T/σ_A on the left of $\mathcal{A}$ by setting

$$[u]([t] \star f) = [ut]f \text{ for all } [u] \in T/\sigma_A.$$

We show that this action is a monoid action. For this let $[t], [u], [v] \in T/\sigma_A$ and $f \in \mathcal{A}$. Then

$$\begin{aligned} [u]\Big([t] \star ([v] \star f)\Big) &= [ut]([v] \star f) \\ &= [(ut)v]f \\ &= [u(tv)]f \\ &= [u]([tv] \star f) \\ &= [u]([t][v] \star f), \end{aligned}$$

so $[t] \star ([v] \star f) = [t][v] \star f$. Also,

$$[u]([1] \star f) = [u1]f = [u]f,$$

implies $[1] \star f = f$. Hence the action is a monoid action.

Finally, we check that the action is by homomorphisms. For this let $[t], [u] \in T/\sigma_A$ and $f, g \in \mathcal{A}$. Then

$$\begin{aligned} [u]([t] \star fg) &= [ut]fg \\ &= [ut]f[ut]g \\ &= [u]([t] \star f)[u]([t] \star g). \end{aligned}$$

Thus $[t] \star fg = ([t] \star f)([t] \star g)$.

For any $a \in A$, we define a map $f_a : T/\sigma_A \to I$ by

$$[u]f_a = \{u' \cdot a : u' \,\sigma_A\, u\}\langle E\rangle.$$

Then $f_a \in \mathcal{A}$. Now define $\varphi : A \to A' = \{f_a : a \in A\}$ by $a\varphi = f_a$. This map is clearly well defined. We check that it is an isomorphism. For this let $a, b \in A$. Then

$$\begin{array}{rrcl} & a\varphi & = & b\varphi \\ \Rightarrow & f_a & = & f_b \\ \Rightarrow & [1]f_a & = & [1]f_b. \end{array}$$

Now as $[1]f_a = \{k \cdot a : k \, \sigma_A \, 1\}\langle E\rangle$, so

$$a = (1 \cdot a)1 \in [1]f_a$$

which implies $a \in [1]f_b$, so that we can write $a = (k \cdot b)e$ where $k \, \sigma_A \, 1$ and $e \in \langle E\rangle$. Now

$$k \, \sigma_A \, 1 \Rightarrow (k \cdot 1)1 = (1 \cdot 1)k \Rightarrow k \cdot 1 = k$$

and so $k \in \langle E\rangle$. Further

$$k \cdot b = (k \cdot b)(k \cdot 1) = (k \cdot b)k = kb.$$

Thus $a = kbe = hb$, where $h = ke \in \langle E\rangle$, because E is central. Similarly $b = h'a$ for some $h' \in \langle E\rangle$. Hence

$$a = hh'a = hh'h'a = h'hh'a = h'hb = h'a = b.$$

It is clear that φ is onto. To see that it is a homomorphism, let $a, b \in A$ and $[u] \in T/\sigma_A$. Then

$$[u]((ab)\varphi) = [u]f_{ab} = \{u' \cdot ab : u \, \sigma_A \, u'\}\langle E\rangle$$

and

$$\begin{array}{rcll} [u](a\varphi b\varphi) & = & ([u](a\varphi))([u](b\varphi)) & \\ & = & ([u]f_a)([u]f_b) & \\ & = & \{u' \cdot a : u \, \sigma_A \, u'\}\langle E\rangle\{u'' \cdot b : u \, \sigma_A \, u''\}\langle E\rangle & \\ & = & \{(u' \cdot a)(u'' \cdot b) : u \, \sigma_A \, u' \, \sigma_A \, u''\}\langle E\rangle & \text{as } E \text{ is central.} \end{array}$$

We need to show that

$$\{u' \cdot ab : u \, \sigma_A \, u'\}\langle E\rangle = \{(u' \cdot a)(u'' \cdot b) : u \, \sigma_A \, u' \, \sigma_A \, u''\}\langle E\rangle.$$

Clearly,

$$\{u' \cdot ab : u \, \sigma_A \, u'\}\langle E\rangle \subseteq \{(u' \cdot a)(u'' \cdot b) : u \, \sigma_A \, u' \, \sigma_A \, u''\}\langle E\rangle.$$

To check the reverse inclusion, let $(u' \cdot a)(u'' \cdot b)g \in \{(u' \cdot a)(u'' \cdot b) : u \,\sigma_A\, u' \,\sigma_A\, u''\}\langle E\rangle$. Then as $u \,\sigma_A\, u' \,\sigma_A\, u''$, we have

$$(u' \cdot 1)u'' = (u'' \cdot 1)u',$$

by the comment succeeding Definition 7.3.8, so that $(u' \cdot 1)(u'' \cdot b) = (u'' \cdot 1)(u' \cdot b)$ using Lemma 7.4.4. Thus

$$\begin{array}{rcll} (u' \cdot a)(u'' \cdot b)g &=& (u' \cdot a)(u' \cdot 1)(u'' \cdot b)g & \\ &=& (u' \cdot a)(u'' \cdot 1)(u' \cdot b)g & \text{because } (u' \cdot 1)(u'' \cdot b) = (u'' \cdot 1)(u' \cdot b) \\ &=& (u' \cdot a)(u' \cdot b)(u'' \cdot 1)g & \text{because } E \text{ is central} \\ &=& (u' \cdot ab)(u'' \cdot 1)g \in \{u' \cdot ab : u \,\sigma_A\, u'\}\langle E\rangle. & \end{array}$$

Thus $(ab)\varphi = f_{ab} = f_a f_b = (a\varphi)(b\varphi)$. Hence φ is an isomorphism.

Next define $\theta : S \to \mathcal{A} \rtimes T/\sigma_A$ by

$$(at)\theta = (f_a, [t]) \quad \text{where } a = a(t \cdot 1).$$

Clearly, θ is well defined because S is (A, T)-proper. Let $at, bu \in S$. Then

$$(at)(bu) = a(tb)u = a(t \cdot b)tu,$$

where $a(t \cdot b)(tu \cdot 1) = a(t \cdot (b(u \cdot 1))) = a(t \cdot b)$, so that

$$\begin{array}{rcl} \big((at)(bu)\big)\theta &=& \big(a(t \cdot b)tu\big)\theta \\ &=& (f_{a(t \cdot b)}, [tu]). \end{array}$$

To prove that $(f_{a(t \cdot b)}, [tu]) = (f_a, [t])(f_b, [u])$ we need to show that $f_{a(t \cdot b)} = f_a([t] \star f_b)$. We see that for any $[u] \in T/\sigma_A$

$$\begin{array}{rcl} [u]\big(f_a([t] \star f_b)\big) &=& [u]f_a[ut]f_b \\ &=& \{u' \cdot a : u' \,\sigma_A\, u\}\langle E\rangle\{w \cdot b : w \,\sigma_A\, ut\}\langle E\rangle \\ &=& \{u' \cdot a : u' \,\sigma_A\, u\}\{w \cdot b : w \,\sigma_A\, ut\}\langle E\rangle \\ &=& \{(u' \cdot a)(w \cdot b) : w \,\sigma_A\, ut \text{ and } u' \,\sigma_A\, u\}\langle E\rangle, \end{array}$$

and

$$\begin{array}{rcl} [u]f_{a(t \cdot b)} &=& \{v \cdot a(t \cdot b) : v \,\sigma_A\, u\}\langle E\rangle \\ &=& \{(v \cdot a)(v \cdot (t \cdot b)) : v \,\sigma_A\, u\}\langle E\rangle \\ &=& \{(v \cdot a)(vt \cdot b) : v \,\sigma_A\, u\}\langle E\rangle. \end{array}$$

Clearly $[u]f_{a(t\cdot b)} \subseteq [u]\Big(f_a([t]f_b)\Big)$.

To check the reverse inclusion, let $(u'\cdot a)(w\cdot b)g \in [u]\Big(f_a([t]\star f_b)\Big)$, where $w\,\sigma_A\,ut$ and $u'\,\sigma_A\,u$, so that $ut\,\sigma_A\,u't$. Now

$$w\,\sigma_A\,u't \Rightarrow (w\cdot 1)u't = (u't\cdot 1)w \quad \text{by the comment succeeding Definition 7.3.8}$$

which further implies that

$$(w\cdot 1)(u't\cdot b) = (u't\cdot 1)(w\cdot b) \quad \text{using Lemma 7.4.4.}$$

Now

$$\begin{aligned}
(u'\cdot a)(w\cdot b)g &= (u'\cdot a(t\cdot 1))(w\cdot b)g && \text{because } a = a(t\cdot 1)\\
&= (u'\cdot a)(u't\cdot 1)(w\cdot b)g\\
&= (u'\cdot a)(w\cdot 1)(u't\cdot b)g && \text{because } (u't\cdot 1)(w\cdot b) = (w\cdot 1)(u't\cdot b)\\
&= (u'\cdot a)(u't\cdot b)(w\cdot 1)g \subseteq [u]f_{a(t\cdot b)} && \text{by our assumption that } E \text{ is central.}
\end{aligned}$$

Thus $f_{a(t\cdot b)} = f_a([t]\star f_b)$ which implies that $(f_{a(t\cdot b)}, [tu]) = (f_a, [t])(f_b, [u])$. Hence

$$((at)(bu))\theta = (at)\theta(bu)\theta.$$

Next to show that θ is one-one, let $s = at, s' = bu \in S$ where $a = a(t\cdot 1), b = b(u\cdot 1)$. Then

$$\begin{aligned}
& (at)\theta = (bu)\theta\\
\Rightarrow\ & (f_a, [t]) = (f_b, [u])\\
\Rightarrow\ & a = b \ \text{ and } \ t\,\sigma_A\,u\\
\Rightarrow\ & at = bu \qquad \text{because } S \text{ is } (A,T)\text{-proper.}
\end{aligned}$$

Hence θ is an embedding.

Finally, as $A' = \{f_a : a \in A\} \cong A$, so clearly $\theta|_A : A \to A' \times \{1\}$ given by $a\theta|_A = (f_a, [1])$ is an isomorphism by above. □

We specialise Theorem 7.4.5 to the following result which appeared in [26] in the context of proper weakly left ample semigroups but can be adapted for left restriction semigroups as decomposed in Theorem 5.3 and the comments in Theorem 7.2 of [6].

Corollary 7.4.6. [6, 26] *Let S be a left restriction monoid with semilattice of projections E and suppose that $S = ES$ where S is acting on E by $s\cdot e = (se)^+$. Suppose that S is proper.*

Then S is embeddable into a semidirect product of a semilattice by a left restriction monoid.

7.5 Covering and embedding theorems for left restriction monoids

In this section we give covering and embedding theorems for certain left restriction monoids. We consider the λ-semidirect product of left restriction monoids and define the notion of (A, T)-properness to prove covering and embedding theorems for left restriction monoids analogous to those in Section 7.4. We comment here that the techniques to prove embedding theorem for left restriction monoids will be little different from those used to prove the embedding theorem (Theorem 7.4.5) for monoids. In Theorem 7.4.5, we chose I to be the set of all subsets of A closed under multiplication with $\langle E \rangle$ (on the right). But in the left restriction case, the elements of I have to satisfy an extra condition.

Let S be a left restriction monoid and A, T be left restriction submonoids of S with identities $1_A, 1_T$, respectively (we drop the subscripts where convenient). Suppose that T acts (as a monoid) on the left of A by (2,1)-endomorphisms. As action is by (2,1)-endomorphisms, so it preserves $^+$, that is for $a \in A$, $t \in T$, we have $(t \cdot a)^+ = t \cdot a^+$.

We know that $t \cdot 1$ is an idempotent for all $t \in T$. Now we notice that $t \cdot 1 \in E_A$ as

$$(t \cdot 1)^+ = t \cdot 1^+ = t \cdot 1.$$

Therefore $\{t{\cdot}1 : t \in T\} \subseteq E_A$ and thus idempotents in E commute. Moreover, $(t{\cdot}1)(u{\cdot}1) \in E_A$ for any $t, u \in T$.

From now on in this chapter, whenever we say that a left restriction semigroup acts on another one, then we mean that the action must preserves $^+$.

From Chapter 6, we know that $A \rtimes^\lambda T = \{(a, t) \in A \times T : t^+ \cdot a = a\}$ is left restriction with semilattice of projections $E_{A \rtimes^\lambda T} = \{(a^+, t^+) : t^+ \cdot a^+ = a^+\}$. Multiplication in $A \rtimes^\lambda T$ is defined by the rule

$$(a, t)(b, u) \;=\; \Big(((tu)^+ \cdot a)(t \cdot b), tu\Big).$$

Now let

$$A' = \{(a, 1) : a \in A\} \;\text{ and }\; T' = \{(t \cdot 1, t) : t \in T\}.$$

It is easy to check that $A' = \{(a, 1) : a \in A\}$ is a left restriction submonoid of $A \rtimes^\lambda T$ with identity $(1_A, 1_T)$ and $A' \cong A$. We now check that $T' = \{(t \cdot 1, t) : t \in T\}$ is also a left restriction submonoid of $A\rtimes^{\lambda}T$ with $T\cong T'$. Clearly $T'\subseteq A\rtimes^{\lambda}T$. Let $(t\cdot 1,t),(u\cdot 1,u)\in T'$. Then

$$\begin{aligned}(t\cdot 1,t)(u\cdot 1,u) &= \Big(((tu)^{+}\cdot(t\cdot 1))(t\cdot(u\cdot 1)),tu\Big)\\ &= \Big(((tu)^{+}\cdot(t\cdot 1))((tu)^{+}(tu)\cdot 1),tu\Big)\\ &= \Big(((tu)^{+}\cdot((t\cdot 1)(tu\cdot 1))),tu\Big)\\ &= \Big(((tu)^{+}\cdot(t\cdot(1(u\cdot 1))),tu\Big)\\ &= \Big(((tu)^{+}\cdot(tu\cdot 1)),tu\Big)\\ &= (tu\cdot 1,tu)\in T'.\end{aligned}$$

It is clear that $t\mapsto(t\cdot 1,t)$ is an embedding, so that T' is a monoid subsemigroup of $A\rtimes^{\lambda}T$.

Now we are going to prove a result analogous to Lemma 7.3.2, where we show that $U=A'T'$ is a monoid subsemigroup of $A\rtimes^{\lambda}T$.

Lemma 7.5.1. *Let A,T be left restriction monoids and let*

$$A\rtimes^{\lambda}T=\{(a,t)\in A\times T: t^{+}\cdot a=a\}$$

be the λ-semidirect product of A and T where T acts (as a monoid) on A by (2,1)-endomorphisms. Then

$$U=\{(a,t)\in A\rtimes^{\lambda}T: a(t\cdot 1)=a\}$$

is a left restriction subsemigroup of $A\rtimes^{\lambda}T$. Also U is a monoid with identity $(1_A,1_T)$ containing the submonoids A',T'. Moreover $U=A'T'$.

Proof. We first check that U is a left restriction submonoid of $A\rtimes^{\lambda}T$. For this let $(a,t),(b,u)\in U$. Then

$$a(t\cdot 1)=a \quad\text{and}\quad b(u\cdot 1)=b.$$

We need to check that $(a,t)(b,u)=\Big(((tu)^{+}\cdot a)(t\cdot b),tu\Big)\in U$. For this we see that

$$\begin{aligned}((tu)^{+}\cdot a)(t\cdot b)(tu\cdot 1) &= ((tu)^{+}\cdot a)(t\cdot b)(t\cdot(u\cdot 1))\\ &= ((tu)^{+}\cdot a)(t\cdot(b(u\cdot 1)))\\ &= ((tu)^{+}\cdot a)(t\cdot b).\end{aligned}$$

Hence $(a,t)(b,u)\in U$.

Next let $(a,t)\in U$. Then as $a=t^{+}\cdot a$ and action preserves $^{+}$, so $a^{+}=t^{+}\cdot a^{+}$. Thus

$$a^{+}(t^{+}\cdot 1)=(t^{+}\cdot a^{+})(t^{+}\cdot 1)=t^{+}\cdot(a^{+}1)=t^{+}\cdot a^{+}=a^{+}$$

and hence $(a,t)^+ = (a^+, t^+) \in U$.

Also for $(a,t) \in U$

$$(a,t)(1,1) = ((t^+ \cdot a)(t \cdot 1), t) = (a(t \cdot 1), t) = (a,t)$$

and

$$(1,1)(a,t) = ((t^+ \cdot 1)a, t) = ((t^+ \cdot 1)(t^+ \cdot a), t) = (t^+ \cdot a, t) = (a,t).$$

Hence U is a submonoid of $A \rtimes^{\lambda} T$. Clearly A' and T' are contained in U. Now for $(a,t) \in U$, we see that

$$\begin{aligned} (a,t) &= (a(t \cdot 1), t) && \text{because } a = a(t \cdot 1) \\ &= ((t^+ \cdot a)(t \cdot 1), t) \\ &= (a,1)(t \cdot 1, t). \end{aligned}$$

Thus each element in U can be written as a product of an element of A' and an element of T'. □

Lemma 7.5.2. *Let A, T be left restriction monoids and let*

$$A \rtimes^{\lambda} T = \{(a,t) \in A \times T : t^+ \cdot a = a\}$$

be the λ-semidirect product of A and T where T acts (as a monoid) on A by (2,1)-endomorphisms. Then T' acts on A' by $(t \cdot 1, t) \odot (a,1) = (t \cdot a, 1)$ such that the action satisfies (LAC).

Proof. The proof that T' acts on A' is similar to Lemma 7.3.4. We now show that the action satisfies (LAC). For this let $(a,1) \in A'$ and $(t \cdot 1, t) \in T'$. Then

$$\begin{aligned} (t \cdot 1, t)(a,1) &= ((t^+ \cdot (t \cdot 1))(t \cdot a), t) \\ &= (t \cdot 1)(t \cdot a), t) \\ &= (t \cdot a, t) \\ &= ((t \cdot a)(t \cdot 1), t) \\ &= (t \cdot a, 1)(t \cdot 1, t) \\ &= \big((t \cdot 1, t) \odot (a,1)\big)(t \cdot 1, t). \end{aligned}$$

Hence the action satisfies (LAC).

□

Definition 7.5.3. Let S be a left restriction monoid with submonoids A, T such that $S = AT$, where T acts (as a monoid) on A by (2,1)-endomorphisms satisfying (LAC). Define

σ_A on T by

$$t\,\sigma_A\,u \Leftrightarrow et = fu \text{ for some } e, f \in E_A.$$

We notice that the above definition is different from Definition 7.3.6 for monoid case in the sense that in Definition 7.3.6, we defined σ_A using $\langle E \rangle$ where $E = \{t \cdot 1 : t \in T\}$. But in the above definition we define σ_A on E_A because we need E to be semilattice later in this section.

It is clear that σ_A is reflexive, symmetric and right compatible. Transitivity of σ_A follows from the fact that idempotents in E_A commute. To check that σ_A is left compatible, let $t\,\sigma_A\,u$. Then

$$et = fu \text{ for some } e, f \in E_A.$$

For $v \in T$

$$\begin{array}{rrcll} & vet & = & vfu & \\ \Rightarrow & (v \cdot e)vt & = & (v \cdot f)vu & \text{using (LAC)} \\ \Rightarrow & (v \cdot e)^+vt & = & (v \cdot f)^+vu & \text{because action preserves }^+ \end{array}$$

and thus $vt\,\sigma_A\,vu$. Hence σ_A is a congruence. We now check that σ_A preserves $^+$. For this let $t, u \in T$ and suppose that $t\,\sigma_A\,u$. Then

$$et = fu \text{ for some } e, f \in E_A.$$

Now $et = fu \Rightarrow et^+ = fu^+$, because S is left restriction and thus, $t^+\,\sigma_A\,u^+$. Hence σ_A preserves $^+$.

To prove covering and embedding theorems for left restriction monoids, we need to suppose that if $s = at \in S$ for $a \in A$ and $t \in T$, where $a = a(t \cdot 1)$, then

$$a(t \cdot 1) = t^+ \cdot a. \tag{P}$$

We see that Condition (P) holds in the following example.

Example 7.5.4. Let S be a left restriction monoid, $A = E_S$, $S = T$. Then S acts on A by $s \cdot e = (se)^+$. From Example 7.3.1 we know that this action is by semigroup morphisms. Now let $e \in E_S$ and $s \in S$. Then

$$e(s \cdot 1) = es^+ = s^+e = (s^+e)^+ = s^+ \cdot e.$$

We now define the notion of (A, T)-properness for left restriction monoids.

Definition 7.5.5. Let S be a left restriction monoid with submonoids A, T such that $S = AT$, where T acts (as a monoid) on A by (2,1)-endomorphisms satisfying (LAC). Suppose that S satisfies condition (P). We say that S is (A,T)-proper if for $a, b \in A$, $t, u \in T$ where $a = a(t \cdot 1)$ and $b = b(u \cdot 1)$

$$at = bu \Leftrightarrow \quad a = b \text{ and } t \,\sigma_A\, u.$$

As S satisfies condition (P), we have

$$a = a(t \cdot 1) = t^+ \cdot a \;\text{ and }\; b = b(u \cdot 1) = u^+ \cdot b.$$

We notice that if S is (A,T)-proper and $t \,\sigma_A\, u$, then putting $a = b = (t \cdot 1)(u \cdot 1)$ we see that

$$a = (t \cdot 1)(u \cdot 1) = (t \cdot 1)(t \cdot 1)(u \cdot 1) = (t \cdot 1)(u \cdot 1)(t \cdot 1) = a(t \cdot 1) = t^+ \cdot a.$$

Similarly $b = b(u \cdot 1) = u^+ \cdot b$. Thus

$$(t \cdot 1)(u \cdot 1)t = (t \cdot 1)(u \cdot 1)u$$

and hence $(u \cdot 1)t = (t \cdot 1)u$.

Lemma 7.5.6. *Let A, T be left restriction monoids and let*

$$A \rtimes^{\lambda} T = \{(a,t) \in A \times T : t^+ \cdot a = a\}$$

be the λ-semidirect product of A and T where T acts (as a monoid) on A by (2,1)-endomorphisms. Then T' acts on A' by

$$(t \cdot 1, t) \odot (a, 1) = (t \cdot a, 1)$$

satisfying (LAC). Let

$$U = \{(a,t) \in A \rtimes^{\lambda} T : a(t \cdot 1) = a\}.$$

Then $U = A'T'$ and U is a monoid left restriction subsemigroup of $A \rtimes^{\lambda} T$. Moreover $\sigma_{A'} = \imath_{T'}$ and U is (A', T')-proper.

Proof. From Lemma 7.5.1 and 7.5.2, we only need to show that $\sigma_{A'} = \imath_{T'}$ and U is (A', T')-proper.

We know that $E = \{t \cdot 1 : t \in T\} \subseteq E_A$, therefore as $A \cong A'$, it is easy to check that

$$E' = (t \cdot 1, 1) : t \in T\}$$

is a set of commuting idempotents, because idempotents in E_A commute. To check that $\sigma_{A'} = \imath_{T'}$, let $(t \cdot 1, t), (u \cdot 1, u) \in T'$ and suppose that $(t \cdot 1, t) \, \sigma_A \, (u \cdot 1, u)$. Then

$$\begin{array}{rrcll} & (e,1)(t \cdot 1, t) & = & (f,1)(u \cdot 1, u) & \quad \text{for some } (e,1),(f,1) \in E_{A'} \\ \Rightarrow & ((t^+ \cdot e)(t \cdot 1), t) & = & ((u^+ \cdot f)(u \cdot 1), u) & \\ \Rightarrow & (t^+ \cdot e)(t \cdot 1) & = & (u^+ \cdot f)(u \cdot 1) & \quad \text{and } t = u \end{array}$$

and thus $(t \cdot 1, t) = (u \cdot 1, u)$. Hence $\sigma_{A'} = \imath_{T'}$.

To show that U is (A', T')-proper let $(a,1)$, $(b,1) \in A'$ and $(t \cdot 1, t)$, $(u \cdot 1, u) \in T'$ such that

$$(a,1) = (a,1)\Big((t \cdot 1, t) \odot (1_A, 1_T)\Big), \quad (b,1) = (b,1)\Big((u \cdot 1, u) \odot (1_A, 1_T)\Big)$$

and

$$(a,1) = (t \cdot 1, t)^+ \odot (a,1), \quad (b,1) = (u \cdot 1, u)^+ \odot (b,1)$$

so that

$$(a,1) = (a,1)(t \cdot 1, 1) = (a(t \cdot 1), 1), \quad (b,1) = (b,1)(u \cdot 1, 1) = (b(u \cdot 1), 1)$$

and

$$(a,1) = (t \cdot 1, t^+) \odot (a,1) = (t^+ \cdot a, 1), \quad (b,1) = (u \cdot 1, u^+) \odot (b,1) = (u^+ \cdot b, 1).$$

Thus

$$a = a(t \cdot 1) = t^+ \cdot a \;\text{ and }\; b = b(u \cdot 1) = u^+ \cdot b.$$

Suppose

$$(a,1)(t \cdot 1, t) = (b,1)(u \cdot 1, u).$$

Then

$$\begin{array}{rrcll} & ((t^+ \cdot a)(t \cdot 1), t) & = & ((u^+ \cdot b)(u \cdot 1), u) & \\ \Rightarrow & (a(t \cdot 1), t) & = & (b(u \cdot 1), u) & \text{because } a = t^+ \cdot a \text{ and } b = u^+ \cdot b \text{ by above} \\ \Rightarrow & a(t \cdot 1) & = & b(u \cdot 1) & \text{and } t = u \\ \Rightarrow & a & = & b & \text{and } t = u \\ \Rightarrow & (a, 1) & = & (b, 1) & \text{and } (t \cdot 1, t) = (u \cdot 1, u). \end{array}$$

Hence $(t \cdot 1, t) \, \sigma_{A'} \, (u \cdot 1, u)$. On the other hand if

$$(a, 1) = (b, 1) \quad \text{and} \quad (t \cdot 1, t) \, \sigma_{A'} \, (u \cdot 1, u)$$

then clearly $(a, 1)(t \cdot 1, t) = (b, 1)(u \cdot 1, u)$ because $\sigma_{A'} = \imath_{T'}$. Hence U is (A', T')-proper. □

Now that we know how to construct a left restriction monoid which is (A', T')-proper, we are in a position to state our covering theorem for left restriction monoids.

Theorem 7.5.7. *Suppose S is a left restriction monoid. Let A, T be submonoids of S such that $S = AT$ where T acts (as a monoid) on A by (2,1)-endomorphisms satisfying (LAC). Let $A \rtimes^{\lambda} T = \{(a, t) \in A \times T : t^+ \cdot a = a\}$ and suppose that $ea = ae$ for all $a \in A, e \in E_T$. Then there is an (A', T')-proper left restriction submonoid U of $A \rtimes^{\lambda} T$ and an onto morphism $\theta : U \to S$ such that $\theta|_{A'} : A' \to A$ is an isomorphism.*

Proof. From Lemma 7.5.6, there is a left restriction submonoid

$$U = \{(a, t) \in A \times T : a(t \cdot 1) = a\}$$

of $A \rtimes^{\lambda} T$ which is (A', T')-proper.

Now define $\theta : U \to S$ by $(a, t)\theta = at$.

To check that θ is onto, let $s \in S$, where $s = at$ for $a \in A$ and $t \in T$. We see that

$$\begin{array}{rcll} at & = & at^+t & \\ & = & t^+at & \text{by supposition that } ea = ae \text{ for all } a \in A, e \in E_T \\ & = & (t^+ \cdot a)t^+t & \text{using (LAC)} \\ & = & (t^+ \cdot a)t & \\ & = & (t^+ \cdot a1)t & \\ & = & (t^+ \cdot a)(t^+ \cdot 1)(t \cdot 1)t. & \end{array}$$

We notice that $at = bt$, where $b = (t^+ \cdot a)(t^+ \cdot 1)(t \cdot 1)$. We see that

$$\begin{aligned} t^+ \cdot b &= t^+ \cdot ((t^+ \cdot a)(t^+ \cdot 1)(t \cdot 1)) \\ &= (t^+ \cdot (t^+ \cdot a))(t^+ \cdot (t^+ \cdot 1))(t^+ \cdot (t \cdot 1)) \\ &= (t^+ \cdot a)(t^+ \cdot 1)(t \cdot 1) \end{aligned}$$

and thus $t^+ \cdot b = b$. Also

$$b(t \cdot 1) = (t^+ \cdot a)(t^+ \cdot 1)(t \cdot 1)(t \cdot 1) = (t^+ \cdot a)(t^+ \cdot 1)(t \cdot 1) = b.$$

Thus we can assume that $s = at$ where $a = t^+ \cdot a = a(t \cdot 1)$. Hence $(a, t) \in U$ and $s = at = (a, t)\theta$ and thus θ is onto.

To check that θ is a morphism, let $(a, t), (b, u) \in U$. Then

$$\begin{aligned} \big((a,t)(b,u)\big)\theta &= \big(((tu)^+ \cdot a)(t \cdot b), tu\big)\theta \\ &= ((tu)^+ \cdot a)(t \cdot b)tu \\ &= ((tu)^+ \cdot a)tbu \\ &= ((tu)^+ \cdot a)tu^+bu && \text{as } u^+b = (u^+ \cdot b)u^+ = bu^+ \\ &= ((tu)^+ \cdot a)(tu)^+tbu \\ &= (tu)^+atbu \\ &= a(tu)^+tbu && \text{by supposition that } ea = ae \text{ for all } a \in A, e \in E_T \\ &= atu^+bu \\ &= atbu \\ &= (a,t)\theta(b,u)\theta. \end{aligned}$$

Hence θ is an onto morphism. Also it is clear that $A' \cong A$. □

We now give a result that is needed in the proof of the embedding theorem which is analogous to Lemma 7.4.4.

Lemma 7.5.8. *Let $S = AT$, where S is a left restriction monoid with submonoids A, T such that T acts (as a monoid) on A by (2,1)-endomorphisms. Suppose that S is (A, T)-proper. Then for $e, f \in E_A$, and $t, u \in T$*

$$et = fu \Rightarrow e(t \cdot a) = f(u \cdot a) \text{ for all } a \in A.$$

Proof. Let $e, f \in E_A$ and $t, u \in T$. Then

$$\begin{array}{rrcll} & et & = & fu & \\ \Rightarrow & eta & = & fua & \\ \Rightarrow & e(t\cdot a)t & = & f(u\cdot a)u & \text{using (LAC)} \\ \Rightarrow & e(t\cdot a)(t\cdot 1)t & = & f(u\cdot a)(u\cdot 1)u & \text{because } t=(t\cdot 1)t \text{ and } u=(u\cdot 1)u \\ \Rightarrow & e(t\cdot a)(t\cdot 1) & = & f(u\cdot a)(u\cdot 1) & \text{because } S \text{ is } (A,T)\text{-proper} \\ \Rightarrow & e(t\cdot a) & = & f(u\cdot a). & \end{array}$$

□

Next we give the main theorem of this section, that is, the embedding theorem for left restriction monoids.

Theorem 7.5.9. *Let S be a left restriction monoid with submonoids A, T such that $S = AT$ where T acts (as a monoid) on A by (2,1)-endomorphisms. Suppose that S is (A, T)-proper. Assume that E_A and E_T are central in A. Then there exists a λ-semidirect product*

$$\mathcal{U} = \mathcal{A} \rtimes^{\lambda} T/\sigma_A$$

where $\mathcal{A}$ contains a submonoid $A' \cong A$ and an embedding $\theta : S \to \mathcal{U}$ such that $\theta|_A : A \to A' \times \{1\}$ is an isomorphism.

Proof. Let

$$I = \{U \subseteq A : E_A U = U, a^+ b = b^+ a \text{ for all } a, b \in U\ \}^1.$$

We first check that I is left restriction with semilattice of projections

$$E_I = \{F : F \text{ is an ideal in } E_A\}$$

and for $U \in I$

$$U^+ = \{u^+ : u \in I\}.$$

First we show that E_I is a semilattice. For this let $F \in E_I$. Then clearly $E_A F = F$ because F is an ideal in E_A. Also for any $f, g \in F$

$$f^+ g = g^+ f.$$

Thus $E_I \subseteq I$. Next let $F, F' \in E_I$. Then

$$E_A(FF') = (E_A F)F' = FF'$$

because F is an ideal in E_A. Next for any $e, f \in F$ and $g, h \in F'$, it is clear that

$$(ef)^+ gh = (gh)^+ ef.$$

Hence E_I is a semilattice.

Next let $U, V \in I$. Then

$$E_A(UV) = (E_A U)V = UV.$$

Let $ab, cd \in UV$ where $a, c \in U$, $b, d \in V$. Then

$$\begin{aligned}
(ab)^+ cd &= (ab)^+ a^+ cd \\
&= (ab)^+ c^+ ad \quad \text{because } a^+ c = c^+ a \\
&= c^+ (ab)^+ ad \\
&= c^+ ab^+ d \\
&= c^+ ad^+ b \quad \text{because } b^+ d = d^+ b \\
&= c^+ (ad)^+ ab \\
&= (c^+ ad)^+ ab \\
&= (a^+ cd)^+ ab \\
&= a^+ (cd)^+ ab \\
&= (cd)^+ ab.
\end{aligned}$$

Thus $UV \in I$.

Next let $U^+ = \{u^+ : u \in U\}$. Then

$$eu^+ = (eu)^+ \in U^+,$$

and so $E_A U^+ \subseteq U^+$. Also for $u^+ \in U^+$, we have

$$u^+ = (u^+ \cdot 1)u^+ \in E_A U^+$$

so that $U^+ \subseteq E_A U^+$. Thus $E_A U^+ = U^+$. Also $e^+ f = f^+ e$ for all $e, f \in U^+$. Hence $U^+ \in I$.

Now we see that

$$U^+ U = \{u^+ v : u, v \in U\} \subseteq E_A U = U.$$

Clearly $U \subseteq U^+U$. Thus $U = U^+U$.

Now if $F \in E_I$ and $FU = U$, then $fu \in U$ for all $f \in F, u \in U$. Therefore

$$(fu)^+ = fu^+ \in U^+ \Rightarrow FU^+ \subseteq U^+.$$

Next let $u \in U$. Then $u = fv$ for some $f \in F$ and $v \in U$, so $u^+ = fv^+ \in FU^+$ and hence $U^+ = FU^+$.

It is easy to see that the left congruence condition holds in I. We check that the left ample condition holds. For this we see that

$$(UV)^+U = \{(uv)^+z : u, z \in U,\, v \in V\}.$$

Then

$$\begin{aligned}(uv)^+z &= (uv)^+u^+z \\ &= (uv)^+z^+u \\ &= z^+(uv)^+u \\ &= z^+uv^+ \\ &\subseteq UV^+.\end{aligned}$$

It is easy to check that $UV^+ \subseteq (UV)^+U$. Hence $(UV)^+U = UV^+$. Thus I is left restriction.

Let $\mathcal{A} = I^{T/\sigma_A} = \{f : T/\sigma_A \to I\}$. Then $\mathcal{A}$ is a left restriction monoid with identity defined by $[t]\bar{1} = 1$ and for all $[t] \in T/\sigma_A$ and for all $f, g \in \mathcal{A}$,

$$[t](fg) = ([t]f)([t]g) \;\text{ and }\; [t]f^+ = ([t]f)^+.$$

Define the action of T/σ_A on $\mathcal{A}$ by

$$[u]([t] \star f) = [ut]f \;\text{ for all } [u] \in T/\sigma_A\ .$$

Similar by the proof of Theorem 7.4.5, it is easy to see that this action is a monoid action. We check that the action preserves $^+$. Let $f \in \mathcal{A}$ and $[u], [t] \in T/\sigma_A$. Then

$$[u]([t] \star f)^+ = \Big([u]([t] \star f)\Big)^+ = ([ut]f)^+ = [ut]f^+ = [u]([t] \star f^+).$$

Thus $[t] \star f^+ = ([t] \star f)^+$. Also $[u]([t] \star \bar{1}) = [ut]\bar{1} = 1 = [u]\bar{1}$ implies $[t] \star \bar{1} = \bar{1}$. Thus the action of T/σ_A on $\mathcal{A}$ is by $(2,1,0)$-morphisms.

Let $f_a : T/\sigma_A \to \mathcal{P}(A)$, where $\mathcal{P}(A)$ is the power set of A, be defined by

$$f_a : [t]f_a = E_A\{t' \cdot a : t' \,\sigma_A\, t\}$$

where $[t] \in T/\sigma_A$. We want to show that $f_a \in \mathcal{A}$, that is, $[t]f_a \in I$ for all $t \in T$. For this let

$$t \,\sigma_A\, t' \,\sigma_A\, t''.$$

Then

$$\begin{array}{rcll} (t' \cdot 1)t'' & = & (t'' \cdot 1)t' & \text{because } S \text{ is } (A,T)\text{-proper} \\ \Rightarrow \quad (t' \cdot 1)(t'' \cdot a) & = & (t'' \cdot 1)(t' \cdot a) & \text{using Lemma 7.5.8.} \end{array}$$

We want $(e(t' \cdot a))^+(f(t'' \cdot a)) = (f(t'' \cdot a))^+ e(t' \cdot a)$, where $e, f \in E_A$. For this we see that

$$\begin{array}{rcll} (f(t'' \cdot a))^+ e(t' \cdot a) & = & (f(t'' \cdot 1)(t'' \cdot a))^+ e(t' \cdot a)^+(t' \cdot a) & \\ & = & (f(t'' \cdot 1)(t'' \cdot a)^+)^+ e(t' \cdot a)^+(t' \cdot a) & \\ & = & f(t'' \cdot 1)(t'' \cdot a)^+ e(t' \cdot a)^+(t' \cdot a) & \\ & = & f(t'' \cdot a)^+(t'' \cdot 1) e(t' \cdot a)^+(t' \cdot a) & \\ & = & (f(t'' \cdot a))^+(e(t' \cdot a))^+(t'' \cdot 1)(t' \cdot a) & \\ & = & (f(t'' \cdot a))^+(e(t' \cdot a))^+(t' \cdot 1)(t'' \cdot a) & \text{because } (t'' \cdot 1)(t' \cdot a) = (t' \cdot 1)(t'' \cdot a) \\ & = & (f(t'' \cdot a))^+(t' \cdot 1)(e(t' \cdot a))^+(t'' \cdot a) & \\ & = & (f(t'' \cdot a))^+((t' \cdot 1)e(t' \cdot a))^+(t'' \cdot a) & \\ & = & (f(t'' \cdot a))^+(e(t' \cdot a))^+(t'' \cdot a) & \\ & = & (e(t' \cdot a))^+(f(t'' \cdot a)). & \end{array}$$

Thus $f_a \in \mathcal{A}$. Define $\theta : S \to \mathcal{A} \rtimes^{\lambda} T/\sigma_A$ by

$$(at)\theta = ([t]^+ \star f_a, [t]) \quad \text{where } a = a(t \cdot 1) = t^+ \cdot a.$$

Clearly θ is well defined because S is (A,T)-proper. Now to show that θ is one-one, let

$$\begin{array}{rcll} (at)\theta & = & (bu)\theta & \text{where } a = a(t \cdot 1) = t^+ \cdot a,\ b = b(u \cdot 1) = u^+ \cdot b \\ \Rightarrow \quad ([t]^+ \star f_a, [t]) & = & ([u]^+ \star f_b, [u]) & \\ \Rightarrow \quad [t]^+ \star f_a & = & [u]^+ \star f_b \text{ and } [t] = [u]\ . & \end{array}$$

As $[1]([t]^+ \star f_a) = E_A\{s \cdot a : s\,\sigma_A\,t^+\}$, we have

$$\begin{array}{rrcll} & a & = & a(t \cdot 1) & \\ & & = & t^+ \cdot a & \text{by condition (P) as } S \text{ is } (A,T)\text{-proper} \\ & & = & 1(t^+ \cdot a) \in [1]([t]^+ \star f_a) & \\ \Rightarrow & a & \in & [1]([u]^+ \star f_b) & \\ \Rightarrow & a & = & g(s \cdot b) & \end{array}$$

where $s\,\sigma_A\,u^+$ and $g \in E_A$. Now

$$s\,\sigma_A\,u^+ \Rightarrow (s \cdot 1)u^+ = (u^+ \cdot 1)s \;\text{ because } S \text{ is } (A,T)\text{-proper}$$

so that $(s \cdot 1)(u^+ \cdot b) = (u^+ \cdot 1)(s \cdot b)$, by Lemma 7.5.8. Also

$$s\,\sigma_A\,u^+ \Rightarrow s^+\,\sigma_A\,u^+\,\sigma_A\,s,$$

so that

$$(s \cdot 1)s^+ = (s^+ \cdot 1)s = (s^+ \cdot 1)s^+s = s^+1s = s^+s = s$$

implies $s = (s \cdot 1)s^+ \in E_S$, so that $s \in E_T$. Therefore

$$\begin{array}{rcll} a & = & g(s \cdot b) & \\ & = & g(su^+ \cdot b) & \\ & = & g(u^+s \cdot b) & \text{because } s, u^+ \in E_T \\ & = & g(u^+ \cdot (1(s \cdot b))) & \\ & = & g(u^+ \cdot 1)(u^+s \cdot b) & \\ & = & g(u^+ \cdot 1)(su^+ \cdot b) & \text{because } s, u^+ \in E_T \\ & = & g(u^+ \cdot 1)(s \cdot b) & \\ & = & g(s \cdot 1)(u^+ \cdot b) & \text{because } (s \cdot 1)(u^+ \cdot b) = (u^+ \cdot 1)(s \cdot b) \\ & = & g(s \cdot 1)b & \end{array}$$

gives $a = eb$ for some $e \in E_A$ and similarly $b = fa$ for some $f \in E_A$. Hence

$$b = fa = feb = efeb = efa = eb = a.$$

Now as S is (A,T)-proper, so $a = b$ and $t\,\sigma_A\,u$ implies that $at = bu$. Hence θ is one-one.

Next to show that θ is a morphism, let $at, bu \in S$ where $a = a(t \cdot 1) = t^+ \cdot a$ and

$b = b(u \cdot 1) = u^+ \cdot b$. Then by the proof of Theorem 7.5.7 we know that $(at)(bu) = ((tu)^+ \cdot a)(t \cdot b)tu$. Also

$$\begin{aligned} ((tu)^+ \cdot a)(t \cdot b)(tu \cdot 1) &= ((tu)^+ \cdot a)(t \cdot b)(t \cdot (u \cdot 1)) \\ &= ((tu)^+ \cdot a)(t \cdot b(u \cdot 1)) \\ &= ((tu)^+ \cdot a)(t \cdot b) \qquad \text{because } b = b(u \cdot 1). \end{aligned}$$

Thus

$$\big((at)(bu)\big)\theta = \big(((tu)^+ \cdot a)(t \cdot b)tu\big)\theta = \big([tu]^+ \star f_{((tu)^+ \cdot a)(t \cdot b)}, [tu]\big). \tag{A}$$

On the other hand,

$$\begin{aligned} (at)\theta(bu)\theta &= ([t]^+ \star f_a, [t])([u]^+ \star f_b, [u]) \\ &= \big((([t][u])^+ \star ([t]^+ \star f_a))([t] \star ([u]^+ \star f_b)), [t][u]\big) \\ &= \big(([tu]^+[t]^+ \star f_a)([t]([u]^+ \star f_b)), [tu]\big), \end{aligned}$$

so that

$$(at)\theta(bu)\theta = \big(([tu]^+ \star f_a)([tu^+] \star f_b), [tu]\big). \tag{B}$$

We want to show that (A)= (B). For this let $[v] \in T/\sigma_A$. Then

$$\begin{aligned} [v]([tu]^+ \star f_{((tu)^+ \cdot a)(t \cdot b)}) &= [v(tu)^+] f_{((tu)^+ \cdot a)(t \cdot b)} \\ &= E_A\{k \cdot (((tu)^+ \cdot a)(t \cdot b)) : k \,\sigma_A\, v(tu)^+\} \\ &= E_A\{(k(tu)^+ \cdot a)(kt \cdot b) : k \,\sigma_A\, v(tu)^+\} \\ &\subseteq E_A\{(k(tu)^+ \cdot a) : k \,\sigma_A\, v(tu)^+\}\{(kt \cdot b) : k \,\sigma_A\, v(tu)^+\}. \end{aligned}$$

Now

$$[v]([tu]^+ \star f_a) = E_A\{k \cdot a : k \,\sigma_A\, v(tu)^+\}$$

and

$$[v]([tu^+] \star f_b) = E_A\{h \cdot b : h \,\sigma_A\, v(tu^+)\}.$$

Also

$$k \,\sigma_A\, v(tu)^+ \Rightarrow k(tu)^+ \,\sigma_A\, v(tu)^+ \,\sigma_A\, k$$

and

$$h \,\sigma_A\, v(tu)^+ \Rightarrow ht \,\sigma_A\, v(tu)^+ t = v(tu^+).$$

Thus $[v]([tu]^+ \star f_{((tu)^+ \cdot a)(t \cdot b)}) \subseteq [v]\big(([tu]^+ \star f_a)([tu^+] \star f_b)\big)$.

To prove the reverse inclusion, let $w \in [v]\big(([tu]^+ \star f_a)([tu^+] \star f_b)\big)$. Then $w = g(k \cdot a)(h \cdot b)$

where

$$k\,\sigma_A\,v(tu)^+ \text{ and } h\,\sigma_A\,v(tu^+).$$

Now $h\,\sigma_A\,v(tu^+) = v(tu)^+t\,\sigma_A\,kt$, so that

$$(h\cdot 1)kt = (kt\cdot 1)h \quad \text{because } S \text{ is } (A,T)\text{-proper}$$

which further implies

$$(h\cdot 1)(kt\cdot b) = (kt\cdot 1)(h\cdot b) \;\text{ using Lemma 7.5.8.}$$

Also

$$k\cdot a = k\cdot(a(t\cdot 1)) = (k\cdot a)(kt\cdot 1).$$

Thus

$$\begin{aligned} w &= g(k\cdot a)(kt\cdot 1)(h\cdot b) \\ &= g(k\cdot a)(h\cdot 1)(kt\cdot b) \quad \text{because } (h\cdot 1)(kt\cdot b) = (kt\cdot 1)(h\cdot b) \\ &= g(h\cdot 1)(k\cdot a)(kt\cdot b) \quad \text{because } E_A \text{ is central} \\ &= g'(k\cdot a)(kt\cdot b). \end{aligned}$$

Now

$$\begin{aligned} kt\cdot b &= kt\cdot(b(u\cdot 1)) \\ &= (kt\cdot b)(ktu\cdot 1) \\ &= (kt\cdot b)(k(tu)^+\cdot(tu\cdot 1)) \\ &= (kt\cdot b)(k(tu)^+\cdot(1(tu\cdot 1))) \\ &= (kt\cdot b)(k(tu)^+\cdot 1)(k(tu)^+\cdot(tu\cdot 1)) \\ &= (k(tu)^+\cdot 1)(kt\cdot b)(kt\cdot(u\cdot 1)) \\ &= (k(tu)^+\cdot 1)(kt\cdot(b(u\cdot 1))) \\ &= (k(tu)^+\cdot 1)(kt\cdot b). \end{aligned}$$

Also as

$$k\,\sigma_A\,v(tu)^+ \Rightarrow k(tu)^+\,\sigma_A\,k,$$

therefore

$$(k(tu)^+\cdot 1)k = (k\cdot 1)k(tu)^+ \quad \text{because } S \text{ is } (A,T)\text{-proper}$$

which implies $(k(tu)^+ \cdot 1)(k \cdot a) = (k \cdot 1)(k(tu)^+ \cdot a)$ using Lemma 7.5.8. Thus

$$\begin{aligned} w &= g'(k \cdot a)(kt \cdot b) \\ &= g'(k \cdot a)(k(tu)^+ \cdot 1)(kt \cdot b) \quad \text{because } kt \cdot b = (k(tu)^+ \cdot 1)(kt \cdot b) \\ &= g'(k \cdot 1)(k(tu)^+ \cdot a)(kt \cdot b) \quad \text{because } (k(tu)^+ \cdot 1)(k \cdot a) = (k \cdot 1)(k(tu)^+ \cdot a) \\ &\subseteq [v]([tu]^+ \star f_{((tu)^+ \cdot a)(t \cdot b)}). \end{aligned}$$

Hence $((at)(bu))\theta = (at)\theta(bu)\theta$. To finish the proof that θ is an embedding, finally we want to show that θ preserves $^+$. For this let $s = at \in S$, where $a = a(t \cdot 1) = t^+ \cdot a$. Then

$$\begin{aligned} a^+ &= (a(t \cdot 1))^+ \\ &= ((t \cdot 1)a)^+ \quad \text{because } E_A \text{ is central in } A \\ &= ((t \cdot 1)a^+)^+ \\ &= (t \cdot 1)a^+ \\ &= a^+(t \cdot 1) \end{aligned}$$

and $a^+ = (a(t \cdot 1))^+ = (t^+ \cdot a)^+ = t^+ \cdot a^+$. Now

$$\begin{aligned} ((at)\theta)^+ &= ([t]^+ \star f_a, [t])^+ \\ &= (([t]^+ \star f_a)^+, [t]^+) \\ &= ([t]^+ \star f_a^+, [t]^+). \end{aligned}$$

Also

$$\begin{aligned} (at)^+\theta &= (a(t \cdot 1)t)^+\theta \\ &= ((t^+ \cdot a)t)^+\theta \quad \text{because } a(t \cdot 1) = t^+ \cdot a \\ &= ((t^+ \cdot a)t^+)^+\theta \\ &= (t^+a)^+\theta \quad \text{using (LAC)} \\ &= (t^+a^+)\theta \\ &= (a^+t^+)\theta \\ &= ([t]^+ \star f_{a^+}, [t]^+). \end{aligned}$$

Now to show that $((at)\theta)^+ = (at)^+\theta$, we need to check that

$$[t]^+ \star f_a^+ = [t]^+ \star f_{a^+}.$$

For this we see that

$$\begin{aligned}[u]([t]^+ \star f_a^+) &= [ut^+]f_a^+ \\ &= ([ut^+]f_a)^+ \\ &= (E_A\{g \cdot a : g\, \sigma_A\, ut^+\})^+ \\ &= E_A\{g \cdot a^+ : g\, \sigma_A\, ut^+\} \\ &= [u]([t]^+ \star f_{a^+}).\end{aligned}$$

Hence $((at)\theta)^+ = (at)^+\theta$ and thus θ preserves $^+$. Hence θ is an embedding. $\square$

The work in this chapter is not polished but forms the basis for further investigations. We are still in process of developing this theory and trying to find some natural examples.

Chapter 8

λ-Zappa-Szép products of restriction semigroups

The Zappa-Szép product of two inverse semigroups is not inverse in general. This is even the case for semidirect products as we have mentioned in Chapter 6. Gilbert and Wazzan generalised the concept of λ-semidirect product to λ-Zappa-Szép products [25, 77]. They found a subset of the Zappa-Szép product $S \bowtie T$ of two inverse semigroups S and T with a modified binary operation such that the resulting subset is a groupoid. In the special case where $S = E$ a semilattice and $T = G$ a group, they ordered this groupoid to become inductive and hence obtained an inverse semigroup. In order to define new modified binary operation on this subset, they used the equivalence of categories between inverse semigroups and inductive groupoids. This equivalence is established by the Ehresmann-Schein-Nambooripad Theorem [45].

In this chapter we consider λ-Zappa-Szép products of restriction semigroups. The Zappa-Szép product of two restriction semigroups S and T need not be restriction in general. By taking a certain set of pairs of $S \times T$, we construct a category. In case where S is a semilattice and T is a monoid, we order this category so that it becomes inductive, yielding thus a restriction semigroup via the use of the standard pseudo-product. Our constructions specialise to those of Gilbert and Wazzan [25] in the inverse case.

8.1 λ-Zappa-Szép product of inverse semigroups

We begin this section by discussing the λ-Zappa-Szép products of inverse semigroups. For convenience of the reader, we give results for λ-Zappa-Szép products of inverse semigroups from Wazzan's Thesis [77], although they can be found in a joint paper of Gilbert and Wazzan [25]. We first see that by considering a special subset of the Zappa-Szép product of any two inverse semigroups S and T we can construct a groupoid.

Theorem 8.1.1. [77, Theorem 4.5.6] *Let $Z = S \bowtie T$ be a Zappa-Szép product of inverse semigroups S and T. Put*

$$B_{\bowtie}(Z) = \{(a,t) \in S \times T : tt^{-1} \cdot a^{-1} = a^{-1},\ tt^{-1} \cdot a^{-1}a = a^{-1}a,\ (t^{-1})^{a^{-1}a} = t^{-1},\ (tt^{-1})^{a^{-1}a} = tt^{-1}\}.$$

Then $B_{\bowtie}(Z)$ is a groupoid under the restriction of the binary operation in Z with set of local identities

$$E(B_{\bowtie}(Z)) = \{(e,f) \in E(S) \times E(T) : f \cdot e = e, f^e = f\},$$

where for $(a,t) \in B_{\bowtie}(Z)$

$$\mathbf{d}(a,t) = (aa^{-1}, (tt^{-1})^{a^{-1}}),\ \mathbf{r}(a,t) = (t^{-1} \cdot (a^{-1}a), t^{-1}t),$$

and

$$(a,t)^{-1} = (t^{-1} \cdot a^{-1}, (t^{-1})^{a^{-1}}).$$

In this general case, it is not possible to order this groupoid to become inductive because the partially ordered set of identities do not form a meet semilattice. Thus the construction does not proceed any further. However Gilbert and Wazzan made this process work in a special case where $S = E$ is a semilattice and $T = G$ is a group. In this case $E(B_{\bowtie}(Z)) \cong E$ and by extending the ordering on E to an ordering on $B_{\bowtie}(Z)$, they obtained an inductive groupoid and hence an inverse semigroup.

Theorem 8.1.2. [77, Proposition 4.5.19] *Let $Z = E \bowtie G$ where E is a semilattice and G is a group. Then*

$$B_{\bowtie}(Z) = \{(e,g) \in E \times G : (g^{-1})^e = g^{-1}\}$$

is an inductive groupoid under the restriction of the binary operation in Z with set of local identities

$$E(B_{\bowtie}(Z)) = \{(e,1) : e \in E\}$$

where

$$\mathbf{d}(e,g) = (e,1),\ \mathbf{r}(e,g) = (g^{-1} \cdot e, 1)\ \textit{ and }\ (e,g)^{-1} = (g^{-1} \cdot e, g^{-1}).$$

Also the partial order on $B_{\bowtie}(Z)$ is defined by the rule

$$(e,g) \leq (f,h) \Leftrightarrow e \leq f \text{ and } g = h^{h^{-1}\cdot e}$$

and as a semilattice, $E(B_{\bowtie}(Z)) \cong E$. For $(e,g) \in B_{\bowtie}(Z)$ and $(f,1) \in E_{B_{\bowtie}(Z)}$, the restriction is defined by

$$_{(f,1)}|(e,g) = (f, g^{g^{-1}\cdot f})$$

and co-restriction is defined by

$$(e,g)|_{(f,1)} = (e(g \cdot f), g^f).$$

By defining a pseudo-product on $B_{\bowtie}(Z)$ we obtain an inverse semigroup which follows from its construction as a pseudo-product on an inductive groupoid.

Theorem 8.1.3. [77, Theorem 4.5.20] *Let E be a semilattice, G be a group and $Z = E \bowtie G$. Then*

$$B_{\bowtie}(Z) = \{(e,g) \in E \times G : (g^{-1})^e = g^{-1}\}$$

is an inverse semigroup with multiplication defined by

$$(e,g)(f,h) = \Big(e(g \cdot f), g^f h^{h^{-1}g^{-1}\cdot e}\Big).$$

8.2 Comparing the inverse and restriction cases

We want to extend the above ideas to Zappa-Szép products of restriction semigroups. For this we need to use the notion of *double action*.

Definition 8.2.1. Let S and T be restriction semigroups and suppose that $Z = S \bowtie T$ is a Zappa-Szép product of S and T. We say that S and T *act doubly* on each other if we have two extra maps

$$S \times T \to T,\ (s,t) \mapsto {}^{s}t \text{ and } S \times T \to S,\ (s,t) \mapsto s \circ t$$

such that for all $s, s' \in S, t, t' \in T$:

$$(1)\ {}^{ss'}t = {}^{s}({}^{s'}t); \quad (2)\ s \circ tt' = (s \circ t) \circ t'$$

and the actions satisfy the following compatibility conditions:

$$\begin{aligned} (t \cdot s) \circ t = &\quad s \circ t^* = t^* \cdot s \\ t \cdot (s \circ t) = &\quad s \circ t^+ = t^+ \cdot s \end{aligned} \tag{CP1}$$

and

$$\begin{aligned} ({}^s t)^s = t^{s^*} = {}^{s^*}t \\ {}^s(t^s) = t^{s^+} = {}^{s^+}t. \end{aligned} \tag{CP2}$$

We will consider the Zappa-Szép product $Z = S \bowtie T$ of two restriction semigroups S and T where S and T are acting doubly on each other. By choosing a certain set of pairs of $S \times T$, we will construct a category. Let

$$V_{\bowtie}(Z) = \{(a,t) \in S \times T : t^+ \cdot a^* = a^*, (t^+)^{a^*} = t^+, {}^a t^+ \cdot a = a, t^{a^* \circ t} = t\}.$$

We first prove that if S and T are inverse semigroups and are regarded as restriction semigroups in the usual way, then $B_{\bowtie}(Z) = V_{\bowtie}(Z)$.

Lemma 8.2.2. *Let S and T be inverse semigroups regarded as restriction semigroups in the usual way. Suppose $Z = S \bowtie T$ is a Zappa-Szép product of S and T. Define two extra maps*

$$S \times T \to S,\ (s,t) \mapsto s \circ t \text{ and } S \times T \to T,\ (s,t) \mapsto {}^s t$$

by $s \circ t = t^{-1} \cdot s$, ${}^s t = t^{s^{-1}}$, respectively. Then (CP1) and (CP2) are satisfied. Moreover $B_{\bowtie}(Z) = V_{\bowtie}(Z)$.

Proof. We first check that (CP1) is satisfied. Let $s \in S$ and $t \in T$. Then

$$(t \cdot s) \circ t = t^{-1} \cdot (t \cdot s) = t^{-1} t \cdot s = t^* \cdot s = s \circ t^*$$

and

$$t \cdot (s \circ t) = t \cdot (t^{-1} \cdot s) = tt^{-1} \cdot s = t^+ \cdot s = s \circ t^+.$$

Hence (CP1) is satisfied. The proof for (CP2) is dual. By definition

$$B_{\bowtie}(Z) = \{(a,t) \in S \times T : tt^{-1} \cdot a^{-1} = a^{-1},\ tt^{-1} \cdot a^{-1}a = a^{-1}a,\ (t^{-1})^{a^{-1}a} = t^{-1},\ (tt^{-1})^{a^{-1}a} = tt^{-1}\}$$

and

$$V_{\bowtie}(Z) = \{(a,t) \in S \times T : tt^{-1} \cdot a^{-1}a = a^{-1}a,\ (tt^{-1})^{a^{-1}a} = tt^{-1},\ (tt^{-1})^{a^{-1}} \cdot a = a,\ t^{t^{-1} \cdot a^{-1}a} = t\}.$$

Let $(a,t) \in B_{\bowtie}(Z)$. From [77, Lemma 4.5.3], we know that

$$t^{t^{-1} \cdot a^{-1}a} = t \text{ and } (tt^{-1})^{a^{-1}} \cdot a = a.$$

Thus $(a,t) \in V_{\bowtie}(Z)$, so that $B_{\bowtie}(Z) \subseteq V_{\bowtie}(Z)$.

Conversely, let $(a,t) \in V_{\bowtie}(Z)$. We want to show that $tt^{-1} \cdot a^{-1} = a^{-1}$ and $(t^{-1})^{a^{-1}a} = t^{-1}$. We see that

$$\begin{aligned} a &= aa^{-1}a \\ &= a(tt^{-1} \cdot a^{-1}a) && \text{as } tt^{-1} \cdot a^{-1}a = a^{-1}a \\ &= a(tt^{-1} \cdot a^{-1})((tt^{-1})^{a^{-1}} \cdot a) && \text{using (ZS2)} \\ &= a(tt^{-1} \cdot a^{-1})a && \text{because } (tt^{-1})^{a^{-1}} \cdot a = a, \end{aligned}$$

and

$$\begin{aligned} (tt^{-1} \cdot a^{-1})a(tt^{-1} \cdot a^{-1}) &= (tt^{-1} \cdot a^{-1})((tt^{-1})^{a^{-1}} \cdot a)(tt^{-1} \cdot a^{-1}) && \text{because } (tt^{-1})^{a^{-1}} \cdot a = a \\ &= (tt^{-1} \cdot a^{-1}a)(tt^{-1} \cdot a^{-1}) && \text{using (ZS2)} \\ &= (tt^{-1} \cdot a^{-1}a)((tt^{-1})^{a^{-1}a} \cdot a^{-1}) && \text{because } tt^{-1} = (tt^{-1})^{a^{-1}a} \\ &= (tt^{-1} \cdot a^{-1}aa^{-1}) && \text{again using (ZS2)} \\ &= tt^{-1} \cdot a^{-1}. \end{aligned}$$

Hence $tt^{-1} \cdot a^{-1} = a^{-1}$. By left-right duality we must also have $(t^{-1})^{a^{-1}a} = t^{-1}$. Checking,

$$\begin{aligned} t &= tt^{-1}t \\ &= (tt^{-1})^{a^{-1}a}t \\ &= t^{t^{-1} \cdot a^{-1}a}(t^{-1})^{a^{-1}a}t && \text{using (ZS4)} \\ &= t(t^{-1})^{a^{-1}a}t && \text{because } t^{t^{-1} \cdot a^{-1}a} = t \end{aligned}$$

and

$$\begin{array}{rcll}(t^{-1})^{a^{-1}a}t(t^{-1})^{a^{-1}a} & = & (t^{-1})^{a^{-1}a}t^{t^{-1}\cdot a^{-1}a}(t^{-1})^{a^{-1}a} & \text{because } t^{t^{-1}\cdot a^{-1}a}=t \\ & = & (t^{-1})^{a^{-1}a}(tt^{-1})^{a^{-1}a} & \text{using (ZS4)} \\ & = & (t^{-1})^{tt^{-1}\cdot a^{-1}a}(tt^{-1})^{a^{-1}a} & \text{because } tt^{-1}\cdot a^{-1}a=a^{-1}a \\ & = & (t^{-1}tt^{-1})^{a^{-1}a} & \text{using (ZS4)} \\ & = & (t^{-1})^{a^{-1}a}. & \end{array}$$

Thus $(t^{-1})^{a^{-1}a}=t^{-1}$ and hence $(a,t)\in B_{\bowtie}(Z)$, so that $V_{\bowtie}(Z)\subseteq B_{\bowtie}(Z)$.

□

Hence in the inverse case, the set of pairs chosen by us are the same as those chosen in [25] and [77]. The following lemma was an important tool to prove that $B_{\bowtie}(Z)$ is a groupoid.

Lemma 8.2.3. [77, Lemma 4.5.2] *Let S and T be inverse semigroups and $Z=S\bowtie T$ be a Zappa-Szép product of S and T.*

(i) If $t^{bb^{-1}}=t$, then $(t\cdot b)^{-1}=t^b\cdot b^{-1}$;
(ii) If $t^{-1}t\cdot b=b$, then $(t^b)^{-1}=(t^{-1})^{t\cdot b}$.

Lemma 8.2.3 gives us the following lemma which was used frequently by Wazzan to prove that $B_{\bowtie}(Z)$ is a groupoid, though she did not mention it explicitly.

Lemma 8.2.4. *Let S and T be inverse semigroups and $Z=S\bowtie T$ be a Zappa-Szép product of S and T. Then*

$$t^{bb^{-1}}=t\Rightarrow\begin{cases}(t\cdot b)^{-1}(t\cdot b)=t^b\cdot b^{-1}b\\ (t\cdot b)(t\cdot b)^{-1}=t\cdot bb^{-1}\end{cases}\quad\textit{and}\quad t^{-1}t\cdot b=b\Rightarrow\begin{cases}(t^b)^{-1}t^b=(t^{-1}t)^b\\ t^b(t^b)^{-1}=(tt^{-1})^{t\cdot b}.\end{cases}$$

Proof. Let $t^{bb^{-1}}=t$. We show that $(t\cdot b)^{-1}(t\cdot b)=t^b\cdot b^{-1}b$ and $(t\cdot b)(t\cdot b)^{-1}=t\cdot bb^{-1}$. From Lemma 8.2.3 we know that $(t\cdot b)^{-1}=t^b\cdot b^{-1}$. Thus

$$\begin{array}{rcll}(t\cdot b)^{-1}(t\cdot b) & = & (t^b\cdot b^{-1})(t\cdot b) & \text{because } (t\cdot b)^{-1}=t^b\cdot b^{-1} \\ & = & (t^b\cdot b^{-1})(t^{bb^{-1}}\cdot b) & \text{because } t^{bb^{-1}}=t \\ & = & t^b\cdot b^{-1}b & \text{using (ZS2),}\end{array}$$

and

$$\begin{array}{rcll}(t\cdot b)(t\cdot b)^{-1} & = & (t\cdot b)(t^b\cdot b^{-1}) & \text{because } (t\cdot b)^{-1}=t^b\cdot b^{-1} \\ & = & t\cdot bb^{-1} & \text{using (ZS2).}\end{array}$$

Dually, we can show that

$$t^{-1}t \cdot b = b \Rightarrow \begin{cases} (t^b)^{-1}t^b = (t^{-1}t)^b \\ t^b(t^b)^{-1} = (tt^{-1})^{t \cdot b}. \end{cases}$$

□

Although Lemma 8.2.4 could be reformulated in the restriction case, its proof involves writing an element $s \in S$ as $ss^{-1}s$ and uses (ZS2) on this decomposition, and similarly for elements of T uses (ZS4). Therefore unlike in the inverse case, we cannot get these results in our case, because we are considering Zappa-Szép products of restriction semigroups and in restriction semigroups we do not have inverses, so we cannot make use of these decompositions.

We can show that if $t^{b^+} = t$, then

$$t \cdot b^+ = t \cdot b^+b^+ = (t \cdot b^+)(t^{b^+} \cdot b^+) = (t \cdot b^+)(t \cdot b^+)$$

and

$$t \cdot b = t \cdot b^+b = (t \cdot b^+)(t^{b^+} \cdot b) = (t \cdot b^+)(t \cdot b).$$

If $t \cdot b^+ \in E_S$, then $t \cdot b^+ \geqslant (t \cdot b)^+$, but we cannot deduce equality. Thus it is reasonable to suppose

$$t^{b^+} = t \Rightarrow \begin{cases} (t \cdot b)^* = t^b \cdot b^* \\ (t \cdot b)^+ = t \cdot b^+ \end{cases} \tag{8.1}$$

and

$$t^* \cdot b = b \Rightarrow \begin{cases} (t^b)^* = (t^*)^b \\ (t^b)^+ = (t^+)^{t \cdot b}. \end{cases} \tag{8.2}$$

as extra conditions to prove that $V_{\bowtie}(Z)$ is a category.

We notice that in the inverse case, if $(a, t) \in B_{\bowtie}(Z)$, then $t^{-1} \cdot (a^{-1}a) \in E(S)$, as

$$\begin{aligned} (t^{-1} \cdot (a^{-1}a))(t^{-1} \cdot (a^{-1}a)) &= (t^{-1} \cdot (a^{-1}a))((t^{-1})^{a^{-1}a} \cdot (a^{-1}a)) && \text{because } (t^{-1})^{a^{-1}a} = t^{-1} \\ &= t^{-1} \cdot \big((a^{-1}a)(a^{-1}a)\big) && \text{using (ZS2)} \\ &= t^{-1} \cdot (a^{-1}a). \end{aligned}$$

Dually $(tt^{-1})^{a^{-1}} \in E(T)$.

8.3 λ-Zappa-Szép product of two restriction semigroup

In this section we define a partial binary operation on $V_{\bowtie}(Z)$ such that under certain extra conditions, it becomes a category.

Theorem 8.3.1. *Let S and T be restriction semigroups and suppose that $Z = S \bowtie T$ is the Zappa-Szép product of S and T where S and T are acting doubly on each other satisfying (CP1) and (CP2). Let*

$$V = V_{\bowtie}(Z) = \{(a,t) \in S \times T : t^+ \cdot a^* = a^*, (t^+)^{a^*} = t^+, {}^{a}t^+ \cdot a = a, t^{a^* \circ t} = t\}.$$

For $(a,t) \in V$, we suppose that $a^* \circ t \in E_S$ and ${}^{a}t^+ \in E_T$. Also suppose that (8.1) and (8.2) hold. Then V is a category under the restriction of the binary operation in Z with set of local identities

$$E_V = \{(e,f) \in E_S \times E_T : f \cdot e = e, f^e = f\}$$

where

$$\mathbf{d}(a,t) = (a^+, {}^{a}t^+) \quad \text{and} \quad \mathbf{r}(a,t) = (a^* \circ t, t^*).$$

Proof. We denote the *restricted product* in V by $\bullet$. We first note that for any $(b,u) \in V$

$$u^+ \cdot b^* = b^*, (u^+)^{b^*} = u^+, {}^{b}u^+ \cdot b = b, u^{b^* \circ u} = u.$$

Now we prove some preliminary results needed to show that V is a category.

Lemma 8.3.2. *Let $(a,t), (b,u) \in V$ be such that $(a,t) \bullet (b,u)$ is defined. Then*

(1) $t^{b^+} = t$ and $t^* \cdot b = b$;
(2) $a^* = t \cdot b^+$ and $(t^*)^b = u^+$;
(3) $(t^b u)^+ = (t^b)^+$;
(4) ${}^{a(t \cdot b)}(t^b u)^+ = {}^{a}t^+$;
(5) $(a(t \cdot b))^+ = a^+$;
(6) $(a(t \cdot b))^* = (t \cdot b)^*$;
(7) $(a(t \cdot b))^* \circ t^b u = b^* \circ u$;
(8) $(t^b u)^* = u^*$.

Proof. As $(a,t) \bullet (b,u)$ is defined, $\mathbf{r}(a,t) = \mathbf{d}(b,u)$, so that $(a^* \circ t, t^*) = (b^+, {}^{b}u^+)$ gives

$$a^* \circ t = b^+ \quad \text{and} \quad t^* = {}^{b}u^+.$$

(1) We have

$$t^{b^+} = t^{a^* \circ t} = t \quad \text{and} \quad t^* \cdot b = {}^{b}u^+ \cdot b = b, \;\; \text{because } (a,t),(b,u) \in V.$$

(2) From $a^* \circ t = b^+$, we have

$$\begin{array}{rrcll} & t \cdot (a^* \circ t) & = & t \cdot b^+ & \\ \Rightarrow & t^+ \cdot a^* & = & t \cdot b^+ & \text{using (CP1)} \\ \Rightarrow & a^* & = & t \cdot b^+ & \text{because } t^+ \cdot a^* = a^* \end{array}$$

and $t^* = {}^{b}u^+$ implies

$$\begin{array}{rrcll} & (t^*)^b & = & ({}^{b}u^+)^b & \\ \Rightarrow & (t^*)^b & = & (u^+)^{b^*} & \text{using (CP2)} \\ \Rightarrow & (t^*)^b & = & u^+. & \end{array}$$

(3) From (1) and (2) proved above , we have that $t^* \cdot b = b$ and $(t^*)^b = u^+$. Hence

$$(t^b u)^+ = (t^b u^+)^+ = \left(t^{t^* \cdot b}(t^*)^b\right)^+ = \left((tt^*)^b\right)^+ = (t^b)^+.$$

(4) We see that

$$\begin{array}{rcll} {}^{a(t\cdot b)}(t^b u)^+ & = & {}^{a(t\cdot b)}(t^b)^+ & \text{by (3) proved above} \\ & = & {}^{a(t\cdot b)}\left((t^+)^{t\cdot b}\right) & \text{using (8.2)} \\ & = & {}^{a}\left({}^{t\cdot b}((t^+)^{t\cdot b})\right) & \\ & = & {}^{a}((t^+)^{(t\cdot b)^+}) & \text{using (CP2)} \\ & = & {}^{a}((t^+)^{t\cdot b^+}) & \text{using (8.1)} \\ & = & {}^{a}((t^+)^{a^*}) & \text{from (2) proved above} \\ & = & {}^{a}({}^{a^*}t^+) & \text{using (CP2)} \\ & = & {}^{aa^*}t^+ & \\ & = & {}^{a}t^+. & \end{array}$$

(5) From (8.1), we know that $(t \cdot b)^+ = t \cdot b^+$ and from (2) proved above, we have that $t \cdot b^+ = a^*$. Hence

$$(a(t \cdot b))^+ = (a(t \cdot b)^+)^+ = (a(t \cdot b^+))^+ = (aa^*)^+ = a^+.$$

Conditions (6), (7) and (8) are dual of (3), (4) and (5), respectively.

$\square$

Now we give the proof of Theorem 8.3.1.

We first check that $\bullet$ is a closed partial binary operation. For this let $(a,t),(b,u) \in V$ such that $(a,t)\bullet(b,u)$ is defined. We have to show that

$$(a,t)\bullet(b,u) = (a(t\cdot b), t^b u) \in V.$$

For this we need to check that

$$\begin{array}{ll} (t^b u)^+ \cdot (a(t\cdot b))^* = (a(t\cdot b))^*, & ((t^b u)^+)^{(a(t\cdot b))^*} = (t^b u)^+, \\ {}^{a(t\cdot b)}(t^b u)^+ \cdot (a(t\cdot b)) = a(t\cdot b) \text{ and } & (t^b u)^{(a(t\cdot b))^* \circ t^b u} = t^b u. \end{array}$$

We now calculate

$$\begin{array}{rcll} (t^b u)^+ \cdot (a(t\cdot b))^* & = & (t^b)^+ \cdot (t\cdot b)^* & \text{from Lemma 8.3.2 ((3) and (6))} \\ & = & (t^b)^+ \cdot (t^b \cdot b^*) & \text{from Lemma 8.3.2 (1) and (8.1)} \\ & = & (t^b)^+ t^b \cdot b^* & \text{using (ZS1)} \\ & = & t^b \cdot b^* & \\ & = & (t\cdot b)^* & \text{by (8.1)} \\ & = & (a(t\cdot b))^* & \text{from Lemma 8.3.2 (6).} \end{array}$$

Also,

$$\begin{array}{rcll} \left((t^b u)^+\right)^{(a(t\cdot b))^*} & = & \left((t^b)^+\right)^{(t\cdot b)^*} & \text{from Lemma 8.3.2 ((3) and (6))} \\ & = & \left((t^+)^{t\cdot b}\right)^{(t\cdot b)^*} & \text{from Lemma 8.3.2 (1) and (8.2)} \\ & = & (t^+)^{t\cdot b} & \\ & = & (t^b)^+ & \text{using (8.2)} \\ & = & (t^b u)^+ & \text{from Lemma 8.3.2 (3).} \end{array}$$

Next,

$$\begin{array}{rcll} {}^{a(t\cdot b)}(t^b u)^+ \cdot (a(t\cdot b)) & = & {}^{a}t^+ \cdot (a(t\cdot b)) & \text{from Lemma 8.3.2 (4)} \\ & = & ({}^{a}t^+ \cdot a)(({}^{a}t^+)^a \cdot (t\cdot b)) & \text{using (ZS2)} \\ & = & a((t^+)^{a^*} \cdot (t\cdot b)) & \text{as } (a,t)\in V \text{ and using (CP2)} \\ & = & a(t^+ \cdot (t\cdot b)) & \text{as } (a,t)\in V \\ & = & a(t\cdot b). & \end{array}$$

Finally,

$$\begin{aligned}
(t^b u)^{(a(t\cdot b))^* \circ t^b u} &= (t^b u)^{b^* \circ u} && \text{from Lemma 8.3.2 (7)}\\
&= (t^b)^{u\cdot(b^*\circ u)} u^{b^*\circ u} && \text{using (ZS4)}\\
&= (t^b)^{u^+ \cdot b^*} u && \text{by (CP1) and as } (b,u)\in V\\
&= (t^b)^{b^*} u && \text{as } (b,u)\in V\\
&= t^b u.
\end{aligned}$$

Hence $(a(t\cdot b), t^b u) \in V$.

Next for $e \in E_S$ and $f \in E_T$, we show that

$$(e,f) \in V \Leftrightarrow f\cdot e = e,\ f^e = f.$$

For this let $(e,f) \in V$. Then clearly

$$f\cdot e = e,\ f^e = f.$$

Conversely, if $f\cdot e = e$ and $f^e = f$, then

$${}^e f \cdot e = f^e \cdot e = f\cdot e = e$$

and

$$f^{e\circ f} = f^{f\cdot e} = f^e = f.$$

Thus $(e,f) \in V$.

Now let $(a,t) \in V$. By definition, $\mathbf{d}(a,t) = (a^+, {}^a t^+)$ and by our assumption, ${}^a t^+ \in E_T$. Thus to show that $\mathbf{d}(a,t) \in V$, we need only to show that

$$({}^a t^+)^{a^+} = {}^a t^+ \text{ and } {}^a t^+ \cdot a^+ = a^+.$$

We see that

$$({}^a t^+)^{a^+} = {}^{a^+}({}^a t^+) = {}^{a^+ a}(t^+) = {}^a t^+$$

and thus

$${}^a t^+ \cdot a^+ = ({}^a t^+ \cdot a)^+ = a^+,$$

using (8.1).

Dually, $\mathbf{r}(a,t) = (a^* \circ t, t^*) \in V$. Next we see that

$$\mathbf{d}(e,f) = (e, {}^{e}f) = (e, f^e) = (e,f)$$

and

$$\mathbf{r}(e,f) = (e \circ f, f) = (f \cdot e, f) = (e,f).$$

Thus

$$\mathbf{d}(e,f) = (e,f) = \mathbf{r}(e,f)$$

where $(e,f) \in E_V$, $e \in E_S$ and $f \in E_T$. Hence E_V is the set of local identities of V.

We now prove that V satisfies the axioms of a category.

(C1) Let $(a,t),(b,u) \in V$ and suppose $\exists (a,t) \bullet (b,u)$, so that $(a^* \circ t, t^*) = (b^+, {}^{b}u^+)$. Thus

$$\begin{aligned}
\mathbf{d}\Big((a,t) \bullet (b,u)\Big) &= \mathbf{d}\Big(a(t \cdot b), t^b u\Big) \\
&= \Big((a(t \cdot b))^+, {}^{a(t \cdot b)}(t^b u)^+\Big) \\
&= (a^+, {}^{a}t^+) && \text{from Lemma 8.3.2 ((5) and (4))} \\
&= \mathbf{d}(a,t),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{r}\Big((a,t) \bullet (b,u)\Big) &= \mathbf{r}\Big(a(t \cdot b), t^b u\Big) \\
&= \Big((a(t \cdot b))^* \circ t^b u, (t^b u)^*\Big) \\
&= (b^* \circ u, u^*) && \text{from Lemma 8.3.2 ((7) and (8))} \\
&= \mathbf{r}(b,u).
\end{aligned}$$

Hence (C1) is satisfied.

(C2) This directly follows from (C1) and the fact that multiplication is associative in a Zappa-Szép product.

(C3) Let $(a,t) \in V$. Then

$$\mathbf{r}(\mathbf{d}(a,t)) = \mathbf{d}(a,t) \text{ because } \mathbf{d}(e,f) = (e,f) = \mathbf{r}(e,f) \text{ for all } (e,f) \in E_V.$$

Hence there exists $\mathbf{d}(a,t) \bullet (a,t)$ and then

$$\begin{aligned}
\mathbf{d}(a,t) \bullet (a,t) &= (a^+, {}^a t^+) \bullet (a,t) \\
&= (a^+({}^a t^+ \cdot a), ({}^a t^+)^a t) \\
&= (a^+ a, (t^+)^{a^*} t) && \text{using (CP2)} \\
&= (a, t^+ t) && \text{because } (a,t) \in V \\
&= (a,t).
\end{aligned}$$

Dually,

$$\mathbf{d}(\mathbf{r}(a,t)) = \mathbf{r}(a,t) \text{ because } \mathbf{d}(e,f) = (e,f) = \mathbf{r}(e,f) \text{ for all } (e,f) \in E_V.$$

Hence there exists $(a,t) \bullet \mathbf{r}(a,t)$ and then

$$\begin{aligned}
(a,t) \bullet \mathbf{r}(a,t) &= (a,t) \bullet (a^* \circ t, t^*) \\
&= (a(t \cdot (a^* \circ t), t^{a^* \circ t} t^*) \\
&= (a(t^+ \cdot a^*), t t^*) && \text{using (CP1) and because } (a,t) \in V, \text{ so } t^{a^* \circ t} = t \\
&= (a a^*, t) && \text{because } (a,t) \in V \\
&= (a,t).
\end{aligned}$$

Hence $V = V_{\bowtie}(Z)$ is a category.

□

8.4 λ-Zappa-Szép product of a semilattice and a monoid

We now specialize Theorem 8.3.1 to the case where we consider the Zappa-Szép product of a semilattice and a monoid. We suppose that in this case (ZS7) $1 \cdot e = e$ and (ZS8)$1^e = 1$ hold. As a monoid T is a reduced restriction semigroup with

$$t^+ = 1 = t^* \quad \text{for all } t \in T,$$

we note that in this special case (8.1) and (8.2) are satisfied trivially.

The existence of the additional actions

$$E \times T \to E, \ \ (e,t) \mapsto e \circ t$$

and

$$E \times T \to T, \ \ (e,t) \mapsto {}^{e}t$$

satisfying (CP1) and (CP2) is equivalent to the action of T on the left of E being by bijections (so that we may define $e \circ t = f$ where $t \cdot f = e$) and ${}^{e}t$ being equal to t^{e}. Thus in this case, it is not strictly necessary to introduce the two new actions. However, we retain the action $\circ$ for convenience. We therefore suppose that

$$e = (t \cdot e) \circ t = t \cdot (e \circ t) \tag{CP3}$$

for all $e \in E$ and $t \in T$. Now

$$V = V_{\bowtie}(Z) = \{(a,t) \in S \times T : t^{+} \cdot a^{*} = a^{*}, (t^{+})^{a^{*}} = t^{+}, {}^{a}t^{+} \cdot a = a, t^{a^{*} \circ t} = t\}$$

reduces to

$$V = V_{\bowtie}(Z) = \{(e,t) \in E \times T : t = t^{e \circ t}\}.$$

We now prove that in this special case, V can be made into an inductive category if the extra condition

$$e \leq f \Rightarrow t^{f} \cdot e = t \cdot e \tag{D}$$

holds, where $e, f \in E$ and $t \in T$.

Theorem 8.4.1. *Let $Z = E \bowtie T$ be a Zappa-Szép product of a semilattice E and a monoid T. Suppose that $1 \cdot e = e$, $1^{e} = 1$ and the action of T on E satisfies (CP3). Also suppose that the condition (D) holds. Then*

$$V = V_{\bowtie}(Z) = \{(e,t) \in E \times T : t = t^{e \circ t}\}$$

is an inductive category under the restriction of the binary operation in Z with set of local identities $E_V = \{(e,1) : e \in E_S\}$, where

$$\mathbf{d}(e,t) = (e,1) \ \ \textit{and} \ \ \mathbf{r}(e,t) = (e \circ t, 1).$$

The partial order $\leq$ on V is defined by

$$(e,s) \leq (f,t) \ \textit{if and only if} \ e \leq f \ \textit{and} \ s = t^{e \circ t}.$$

Also for $(e,1) \in E$, $(f,t) \in V$ where $(e,1) \leq \mathbf{d}(f,t)$, the restriction is defined by

$$_{(e,1)}|(f,t) = (e, t^{e \circ t})$$

and where $(e,1) \leq \mathbf{r}(f,t)$, the co-restriction is defined as:

$$(f,t)|_{(e,1)} = (f(t \cdot e), t^{e}).$$

Proof. Before giving the proof of Theorem 8.4.1, we record some preliminary computations.

Lemma 8.4.2. *Let $(e,s),(f,t) \in V_{\bowtie}(Z)$, then;*

(1) $ef \circ s = (e \circ s)(f \circ s)$;

(2) if $(e,s) \leq (f,t)$, then $e \circ s = e \circ t$ and $e \circ t \leq f \circ t$;

(3) for $e, f \in E$ and $t \in T$, if $e \leq f$, then $t \cdot e \leq t \cdot f$.

Proof. (1) We have.

$$\begin{aligned} (e \circ s)(f \circ s) &= \Big(s \cdot ((e \circ s)(f \circ s))\Big) \circ s && \text{using (CP3)} \\ &= \Big((s \cdot (e \circ s))(s^{e \circ s} \cdot (f \circ s))\Big) \circ s && \text{using (ZS2)} \\ &= \Big(e(s \cdot (f \circ s))\Big) \circ s && \text{because } s = s^{e \circ s} \text{ and using (CP3)} \\ &= ef \circ s && \text{using (CP3).} \end{aligned}$$

(2) Suppose that $(e,s) \leq (f,t)$. Then $e \leq f$ and $s = t^{e \circ t}$. To prove that $e \circ s = e \circ t$, we see that

$$\begin{aligned} (e \circ s)(e \circ t) &= \Big(s \cdot ((e \circ s)(e \circ t))\Big) \circ s && \text{using (CP3)} \\ &= \Big((s \cdot (e \circ s))(s^{e \circ s} \cdot (e \circ t))\Big) \circ s && \text{using (ZS2)} \\ &= \Big(e(s \cdot (e \circ t))\Big) \circ s && \text{using (CP3) and } s = s^{e \circ s} \\ &= \Big((t \cdot (e \circ t))(t^{e \circ t} \cdot (e \circ t))\Big) \circ s && \text{using (CP3) and } s = t^{e \circ t} \\ &= \Big(t \cdot ((e \circ t)(e \circ t))\Big) \circ s && \text{using (ZS2)} \\ &= e \circ s && \text{using (CP3)} \\ \Rightarrow \quad e \circ s &\leq (e \circ t). \end{aligned}$$

and

$$\begin{array}{rcll} (e\circ t)(e\circ s) & = & \Big(t\cdot((e\circ t)(e\circ s))\Big)\circ t & \text{using (CP3)} \\ & = & \Big((t\cdot(e\circ t))(t^{e\circ t}\cdot(e\circ s))\Big)\circ t & \text{using (ZS2)} \\ & = & \Big(e(s\cdot(e\circ s))\Big)\circ t & \text{because } s=t^{e\circ t} \text{ and using (CP3)} \\ & = & e\circ t & \text{using (CP3)} \\ \Rightarrow \qquad e\circ t & \leq & e\circ s. & \end{array}$$

Thus $e\circ t = e\circ s$. Next we see that

$$\begin{array}{rcll} (e\circ t)(f\circ t) & = & ef\circ t & \text{from (1) above} \\ & = & e\circ t & \text{because } e\leq f. \end{array}$$

Hence $e\circ t\leq f\circ t$.

(3) Suppose $e\leq f$. Then $e=ef$, so that

$$\begin{array}{rcll} t\cdot e & = & t\cdot(ef) & \\ & = & t\cdot(fe) & \\ & = & (t\cdot f)(t^f\cdot e) & \text{using (ZS2)} \end{array}$$

and hence $t\cdot e\leq t\cdot f$. □

We now give the proof of Theorem 8.4.1.

From Theorem 8.3.1, V is a category with set of local identities $E_V=\{(e,1):e\in E_S\}$ where

$$\mathbf{d}(e,t)=(e,1) \text{ and } \mathbf{r}(e,t)=(e\circ t,1).$$

We notice that for $(e,1)$, $(f,1)\in E_V$

$$(e,1)\leq(f,1)\Leftrightarrow e\leq f.$$

Now we show that V is an inductive category. First we prove that $\leq$ is a partial order. It is clear that $\leq$ is reflexive. Let (e,s), $(f,t)\in V$ and suppose that

$$(e,s)\leq(f,t) \text{ and } (f,t)\leq(e,s).$$

Then $e \le f \le e$ implies $e = f$ and

$$s = t^{e \circ t} = t^{f \circ t} = t.$$

Hence $\le$ is antisymmetric.

To check transitivity, let (e,s), (f,t), $(g,u) \in V$ and suppose that

$$(e,s) \le (f,t) \text{ and } (f,t) \le (g,u).$$

Then $e \le f \le g$, $s = t^{e \circ t}$ and $t = u^{f \circ u}$. Now $e \le f \le g$ implies that $e \le g$ and

$$\begin{aligned}
s &= t^{e \circ t} \\
&= (u^{f \circ u})^{(e \circ u^{f \circ u})} && \text{because } t = u^{f \circ u} \\
&= u^{(f \circ u)(e \circ u^{f \circ u})} \\
&= u^{\left(u \cdot ((f \circ u)(e \circ u^{f \circ u}))\right) \circ u} && \text{using (CP3)} \\
&= u^{\left((u \cdot (f \circ u))(u^{f \circ u} \cdot (e \circ u^{f \circ u}))\right) \circ u} && \text{using (ZS2)} \\
&= u^{f e \circ u} && \text{using (CP3)} \\
&= u^{e \circ u}.
\end{aligned}$$

Hence $(e,s) \le (g,u)$ and thus $\le$ is a partial order on V.

We now show that V satisfies the axioms of an inductive category.

(IC1) Let (e,s), $(f,t) \in V$ and suppose that $(e,s) \le (f,t)$. Then $e \le f$ and $s = t^{e \circ t}$. Clearly $\mathbf{d}(e,s) \le \mathbf{d}(f,t)$. Now to show that $\mathbf{r}(e,s) \le \mathbf{r}(f,t)$, we need to check that $e \circ s \le f \circ t$. From Lemma 8.4.2, we see that

$$e \circ s = e \circ t \le f \circ t,$$

as required.

(IC2) Let $(e,s), (f,t), (g,u), (h,v) \in V$ and $(e,s) \le (f,t)$, $(g,u) \le (h,v)$. Then

$$e \le f,\ s = t^{e \circ t} \text{ and } g \le h,\ u = v^{g \circ v}.$$

Suppose that $\exists (e,s) \bullet (g,u)$ and $\exists (f,t) \bullet (h,v)$. Then

$$\mathbf{r}(e,s) = \mathbf{d}(g,u) \text{ and } \mathbf{r}(f,t) = \mathbf{d}(h,v),$$

so that

$$(e \circ s, 1) = (g, 1) \text{ and } (f \circ t, 1) = (h, 1).$$

To prove that $(e, s) \bullet (g, u) \leq (f, t) \bullet (h, v)$, that is, $(e(s \cdot g), s^g u) \leq (f(t \cdot h), t^h v)$, we need to show

$$e(s \cdot g) \leq f(t \cdot h) \text{ and } s^g u = (t^h v)^{e(s \cdot g) \circ t^h v}.$$

We see that

$$e(s \cdot g) = e(s \cdot (e \circ s)) = ee = e,$$

and

$$f(t \cdot h) = f(t \cdot (f \circ t)) = ff = f,$$

so that $e(s \cdot g) \leq f(t \cdot h)$. Now

$$\begin{array}{rcll} s^g u & = & s^{e \circ s} u & \\ & = & su & \text{because } s = s^{e \circ s} \\ & = & t^{e \circ t} v^{g \circ v} & \text{because } s = t^{e \circ t},\ u = v^{g \circ v} \\ & = & t^{v \cdot ((e \circ t) \circ v)} v^{(e \circ s) \circ v} & \text{using (CP3) and } g = e \circ s \\ & = & t^{v \cdot ((e \circ t) \circ v)} v^{(e \circ t) \circ v} & \text{because } e \circ s = e \circ t \text{ from Lemma 8.4.2 (2)} \\ & = & (tv)^{(e \circ t) \circ v} & \text{using (ZS4)} \\ & = & (t^{f \circ t} v)^{e \circ t^{f \circ t} v} & \text{because } t = t^{f \circ t} \\ & = & (t^h v)^{(e(s \cdot g)) \circ t^h v} & \text{because } h = f \circ t \text{ and } e = e(s \cdot g). \end{array}$$

Hence $(e, s) \bullet (g, u) \leq (f, t) \bullet (h, v)$.

(IC3) Let $(e, 1) \in E_V$ and $(f, t) \in V$ be such that $(e, 1) \leq \mathbf{d}(f, t)$. Then $(e, 1) \leq (f, 1)$, so $e \leq f$. We first note that $(e, t^{e \circ t}) \in V$ as

$$\begin{array}{rcll} (t^{e \circ t})^{(e \circ t^{e \circ t})} & = & t^{(e \circ t)(e \circ t^{e \circ t})} & \\ & = & t^{\left(t \cdot ((e \circ t)(e \circ t^{e \circ t}))\right) \circ t} & \text{using (CP3)} \\ & = & t^{\left((t \cdot (e \circ t))(t^{e \circ t} \cdot (e \circ t^{e \circ t}))\right) \circ t} & \text{using (ZS2)} \\ & = & t^{e \circ t}. & \end{array}$$

Clearly $\mathbf{d}(e, t^{e \circ t}) = (e, 1)$ and $(e, t^{e \circ t}) \leq (f, t)$.

We show that $(e, t^{e \circ t})$ is the unique element with these properties. Let (g, u) be another element such that $(g, u) \leq (f, t)$ and $\mathbf{d}(g, u) = (e, 1)$. Then

$$g \leq f,\ u = t^{g \circ t} \text{ and } (g, 1) = (e, 1).$$

Thus $g = e$ and $u = t^{g \circ t} = t^{e \circ t}$. Hence $(e, t^{e \circ t})$ is unique and we thus define the restriction

$$_{(e,1)}|(f, t) = (e, t^{e \circ t}).$$

(IC4) Let $(e, 1) \in E$ and $(f, t) \in V$ be such that $(e, 1) \leq \mathbf{r}(f, t)$. Then $(e, 1) \leq (f \circ t, 1)$, so that $e \leq f \circ t$. We want to show that $(f(t \cdot e), t^e)$ is the unique element in V such that

$$(f(t \cdot e), t^e) \leq (f, t) \text{ and } \mathbf{r}(f(t \cdot e), t^e) = (e, 1).$$

We first check that $(f(t \cdot e), t^e) \in V$. First we notice that as $e \leq f \circ t$, so $t \cdot e \leq t \cdot (f \circ t) = f$, by Lemma 8.4.2 (3). Next we see that

$$\begin{aligned} (t^e)^{(f(t \cdot e)) \circ t^e} &= (t^e)^{(t \cdot e) \circ t^e} && \text{because } t \cdot e \leq f, \text{ so } f(t \cdot e) = t \cdot e \\ &= (t^e)^{(t^e \cdot e) \circ t^e} && \text{since } e \leq e, \text{ then } t^e \cdot e = t \cdot e \text{ by our assumption} \\ &= (t^e)^e && \text{using (CP3)} \\ &= t^e. \end{aligned}$$

Hence $(f(t \cdot e), t^e) \in V$. Clearly $f(t \cdot e) \leq f$ and

$$\begin{aligned} t^{(f(t \cdot e)) \circ t} &= t^{(t \cdot e) \circ t} && \text{because } t \cdot e \leq f \\ &= t^e. \end{aligned}$$

Hence $(f(t \cdot e), t^e) \leq (f, t)$. Also

$$\begin{aligned} \mathbf{r}(f(t \cdot e), t^e) &= ((f(t \cdot e)) \circ t^e, 1) \\ &= ((t \cdot e) \circ t^e, 1) && \text{because } t \cdot e \leq f \\ &= ((t^e \cdot e) \circ t^e, 1) && \text{since } e \leq e, \text{ then } t^e \cdot e = t \cdot e \text{ by our assumption} \\ &= (e, 1). \end{aligned}$$

Next to prove the uniqueness of $(f(t \cdot e), t^e)$, let (g, u) be another element in V such that $(g, u) \leq (f, t)$ and $\mathbf{r}(g, u) = (e, 1)$. Then

$$g \leq f,\ u = t^{g \circ t} \text{ and } g \circ u = e.$$

Now

$$\begin{aligned} f(t\cdot e) &= f(t\cdot(g\circ u)) \quad \text{because } e=g\circ u \\ &= f(t\cdot(g\circ t)) \quad \text{because } g\circ u = g\circ t \text{ from Lemma 8.4.2 (2)} \\ &= fg \\ &= g, \end{aligned}$$

and

$$u = t^{g\circ t} = t^{g\circ u} = t^e.$$

Hence $(f(t\cdot e), t^e)$ is unique and we thus define co-restriction

$$(f,t)|_{(e,1)} = (f(t\cdot e), t^e).$$

(IC5) It is clear that $E_V = \{(e,1) : e \in E\}$ is a meet semilattice with

$$(e,1)\wedge(f,1) = (ef,1).$$

Hence V is an inductive category.

□

Next we define a pseudo-product on our inductive category to obtain a restriction semigroup.

Theorem 8.4.3. *Let E be a semilattice and T be a monoid and $Z = E \bowtie T$. Suppose that $1\cdot e = e$, $1^e = 1$ and the action of T on E satisfies (CP3). Also suppose that the condition (D) holds. Suppose that $(V, \bullet, \mathbf{d}, \mathbf{r}, \leq)$ is the inductive category as defined in Theorem 8.4.1. Let $(e,s),(f,t) \in V$. Defining the pseudo product on V by the usual rule*

$$(e,s)\otimes(f,t) = ((e,s)|_{\mathbf{r}(e,s)\wedge\mathbf{d}(f,t)})(_{\mathbf{r}(e,s)\wedge\mathbf{d}(f,t)}|(f,t)),$$

V is a restriction semigroup. Here the multiplication simplifies to

$$(e,s)\bullet(f,t) = \Big(e(s\cdot f), s^f t^{(e\circ st)}\Big).$$

Proof. We first see that

$$\mathbf{r}(e,s)\wedge\mathbf{d}(f,t) = (e\circ s,1)\wedge(f,1) = ((e\circ s)f,1).$$

Now

$$\begin{aligned}(e,s)|_{\mathbf{r}(e,s)\wedge\mathbf{d}(f,t)} &= (e,s)|_{((e\circ s)f,1)}\\ &= \Big(e(s.((e\circ s)f)), s^{(e\circ s)f}\Big)\\ &= \Big(e((s\cdot(e\circ s))(s^{e\circ s}\cdot f), s^f\Big)\\ &= (e(s\cdot f), s^f).\end{aligned}$$

and

$$\begin{aligned}{}_{\mathbf{r}(e,s)\wedge\mathbf{d}(f,t)}|(f,t) &= {}_{((e\circ s)f,1)}|(f,t)\\ &= ((e\circ s)f, t^{((e\circ s)f)\circ t}).\end{aligned}$$

Thus

$$\begin{aligned}(e,s)\otimes(f,t) &= (e(s\cdot f), s^f)((e\circ s)f, t^{((e\circ s)f)\circ t})\\ &= \Big(e(s\cdot f)(s^f\cdot((e\circ s)f)), (s^f)^{(e\circ s)f}t^{((e\circ s)f)\circ t}\Big)\\ &= \Big(e(s\cdot(f^2(e\circ s)), s^{(e\circ s)f}t^{((e\circ s)f)\circ t}\Big) && \text{using (ZS2)}\\ &= \Big(e(s\cdot(f(e\circ s)), s^{(e\circ s)f}t^{((e\circ s)f)\circ t}\Big)\\ &= \Big(e(s\cdot((e\circ s)f), s^{(e\circ s)f}t^{((e\circ s)f)\circ t}\Big)\\ &= \Big(e(s\cdot(e\circ s))(s^{e\circ s}\cdot f), s^f t^{((e\circ s)f)\circ t}\Big) && \text{using (ZS2) and (CP3)}\\ &= \Big(e(s\cdot f), s^f t^{((e\circ s)\circ t)(f\circ t)}\Big) && \text{from Lemma 8.4.2 (1)}\\ &= \Big(e(s\cdot f), s^f (t^{f\circ t})^{(e\circ st)}\Big)\\ &= \Big(e(s\cdot f), s^f t^{(e\circ st)}\Big) && \text{because } t^{f\circ t}=t\ .\end{aligned}$$

Hence the fact that V is a restriction semigroup follows from its construction as the pseudo-product on an inductive category. □

From Lemma 8.2.2, we know that the map

$$S\times T\to S,\ (s,t)\mapsto s\circ t$$

is defined by $s\circ t = t^{-1}\cdot s$. Thus we see that

$$\begin{aligned}(e,s)\otimes(f,t) &= \Big(e(s\cdot f), s^f t^{(e\circ st)}\Big)\\ &= \Big(e(s\cdot f), s^f t^{((st)^{-1}\cdot e)}\Big) && \text{because } s\circ t = t^{-1}\cdot s\\ &= \Big(e(s\cdot f), s^f t^{t^{-1}s^{-1}\cdot e}\Big).\end{aligned}$$

Which is the same multiplication as we have seen in Theorem 8.1.3 for inverse semigroups.

Now we consider the usual Zappa-Szép product $E\bowtie T$ of a semilattice E and a monoid

T. By defining two new actions we construct a Zappa-Szép product $T \bowtie E$.

Lemma 8.4.4. *Let $E \bowtie T$ be a Zappa-Szép product of a semilattice E and a monoid T. Suppose that there are two extra maps*

$$E \times T \to T,\ (e,t) \mapsto {}^{e}t \text{ and } E \times T \to E,\ (e,t) \mapsto e \circ t$$

and suppose that the action of T on E satisfies (CP3). We define a new action of E on T by the following.

$$e \diamond t = t^{e \circ t}$$

Then the actions $(e,t) \mapsto e \diamond t$ and $(e,t) \mapsto e \circ t$ satisfy (ZS1)-(ZS4) and we can form a Zappa-Szép product $T \bowtie E$.

Proof. We show that these actions satisfy the Zappa-Szép axioms.

(ZS1) Let $e, f \in E$ and $t \in T$. Then

$$\begin{aligned}
e \diamond (f \diamond t) &= e \diamond (t^{f \circ t}) \\
&= \left(t^{f \circ t}\right)^{(e \circ t^{f \circ t})} \\
&= t^{\left(t \cdot ((f \circ t)(e \circ t^{f \circ t}))\right) \circ t} && \text{using (CP3)} \\
&= t^{\left((t \cdot (f \circ t))(t^{f \circ t} \cdot (e \circ t^{f \circ t})\right) \circ t} && \text{using (ZS2) as } E \bowtie T \text{ is a Zappa-Szép product} \\
&= t^{fe \circ t} && \text{using (CP3)} \\
&= t^{ef \circ t} \\
&= ef \diamond t.
\end{aligned}$$

Hence (ZS1) is satisfied.

(ZS2) Let $e \in E$ and $t, u \in T$. Then

$$\begin{aligned}
e \diamond tu &= (tu)^{e \circ tu} \\
&= t^{u \cdot (e \circ tu)} u^{e \circ tu} && \text{using (ZS4) as } E \bowtie T \text{ is a Zappa-Szép product} \\
&= t^{u \cdot ((e \circ t) \circ u)} u^{(e \circ t) \circ u} \\
&= t^{e \circ t} u^{(e \circ t) \circ u} && \text{using (CP3)} \\
&= (e \diamond t)((e \circ t) \diamond u).
\end{aligned}$$

(ZS3) Let $e \in E$ and $t, u \in T$. Then by assumption,

$$e \circ tu = (e \circ t) \circ u.$$

(ZS4) Let $e, f \in E$ and $t \in T$. Then

$$\begin{aligned} ef \circ t &= fe \circ t \\ &= \Big((t \cdot (f \circ t))(t^{f \circ t} \cdot (e \circ t^{f \circ t}))\Big) \circ t && \text{using (CP3)} \\ &= (t \cdot ((f \circ t)(e \circ t^{f \circ t}))) \circ t && \text{using (ZS2)} \\ &= (f \circ t)(e \circ t^{f \circ t}) && \text{using (CP3)} \\ &= (e \circ t^{f \circ t})(f \circ t) && \text{because } E \text{ is a semilattice} \\ &= (e \circ (f \diamond t))(f \circ t). \end{aligned}$$

Hence $T \bowtie E$ is a Zappa-Szép product with multiplication defined by

$$(t, e)(u, f) = (tu^{e \circ u}, (e \circ u)f)$$

for all $(t, e), (u, f) \in T \bowtie E$.

□

The results in this chapter can be specialised to inverse semigroups but we are looking for some more concrete examples.

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