# Similarity Reductions and Integrable Lattice Equations

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*The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.*

# Abstract

In this thesis I extend the theory of integrable partial difference equations (P$\Delta$Es) and reductions of these systems under scaling symmetries. The main approach used is the *direct linearization method* which was developed previously and forms a powerful tool for dealing with both continuous and discrete equations. This approach is further developed and applied to several important classes of integrable systems.

Whilst the theory of continuous integrable systems is well established, the theory of analogous difference equations is much less advanced. In this context the study of symmetry reductions of integrable (P$\Delta$Es) which lead to ordinary difference equations (O$\Delta$Es) of Painlevé type, forms a key aspect of a more general theory that is still in its infancy.

The first part of the thesis lays down the general framework of the direct linearization scheme and reviews previous results obtained by this method. Most results so far have been obtained for lattice systems of KdV type. One novel result here is a new approach for deriving Lax pairs. New results in this context start with the embedding of the lattice KdV systems into a multi-dimensional lattice, the reduction of which leads to both continuous and discrete Painlevé hierarchies associated with the Painlevé VI equation.

The issue of multidimensional lattice equations also appears, albeit in a different way, in the context of the lattice KP equations, which by dimensional reduction lead to new classes of discrete equations.

This brings us in a natural way to a different class of continuous and discrete systems, namely those which can be identified to be of Boussinesq (BSQ) type. The development of this class by means of the direct linearization method forms one of the major parts of the thesis. In particular, within this class we derive new differential-difference equations and exhibit associated linear problems (Lax pairs). The consistency of initial value problems on the multi-dimensional lattice is established. Furthermore, the similarity constraints and their compatibility with the lattice systems guarantee the consistency of the reductions that are considered. As such the resulting systems of lattice equations are conjectured to be of Painlevé type.

The final part of the thesis contains the general framework for lattice systems of AKNS type for which we establish the basic equations as well as similarity constraints.

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# Contents

# List of Figures

# Chapter 1

# Introduction

The notion of a "soliton" was first introduced in 1965 by Zabusky and Kruskal [1] with a numerical solution of the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.0.1}$$

Two years later, in 1967, Gardner, Greene, Kruskal and Miura [2], were able to give an exact solution of the KdV Equation by means a new approach which became known as the Inverse Scattering Transform (IST). These discoveries heralded the birth of the subject now known as Integrable systems - nonlinear dynamical systems with exact solutions. Since then the subject has grown to become a major field of research and there are many textbooks outlining the history and general principles of these systems, see for example [3, 4, 5]. One of the main reasons for this growth is that integrable systems have found applications in a wide variety of fields including quantum field theory, string theory, condensed matter, optics and biology, see [6]. Recently, one of the most fruitful branches of this field has been the study of discrete integrable systems and it is these which form the major part of this thesis. However, before concentrating on discrete systems, some background is required on integrable systems in the continuous case.

## 1.1 Integrability in the Continuous Case

### 1.1.1 Classical Integrability

In classical mechanics the question of integrability for a dynamical system with finite degrees of freedom is well understood. For such a system, defined in terms of generalized coordinates $q_i$, conjugate momenta $p_i$ and a Hamiltonian $H(q_1, \dots, q_N, p_1, \dots, p_N)$, where $N$ is the number of degrees of freedom, if

i) There exist $N$ functionally independent integrals of motion $I_i(q_1, \dots, q_N, p_1, \dots, p_N)$ which do not depend explicitly on time,

ii) These integrals of motion are in evolution w.r.t. the Poisson structure, i.e. $\{I_i, I_j\} = 0$,

then the system is said to be completely integrable and can be solved in quadratures [7]. Unfortunately, this definition cannot be used when dealing with partial differential equations which have infinite degrees of freedom and, although much work has been done in this area, as of yet there is no definitive test for integrability in this case. There are, however, several features which all of the known integrable systems have in common, each of which is generally taken as a strong indicator of integrability.

### 1.1.2 Conservation Laws and Inverse Scattering

The most closely related to the classical case is the existence of an infinite number of conservation laws. A conservation law is an equation of the form

$$\frac{\partial D}{\partial t} + \frac{\partial F}{\partial x} = 0,$$

where $D$ (the density) and $F$ (the flux) are functions of $x, t$, the dependent variable and derivatives w.r.t. $x$, but not $t$, of the dependent variable.

Then, if $F \to$ constant as $|x| \to \infty$,

$$\int_{-\infty}^{\infty} D\,dx = \text{constant}.$$

i.e. the integral of $D$, over all $x$, is a constant of the motion.

In 1968, it was proved that the KdV equation had an infinite number of constants of motion [8]. This was done with the help of the following transformation,

$$u = w - \epsilon w_x - \epsilon^2 w^2, \tag{1.1.1}$$

which, after expanding $w$ as a power series in $\epsilon$ leads recursively to an infinite sequence of conserved densities. This, in light of the classical case, strongly suggests that the corresponding equation is integrable.

Equation (1.1.1) is a generalization of the Miura transformation

$$u = -(v^2 + v_x). \tag{1.1.2}$$

This transformation allows us to obtain a solution of the KdV equation from solutions of the modified KdV equation, namely

$$v_t - 6v^2 v_x + v_{xxx} = 0, \tag{1.1.3}$$

and it is this transformation which forms the basis of the Inverse Scattering Transform. Equation (1.1.2) is a Riccati equation and as such can be linearized. This is achieved by the substitution

$$v = \psi_x/\psi,$$

which, after noting that the KdV equation is invariant under Galilean transformations, gives the following eigenvalue problem for the linear operator $L$

$$L\psi = \lambda\psi \qquad \text{where} \quad L \equiv \partial_x^2 + u, \tag{1.1.4}$$

for some real $\lambda = \lambda(t)$.

Now, equation (1.1.4) is the time-independent Schrödinger equation with potential $u$ and eigenvalue $\lambda$. From it, we are able to obtain the scattering data $S(0)$, consisting of the discrete spectrum, $\kappa_n$, the normalization constants, $c_n$ and the reflection coefficient, $b(k)$, for some initial conditions $u(x, 0)$. The time evolution of this scattering data is well behaved and thus we can calculate $S(t)$ for any subsequent time. Having found this, the inverse problem of reconstructing the potential involves the solution of a linear integral equation

$$K(x,y) + B(x+y) + \int_x^\infty K(x,z)B(y+z)dz \;=\; 0, \tag{1.1.5}$$

where

$$B(\xi) \equiv \frac{1}{2\pi}\int_\infty^\infty b(k)\exp(ik\xi)dk + \sum_{n=1}^{N} c_n^2 \exp(\kappa_n \xi),$$

known as the Gel'fand-Levitan-Marchenko equation, in which $B(\xi)$ is a function of the scattering data. Solving (1.1.5) for $K$ we obtain a solution for our potential $u(x, t)$ via

$$u(x,t) \;=\; 2\frac{d}{dx}K(x,x).$$

This is the essence of the Inverse Scattering Transform (IST). It was first introduced by Gardner, Greene, Kruskal and Miura in their seminal paper of 1965, [2], and there now exists an extensive literature in which the basic theory is explained, see for example [3, 4, 5]. The IST is in effect a nonlinear version of a Fourier transform. In practice, the rigorous implementation of this method can pose major technical difficulties, especially with regards to the class of initial conditions one wishes to consider. However, it does reduce the solution of a nonlinear partial differential equation to the solution of a linear ordinary differential equation (ODE) and a linear ordinary integral equation.

### 1.1.3 Lax Equations

In 1968, Lax [9] reformulated the IST by giving it in the form of two linear differential operators, $L$ and $M$, the first defining the spectral problem and the second giving the time evolution of the eigenfunctions, i.e.

$$L \cdot \psi = \lambda \psi, \tag{1.1.6a}$$

$$\psi_t = M \cdot \psi. \tag{1.1.6b}$$

The nonlinear PDE then arises as the compatibility condition of these two operators, namely

$$L_t + [L, M] = 0, \qquad \text{where} \quad [L, M] \equiv L \cdot M - M \cdot L,$$

the above equation being Lax's Equation and the operators $L$ and $M$ being the Lax Pair. This technique relied upon the fact that for the operator $L$ the spectrum is preserved and Lax showed that there was and infinite number of $M$'s for which this was the case, thus giving an infinite hierarchy of compatible flows for the KdV equation. In 1972, Zakharov and Shabat extended this technique further, [10], by giving a Lax pair of the form

$$\begin{aligned} \boldsymbol{\psi}_x &= \boldsymbol{L}\cdot\boldsymbol{\psi}, & (1.1.7a)\\ \boldsymbol{\psi}_t &= \boldsymbol{M}\cdot\boldsymbol{\psi}, & (1.1.7b) \end{aligned}$$

where $\boldsymbol{\psi}$ is now an $n$-dimensional vector and $\boldsymbol{L}$ and $\boldsymbol{M}$ are $n \times n$ matrices. In the case of the modified KdV equation these matrices are the following $2 \times 2$ matrices

$$L = \begin{pmatrix} -i\lambda & iv \\ -iv & -i\lambda \end{pmatrix}, \quad (1.1.8a)$$

$$M = \begin{pmatrix} -4i\lambda^3 - 2i\lambda v^2 & 4i\lambda^2 v - 2\lambda v_x - iv_{xx} + 2iv^3 \\ -4i\lambda^2 v - 2\lambda v_x + iv_{xx} - 2iv^3 & 4i\lambda^3 + 2i\lambda v^2 \end{pmatrix} \quad (1.1.8b)$$

The compatibility condition, which is now given by

$$\boldsymbol{L}_t - \boldsymbol{M}_x + [\boldsymbol{L}, \boldsymbol{M}] = 0,$$

yields the modified KdV equation (1.1.3).

This reformulation of the IST facilitated the discovery of many more nonlinear equations which could be solved via the IST. First, Zakharov and Shabat gave the Lax pair for the nonlinear Schrödinger equation (NLS), [10], followed in 1973 by Ablowitz, Kaup, Newell and Segur (AKNS) who solved the sine-Gordon equation [11]. Since then a large class of nonlinear PDEs have been shown to be solvable my means of the IST and are generally considered to be integrable.

### 1.1.4 Bäcklund Transformations

Another defining property of integrable systems is the existence of Bäcklund transformations which relate two solutions of the integrable PDE to each other, [12]. They arise from the so called dressing procedure whereby the linear operator of the relevant spectral problem is factorized, commutation of these factors then gives rise to alternative solutions of the PDE, [13].

For example, if we consider the spectral problem associated with the KdV equation

$$L\phi = (\partial_{xx} + u(x,t))\phi = \lambda\phi.$$

The linear operator $L$ can be factorized into two linear operators in the following way

$$L = L_1 \cdot L_2 = (\partial_x + v(x,t))(\partial_x - v(x,t)),$$

from which we immediately obtain the Miura transformation (1.1.2). However, if we commute the two operators $L_1$ and $L_2$ we obtain the alternative Miura transformation

$$\widetilde{u} = -(v^2 - v_x),$$

which relates solutions of the modified KdV equation (mKdV) (1.1.3) to an alternative solution, $\widetilde{u}$, of the KdV equation. (note: This alternative Miura transformation can also be obtained by noticing that the mKdV equation is invariant under the transformation $v \mapsto -v$). If we then define $u = w_x$ we obtain the following set of equations

$$\widetilde{w}_x + w_x = 2\lambda - \frac{1}{2}(\widetilde{w} - w)^2, \tag{1.1.9a}$$

$$\widetilde{w}_t - w_t = \frac{3}{2}(\widetilde{w} - w)^2(\widetilde{w}_x - w_x) - \widetilde{w}_{xxx} + w_{xxx}\,, \tag{1.1.9b}$$

which is the Bäcklund transformation for the (potential) KdV equation. Using (1.1.9) we are able to build up more complicated solutions of the (potential) KdV equation from a known simpler one (even the trivial solution $w(x,t) = 0$).

So although, as we have mentioned earlier, there is no definitive test for integrability, there are many features which a given PDE can possess which suggest that it is integrable: infinite conservation laws and commuting higher order flows, Miura and Bäcklund transformations, solvability by the IST and a Lax representation.

## 1.2 Painlevé Equations

Closely related to the subject of integrable systems are a class of ODEs known as Painlevé equations. In this section we shall give a brief review of these equations and show how they are linked to the study of integrable systems.

### 1.2.1 The Painlevé Property

When dealing with an ODE it is natural to consider whether its solution has any singularities and, if so, whether these singularities are fixed or movable. A fixed singularity is one whose position does not depend upon initial conditions whereas a movable singularity is one whose position does depend upon the initial conditions. It can be shown that for a linear ODE, all the singularities are fixed, see for example [14, 15]. However, for nonlinear ODEs this is not the case and their solutions may contain movable singularities. In 1884, L. Fuchs [16] showed that for a first order ODE of the form

$$\frac{dw}{dz} = \frac{f(z,w)}{g(z,w)},$$

where $f$ and $g$ are polynomials in $w$ with coefficients analytic in $z$, the only equation with no movable singularities other than poles is the generalized Riccati equation

$$\frac{dw}{dz} = p_0(z) + p_1(z)w + p_2(z)w^2.$$

Using the transformation

$$w(z) = -\frac{du}{dz}/(p_2(z)u),$$

the Riccati equation can be transformed into a second order linear ODE with no movable singularities but possibly movable zeros, as a result, the only movable singularities of the Riccati equation will be poles, see [14]. In 1887, Picard [17] extended the problem to second order ODEs and attempted to deduce all equations of the form

$$\frac{d^2w}{dz^2} = F(z, w, \frac{dw}{dz}),$$

with $F$ rational in $\frac{dw}{dz}$, algebraic in $w$ and analytic in $z$, whose solutions had no movable singularities other than poles, a condition which has become known as the *Painlevé Property.* The task was completed by Painlevé, Picard, Gambier and Fuchs in the period 1887-1910, [18, 19, 20, 21, 22, 23] with fifty canonical equations being shown to possess the Painlevé property (see [14] for a full list), among these fifty were six irreducible equations, i.e. ones whose solutions could not be expressed in terms of known functions. The six equations are

$$\begin{aligned}
\frac{d^2w}{dz^2} &= 6w^2 + z, \\
\frac{d^2w}{dz^2} &= 2w^3 + zw + \alpha, \\
\frac{d^2w}{dz^2} &= \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w}, \\
\frac{d^2w}{dz^2} &= \frac{1}{2w}\left(\frac{dw}{dz}\right)^2 + \frac{3w^3}{2} + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \\
\frac{d^2w}{dz^2} &= \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \\
\frac{d^2w}{dz^2} &= \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{dw}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-w}\right)\frac{dw}{dz} \\
&\quad + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right),
\end{aligned}$$

in which $\alpha, \beta, \gamma$ and $\delta$ are arbitrary parameters.

Of these equations, now known as the Painlevé Equations, ($P_I$-$P_{VI}$), only the first three were discovered by Painlevé [20], the fourth and fifth were discovered by Gambier [23] and the sixth by R. Fuchs [21]. Their solutions define new transcendental functions known as the Painlevé Transcendents. It should be noted at this point that the proof that these six equations are irreducible was found to contain flaws. Painlevé first attempted to prove the irreducibility of $\mathrm{P_I}$ [24]. For this he used a definition of irreducibility first given by Drach in [25] and later in [26], however this defintion was not complete and hence Painlevé's proof was incorrect, see for example Pommaret [27] or Umemura [28]. Attempts are currently under way to give a complete proof of their irreducibility. This has been achieved for the first equation $P_{\mathrm{I}}$, see [29], and it is hoped that a rigorous proof of the irreducibility of all six equations will be completed in the near future.

The sixth equation, $\mathrm{P_{VI}}$, is the richest equation as it contains the other five in a coalescence chain. For example, if we replace

$$z \quad \text{by} \quad 1+\epsilon z \qquad , \qquad \delta \quad \text{by} \quad \frac{\delta}{\epsilon^2} \qquad , \qquad \gamma \quad \text{by} \quad \frac{\gamma}{\epsilon} - \frac{\delta}{\epsilon^2}$$

and take the limit $\epsilon \to 0$, then we obtain $\mathrm{P_V}$. Similar limiting procedures take you down to $\mathrm{P_I}$ in the manner indicated schematically below,

$$\begin{array}{ccccccccc} & & & \nearrow & \mathrm{P_{IV}} & \searrow & & & \\ \mathrm{P_{VI}} & \to & \mathrm{P_V} & & & & \mathrm{P_{II}} & \to & \mathrm{P_I}. \\ & & & \searrow & \mathrm{P_{III}} & \nearrow & & & \end{array}$$

Although the Painlevé equations are irreducible, it is possible to solve them for certain values of the arbitrary parameters. There is currently much interest in higher order equations which posses the Painlevé Property, see for example [30, 31, 32, 33], and also the search for their discrete counterparts [34] and this will form part of this thesis.

### 1.2.2 ARS-conjecture

In 1977, Ablowitz and Segur first showed that self-similar solutions of partial differential equations, solvable by the IST, were of Painlevé type, [35]. This led them, along with Ramani to conjecture that *every nonlinear ODE obtained by an exact reduction of a nonlinear PDE of IST class is of Painlevé-type*, [36]. This is now known as the ARS-conjecture and is commonly used as a further test for integrability. (In certain cases it is necessary to use a slightly weaker version of the Painlevé property, see [37]).

For example, The modified KdV equation (1.1.3) is invariant under the scaling transformation

$$v(x,t) \mapsto \beta v(\beta x, \beta^3 t). \tag{1.2.1}$$

Hence, if we set

$$v(x,t) = \frac{1}{(3t)^{1/3}} f_1(\xi), \qquad \text{with} \qquad \xi = \frac{x}{(3t)^{1/3}}, \tag{1.2.2}$$

then equation (1.1.3) gives us after one integration, the Painlevé II equation.

For any particular integrable PDE there may be many different types of self-similar solutions, e.g. translations in time or space, Galilean transformations or scaling transformations. These can be obtained systematically by various methods, the most common being the Lie group approach, an account of which can be found in numerous texts, e.g. [38], [39] or [5]. This method, when applied to an $n$th order PDE

$$\Delta(x, t, u, u_x, u_t, \ldots) = 0, \tag{1.2.3}$$

considers a one-parameter Lie group of infinitesimal point transformations

$$\begin{aligned} x^* &= x + \epsilon X(x,t,u) + O(\epsilon^2), \\ t^* &= t + \epsilon T(x,t,u) + O(\epsilon^2), \\ u^* &= u + \epsilon U(x,t,u) + O(\epsilon^2), \end{aligned}$$

(described as point transformations as they depend only on the independent variables and the dependent variable of the PDE but not derivatives of the dependent variable). Requiring that the PDE (1.2.3) is invariant under these transformations provides a set of determining equations for the infinitesimals $X(x,t,u)$, $T(x,t,u)$ and $U(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$V = X\partial_x + T\partial_t + U\partial_u,$$

where $\partial_x = \frac{\partial}{\partial x}$ etc.

and the similarity variable and form are obtained by solving the characteristic equations

$$\frac{dx}{X} = \frac{dt}{T} = \frac{du}{U}.$$

For example, using this method, we can obtain for the Boussinesq equation

$$u_{tt} = u_{xx} - \frac{1}{2}(u^2)_{xx} - u_{xxxx}, \tag{1.2.4}$$

the following infinitesimals

$$X = \alpha x + \beta\ ,\ T = 2\alpha t + \gamma\ ,\ U = -2\alpha(u-1). \tag{1.2.5}$$

Setting $\alpha = 0$ in (1.2.5) we obtain the traveling wave solution

$$u(x,t) = f_2(\xi), \qquad \text{with} \qquad \xi = z - ct.$$

Applying this transformation to (1.2.4) we obtain a fourth order ODE. Integrating this ODE twice gives

$$f_2'' + (c^2-1)f_2 + \frac{1}{2}f_2^2 = k_1\xi + k_2, \qquad \left(f_2' \equiv \frac{df}{d\xi}\right),$$

with $k_1$, $k_2$ constants, which is solvable in terms of either the first Painlevé equation ($k_1 \neq 0$) or elliptic functions ($k_1 = 0$).

Setting $\beta = \gamma = 0$ in (1.2.5) gives the scaling reduction

$$u(x,t) = t^{-1}\, f_3(\xi) + 1, \qquad \text{with} \qquad \xi = \frac{x}{t^{1/2}}.$$

Applying this second transformation to (1.2.4) we obtain the following fourth order ODE

$$2f_3 + \frac{7}{4}\xi f_3' + \frac{1}{4}\xi^2 f_3'' + (f_3')^2 + f_3 f_3'' + f_3'''' = 0,$$

which is solvable in terms of the fourth Painlevé equation, see [40].

However, although this method provides a systematic way of finding similarity reductions, it is not exhaustive. For example, Quispel, Nijhoff and Capel, [41] and Nishitani and Tajiri [42], showed that the Boussinesq equation also possesses the reduction

$$u(x,t) = f_4(\xi) - 4c^2t^2, \qquad \text{with} \qquad \xi = x + ct^2,$$

Applying this second transformation to (1.2.4) and integrating once we obtain the following third order ODE

$$2cf_4 - 8c^2\xi - f_4' + f_4f_4' + f_4''' = k_3,$$

with $k_3$ a constant, which, after a further transformation gives the second Painlevé equation, see [42].

An alternative method, first proposed by Clarkson and Kruskal, [40, 43], involved looking for solutions of the form

$$u(x,t) = U(x,t,w(z(x,t))),$$

substituting this into the PDE and requiring that the the result is an ODE in $w(z)$ imposes conditions on $U, z$ and their derivatives, the solution of which gives the similarity reductions. By this so-called direct method, Clarkson and Kruskal were able to find six different similarity reductions of the Boussinesq equation, only two of which were obtainable by the classical Lie group approach. The direct method was, however, shown to be equivalent to the nonclassical approach of Bluman and Cole, see [44, 45], which adds an extra surface condition which must also be invariant under the group of infinitesimal point transformations.

The reduction to self-similar solutions can also be expressed through a differential relationship, for example, (1.2.1) can be written

$$v + xv_x + 3tv_t = 0. \tag{1.2.6}$$

As such (1.2.6) can be considered as a differential constraint on the original PDE which can be used to reduce the number of independent variables in the PDE. We shall therefore refer to equations like (1.2.6) as *similarity constraints.*

As a test for integrability the ARS-conjecture can provide a useful tool. Having found a self-similar solution of a PDE it is a relatively straightforward procedure to test whether or not the resulting reduction is of Painlevé type, see [36]. Unfortunately this is only a necessary condition for integrability as it is possible to find PDEs which although having self-similar solutions of Painlevé type do not appear to be integrable. Hence, the ARS-conjecture is a necessary but not sufficient condition for integrability.

### 1.2.3 Isomonodromic Deformation

The self-similar solutions of integrable equations also provide a way to obtain a linear system, the compatibility condition of which is an an equation of Painlevé type. This was first demonstrated in 1980 by Flaschka and Newell, [46].

Applying the similarity reduction

$$v(x,t) = \frac{1}{(3t)^{1/3}} f(\xi), \tag{1.2.7a}$$

$$\psi(x,t:\lambda) = \Psi(\xi:k), \tag{1.2.7b}$$

$$\text{with} \qquad \xi = \frac{x}{(3t)^{1/3}} \quad \text{and} \quad k = (3t)^{1/3}\lambda, \tag{1.2.7c}$$

(cf. (1.2.2)), to the Lax pair (1.1.7) we get

$$\begin{aligned} \psi_x &= (3t)^{-\frac{1}{3}}\Psi_\xi, \\ \psi_t &= \frac{k}{3t}\frac{d}{dk}\Psi - \frac{\xi}{3t}\Psi_x. \end{aligned}$$

Thus, applying (1.2.7) to the Lax matrices for the modified KdV equation (1.1.8), we get the following linear system

$$\Psi_\xi = A\Psi, \tag{1.2.8a}$$

$$k\frac{d}{dk}\Psi = B\Psi, \tag{1.2.8b}$$

where

$$\begin{aligned} A &= \begin{pmatrix} -ik & if \\ -if & -ik \end{pmatrix}, \\ B &= \begin{pmatrix} -4ik^3 - 2ikf^2 - ik\xi & 4ik^2 f - 2kf_\xi - i\nu \\ -4ik^2 f - 2kf_\xi + i\nu & 4ik^3 + 2ikf^2 + ik\xi \end{pmatrix}, \end{aligned}$$

for which the compatibility relation

$$k\frac{d}{dk}A - B_\xi + [A, B] \;=\; 0, \tag{1.2.9}$$

is only satisfied if $f$ obeys $\mathrm{P_{II}}$, see [47].

We will refer to the linear system (1.2.8) as a monodromy problem. It is part of the theory of isomonodromic deformation, which dates back to the work of R. Fuchs and Schlesinger [48, 49] and extended more recently by Jimbo, Miwa and Ueno [46, 50, 51, 52]. This theory can be seen as an alternative to the IST method whereby the solution of a nonlinear PDE is is obtained from the monodromy data of the linear system (1.2.8).

## 1.3 Integrability in the Discrete Case

Having given a brief summary of the main notions of integrability for continuous integrable systems we now turn our attentions to discrete integrable systems. On the whole, the general field of discrete systems and difference equations is somewhat underdeveloped in comparison to its continuous counterpart. However, in recent years, interest in these systems has increased dramatically, especially with regards to integrability and many international conferences are now dedicated exclusively to discrete systems, most notably the SIDE (Symmetries and Integrability of Difference Equations) meetings. Integrable lattice equations have also found applications in the field of numerical analysis, for example as convergence accelerators algorithms, see [53, 54]. In this section we give a short review of how discrete systems are obtained and integrability in the discrete sense.

### 1.3.1 Integrable Difference Equations

An early example of an integrable difference equation was the Toda lattice

$$\frac{\partial^2 u_n}{\partial t^2} = \exp(u_{n-1} - u_n) - \exp(u_n - u_{n+1}), \tag{1.3.1}$$

given by Toda in 1967, [55]. This is an example of a partial differential-difference equation in which the dependent variable $u$ is a function of one continuous independent variable $t$ and one discrete independent variable $n$. The integrability of this equation was demonstrated by Flaschka in 1974, [56, 57], who solved by means of the IST.

Since then many techniques have been used to derive both partial differential-difference equations and partial difference-difference equations.

For example, in 1981, Hirota [58] gave the following bilinear generalization of the Toda equation

$$[z_1 \exp(D_1) + z_2 \exp(D_2) + z_3 \exp(D_3)] f \cdot f = 0,$$

where $z_1, z_2, z_3$ are arbitrary parameters and $D_1, D_2, D_3$ are linear combinations of binary operators $D_x, D_y, D_t, D_n$, etc., defined by

$$D_x f \cdot g = \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} f(x+\epsilon) g(x-\epsilon) = f_x g - f g_x.$$

Using this equation, Hirota was able to derive discrete versions of many equations including KdV, modified KdV and Sine-Gordon. The discretization in this case arising from the exponentiation of a differential operator.

Earlier, in 1976, Ablowitz and Ladik, [59, 60], using a discretization of the Lax pair, gave discrete versions of the nonlinear Schrödinger equation. This method has applications in numerical analysis [61] and has recently been refined by Suris, [62, 54].

Much work has also been done by Bobenko and Pinkall in connecting integrable systems with discrete geometries, see [63]. The integrable equations in this case derive from conditions on quadrilaterals.

Discrete, integrable equations were also derived by Quispel, Nijhoff et al. using the direct linearization technique. This approach was based on a linear integral equation given by Fokas and Ablowitz in 1981, [64], and as this technique is used throughout this thesis we shall now give a short review of this paper.

### 1.3.2 Direct Linearization

In 1981, Fokas and Ablowitz presented an integral equation which linearized the KdV equation [64]. Specifically, they showed that if $\phi(k;x,t)$ was the solution of the integral equation

$$\phi(k;x,t) + i\exp[i(kx+k^3t)]\int_L \frac{\phi(l;x,t)}{l+k}d\lambda(l) \quad = \quad \exp[i(kx+k^3t)], \tag{1.3.2}$$

then

$$u \quad = \quad -\frac{\partial}{\partial x}\int_L \phi(k;x,t)d\lambda(k), \tag{1.3.3}$$

solved the KdV equation (1.0.1). In equation (1.3.2), $d\lambda(k)$ and $L$ are an arbitrary contour and measure which satisfy the conditions:

i) differentiation with respect to $x$ and $t$ can be interchanged with $\int_L$.

ii) the homogeneous integral equation has zero solution.

By specifying the exact form of the contour and measure they were also able to show the following results:

i) Setting the measure to be

$$d\lambda(k) \quad = \quad b(k)(\frac{1}{2}k)\frac{dk}{2\pi},$$

where $b(k)$ is the reflection coefficient of $u(x,0)$ and letting the contour $L$ go over all the poles of $b(k)$ the integral equation (1.3.2) yields the Gel'fand-Levitan Marchenko equation (1.1.5).

ii) Setting the measure to be

$$d\lambda(k) = \sum_{j=1}^{N} c_j \delta(k - i\kappa_j) dk,$$

and letting $L$ pass through the $k = i\kappa_j$ then the integral equation (1.3.2) gives the N-soliton solution.

iii) Imposing a self-similar reduction on (1.3.2) and (1.3.3), solutions to the second Painlevé equation can be derived.

In 1983, Quispel, Nijhoff Capel and J. van der Linden [65, 66], showed how the linear integral equation (1.3.2) could be used to derive nonlinear difference-difference equations via Bäcklund transformations. We shall save a more detailed explanation of this procedure for the next section.

### 1.3.3 Discrete Painlevé Equations and Singularity Confinement

As well as discrete versions of integrable partial difference equations, there are also discrete versions of Painlevé equations (dP's). These first appeared as nonlinear recurrence relations for the coefficients in the linear recurrence relations of orthogonal polynomials with the relation

$$a_n^2(a_{n-1}^2 + a_n^2 + a_{n+1}^2) + 2ta_n^2 = n. \tag{1.3.4}$$

This equation, with the transformation $a_n^2 = u_n$, is a discrete version of $\mathrm{P_I}$. However, although (1.3.4) was first given by Shohat [67], this fact was not realised until much later, see [68, 69]. Since then, discrete versions of all six of the Painlevé equations have been found. For example, the following list of difference equations gives some of the most common versions of discrete Painlevé equations, [70].

$$\begin{aligned}
\text{dPI} &: \quad x_{n+1}+x_n+x_{n-1}=\frac{\zeta_n}{x_n}+a,\\
\text{dPII} &: \quad x_{n+1}+x_{n-1}=\frac{\zeta_n x_n+a}{1-x_n^2},\\
\text{dPIII} &: \quad x_{n+1}x_{n-1}=\frac{(x_n+a)(x_n+b)}{(cq^n x_n+1)(dq^n x_n+1)},\\
\text{dPIV} &: \quad (x_{n+1}+x_n)(x_n+x_{n-1})=\frac{(x_n^2-a^2)(x_n-b^2)}{(x_n+\zeta_n)^2-c^2},\\
\text{dPV} &: \quad (x_{n+1}x_n-1)(x_nx_{n-1}-1)=\eta_n\theta_n\frac{(x_n+a)(x_n+a^{-1})(x_n+b)(x_n+b^{-1})}{(x_n+\eta_n)(x_n+\theta_n)},\\
\text{dPVI} &: \quad \frac{(x_{n+1}x_n-\eta_n\eta_{n+1})(x_nx_{n-1}-\eta_n\eta_{n-1})}{(x_{n+1}x_n-1)(x_nx_{n-1}-1)}=\\
&\qquad \frac{(x_n-\eta_n a)(x_n-\eta_n/a)(x_n-\eta_n b)(x_n-\eta_n/b}{(x_n-c)(x_n-1/c)(x_n-d)(x_n-1/d)},
\end{aligned}$$

(where $\zeta_n=\alpha n+\beta$, $\eta_n=\gamma q^n$, $\theta_n=\delta q^n$, $\alpha$,$\beta$,$\gamma$,$\delta$,$a$,$b$ and $c$ being constants).

These equations were derived by a variety of methods. The first is simply equation (1.3.4) which we have already discussed, the second was derived in [71] using matrix-models of two-dimensional quantum gravity and also by Nijhoff and Papageorgiou [72] as a similarity reduction of a lattice equation, the third, fourth and fifth were given by Ramani, Grammaticos and Hietarinta [70] using singularity confinement (of which more later) and the last was given by Grammaticos and Ramani [73] through "intuition and inspiration".

The numbering of the above discrete equations arises from the fact that in a continuous limit each of the equations reduces to the relevant continuous Painlevé equation. However, this correspondence is not unique as there are other discrete equations which also, in the continuous limit, reduce to continuous Painlevé equations. For example, alternative versions of $dP_I$ and $dP_{II}$ can be given by

$$\begin{aligned}
\text{alt}-\text{dPI} \quad &: \quad \frac{n+1}{x_{n+1}+x_n}+\frac{n}{x_n+x_{n-1}}=n+a+bx_n^2, \\
\text{alt}-\text{dPII} \quad &: \quad \frac{n+1}{x_{n+1}x_n+1}+\frac{n}{x_nx_{n-1}+1}=n+a+b\left(x_n-\frac{1}{x_n}\right).
\end{aligned}$$

The first of these equations was given by Jimbo and Miwa [52]. The second was given, along with the first, by Ramani, Grammaticos and Hietarinta [74] who derived them from Bäcklund transformations of continuous Painlevé equations. Unlike the continuous Painlevé equations, there is currently no complete classification of discrete Painlevé equations although Sakai's recent paper [75] does attempt such a classification in terms of affine root systems.

The main problem concerning discrete Painlevé equations is that there is no discrete analogue of the Painlevé property. In 1991, Ramani, Grammaticos and Papageorgiou [76] proposed the *singularity confinement method* as a possible candidate. This test involves determining whether singularities which arise from the initial data of a given integrable mapping remain confined, (i.e. they do no propagate indefinitely as the map is iterated). As already mentioned, this method has been very successful in providing further examples of discrete Painlevé equations but as a test for integrability it is not complete. It is, for example, only able to deal rational maps and recently, Hietarinta and Viallet [77] showed that the following equation

$$x_{n+1}+x_{n-1} \quad = \quad x_n+\frac{a}{u_n^2}, \tag{1.3.5}$$

does pass the singularity confinement test but is also seen to be numerically chaotic, hence, as a test for integrability singularity confinement is not sufficient.

Other approaches proposed as a discrete analogue of the Painlevé property include Nevanlinna theory and the discrete Painlevé property of Conte and Musette, see [78] for a review of these methods.

This concludes a brief introduction into some of the main aspects of integrable systems, in particular, those which shall be of relevance throughout the rest of the thesis. In the next chapter we shall focus more closely on the direct linearization of the KdV equation, describing the main results obtainable by this method.

# Chapter 2

# The KdV System

In this section we shall review the work of Quispel, Nijhoff, Capel, Papageorgiou, and J. van der Linden on the direct linearization scheme, in particular, with reference to the KdV system [79, 65, 80, 66, 72, 81, 82, 83]. We shall show how the linear integral equation (1.3.2) of Fokas and Ablowitz [64] can be generalized to include the modified KdV (mKdV) and the Schwarzian KdV (SKdV) equations as well as the KdV equation and how the scheme can be used to derive discrete lattice equations, similarity constraints and differential-difference equations.

## 2.1 Direct Linearization of the KdV System

The Direct Linearization approach of Fokas and Ablowitz can be generalized in such a way that the dependent variable $\boldsymbol{u}_k$ is an infinite vector, rather than a scalar [65, 80]. Thus we have the following integral equation

$$\boldsymbol{u}_k + \int_\Gamma d\lambda(l) \frac{\rho_k \boldsymbol{u}_l}{k+l} = \rho_k \boldsymbol{c}_k, \tag{2.1.1}$$

where $\boldsymbol{c}_k$ is an infinite vector with components $(\boldsymbol{c}_k)_j = k^j$,

$\rho_k$ is a plane wave factor,

$\Gamma$ and $d\lambda$ are arbitrary contour and measure.

Equation (2.1.1) can now be used to derive various integrable systems. The choice of the plane plane wave factor $\rho_k$ will determine number of independent variables and whether these variables are discrete or continuous, while a choice of entry in the dependent variable will isolate specific members of a Miura chain.

In order to achieve this we must first develop an Infinite Matrix structure. For this we will treat the integral equation (2.1.1) as a purely formal tool and not specify any particular measures or contours on the integration. We do this in order to derive the equations and provide some insight into the algebraic structure underlying the system.

## 2.2 Infinite Matrix Formalization

Firstly we require an infinite $(\mathbb{Z} \times \mathbb{Z})$ matrix $C$, of the form

$$C = \int_\Gamma d\lambda(l)\rho_l \boldsymbol{c}_l\, {}^t\boldsymbol{c}_l. \tag{2.2.1}$$

As already mentioned, the specific contour and measure will not be defined, but it is assumed they can be chosen such that all subsequent objects are well defined. From the definition it is clear that this matrix has the symmetry $C = {}^tC$ (where the left superscript $t$ indicates matrix transposition). Using the matrix $C$ we shall derive various linear relationships whose form depends on the choice of the wave factor $\rho_k$. In order to obtain the more important nonlinear equations we require a second infinite $(\mathbb{Z} \times \mathbb{Z})$

matrix $U$ defined by

$$U = \int_\Gamma d\lambda(l) \boldsymbol{u}_l \, {}^t\boldsymbol{c}_l, \tag{2.2.2}$$

and it is specific entries of this matrix which shall provide us with closed form scalar equations. Again, by definition, this matrix has the symmetry $U = {}^tU$ and it is this property that defines the KdV system.

These are the two main ingredients, but we also require the following infinite matrices

i) Index raising matrices $\Lambda$ and ${}^t\Lambda$, used for left and right multiplication respectively. These play the role of either stepping up or down or left or right through a matrix, e.g.

$$(\Lambda \cdot \boldsymbol{c}_k)_j = k(\boldsymbol{c}_k)_j \quad \text{and} \quad ({}^t\boldsymbol{c}_k \cdot {}^t\Lambda)_j = k(\boldsymbol{c}_k)_j. \tag{2.2.3a}$$

ii) Index counting matrices $I$ and ${}^tI$, again left and right multipliers respectively, which operate on a matrix in the following way

$$(I \cdot \boldsymbol{c}_k)_j = j(\boldsymbol{c}_k)_j \quad \text{and} \quad ({}^t\boldsymbol{c}_k \cdot {}^tI)_j = j(\boldsymbol{c}_k)_j. \tag{2.2.3b}$$

iii) Projection matrix $O$ which picks out the central element of a matrix, e.g.

$$(O \cdot \boldsymbol{c}_k)_j = \delta_{0,j}\boldsymbol{c}_j \quad \text{and} \quad ({}^t\boldsymbol{c}_k \cdot O)_j = \delta_{j,0}\, {}^t\boldsymbol{c}_k. \tag{2.2.3c}$$

iv) Infinite matrix $\Omega$ which we define in terms of the way it relates to the other infinite matrices in the scheme, namely, for the index raising matrices it obeys the relationship

$$\Omega \cdot \Lambda^j - (-\,{}^t\Lambda)^j \cdot \Omega = O_j, \tag{2.2.3d}$$

$$\text{where} \qquad O_k = \sum_{j=0}^{k-1} (-\,{}^t\Lambda)^j \cdot O \cdot \Lambda^{k-1-j}, \tag{2.2.3e}$$

and for the index counting matrices it obeys the relationship

$${}^tI \cdot \Omega + \Omega \cdot I + \Omega = 0. \tag{2.2.3f}$$

Using these objects, the original integral equation can be expressed in infinite matrix form

$$U = C \cdot (1 + \Omega \cdot C)^{-1}. \tag{2.2.4}$$

We are now free to define the wave factor $\rho_k$, which as mentioned can depend on either discrete or continuous variables. For this section we shall concentrate on discrete variables as it is these which shall be used throughout this thesis.

## 2.3 Discrete Case

If we choose to work with a discrete system then $\rho_k$ takes on the form

$$\rho_k = \prod_{\nu} \left( \frac{p_\nu + k}{p_\nu - k} \right)^{n_\nu}, \tag{2.3.1}$$

where $\nu = 1, 2, .., N$ with $N$ being the number of dimensions of the system.

The factor $\rho_k$ then consists of $N$ independent discrete variables $n_\nu$ (these may in fact take on non-integer values but are discrete in that they are required to shift by integers, i.e. $n_\nu = \theta_\nu + \mathbb{Z}, \theta_\nu \in \mathbb{R}$), each of which has an associated lattice parameter $p_\nu$ which are in fact the Bäcklund parameters and a parameter $k$ which takes on the role of the spectral parameter.

Now, from this choice of $\rho_k$ we are able to impose three different relationships on $C$ and hence also on $U$.

First we have the discrete evolution, which for $\rho_k$ is given by

$$T_\nu \rho_k = \left( \frac{p_\nu + k}{p_\nu - k} \right) \rho_k, \tag{2.3.2}$$

where $T_\nu$ represents the operation of shifting a function i.e. $T_\nu f(n_\nu) = f(n_\nu + 1)$.

In [84, 65, 80] it was shown that this shift is equivalent to a singular transformation of the measure in the integral equation (2.1.1) and as such represents a Bäcklund transformation. The resulting lattice equations can therefore be viewed as a sequence of Bäcklund transformations.

This leads us to the following two linear relationships for $C$

$$(T_\nu C) \cdot (p_\nu - {}^t\Lambda) = (p_\nu + \Lambda) \cdot C, \tag{2.3.3a}$$

$$(p_\nu - \Lambda) \cdot T_\nu C = C \cdot (p_\nu + {}^t\Lambda), \tag{2.3.3b}$$

which, given that $C$ is symmetric i.e. $C = {}^tC$, are equivalent.

Equations (2.2.4) and (2.2.3d) then give the nonlinear relationships in $U$

$$(T_\nu U) \cdot (p - {}^t\Lambda) = (p + \Lambda) \cdot U - (T_\nu U) \cdot O \cdot U, \tag{2.3.4a}$$

$$(p - \Lambda) \cdot T_\nu U = U \cdot (p + {}^t\Lambda) - U \cdot O \cdot T_\nu U. \tag{2.3.4b}$$

Equations (2.3.4a) and (2.3.4b) are matrix Riccati equations from which the lattice equations can be derived.

Again, as we have the condition $U = {}^tU$ equations (2.3.4a) and (2.3.4b) are equivalent, and by transposing either of the equations we can eliminated the shifts to produce the following, purely algebraic relationship

$$U \cdot {}^t\Lambda^2 = \Lambda^2 \cdot U - U \cdot O \cdot \Lambda \cdot U + U \cdot {}^t\Lambda \cdot O \cdot U. \tag{2.3.5}$$

In order to derive similarity constraints for the system we must impose a scaling invariance on the integral equation. In [72] it was shown that this could be achieved by imposing the following constraint on the measure and contour

$$\int_\Gamma d\lambda(l) \frac{d}{dl} f(l) = 0, \tag{2.3.6}$$

for some function $f(l)$ which is a solution of the integral equation. In order to implement (2.3.6) on the level of the infinite matrices the function we must consider is $f(l)\rho_l \boldsymbol{c}_l \, {}^t\boldsymbol{c}_l$

and we therefore need to know how the plane-wave factor $\rho_k$ behaves under the action of the differential operator $k\frac{\partial}{\partial k}$. For this we find that

$$k\frac{\partial}{\partial k}\rho_k = \sum_\nu n_\nu p_\nu \left(\frac{1}{p_\nu - k} - \frac{1}{p_\nu + k}\right)\rho_k, \tag{2.3.7}$$

and, noting that

$$k\frac{\partial}{\partial k}c_k = I \cdot c_k \quad \text{and} \quad k\frac{\partial}{\partial k}{}^t c_k = {}^t c_k \cdot {}^t I,$$

we find that in infinite matrix notation

$$\begin{aligned} C + I \cdot C + C \cdot {}^t I &= [l\rho_l c_l\, {}^t c_l]_{\partial\Gamma} + \\ &\sum_\nu n_\nu p_\nu \left(\frac{1}{p_\nu + \Lambda} \cdot C - C \cdot \frac{1}{p_\nu - {}^t\Lambda}\right). \end{aligned} \tag{2.3.8}$$

Note: The term $\frac{1}{p_\nu+\Lambda}$ and all subsequent terms of this type should be taken to mean the inverse of $p_\nu(\text{id}) + \Lambda$ in the sense of a formal infinite series, where (id) is the $2 \times 2$ identity matrix. In all the following calculations involving these terms we shall only be interested in the combinatorics of the relevant system and hence do not need to calculate the explicit form of these inverses. Also, the term in square brackets represents the boundary conditions arising from (2.3.6).

Again, this is a linear equation in $C$ which we can use to derive the following nonlinear equation in terms of the variable $U$

$$\begin{aligned} U + I \cdot U + U^t \cdot I &= [l\rho_l^{-1} u_l\, {}^t u_l]_{\partial\Gamma} + \\ &\sum_\nu n_\nu p_\nu \left(\frac{1}{p_\nu + \Lambda} \cdot U - U \cdot \frac{1}{p_\nu - {}^t\Lambda} + U \cdot \frac{1}{p_\nu - {}^t\Lambda} \cdot O \cdot \frac{1}{p_\nu + \Lambda} \cdot U\right), \end{aligned} \tag{2.3.9}$$

and from equation (2.3.9) we shall derive similarity constraints for the system.

Finally, we obtain relationships for how the factor $\rho_k$ depends on the lattice variable $p_\nu$.

$$\frac{\partial}{\partial p_\nu}\rho_k = n_\nu \left(\frac{1}{p_\nu + k} - \frac{1}{p_\nu - k}\right)\rho_k, \tag{2.3.10}$$

which gives us, in terms of the infinite matrix $C$, the relation

$$\frac{\partial}{\partial p_\nu} C = n_\nu \left( \frac{1}{p_\nu + \Lambda} \cdot C - C \cdot \frac{1}{p_\nu - {}^t\Lambda} \right), \tag{2.3.11}$$

and thus for $U$ we have

$$\frac{\partial}{\partial p_\nu} U = n_\nu \left( \frac{1}{p_\nu + \Lambda} \cdot U - U \cdot \frac{1}{p_\nu - {}^t\Lambda} + U \cdot \frac{1}{p_\nu - {}^t\Lambda} \cdot O \frac{1}{p_\nu + \Lambda} \cdot U \right). \tag{2.3.12}$$

This equation gives us the differential-difference relations.

We should note here that the operation of shifting and the dependence on the lattice parameters commute, namely

$$T_\nu \frac{\partial}{\partial p_\nu} \rho_k = \frac{\partial}{\partial p_\nu} T_\nu \rho_k, \tag{2.3.13}$$

thus ensuring that the lattice equations and the differential-difference equations are compatible.

## 2.4 Closed Form Scalar Equations

At this stage we are still dealing with an arbitrary number of dimensions $N$, but for the remainder of this section we shall limit ourselves to the 2-dimensional case and make the following identifications

$$n_1 = n \quad , \quad n_2 = m \quad , \quad p_1 = p \quad , \quad p_2 = q.$$

Along with two different transformations of the factor $\rho_k$

$$\rho_k \mapsto T_1 \rho_k = \widetilde{\rho}_k \qquad , \qquad \rho_k \mapsto T_2 \rho_k = \widehat{\rho}_k, \tag{2.4.1}$$

each of the same type but with different parameters $p$ and $q$ respectively. Transformations of the objects in (2.2.3) are then interpreted as transformations of functions on a two-dimensional lattice with grid points $(n,m)$.

In order to derive closed form lattice equations we must introduce the following objects

$$u = U_{0,0}, \qquad s_{\alpha,\beta} = \left(\tfrac{1}{\alpha+\Lambda}\cdot U\cdot\tfrac{1}{\beta+{}^t\Lambda}\right)_{0,0},$$
$$v_\alpha = 1-\left(\tfrac{1}{\alpha+\Lambda}\cdot U\right)_{0,0}, \quad s_\alpha = \alpha-\left(\tfrac{1}{\alpha+\Lambda}\cdot U\cdot{}^t\Lambda\right)_{0,0},$$

where $\alpha$ and $\beta$ are arbitrary parameters which can take on any value we require.

The main equations (2.3.4) then gives us the following Miura type relations which relate $u$ to $v_\alpha$

$$\begin{aligned}
p-q+\widetilde{u}-\widehat{u} &= (p-\alpha)\frac{\widetilde{v}_\alpha}{\widehat{\widetilde{v}}_\alpha}-(q-\alpha)\frac{\widehat{v}_\alpha}{\widehat{\widetilde{v}}_\alpha},\\
&= (p+\beta)\frac{\widetilde{v}_\beta}{v_\beta}-(q+\beta)\frac{\widehat{v}_\beta}{v_\beta},\\
p+q+u-\widehat{\widetilde{u}} &= (p-\alpha)\frac{v_\alpha}{\widetilde{v}_\alpha}+(q+\alpha)\frac{\widehat{\widetilde{v}}_\alpha}{\widetilde{v}_\alpha},\\
&= (p+\beta)\frac{\widehat{\widetilde{v}}_\beta}{\widehat{v}_\beta}+(q-\beta)\frac{v_\beta}{\widehat{v}_\beta},
\end{aligned}$$

along with another set relating $v_\alpha$ to $s_{\alpha,\beta}$

$$1-(p+\beta)\widetilde{s}_{\alpha,\beta}+(p-\alpha)s_{\alpha,\beta}=\widetilde{v}_\alpha v_\beta, \tag{2.4.2}$$

$$1-(q+\beta)\widehat{s}_{\alpha,\beta}+(q-\alpha)s_{\alpha,\beta}=\widehat{v}_\alpha v_\beta. \tag{2.4.3}$$

A process of elimination then gives us various closed form lattice equations. For example, for the variable $u=u_{n,m}$ we have

$$(p-q+u_{n,m+1}-u_{n+1,m})(p+q-u_{n+1,m+1}+u_{n,m})=p^2-q^2, \tag{2.4.4}$$

which is the lattice version of the (potential) KdV eqation. For the variable $v_\alpha=v_{n,m}$ we have

$$(p-\alpha)\frac{v_{n,m+1}}{v_{n+1,m+1}}-(q-\alpha)\frac{v_{n+1,m}}{v_{n+1,m+1}}=(p+\alpha)\frac{v_{n+1,m}}{v_{n,m}}-(q+\alpha)\frac{v_{n,m+1}}{v_{n,m}},$$

which, on setting $\alpha = 0$ gives us

$$pv_{n,m}v_{n,m+1} + qv_{n,m+1}v_{n+1,m+1} = qv_{n,m}v_{n+1,m} + pv_{n+1,m}v_{n+1,m+1}, \tag{2.4.5}$$

which is the lattice version of the (potential) modified KdV equation, and for the variable $s_{\alpha,\beta} = s_{n,m}$ we have

$$\frac{1-(p+\beta)s_{n+1,m}+(p-\alpha)s_{n,m}}{1-(q+\beta)s_{n,m+1}+(q-\alpha)s_{n,m}} = \frac{1-(q+\alpha)s_{n+1,m+1}+(q-\beta)s_{n+1,m}}{1-(p+\alpha)s_{n+1,m+1}+(p-\beta)s_{n,m+1}},$$

which on setting $\alpha = \beta = 0$ and defining $z_{n,m} \equiv s_{0,0} - \frac{n}{p} - \frac{m}{q}$ gives us

$$\frac{(z_{n,m}-z_{n+1,m})(z_{n,m+1}-z_{n+1,m+1})}{(z_{n,m}-z_{n,m+1})(z_{n+1,m}-z_{n+1,m+1})} = \frac{q^2}{p^2}, \tag{2.4.6}$$

which is the lattice version of the (potential) Schwarzian KdV equation.

Now, using equation (2.3.9), we are able to derive the closed form similarity constraints for each of the variables $u$, $v_0$ and $z$

$$(\lambda(-1)^{n+m} + \frac{1}{2})(u - np - mq) = \frac{np^2}{2p - \widetilde{u} + \underset{\sim}{u}} + \frac{mq^2}{2q - \widehat{u} + \underset{\frown}{u}}, \tag{2.4.7a}$$

$$\mu - \lambda(-1)^{n+m} = n\frac{\widetilde{v}_0 - \underset{\sim}{v}_0}{\widetilde{v}_0 + \underset{\sim}{v}_0} + m\frac{\widehat{v}_0 - \underset{\frown}{v}_0}{\widehat{v}_0 + \underset{\frown}{v}_0}, \tag{2.4.7b}$$

$$z(1+2\mu) = 2n\frac{(\widetilde{z}-z)(z-\underset{\sim}{z})}{\widetilde{z}-\underset{\sim}{z}} + 2m\frac{(\widehat{z}-z)(z-\underset{\frown}{z})}{\widehat{z}-\underset{\frown}{z}}, \tag{2.4.7c}$$

where $\widetilde{u}$ represents a forward shift through the lattice in the relevant dimension and $\underset{\sim}{u}$ represents a backward shift.

And from equation (2.3.12) we can derive the closed form differential-difference relations

$$\frac{\partial u}{\partial p} = n\left(1 - \frac{2p}{2p + \underset{\sim}{u} - \widetilde{u}}\right), \tag{2.4.8a}$$

$$\frac{\partial}{\partial p}\log v_0 = -\frac{n}{p}\frac{\widetilde{v}_0 - \underset{\sim}{v}_0}{\widetilde{v}_0 + \underset{\sim}{v}_0}, \tag{2.4.8b}$$

$$\frac{\partial z}{\partial p} = -\frac{2n}{p}\frac{(\widetilde{z}-z)(z-\underset{\sim}{z})}{\widetilde{z}-\underset{\sim}{z}}, \tag{2.4.8c}$$

We now have a complete set of equations for the KdV system, derived from the integral equation (2.1.1). These are the lattice equations (2.4.4), (2.4.5) and (2.4.6), the similarity constraints (2.4.7) and the differential difference relations (2.4.8). We shall now derive the associated linear problems for these equations from the basic equations of the infinite matrix structure.

## 2.5 Lax Pairs

Each of the preceding types of relationships can also be derived from an associated linear problem. For this we must introduce the infinite vector

$$\boldsymbol{u}_k \quad = \quad \rho_k(\mathbf{1} - \boldsymbol{U} \cdot \boldsymbol{\Omega}) \cdot \boldsymbol{c}_k. \tag{2.5.1}$$

The linear problem is then derived by once again picking out certain entries in this vector. For the (potential) KdV this is given in terms of the two-component vector

$$\phi_k \equiv (p-k)^n (q-k)^m \begin{pmatrix} u_k^0 \\ u_k^1 \end{pmatrix}, \tag{2.5.2}$$

which leads to the Lax pair

$$\widetilde{\phi}_k \quad = \quad L_k^{(KdV)} \cdot \phi_k, \tag{2.5.3a}$$

$$\widehat{\phi}_k \quad = \quad M_k^{(KdV)} \cdot \phi_k, \tag{2.5.3b}$$

where

$$L_k^{(KdV)} \equiv \begin{pmatrix} p - \widetilde{u} & 1 \\ k^2 - p^2 + * & p + u \end{pmatrix}, \tag{2.5.4}$$

$* =$ product of the diagonal entries

and $M_k^{(KdV)}$ is given by replacing $\widetilde{u}$ with $\widehat{u}$ and $p$ with $q$.

The lattice (potential) KdV equation (2.4.4) is then the compatibility condition

$$\widehat{L}_k^{(KdV)} \cdot M_k^{(KdV)} = \widetilde{M}_k^{(KdV)} \cdot L_k^{(KdV)}.$$

For the modified (potential) KdV equation, the linear problem is given in terms of the vector

$$\psi_k \equiv (p-k)^n (q-k)^m \begin{pmatrix} u_k^{(\alpha)} \\ u_k^0 \end{pmatrix}, \tag{2.5.5}$$

where $u_k^{(\alpha)}$ is given by

$$u_k^{(\alpha)} \equiv \left(\frac{1}{\alpha+\Lambda} \cdot u_k\right)_0 . \tag{2.5.6}$$

This leads to the Lax pair

$$\widetilde{\psi}_k = L_k^{(mKdV)} \cdot \psi_k, \tag{2.5.7a}$$

$$\widehat{\psi}_k = M_k^{(mKdV)} \cdot \psi_k, \tag{2.5.7b}$$

where

$$L_k^{(mKdV)} \equiv \begin{pmatrix} p-\alpha & \widetilde{v}_\alpha \\ \frac{k^2-\alpha^2}{v_\alpha} & (p+\alpha)\frac{\widetilde{v}_\alpha}{v_\alpha} \end{pmatrix}, \tag{2.5.7c}$$

and $M_k^{(mKdV)}$ is again given by replacing $\widetilde{v}$ with $\widehat{v}$ and $p$ with $q$.

The compatibility condition of this Lax pair then leads to the lattice (potential) modified KdV equation(2.4.5) as follows from a direct calculation.

The lax representations (2.5.3) and (2.5.7) are related to each other via the following gauge transformation

$$L_k^{(mKdV)} = \widetilde{U}_k \cdot L_k^{(KdV)} \cdot U_k^{-1}. \tag{2.5.8}$$

where

$$U_k = \begin{pmatrix} -s_\alpha & v_\alpha \\ k^2 - \alpha^2 & 0 \end{pmatrix}, \tag{2.5.9}$$

(and similarly for the $M_k$ matrices).

For the Schwarzian (potential) KdV equation we must perform a further gauge transformation

$$L_k^{(mKdV)} = \widetilde{V}_k \cdot L_k^{(KdV)} \cdot V_k^{-1}, \tag{2.5.10}$$

where

$$V_k = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{v_\alpha} \end{pmatrix}. \tag{2.5.11}$$

This leads to the Lax pair

$$\widetilde{\chi}_k = L_k^{(SKdV)} \cdot \chi_k, \tag{2.5.12a}$$

$$\widehat{\chi}_k = M_k^{(SKdV)} \cdot \chi_k, \tag{2.5.12b}$$

where

$$L_k^{(SKdV)} \equiv \begin{pmatrix} 1 & z - \widetilde{z} \\ \frac{k^2}{p^2(z-\widetilde{z})} & 1 \end{pmatrix}, \tag{2.5.13}$$

and once again $M_k^{(SKdV)}$ is given by replacing $\widetilde{z}$ with $\widehat{z}$ and $p$ with $q$.

The compatibility condition of this Lax pair then leads to the lattice (potential) Schwarzian KdV equation(2.4.6).

Using similar methods we are able to derive monodromy problems for each of the variables from the basic equations (2.3.9).

For the vector $\phi_k$ we obtain

$$k\frac{d}{dk}\phi_k = \begin{pmatrix} n+m & 0 \\ -np-mq & n+m+1 \end{pmatrix}\phi_k + \lambda(-1)^{n+m}\begin{pmatrix} 1 & 0 \\ 2np+2mq & -1 \end{pmatrix}\phi_k$$
$$- \frac{2np^2}{2p+\underset{\sim}{u}-\widetilde{u}}\begin{pmatrix} 1 & 0 \\ -p+\widetilde{u} & 0 \end{pmatrix}\underset{\sim}{\phi_k} - \frac{2mq^2}{2q+\underset{\frown}{u}-\widehat{u}}\begin{pmatrix} 1 & 0 \\ -q+\widehat{u} & 0 \end{pmatrix}\underset{\frown}{\phi_k}, \quad (2.5.14)$$

which, on using the inverse of the Lax pair (2.5.3) to express $\underset{\sim}{\phi_k}$ and $\underset{\frown}{\phi_k}$ in term of $\phi_k$, gives a differential equation in terms of the spectral parameter $k$.

For the vector $\psi_k$ we obtain

$$k\frac{d}{dk}\psi_k = \begin{pmatrix} 0 & 0 \\ 0 & n+m \end{pmatrix}\psi_k + \begin{pmatrix} -(\mu+1) & 0 \\ 0 & \lambda(-1)^{n+m} \end{pmatrix}\psi_k$$
$$+\frac{2nv_0}{\underset{\sim}{v_0}+\widetilde{v}_0}\begin{pmatrix} 0 & \widetilde{v}_0 \\ 0 & -p \end{pmatrix}\underset{\sim}{\psi_k} + \frac{2mv_0}{\underset{\frown}{v_0}+\widehat{v}_0}\begin{pmatrix} 0 & \widehat{v}_0 \\ 0 & -q \end{pmatrix}\underset{\frown}{\psi_k}. \quad (2.5.15)$$

Again, the backward shifts can be eliminated using the inverse of the Lax pair (2.5.7), giving the purely differential equation.

For the dependence on the Bäcklund parameter we have the following linear system for the $u$ variable

$$\frac{\partial\phi}{\partial p} = A_1\cdot\phi + A_2\cdot\underset{\sim}{\phi}, \quad (2.5.16a)$$
$$\frac{\partial\phi}{\partial q} = B_1\cdot\phi + B_2\cdot\underset{\sim}{\phi}, \quad (2.5.16b)$$

where

$$A_1 = n\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad\text{and}\quad A_2 = \frac{2np}{2p+\underset{\sim}{u}-\widetilde{u}}\begin{pmatrix} 1 & 0 \\ -(p-\widetilde{u}) & 0 \end{pmatrix},$$

and $B_1$ and $B_2$ are given by replacing $p$ with $q$ and $n$ with $m$.

While, for the $v$ variable we have the system

$$\frac{\partial \psi}{\partial p} = C_1 \cdot \psi + C_2 \cdot \underset{\sim}{\psi}, \tag{2.5.17a}$$

$$\frac{\partial \psi}{\partial q} = D_1 \cdot \psi + D_2 \cdot \underset{\sim}{\psi}, \tag{2.5.17b}$$

where

$$C_1 = \frac{n}{p}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad C_2 = \frac{2nv_0}{\widetilde{v}_0 + \underset{\sim}{v}_0}\begin{pmatrix} 0 & -\frac{1}{p}\widetilde{v}_0 \\ 0 & 1 \end{pmatrix}, \tag{2.5.17c}$$

with $D_1$ and $D_2$ given by the usual replacements.

Finally, by applying the gauge transformation, (2.5.11), we obtain the the following linear system for the $z$ variable

$$\frac{\partial \chi}{\partial p} = \frac{n}{p}\begin{pmatrix} 1 & 0 \\ 0 & \frac{2z-\widetilde{z}-z}{\underset{\sim}{z}-\widetilde{z}} \end{pmatrix}\chi + \frac{2n}{p(k^2-p^2)(\underset{\sim}{z}-\widetilde{z})}\begin{pmatrix} 2k^2(\widetilde{z}-z) & (\widetilde{z}-z)(z-\underset{\sim}{z}) \\ k^2 & -p^2(\underset{\sim}{z}-z) \end{pmatrix}\chi, \tag{2.5.18}$$

with a similar expression for the dependence on $q$ with the usual replacements.

### 2.5.1 Alternative Lax Pairs for the lattice equations

Lax pairs for the discrete equations can also be obtained directly from the lattice equations. This is a new approach which we shall illustrate using the SKdV lattice equation (2.4.6).

If we introduce an auxiliary shift of the same type as (2.4.1), namely

$$\rho_k \mapsto T_3\rho_k = \overline{\rho}_k,$$

with an associated lattice parameter $r$. If we substitute this shift for the $\widehat{\cdot}$, equation (2.4.6) can be re-expressed in the following form

$$\widetilde{w} = \frac{(z-\widetilde{z})w - \frac{r^2}{p^2}(z-w)\widetilde{z}}{(z-\widetilde{z}) + \frac{r^2}{p^2}(z-w)}, \tag{2.5.19}$$

where we have used an auxiliary variable $w = \overline{z}$. This equation is a Riccati equation in terms of $w$ and can be linearized using the transformation $w = \frac{f}{g}$, giving us the following set of equations

$$\begin{aligned}\widetilde{f} &= \Gamma\big[(z-\widetilde{z}+\frac{r^2}{p^2}\widetilde{z})f - \frac{r^2}{p^2}z\widetilde{z}g\big],\\ \widetilde{g} &= \Gamma\big[\frac{r^2}{p^2}f + (z-\widetilde{z}-\frac{r^2}{p^2}\widetilde{z})g\big],\end{aligned}$$

with $\Gamma$ being an arbitrary factor. A similar set of equations can also be obtained if we make the substitutions $\widetilde{\cdot} \mapsto \widehat{\cdot}$ and $p \mapsto q$, giving us the following linear system

$$\widetilde{\phi} = L_2^{(SKdV)} \cdot \phi, \tag{2.5.20a}$$

$$\widehat{\phi} = M_2^{(SKdV)} \cdot \phi. \tag{2.5.20b}$$

The factor $\Gamma$ must be chosen so that the condition

$$\det(\widehat{L}_2^{(SKdV)} \cdot M_2^{(SKdV)}) = \det(\widetilde{M}_2^{(SKdV)} \cdot L_2^{(SKdV)}),$$

is satisfied, which in this case gives us

$$\Gamma = \frac{1}{z-\widetilde{z}}.$$

The Lax matrix $L_2^{(SKdV)}$ is then

$$L_2^{(SKdV)} = \begin{pmatrix} 1+\frac{r^2}{p^2}\frac{\widetilde{z}}{z-\widetilde{z}} & -\frac{r^2}{p^2}\frac{z\widetilde{z}}{z-\widetilde{z}} \\ \frac{r^2}{p^2}\frac{1}{z-\widetilde{z}} & 1-\frac{r^2}{p^2}\frac{z}{z-\widetilde{z}} \end{pmatrix},$$

and $M_2^{(SKdV)}$ given by the substitutions $\widetilde{\cdot} \mapsto \widehat{\cdot}$ and $p \mapsto q$. The compatibility of the Lax pair (2.5.20) again gives the SKdV equation (2.4.6).

Using this method, alternative Lax Pairs can also be derived for the KdV and modified KdV lattice equation. For the KdV lattice equation the Lax pair is given by the matrices

$$L_2^{(KdV)} = \begin{pmatrix} p+r+u & (p+r+u)(p-r-\widetilde{u}) \\ 1 & p-r-\widetilde{u} \end{pmatrix},$$

and $M_2^{(KdV)}$ given by the substitutions $\widetilde{\cdot} \mapsto \widehat{\cdot}$ and $p \mapsto q$. While for the modified KdV lattice equation the Lax pair is given by

$$L_2^{(mKdV)} = \begin{pmatrix} p & r\widetilde{v}_0 \\ \frac{r}{v_0} & p\frac{\widetilde{v}_0}{v_0} \end{pmatrix},$$

and $M_2^{(mKdV)}$ given by the substitutions $\widetilde{\cdot} \mapsto \widehat{\cdot}$ and $p \mapsto q$. In all these alternative Lax pairs $r$ has assumed the role of spectral parameter and the compatibility condition gives the relevant lattice equation.

## 2.6 Schwarzian PDE

The linear problem for the dependence of $z$ on the lattice parameters (2.5.18) can be rewritten in the following form

$$2t\frac{\partial \chi}{\partial t} = \frac{n}{p}\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\chi + \frac{1}{k^2-p^2}\begin{pmatrix} -kn(1+a) & 2t^2 z_t \\ kn^2(1-a^2)/(2tz_t) & -nt(1-a) \end{pmatrix}\chi, \quad (2.6.1a)$$

$$2t\frac{\partial \chi}{\partial t} = \frac{n}{p}\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\chi + \frac{1}{k^2-p^2}\begin{pmatrix} -kn(1+a) & 2t^2 z_t \\ kn^2(1-a^2)/(2tz_t) & -nt(1-a) \end{pmatrix}\chi, \quad (2.6.1b)$$

where $a$ and $b$ are auxiliary variables. If we cross differentiate (2.6.1a) and (2.6.1b) we obtain the following compatibility conditions

$$nsa_s = mtb_t = \frac{1}{2(s-t)}\left[n^2\frac{s^2}{t}\frac{z_s}{z_t}(1-a^2) - m^2\frac{t^2}{s}\frac{z_t}{z_s}(1-b^2)\right], \quad (2.6.2a)$$

$$stz_{st} = \frac{mt^2 z_t b - ns^2 z_s a}{s-t}. \quad (2.6.2b)$$

By differentiating (2.6.2b) with respect to $s$ and $t$ we obtain expressions for $a_t$ and $b_s$, then, by cross differentiating either the $a$ or the $b$, for example $\partial_s a_t = \partial_t a_s$, we are able to eliminate the variable $a$ and, surprisingly, also the variable $b$ to obtain the following PDE for the variable $z$

$$\begin{aligned} z_{sstt} &= z_{sst}\left(\frac{z_{st}}{z_s}+\frac{z_{tt}}{z_t}\right)+z_{stt}\left(\frac{z_{st}}{z_t}+\frac{z_{ss}}{z_s}\right)-z_{st}\left(\frac{z_{st}z_{ss}}{z_s^2}+\frac{z_{st}z_{tt}}{z_t^2}+\frac{z_{ss}z_{tt}}{z_s z_t}\right)\\ &\quad+\frac{1}{s-t}\left[\frac{s}{t}\left(z_{sst}-\frac{z_{st}z_{ss}}{z_s}-\frac{1}{2}\frac{z_{st}^2}{z_t}\right)-\frac{t}{s}\left(z_{stt}-\frac{z_{st}z_{tt}}{z_t}-\frac{1}{2}\frac{z_{st}^2}{z_s}\right)\right]\\ &\quad-\frac{1}{(s-t)^2}\left[n^2\frac{s^2}{t^2}\frac{z_s}{z_t}\left(z_{st}-\frac{z_s z_{tt}}{z_t}\right)+m^2\frac{t^2}{s^2}\frac{z_t}{z_s}\left(z_{st}-\frac{z_t z_{ss}}{z_s}\right)\right]\\ &\quad-\frac{1}{2}\frac{1}{(s-t)^3}\left[n^2\frac{s}{t}z_s\left(1+\frac{(4t-3s)s}{t^2}\frac{z_s}{z_t}\right)-m^2\frac{t}{s}z_t\left(1+\frac{(4s-3t)t}{s^2}\frac{z_t}{z_s}\right)\right] . \end{aligned} \tag{2.6.3}$$

This equation was the subject of a paper by Nijhoff, Hone and Joshi [85]. It was shown to be integrable in the sense that it has a Lax pair and an infinite number of conservation laws as well as forming a compatible parameter-family of equations. It was also shown that by an expansion of the independent variables one could obtain the Lagrangians for the whole of the Schwarzian KdV hierarchy and that under a scaling symmetry it reduces to the full $\mathrm{P_{VI}}$ equation. In chapter 5 we shall attempt to obtain a similar PDE for the Boussinesq system.

In this chapter we have given a relatively brief review of the direct linearization scheme, from it we have derived closed form lattice equations for the discrete KdV system, the modified KdV system and the Schwarzian KdV system. For each of these systems we also have closed form similarity constraints and differential-difference relations. The integrability of these equations is suggested by the existence of associated linear problems. In the next section the integrability of these systems will be investigated further as we shall show how the lattice equations can be embedded in a multi-dimensional lattice and how the constraints can be used to reduce the system to Painlevé type equations.

# Chapter 3

# Higher Order PVI Equations

## 3.1 Introduction

In the previous chapter we derived two-dimensional nonlinear partial difference equations (P$\Delta$Es) for the KdV system along with the corresponding two-dimensional similarity constraints and differential-difference relations. In this chapter we shall first show that the lattice equations can be consistently embedded in a multi-dimensional lattice and how both the similarity constraints and the differential-difference equations can also be extended to the multi-dimensional case. Having done that, we then show, in general, how to reduce the modified P$\Delta$E to either a coupled system of ordinary difference equations (O$\Delta$Es) using the constraint or, if we also employ the differential-difference relations, a coupled system of ordinary differential equations (ODEs). We shall then go on to give explicit examples of these reductions in the special cases $N = 2$ and $N = 3$. As both these systems will have been derived by a similarity reduction of an integrable equation, they should both be of Painlevé type. Indeed in the case $N = 2$ we do in fact obtain the full $\mathrm{P_{VI}}$ equation as well as a discrete analogue of the $\mathrm{P_{VI}}$ equation.

## 3.2 Lattice Equations in Multi-Dimensions

In section (2.4) we derived closed form lattice equations for the KdV family, namely the lattice KdV equation (2.4.4), the lattice modified KdV equation (2.4.5) and the lattice Schwarzian KdV equation (2.4.6). In each of these we considered the dependent variable to be a function of two discrete independent variables $n$ and $m$ each with an associated lattice parameter, $p$ and $q$ respectfully. In fact, each of these lattice equations actually represents a *compatible parameter-family of partial difference equations*, cf.[81].

If we take for example the lattice mKdV equation

$$p v_{n,m} v_{n,m+1} + q v_{n,m+1} v_{n+1,m+1} = q v_{n,m} v_{n+1,m} + p v_{n+1,m} v_{n+1,m+1}, \tag{3.2.1}$$

this means that we can embed the equation (3.2.1) into a multidimensional lattice by imposing a copy of (3.2.1) with different parameters on any two-dimensional sublattice, identifying each lattice direction with a corresponding lattice parameter $p_i \in \mathbb{C}$ in which direction the sites are labelled by discrete variables $n_i$ (noting that these are not necessarily integer-valued, but they *shift* by units, i.e. $n_i \in \theta_i + \mathbb{Z}$, $\theta_i \in \mathbb{C}$). Thus, combining two different lattice directions, labelled by $(i,j)$, we can write the lattice equation (3.2.1) on the corresponding sublattice as

$$p_i v v^j + p_j v^j v^{ij} = p_j v v^i + p_i v^i v^{ij}, \tag{3.2.2}$$

in which we use the right superscripts $i, j$ to denote the shifts in the corresponding directions, whereas we will use left subscripts $i, j$ denote shifts in the reverse direction, i.e.

$$v = v(\boldsymbol{n};\boldsymbol{p}) \ , \ \ v^j = T_j v(\boldsymbol{n};\boldsymbol{p}) = v(\boldsymbol{n}+\boldsymbol{e}_j;\boldsymbol{p}) \ , \ \ {}_j v = T_j^{-1} v(\boldsymbol{n};\boldsymbol{p}) = v(\boldsymbol{n}-\boldsymbol{e}_j;\boldsymbol{p}) \ ,$$

where $\boldsymbol{n}$ denotes the vector of the discrete variables $n_i$ for all lattice directions labelled by $i$, each corresponding to the component $p_i$ of the vector $\boldsymbol{p}$ of lattice parameters. We use the vector $\boldsymbol{e}_j$ to denote the vector with single nonzero entry equal to unity in its $j^{\text{th}}$ component.

The consistency of the lattice equation (3.2.2) along the multi-dimensional lattice is illustrated by figure 3.1. Considering the three-dimensional sublattice with elementary directions $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, then on each elementary cube in this lattice the iteration of initial data proceeds along the six faces of this cube, on each of which we have an equation of the form (3.2.2). Thus, starting from initial data $v$, $v^1$, $v^2$, $v^3$ we can then uniquely calculate the values of $v^{12}$, $v^{13}$ and $v^{23}$ by using the equation. However, proceeding further there are in principle three different ways to calculate the value of $v^{123}$, unless the equation satisfies (as is the case for the equation (3.2.2)) the special property that these three different ways of calculating this point actually lead to one and the same value. It is indeed at this point that the consistency of the embedding of the lattice mKdV into the multidimensional lattice is tested. In fact, by using equation (3.2.2) to eliminate all terms shifted in two directions in favour of terms shifted in only one direction we find that this value is given by

$$v^{ijk} = \frac{(p_i-p_k)(p_i+p_k)p_j v^i v^k + (p_j-p_i)(p_j+p_i)p_k v^j v^i + (p_k-p_j)(p_k+p_j)p_i v^j v^k}{(p_i-p_k)(p_i+p_k)p_j v^j + (p_j-p_i)(p_j+p_i)p_k v^k + (p_k-p_j)(p_k+p_j)p_i v^i} .$$

$$i,j,k = 1,2,3$$

This is clearly invariant for any permutation of the labels $ijk$ and therefore the value of $v^{123}$ is independent of the way in which we calculate it! Thus, the equation (3.2.2) can be simultaneously imposed on functions $v(n_1, n_2, n_3, \ldots)$ of the lattice sites. This is precisely the discrete analogue of the hierarchy of commuting higher-order flows of the (modified) KdV equation!

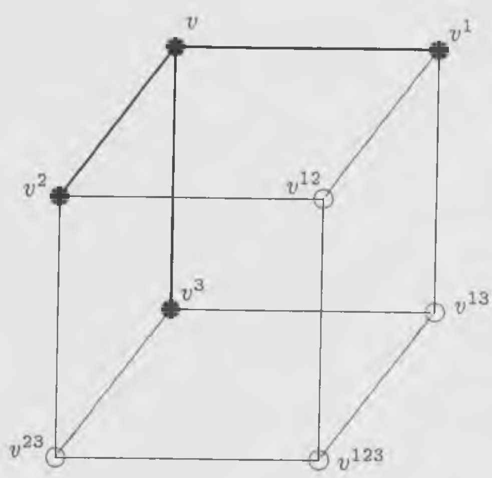



Figure 3.1: *Consistency of the lattice equation.*

As a consequence of this compatibility we will call the system (3.2.2) a *holonomic system of partial difference equations*, [86].

For the lattice KdV equation (2.4.4), the lattice equation on a two-dimensional sublattices is given by

$$(p_i - p_j + u^j - u^i)(p_i + p_j - u^{ij} + u) \quad = \quad p^2 - q^2. \tag{3.2.3}$$

The consistency of this equation in the multi-dimensional lattice follows an analogous route to that of the above modified equation. In this case we find that the value for $u^{123}$

is given by

$$
\begin{aligned}
u^{ijk} = & \Big[(p_j^2 - p_i^2)u^i u^j + (p_k^2 - p_j^2)u^j u^k + (p_i^2 - p_k^2)u^k u^i + (p_i + p_j)(p_j - p_k)(p_k + p_i)u^i \\
& + (p_j + p_k)(p_k - p_i)(p_i + p_j)u^j + (p_k + p_i)(p_i - p_k)(p_k + p_j)u^k\Big] \quad / \\
& \Big[u^i(p_j^2 - p_k^2) + u^j(p_k^2 - p_i^2) + u^k(p_i^2 - p_j^2) + (p_i - p_j)(p_j - p_k)(p_k - p_i)\Big], \quad (3.2.4)
\end{aligned}
$$

which is again clearly invariant for any permutation of the labels $i, j, k$ and hence equation (3.2.3) is also a holonomic system of partial difference equations.

Finally, for the lattice Schwarzian KdV equation (2.4.6), the lattice equation on a two-dimensional sublattices is given by

$$\frac{(z - z^i)(z^i - z^{ij})}{(z - z^j)(z^j - z^{ij})} = \frac{p_j^2}{p_i^2}. \tag{3.2.5}$$

Again, the consistency of this equation in the multi-dimensional lattice follows a similar route to that above, and for this equation the value for $z^{123}$ is given by

$$z^{ijk} = \frac{p_i^2 z^i(z^k - z^j) + p_j^2 z^j(z^i - z^k) + p_k^2 z^i(z^j - z^i)}{p_i^2(z^k - z^j) + p_j^2(z^i - z^k) + p_k^2(z^j - z^i)},$$

which, once again, is clearly invariant for any permutation of the labels $i, j, k$ and hence equation (3.2.5) is also a holonomic system of partial difference equations.

The continuous equation for the PVI hierarchy follows from the differential equations with respect to the lattice parameters $p_i$ given in (2.4.8), which for the variable $v$ now reads

$$-p_i \frac{\partial}{\partial p_i} \log v = n_i a_i \ , \tag{3.2.6}$$

in which the variable $a_i$ is given by

$$a_i \equiv \frac{v^i - {}_i v}{v^i + {}_i v} \ . \tag{3.2.7}$$

It can be shown that the differential relations (3.2.6) are actually compatible not only amongst themselves, but also with the discrete equations on the lattice (3.2.2), i.e. the discrete and continuous flows are commuting

$$\frac{\partial}{\partial p_i}\left(\frac{\partial v}{\partial p_j}\right) = \frac{\partial}{\partial p_j}\left(\frac{\partial v}{\partial p_i}\right) \quad , \quad \frac{\partial v^i}{\partial p_j} = T_i\left(\frac{\partial v}{\partial p_j}\right) \ ,$$

This can actually be demonstrated by explicit calculation exploiting the discrete relations (3.3.6) below, but we will not give the details here (which follow closely the pattern of calculations of [83]). Thus, we have here a large multidimensional system of equations with discrete (in terms of the variables $n_i$) as well as continuous (in terms of the parameters $p_i$) commuting flows, in terms of which compatible equations of three different types (partial difference, differential-difference and partial differential) figure in one and the same framework: the partial difference equations are precisely the lattice equations (3.2.2), the differential-difference equations are the relations (3.2.6), whilst for the partial differential equations in the scheme we refer back to equation (2.6.3) given in section 2.6 and to the results given in [85].

## 3.3 The $\mathrm{P_{VI}}$ Hierarchy

Now, we turn to the issue of the symmetry reduction of the multidimensional lattice in the sense of [87]. It follows from the general framework of [83] that the similarity constraint for the multidimensional lattice mKdV system is as follows:

$$\sum_i n_i a_i = \mu - \nu \quad , \quad \nu = \lambda(-1)^{\sum_i n_i} \ , \tag{3.3.1}$$

$\mu$ and $\lambda$ being constants. The sum in (3.3.1) is over all the $i$ labelling the lattice directions, hence, for each dimension in the lattice we get a term of the form $n_i a_i$ on the

LHS of the constraint (3.3.1).

To analyse the reduction it is convenient to introduce the following variables:

$$x_{ij} \equiv \frac{v}{v^{ij}} \quad \Rightarrow \quad {}_i x_{ij} \equiv T_i^{-1} x_{ij} = \frac{{}_i v}{v^j} \ , \tag{3.3.2}$$

$$X_{ij} \equiv \frac{v^i}{v^j} \quad \Rightarrow \quad {}_i X_{ij} \equiv T_i^{-1} X_{ij} = \frac{v}{{}_i v^j} . \tag{3.3.3}$$

The variables $x_{ij} = x_{ji}$ and $X_{ij} = 1/X_{ji}$ are not independent, but related via

$$X_{ij} = \frac{p_i x_{ij} + p_j}{p_j x_{ij} + p_i} \quad \Leftrightarrow \quad x_{ij} = \frac{-p_i X_{ij} + p_j}{p_j X_{ij} - p_i} \ , \tag{3.3.4}$$

which follow directly from the definitions and (3.2.2). Furthermore, we have

$$\frac{T_i^{-1} x_{ij}}{X_{ij}} = \frac{{}_i v}{v^i} = \frac{1 - a_i}{1 + a_i} \ , \tag{3.3.5}$$

which follows from the definitions of $a_i$, (3.2.7) and those of $x_{ij}$ and $X_{ij}$, (3.3.2) and (3.3.3). In order to obtain explicit equations from the reduction given by the constraint (3.3.1) we need a number of relations for the objects $a_i$ which follow from (3.2.2), namely

$$1 + a_j^i \ = \ \frac{(p_i X_{ij} - p_j)(a_j + 1) + 2p_j}{p_i x_{ij} + p_j} \quad , \quad i \neq j \tag{3.3.6}$$

which expresses shifted $a_j$ in terms of unshifted objects, and, from the definition together with (3.3.5)

$$\begin{aligned} a_i \ &= \ \frac{p_j \, {}_i X_{ij} \, X_{ij} + p_i ( {}_i X_{ij} - X_{ij}) - p_j}{p_j \, {}_i X_{ij} \, X_{ij} - p_i ( {}_i X_{ij} + X_{ij}) + p_j} \\ &= \ \frac{-p_j \, {}_i x_{ij} \, x_{ij} - p_i ( {}_i x_{ij} - x_{ij}) + p_j}{p_j \, {}_i x_{ij} \, x_{ij} + p_i ( {}_i x_{ij} + x_{ij}) + p_j} \quad , \quad i \neq j \end{aligned} \tag{3.3.7}$$

which expresses $a_i$ in terms of shifted $x_{ij}$ or $X_{ij}$.

First we will focus now on the reductions under the symmetry constraint (3.3.1) to derive closed-from *ordinary* differential equations (ODEs) choosing one particular lattice parameter $p_i$ as our independent variable. To implement this reduction explicitly we first need to derive differential relations for the $a_j$. By using (3.2.6) in combination with (3.3.5) we easily obtain

$$-p_i \frac{\partial}{\partial p_i} \log \left( \frac{1-a_j}{1+a_j} \right) = n_i \left( {}_j a_i - a_i^j \right) \ , \tag{3.3.8}$$

which, on using (3.3.6), in combination with (3.3.4), to eliminate the forward and backward shifted $a_i$, yields after a lengthy calculation the following differential relation:

$$\frac{\partial a_j}{\partial p_i} = \frac{n_i p_j}{(p_j - p_i)(p_j + p_i)} \left[ (1+a_i)(1-a_j) X_{ji} - (1+a_j)(1-a_i) X_{ij} \right] \ . \ (i \neq j) \tag{3.3.9}$$

We also require a differential relation for the variables $X_{ij}$, so, using the definition (3.3.3) and equation (3.2.6) we get

$$-p_i \frac{\partial}{\partial p_i} \log X_{ij} \ = \ (n_i + 1) a_i^i - n_i a_i^j ,$$

using the constraint (3.3.1) to eliminate the $a_i^i$ gives

$$-p_i \frac{\partial}{\partial p_i} \log X_{ij} \ = \ \mu + \nu - \sum_{k \neq i} n_k a_k^i - n_i a_i^j ,$$

and eliminating the shifts in the $a$'s using (3.3.6), yields the following relations for the reduced variables $X_{ij}$

$$\begin{aligned} \mu + \nu + p_i \frac{\partial}{\partial p_i} \log X_{ij} \ &= \ n_i \mathcal{X}_{ji} a_i + \sum_{k \neq i} n_k \mathcal{X}_{ik} a_k + n_i \frac{p_i p_j}{(p_i - p_j)(p_i + p_j)} (X_{ji} - X_{ij}) \\ &\quad + \sum_{k \neq i} n_k \frac{p_k p_i}{(p_k - p_i)(p_k + p_i)} (X_{ik} - X_{ki}), \end{aligned} \tag{3.3.10}$$

in which we have abbreviated

$$\mathcal{X}_{ij} \equiv \frac{(p_i X_{ij} - p_j)(p_j - p_i X_{ji})}{(p_j - p_i)(p_j + p_i)} = -\mathcal{X}_{ji} \ . \tag{3.3.11}$$

Using (3.3.9) in conjunction with (3.3.10) and using the similarity constraint (3.3.1) to eliminate the $a_i$, we obtain a coupled first-order system of ODEs w.r.t. the independent variable $t_i = p_i^2$ in terms of the $2N-2$ variables $a_k$, $X_{ik}$ , $(k \neq i)$. Solving the variables

$a_k$ from the linear system given by the equations (3.3.10) and inserting them into (3.3.9) we obtain a coupled set of second-order nonlinear ODEs for the variables $X_{ik}$. It is this set of equations which makes up the continuous PVI hierarchy.

Second, turning our attention to the discrete case, we note that since the left-hand side of (3.3.7) depends only on the label $i$ but not on $j$, for fixed $i$ this represents a set of $N-2$ coupled first-order ordinary difference equations (O$\Delta$Es) with respect to the shift in the discrete variable $n_i$ between the $N-1$ variables $X_{ij}$, $j \neq i$. Furthermore, the relations (3.3.6), for the same fixed label $i$, provide us with a set of $N-1$ first-order relations between the variables $a_j$, $j \neq i$, and thus together with the similarity constraint (3.3.1) where $a_i$ is substituted by (3.3.7) we obtain a set of $2(N-1)$ first-order nonlinear O$\Delta$Es for the $2(N-1)$ variables $X_{ij}$, $a_j$, $j \neq i$, which together form our higher-order discrete system.

## 3.4 Special Cases: N=2, N=3

In the previous section we described how the symmetry constraint (3.3.1) could be used to reduce the partial difference equations (3.2.2) and the differential-difference equations (3.2.6) to coupled systems of ODEs (in the continuous case) and O$\Delta$Es (in the discrete case). We will now give explicit examples of these coupled systems in the particular cases $N=2$ and $N=3$.

### 3.4.1 N=2:

For $N = 2$ we are not, strictly speaking, dealing with a higher dimensional case, but the procedure developed in the previous section will work nonetheless. The results for this special case were first given in [83] but we reproduce them here to show that they are a special case of larger multi-dimensional system.

So, for this case we have a single lattice equation of the type (3.2.2) along with a similarity constraint (3.3.1) which are represented symbolically in figure 3.2.

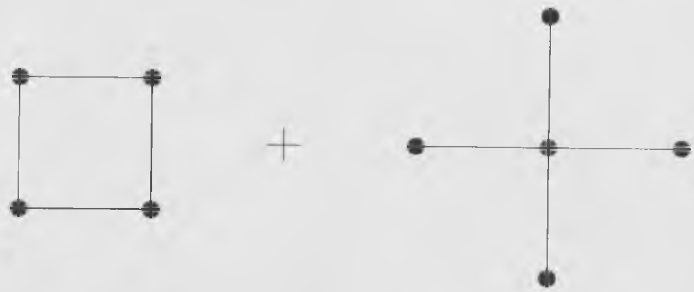


Figure 3.2: *symbolic representation of lattice equation and similarity constraint,* $N = 2$.

As already mentioned, although the lattice equation and the similarity constraint are, in general, compatible, we must show, in each specific case, how an initial value problem (IVP) can be formulated consistently, and to demonstrate that this IVP is well-posed, i.e. that given a suitable choice of initial data, we are able to iterate the solution through the whole lattice and that the corresponding solution is single-valued at each lattice site.

In figure 3.3 we have indicated how the iteration of the system proceeds. From a given configuration of initial data (located at the vertices indicated by $\bullet$) we move through the lattice by calculating each point by means of either the lattice equation (points indicated by $\circ$) or the similarity constraint (points indicated by $\times$). The first point where a possible conflict arises, due to the fact that the corresponding values of the dependent variable can be calculated in more than one way, is indicated by $\otimes$. It is at such points that the consistency of the similarity reduction needs to be verified by explicit computation. This was verified in [83] for this two-dimensional case using MAPLE. However, even for this case, the iteration involves too many steps and the expression are too large to reproduce here.

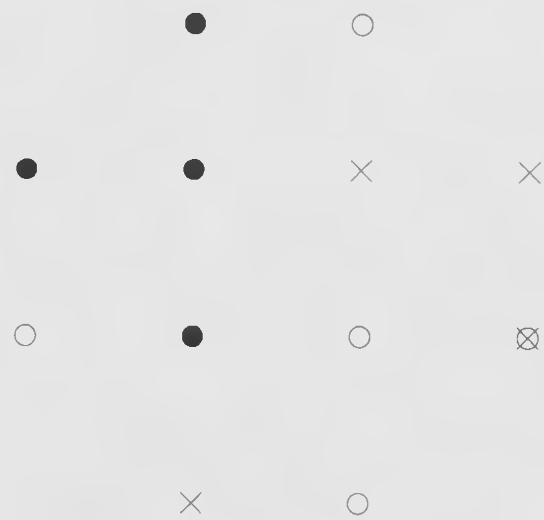


Figure 3.3: *consistency of the constrained lattice, $N = 2$.*

Having proved the compatibility of the lattice equation and the similarity constraint we can now proceed to reduce the system to either an ODE or an O$\Delta$E in one dependent variable. In order to simplify the equations we shall adopt a slightly more convenient notation, for the dependent variables we use,

$$a_1 = a \quad , \quad a_2 = b \quad , \quad x_{12} = x \quad , \quad X_{12} = X$$

while, for the discrete independent variables and the lattice parameters we use

$$n_1 = n \quad , \quad n_2 = m \quad , \quad p_1 = p \quad , \quad p_2 = q$$

Choosing first to reduce to an O$\Delta$E and taking $n$ as our independent discrete variable, equations (3.3.7) and (3.3.1) now give us

$$\frac{\mu-\nu-mb}{n} = \frac{q\underset{\sim}{X}X+p(\underset{\sim}{X}-X)-q}{q\underset{\sim}{X}X-p(\underset{\sim}{X}+X)-q}, \tag{3.4.1}$$

while equation (3.3.6) gives

$$1+\widetilde{b} = \frac{(pX-q)(b+1)+2q}{px+q}, \tag{3.4.2}$$

in which the variable $x$ can be eliminated using (3.3.4). Solving (3.4.1) for $b$ and substituting this into (3.4.2) gives a second order nonlinear non-autonomous difference equation for the variable $X$. In [83] this equation was given as:

$$\begin{aligned} &\frac{2(n+1)}{1-y_{n+1}y_n}+\frac{2n}{1-y_ny_{n-1}} = \mu+\lambda(-1)^n+2n+1+ \\ &+\frac{(\mu-\lambda(-1)^n)(r^2-1)y_n+r(1-y_n^2)\left[(n+\frac{1}{2})-(m+\frac{1}{2})(-1)^n\right]}{(r+y_n)(1+ry_n)}, \end{aligned} \tag{3.4.3}$$

(the notation deviates slightly from the one of [83]), where $r = p/q$ and where the variables $y_n$ are related to the $X$ by the prescription: $y_{2n} = X(2n)$ for the even sites, whilst $y_{2n+1} = -1/X(2n+1)$ for the odd lattice sites (the latter choice being mainly motivated by the wish to cast the equation into a convenient shape). It was pointed out in [83] that whilst a continuum limit of (3.4.3) yields the $\mathrm{P_V}$ equation, its general solution can be expressed in terms of $\mathrm{P_{VI}}$ transcendents (noting its dependence on four arbitrary parameters, $\mu$, $\lambda$, $r$ and $m$).

Turning now to the reduction to an ODE and choosing the lattice parameter $p$ as our independent variable, equation (3.3.9) gives us

$$\frac{\partial b}{\partial p} = \frac{nq}{q^2-p^2}\left[(1+a)(1-b)\frac{1}{X}-(1+b)(1-a)X\right], \tag{3.4.4}$$

while equation (3.3.10) gives

$$\begin{aligned}\mu+\nu+p\frac{\partial}{\partial p}\log X &= (na-mb)\frac{(pX-q)(q-p\frac{1}{X})}{p^2-q^2}\\ &+(n+m)\frac{pq}{p^2-q^2}(\frac{1}{X}-X).\end{aligned} \tag{3.4.5}$$

Using the similarity constraint (3.3.1) to eliminate all the $a$'s from these two equations we can solve (3.4.5) for $b$ and substitute this into (3.4.4). This gives us the following continuous equation for the variable $X$ in terms of $p$.

$$\begin{aligned}&p(p^2-q^2)^2X(qX-p)(pX-q)\frac{\partial^2 X}{\partial p^2}=\\ &\frac{1}{2}p(p^2-q^2)^2\left[pq(3X^2+1)-2(p^2+q^2)X\right]\left(\frac{\partial X}{\partial p}\right)^2+\\ &(q^2-p^2)\left[2p^2X(pX-q)(qX-p)+(q^2-p^2)^2X^2\right]\frac{\partial X}{\partial p}+\\ &\frac{1}{2}q\left[(\alpha X^2-\beta)(pX-q)^2(qX-p)^2+(p^2-q^2)X^2\left((\gamma-1)(qX-p)^2-(\delta-1)(pX-q)^2\right)\right],\end{aligned} \tag{3.4.6}$$

and it is not difficult to show that this is actually the $\mathrm{P_{VI}}$ equation through the identification $w(t)=pX(p)$, where $t=p^2$, and setting $q=1$, leading to

$$\begin{aligned}\frac{d^2w}{dt^2}= &\frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right)\left(\frac{dw}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{w-t}\right)\frac{dw}{dt}\\ &+\frac{w(w-1)(w-t)}{8t^2(t-1)^2}\left(\alpha-\beta\frac{t}{w^2}+\gamma\frac{t-1}{(w-1)^2}-(\delta-4)\frac{t(t-1)}{(w-t)^2}\right)\end{aligned} \tag{3.4.7a}$$

with the identification of the parameters $\alpha$,$\beta$,$\gamma$,$\delta$ as follows:

$$\alpha=(\mu-\nu+m-n)^2, \tag{3.4.7b}$$

$$\beta=(\mu-\nu-m+n)^2, \tag{3.4.7c}$$

$$\gamma=(\mu+\nu-m-n-1)^2, \tag{3.4.7d}$$

$$\delta=(\mu+\nu+m+n+1)^2. \tag{3.4.7e}$$

Equation (3.4.7) is interesting in its own right since it provides us with a covariant way of writing $P_{VI}$, noting its invariance under the transformations:

$$n \leftrightarrow m \ , \ p \leftrightarrow q \ , \ X \leftrightarrow 1/X \ .$$

### 3.4.2 N=3

This first higher-order case deals with the first genuinely multidimensional situation of three two-dimensional sublattices. This is an extension of the work done in [83] and the results were first given in our recent paper [86].

On each two-dimensional sublattice we impose a copy of the lattice mKdV equation (3.2.2). In addition there is also the similarity constraint (3.3.1) which couples the three lattice directions. Thus, for the three-dimensional case we have a coupled system of equations whose symbolic representation is shown in figure 3.4.

Figure 3.4: *symbolic representation of lattice equation and similarity constraint, N=3.*

In the previous section we have already demonstrated the consistency of the three copies of the lattice equation (3.2.2) amongst themselves. What remains, is to show that in this higher dimensional case we can still formulate a well-posed IVP. In figure 3.5 we show how the iteration scheme works in three dimensions. Again, the initial data points are indicated by a $\bullet$, points calculated using the lattice equation by a $\circ$, those calculated by the similarity constraint by a $\times$ and the first point of potential conflict (i.e. that can be calculated by either the lattice equation or the constraint) by a $\otimes$. The consistency of this IVP has been verified using MAPLE.

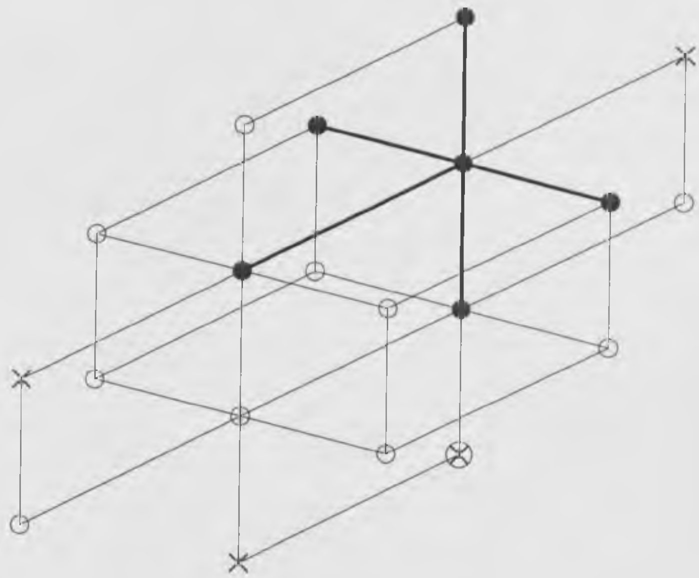


Figure 3.5: *Consistency of the constrained lattice system.*

With the consistency of the lattice equation and similarity constraint confirmed we can proceed to analyse the explicit reduction to coupled systems of either ODEs or O$\Delta$Es. we redefine the following objects

$$a_1 = a \ , \ a_2 = b \ , \ a_3 = c$$

$$X_{12} = X \ , \ X_{13} = Y \ , \ \mathcal{X}_{12} = \mathcal{X} \ , \ \mathcal{X}_{13} = \mathcal{Y}$$

using also $n_1 = n$, $n_2 = m$, $n_3 = h$, as well as $p_1 = p$, $p_2 = q$ and $p_3 = r$ to simplify the notation.

To start with the continuous equations, fixing the independent variable to be $p$ we obtain the following linear system for the quantities $b$ and $c$ from eq. (3.3.10)

$$\begin{bmatrix} 2\mathcal{X} & \mathcal{X}+\mathcal{Y} \\ \mathcal{X}+\mathcal{Y} & 2\mathcal{Y} \end{bmatrix} \begin{bmatrix} mb \\ hc \end{bmatrix} = \begin{bmatrix} \mu+\nu+p\frac{\partial}{\partial p}\log X \\ \mu+\nu+p\frac{\partial}{\partial p}\log Y \end{bmatrix} + \begin{bmatrix} (\mu-\nu)\mathcal{X}+(n+m)\frac{pq}{p^2-q^2}(\frac{1}{X}-X)+h\frac{pr}{p^2-r^2}(\frac{1}{Y}-Y) \\ (\mu-\nu)\mathcal{Y}+(n+h)\frac{pr}{p^2-r^2}(\frac{1}{Y}-Y)+m\frac{pq}{p^2-q^2}(\frac{1}{X}-X) \end{bmatrix}, \tag{3.4.8}$$

where we have used the similarity constraint to eliminate the quantity $a$. Furthermore, from (3.3.9) we obtain the differential relations

$$\begin{aligned} \frac{\partial(mb)}{\partial p} &= \frac{mq}{q^2-p^2}[(n+\mu-\nu-mb-hc)(1-b)\frac{1}{X} \\ &\quad -(1+b)(n-\mu+\nu+mb+hc)X], \end{aligned} \tag{3.4.9a}$$

$$\begin{aligned} \frac{\partial(hc)}{\partial p} &= \frac{hr}{r^2-p^2}[(n+\mu-\nu-mb-hc)(1-c)\frac{1}{Y} \\ &\quad -(1+c)(n-\mu+\nu+mb+hc)Y]\ . \end{aligned} \tag{3.4.9b}$$

Solving $b$ and $c$ from the linear system (3.4.8), and substituting the results in the differential relations (3.4.9a) and (3.4.9b), we obtain a coupled system of second-order nonlinear ODEs of the form:

$$\begin{aligned} \frac{\partial^2 X}{\partial p^2} &= A_1\left(\frac{\partial X}{\partial p}\right)^2 + A_2\left(\frac{\partial Y}{\partial p}\right)^2 + A_3\left(\frac{\partial X}{\partial p}\right)\left(\frac{\partial Y}{\partial p}\right) + A_4\left(\frac{\partial X}{\partial p}\right) + A_5\left(\frac{\partial Y}{\partial p}\right) + A_6, \\ \frac{\partial^2 Y}{\partial p^2} &= B_1\left(\frac{\partial X}{\partial p}\right)^2 + B_2\left(\frac{\partial Y}{\partial p}\right)^2 + B_3\left(\frac{\partial X}{\partial p}\right)\left(\frac{\partial Y}{\partial p}\right) + B_4\left(\frac{\partial X}{\partial p}\right) + B_5\left(\frac{\partial Y}{\partial p}\right) + B_6, \end{aligned} \tag{3.4.10}$$

where each of the coefficients $A_1..A_6$, $B_1..B_6$ are functions of $X, Y, p$ and six free parameters, namely $\mu$, $\nu$, $n$, $m$, $h$ and $q/r$, these functions have been calculated using Maple and although too large to give here are given explicitly in the Appendix.

Alternatively, we can derive a system of second-order ordinary difference equations by fixing one of the discrete variables, say $n = n_1$, and using the relations (3.3.6) to obtain the equations

$$(pX-q)b+pX+q = \frac{(q^2-p^2)X}{qX-p}(\widetilde{b}+1), \tag{3.4.11a}$$

$$(pY-r)c+pY+r = \frac{(r^2-p^2)Y}{rY-p}(\widetilde{c}+1)\ , \tag{3.4.11b}$$

where the tilde denotes the shift in the lattice direction associated with the variable $n$. Using the similarity constraint

$$na+mb+hc=\mu-\nu \quad , \quad \nu=\lambda(-1)^{n+m+h}, \tag{3.4.12}$$

to eliminate the variables $c$, we obtain the following linear system in terms of $\widetilde{b}$ and $b$

$$\begin{bmatrix} (q^2-p^2)X & -(pX-q)(qX-p) \\ -m(r^2-p^2)Y & m(pY-r)(rY-p) \end{bmatrix} \begin{bmatrix} \widetilde{b} \\ b \end{bmatrix} = \tag{3.4.13}$$

$$\begin{bmatrix} (pX+q)(qX-p)-(q^2-p^2)X \\ (rY-p)((pY-r)(\mu-\nu-na)+h(pY+r))-(r^2-p^2)Y(h+\mu+\nu-(n+1)\widetilde{a}) \end{bmatrix},$$

where the $a$ and $\widetilde{a}$ can be expressed in terms of $X$ and $Y$ by

$$a=\frac{qX\underset{\sim}{X}+p(\underset{\sim}{X}-X)-q}{qX\underset{\sim}{X}-p(\underset{\sim}{X}+X)+q}=\frac{rY\underset{\sim}{Y}+p(\underset{\sim}{Y}-Y)-r}{rY\underset{\sim}{Y}-p(\underset{\sim}{Y}+Y)+r}, \tag{3.4.14}$$

(where the undertilde denotes the backward shift with respect to the discrete variable). The system of equations (3.4.11), (3.4.12) and (3.4.14) – or, equivalently, (3.4.13) together with (3.4.14) leads effectively to a fourth order ordinary difference equation in one variable. In fact, solving $b$ and $\widetilde{b}$ from (3.4.13) and then eliminating $b$ altogether by a shift in the independent variable $n$ we get a coupled system containing one equation in terms of $X$, $\widetilde{X}$,$\widetilde{\widetilde{X}}$, $\underset{\sim}{X}$ and $Y$, and the equation (3.4.14) which is first order in the both $X$ and $Y$ with respect to the shift in the variable $n$. This system of equations depends on six free parameters, namely $\mu$, $\nu$, $m$, $h$, $q/p$ and $r/p$ and is quadratic in $Y$, hence we cannot explicitly reduce it to a single fourth order ordinary difference equation in terms of $X$ and $n$ without evoking algebraic expressions. However it can be shown that the IVP for the coupled system of equations (3.4.13) and (3.4.14) is well posed, i.e. given initial data points $\utilde{X}, X, \widetilde{X}$ and $Y$ at a fixed value of the discrete variable $n$, this system allows the subsequent iterates, $\widetilde{\widetilde{X}}$ and $\widetilde{Y}$, to be calculated uniquely by the following equations

$$\widetilde{Y} = -\frac{\widetilde{X}(qX-p)(pY-r)}{p\widetilde{X}(qX-p)+rY(pX-q)}, \tag{3.4.15}$$

and

$$\widetilde{\widetilde{X}} = \frac{[E_1 - E_2 + E_3 + E_4 E_5](p\widetilde{X} - q) + [E_4(n+2)(p\widetilde{X} - q)]}{[E_1 - E_2 + E_3 + E_4 E_5](q\widetilde{X} - p) - [E_4(n+2)(q\widetilde{X} - p)]}, \tag{3.4.16}$$

where

$$\begin{aligned}
E_1 &= (p\,X - q)\,(q\,X - p) \\
&\quad \left[(r\,Y - p)\left[p\,Y - r)\left[\mu - \nu - \frac{n\,(q\,\utilde{X}\,X + p\,(\utilde{X} - X) - q)}{q\,\utilde{X}\,X - p\,(\utilde{X} + X) + q}\right] + h\,(p\,Y + r)\right]\right. \\
&\quad \left.- (r^2 - p^2)\,Y\left[h + \mu + \nu - \frac{(n+1)\,(q\,X\,\widetilde{X} + p\,(X - \widetilde{X}) - q)}{q\,X\,\widetilde{X} - p\,(X + \widetilde{X}) + q}\right]\right], \\
E_2 &= (q^2 - p^2)\,\widetilde{X}(r\,\widetilde{Y} - p) \\
&\quad \left[(p\,\widetilde{Y} - r)\left[\mu - \nu - \frac{(n+1)\,(q\,X\,\widetilde{X} + p\,(X - \widetilde{X}) - q)}{q\,X\,\widetilde{X} - p\,(X + \widetilde{X}) + q}\right] + h\,(p\,\widetilde{Y} + r)\right], \\
E_3 &= mpq\left[(pY - r)(rY - p)(X^2 - 1) - (r^2 - p^2)(\widetilde{X}^2 - 1)\widetilde{Y}\right], \\
E_4 &= (q^2 - p^2)(r^2 - p^2)(r\widetilde{Y} - p)\widetilde{X}\widetilde{Y}, \\
E_5 &= h + \mu + \nu.
\end{aligned}$$

So although the system of equations (3.4.13) and (3.4.14) appears to be quadratic, its evolution is, surprisingly, linear.

## 3.5 Isomonodromic Deformation Problem

The isomonodromic deformation problem for the multidimensional lattice system is of Schlesinger type, [49]. In the two-dimensional case it was already presented in section 2.5. The extension from the two-dimensional to the multidimensional lattice is immediate: one only needs to introduce additional terms of similar form for each additional lattice direction. Thus, the Lax representation consists on the one hand of the linear shifts on the lattice of the form

$$\psi^{i}(k) = T_i\psi(k) = L_i(k)\cdot\psi(k) \ , \tag{3.5.1}$$

in which $k$ is a spectral parameter, and where the Lax matrices $L_i$ are given by

$$L_i(k) = \begin{pmatrix} p_i & v^i \\ \frac{k}{v} & p_i\frac{v^i}{v} \end{pmatrix} , \tag{3.5.2}$$

(cf. (2.5.7)). Leading to the Lax equations

$$L_i^j\cdot L_j = L_j^i\cdot L_i \ , \tag{3.5.3}$$

which lead to a copy of the lattice MKdV equation on each two-dimensional sublattice labelled by the indices $(i,j)$. On the other hand we have the linear differential equation for $\psi(k)$ with respect to its dependence on the spectral variable $k$, cf. (2.5.15)

$$\begin{aligned} k\frac{d}{dk}\psi(k) \ &= \ \frac{1}{2}\begin{pmatrix} -(1+\mu) & 0 \\ 0 & \lambda(-1)^{\sum_i n_i}+\sum_i n_i \end{pmatrix}\psi(k) \\ &+\sum_i \frac{n_i\ v}{v^i+{}_iv}\begin{pmatrix} 0 & v^i \\ 0 & -p_i \end{pmatrix} T_i^{-1}\psi(k) \ , \end{aligned} \tag{3.5.4}$$

the compatibility of which with (3.5.1) leads to the similarity constraint (3.3.1). In addition, we have differential equations for $\psi$ in terms of its dependence on the lattice parameters $p_i$ which are of the form

$$\frac{\partial \psi}{\partial p_i} = \frac{n_i}{p_i} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \psi + \frac{2n_i v}{v^i + {}_i v} \begin{pmatrix} 0 & -\frac{1}{p_i} v^i \\ 0 & 1 \end{pmatrix} T_i^{-1} \psi \,, \tag{3.5.5}$$

for each of the variable $p_i$, cf. (2.5.17a). It is the variables $t_i = p_i^2$ that play the role as independent variables in the continuous $\mathrm{P_{VI}}$ hierarchy.

The elimination of the back-shifted vectors $T_i^{-1}\psi$ by using the inverse of the Lax relations (3.5.1) lead to the following linear differential equation for $\psi$

$$\frac{\partial \psi}{\partial k} = \left( \frac{A_0}{k} + \sum_i \frac{A_i}{k - t_i} \right) \cdot \psi, \tag{3.5.6}$$

thus leading to the problem in the Schlesinger form, with regular singularities at $0, \infty, \{t_i\}$. The matrices $A_0$ and $A_i$ are given by

$$A_0 = \frac{1}{2} \begin{pmatrix} -(\mu + 1) & \sum_i \frac{n_i}{p_i}(1 - a_i) v^i \\ 0 & \lambda^{\sum_i n_i} + \sum_i n_i a_i \end{pmatrix},$$

$$A_i = n_i \begin{pmatrix} \frac{1}{2}(1 + a_i) & -\frac{1}{2p_i} v^i (1 - a_i) \\ -\frac{p_i}{2v^i}(1 + a_i) & \frac{1}{2}(1 - a_i) \end{pmatrix}.$$

The continuous isomonodromic deformation is provided by the linear differential equations in terms of the lattice parameters, namely

$$\frac{\partial \psi}{\partial t_i} = \left( P_i - \frac{A_i}{k - t_i} \right) \cdot \psi \,, \tag{3.5.7}$$

where

$$P_i = \frac{n_i}{2p_i} \begin{pmatrix} -\frac{1}{p_i} a_i & 0 \\ \frac{1}{v^i}(1 + a_i) & 0 \end{pmatrix}.$$

Eq. (3.5.7) is not quite in standard form, and we need to apply a gauge transformation of the form

$$\overline{\psi} \equiv V \cdot \psi \quad , \quad V = \begin{pmatrix} 1/v & 0 \\ U/v & 1 \end{pmatrix} , \tag{3.5.8}$$

to remove the term with $P_i$. With this gauge, the continuous isomonodromic deformation (3.5.7) becomes the standard form

$$\frac{\partial \overline{\psi}}{\partial t_i} = -\frac{\overline{A}_i}{\kappa - t_i} \cdot \overline{\psi} \quad , \quad \overline{A}_i = V \cdot A_i \cdot V^{-1} , \tag{3.5.9}$$

whilst the discrete isomonodromic condition is readily obtained from the Lax representation of (3.5.1).

## 3.6 Connection with Garnier Systems

M.R. Garnier in his seminal paper of 1912, [88], embarked on the question of finding higher-order analogues of the PVI equation, adopting the method that was proposed somewhat earlier by R. Fuchs, in [21], which can be identified with the isomonodromic deformation approach, cf. also [49]. Garnier gave a general construction of such higher-order equations constituting coupled systems of partial differential equations, which are the isomonodromic *Garnier systems.* As a particular example, he wrote down explicitly in [88] the first higher-order PVI equation in terms of the following coupled system, consisting of the second order ODE in terms of two dependent variables $w = w(t, s)$ and

$z = z(t,s)$

$$\begin{aligned}
\frac{\partial^2 w}{\partial t^2} &= \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}+\frac{1}{w-s}-\frac{1}{w-z}\right)\left(\frac{\partial w}{\partial t}\right)^2 \\
&-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{t-s}-\frac{1}{t-w}-\frac{1}{t-z}\right)\frac{\partial w}{\partial t} \\
&+\frac{1}{2}\frac{w(w-1)(w-s)(z-t)}{z(z-1)(z-s)(w-t)(z-w)}\left(\frac{\partial z}{\partial t}\right)^2-\frac{w-t}{(z-t)(z-w)}\left(\frac{\partial w}{\partial t}\right)\left(\frac{\partial z}{\partial t}\right) \\
&+\frac{2w(w-1)(w-t)(w-s)(z-t)^2}{t^2(t-1)^2(t-s)^2(w-z)}\times \\
&\qquad\times\left[\alpha+\beta+\gamma+\delta+k+\frac{7}{4}-\frac{ts}{z}\frac{\alpha+\frac{1}{4}}{w^2}+\frac{(t-1)(s-1)}{(z-1)}\frac{\beta+\frac{1}{4}}{(w-1)^2}\right. \\
&\qquad\left.+\frac{t(t-1)(t-s)}{(z-t)}\frac{\gamma}{(w-t)^2}+\frac{s(s-1)(s-t)}{(z-s)}\frac{\delta}{(w-s)^2}\right],
\end{aligned} \tag{3.6.1a}$$

together with coupled first order PDEs

$$\frac{t(t-1)}{t-z}\frac{\partial w}{\partial t}+\frac{s(s-1)}{s-z}\frac{\partial w}{\partial s}=\frac{w(w-1)}{w-z}, \tag{3.6.1b}$$

$$\frac{t(t-1)}{t-w}\frac{\partial z}{\partial t}+\frac{s(s-1)}{s-w}\frac{\partial z}{\partial s}=\frac{z(z-1)}{z-w}. \tag{3.6.1c}$$

It should be pointed out that the system consisting of (3.6.1a), (3.6.1b) and (3.6.1c) amounts actually to a fourth order ODE in terms of $w = w(t)$ only, and as such can be rightly considered to be the first higher-order member of the Painlevé VI hierarchy.

Subsequent work on the Garnier systems was done mostly by K. Okamoto and his school, cf. e.g. [89, 90]. However, it seems that in most of these works these systems were treated as an overdetermined system of PDEs rather than (as Garnier himself clearly had in mind) as a consistent system of ODEs. Although it is not easy to find the explicit transformation of the lattice system exposed in section 3.4.2 to the systems that Garnier wrote down, in particular to find the explicit relation between the above system (3.6.1) and the system consisting of (3.4.8) and (3.4.9), it is to be expected that such a mapping exists. The identification is probably most readily obtained via the transformation of the corresponding Schlesinger type of system as given in section 3.5 and the linear system that Garnier exploited in [88]. However, the search for such an identification will be left to a future study.

# Chapter 4

# The KP System

## 4.1 The KP Lattice Systems

In this chapter we shall show how the integral equation (2.1.1) can be extended to higher dimension and how this leads to either continuous PDEs or lattice equations of Kadomtsev-Petviashvili (KP) type. As this was the subject of a 1984 paper by Nijhoff, Capel, Quispel and Wiersma, [92] we shall keep the derivation of these equations brief. The main aim of this chapter is to show how the KP system is related to the Gel'fand-Dikii hierarchy by means of a dimensional reduction. Particular members of which are the KdV system and the Boussinesq system which is the subject of the next chapter.

The integral equation for the infinite component wave-vector $\boldsymbol{u}_k$ is, in this case

$$\mathbf{u}_k + \rho_k \int\int_D d\zeta(\ell, \ell') \frac{\sigma_{\ell'} \mathbf{u}_\ell}{k + \ell'} = \rho_k \, \mathbf{c}_k \, . \tag{4.1.1}$$

In equation (4.1.1), $\boldsymbol{c}_k$ is the same infinite component vector defined in (2.1.1). However, the integration is now performed over an arbitrary region in the complex hyper-plane

$\mathbb{C}^2$ of $\ell$ and $\ell'$, with integrations over suitable measure $d\zeta(\ell,\ell')$. Again we shall not be specific about the exact choice of the domain of integration or the measure as the integral equation is used merely as a formal tool in order to investigate the algebraic structure of the resulting system. To complete the system we must also consider an "adjoint" version of the integral equation (4.1.1), namely

$$ {}^t\mathbf{u}_{k'} + \sigma_{k'} \int\!\!\int_D d\zeta(\ell,\ell') \frac{{}^t\mathbf{u}_{\ell'}\rho_\ell}{k'+\ell} = \sigma_{k'}\,{}^t\mathbf{c}_k\,, \tag{4.1.2} $$

in terms of an "adjoint" wave-vector ${}^t\mathbf{u}_{k'}$. Now, equations (4.1.1) and (4.1.2) each contain a plane wave factor, $\rho_k$ and $\sigma_{k'}$ respectively. Again we are free to choose whether these depend on continuous or discrete variables.

As with the KdV system we shall develop an infinite matrix formalism in order to simplify the algebra and derive closed form equations.

Firstly we require the infinite $(\mathbb{Z}\times\mathbb{Z})$ matrix

$$ C = \int\!\!\int_D d\zeta(\ell,\ell')\rho_k\sigma_{k'}\mathbf{c}_\ell\,{}^t\mathbf{c}_{\ell'}, \tag{4.1.3} $$

which shall be used to derive the linear relationships of the system. While for the nonlinear relationships we now require the following two infinite $(\mathbb{Z}\times\mathbb{Z})$ matrices,

$$ U = \int\!\!\int_D d\zeta(\ell,\ell')\,\mathbf{u}_\ell\,{}^t\mathbf{c}_\ell\sigma_{\ell'}, \tag{4.1.4a} $$

$$ {}^tU = \int\!\!\int_D d\zeta(\ell,\ell')\,\mathbf{c}_\ell\,{}^t\mathbf{u}_{\ell'}\rho_\ell. \tag{4.1.4b} $$

This highlights the main difference between the KP system and the KdV system, in the KdV system we needed only to define the matrix $U$ as we had the added symmetry $U = {}^tU$.

To complete the infinite matrix formalisation we again require index raising matrices $\Lambda$

and ${}^t\Lambda$, index counting matrices $I$ and ${}^tI$, a projection matrix $O$ and a matrix $\Omega$, but these are the same as in the KdV system and are defined in (2.2.3).

This completes the infinite matrix formalisation and using the objects defined above we can write equation (4.1.1) in the following way

$$U = C \cdot (\mathbf{1} + \Omega \cdot C)^{-1}, \tag{4.1.5}$$

with a similar expression for equation (4.1.2).

We must now choose the form of the plane-wave factors $\rho_k$ and $\sigma_{k'}$. We shall first define them in terms of continuous time and space variables in order to demonstrate how this formalisation can be used to derive the continuous KP hierarchy. We shall then go on to define them in terms of discrete variables and derive the lattice equations and similarity constraints for the KP system.

## 4.2 Continuous Case

if we choose to work with continuous variables, then the plane wave factor $\rho_k$ takes the following form:

$$\rho_k = \exp\left(\sum_j k^j x_j\right) \quad , \quad \sigma_{k'} = \exp\left(-\sum_j (-k')^j x_j\right) \quad , \tag{4.2.1}$$

where $j = 1, 2, .., N$.

In this case we are now dealing with an N-dimensional system which depends on $N$ independent continuous variables $x_1, x_2, .., x_N$.

Differentiating these with respect to any any of the continuous variables we get

$$\frac{\partial \rho_k}{\partial x_j} = k^j \rho_k \quad , \quad \frac{\partial \sigma_{k'}}{\partial x_j} = -(-k')^j \sigma_{k'}$$

which leads us to the following linear relationship for $C$

$$\partial_j C = \Lambda^j \cdot C - C \cdot (-{}^t\Lambda)^j. \tag{4.2.2}$$

which gives us the nonlinear relationship for $U$

$$\partial_j U = \Lambda^j \cdot U - U \cdot {}^t\Lambda^j - U \cdot O_j \cdot U. \tag{4.2.3}$$

We now deduce recursive definitions of $O_{i+j}$ and $\partial_{x_{i+j}} U$ by substituting $i+j$ for $j$ in equations (2.2.3e) and (4.2.3) Thus, equation (2.2.3e) becomes

$$O_{i+j} = O_i \cdot \Lambda^j + (-{}^t\Lambda^i) \cdot O_j. \tag{4.2.4}$$

Similarly, equation (4.2.3) becomes

$$(\partial_{i+j} + \partial_i \partial_j) U = (\Lambda^i - U \cdot O_i) \cdot \partial_j U + (\Lambda^j - U \cdot O_j) \cdot \partial_i U. \tag{4.2.5}$$

We now consider

$$\begin{aligned}(\Lambda^k - U \cdot O_k) \cdot (\partial_{i+j} + \partial_i \partial_j) U &= (\Lambda^{k+i} - U \cdot O_{k+i} - (\partial_k U) \cdot O_i) \cdot \partial_j U \\ &\quad + (\Lambda^{k+j} - U \cdot O_{k+j} - (\partial_k U) \cdot O_j) \cdot \partial_i U.\end{aligned}$$

Adding cyclic permutations of this equation we obtain

$$\begin{aligned}(\Lambda^i - U \cdot O_i) \cdot \partial_{k+j} U + \text{cycl.} &= 2\partial_{i+j+k} U + \frac{1}{2}(\partial_i \partial_{k+j} + \partial_j \partial_{i+k} + \partial_k \partial_{i+j}) U \\ &- \frac{1}{2}\partial_i \partial_j \partial_k U - \frac{1}{2}\left[(\partial_j U) \cdot O_k \cdot \partial_i U + ((\partial_i U) \cdot O_k \cdot \partial_j U + \text{cycl.}\right].\end{aligned}$$

Differentiating this with respect to another independent variable $x_l$ and again adding cyclic permutations we eliminate all the $\Lambda$'s to give

$$\begin{aligned}&2(\partial_{i+j+k}\partial_l + \partial_{j+k+l}\partial_i + \partial_{k+l+i}\partial_j + \partial_{l+i+j}\partial_k) U \\ &-2\partial_i \partial_j \partial_k \partial_l U - 2(\partial_{i+j}\partial_{k+l} + \partial_{i+k}\partial_{j+l} + \partial_{i+l}\partial_{k+j}) U = \\ &\quad \frac{1}{2}\Big[\partial_l \left[(\partial_i U) \cdot O_k \cdot \partial_j U + (\partial_j U) \cdot O_k \cdot \partial_i U + \text{cycl(ijk)}\right] + \text{cycl(ijkl)}\Big] \\ &\quad + \frac{1}{2}\Big[\left[(\partial_{i+j} U) \cdot O_k \cdot \partial_l U - (\partial_l U) \cdot O_k \cdot \partial_{i+j} U + \text{cycl(ijk)}\right] + \text{cycl(ijkl)}\Big],\end{aligned} \tag{4.2.6}$$

which can be considered as a generating equation for the KP hierarchy. For example, if we limit ourselves to the simplest case, namely $i = j = k = l = 1$, equation (4.2.6) gives us

$$4\partial_{31}U - 6\partial_1^2 U \cdot O \cdot \partial_1 U - 6\partial_2 U \cdot O \cdot \partial_1 U + 6\partial_1 U \cdot O \cdot \partial_2 U - 6\partial_1 U \cdot O \cdot \partial_1^2 U - \partial_1^4 U = 3\partial_2^2 U.$$

If we now rename our independent variables as follows

$$x_1 = x\ ,\ x_2 = y\ ,\ x_3 = t\ ,\ U_{0,0} = u$$

and isolate the central element of equation (4.2.7) we obtain

$$(4u_t - 6u_x^2 - u_{xxx})_x = 3u_{yy}, \tag{4.2.7}$$

which is the first of the (potential) KP hierarchy.

In order to dimensionally reduce this system to the KdV system we must impose the symmetry

$$U \quad = \quad {}^tU. \tag{4.2.8}$$

Using this symmetry, equation (4.2.3) provides the further, purely algebraic relation

$$U \cdot (-{}^t\Lambda)^j \quad = \quad \Lambda^j \cdot U - U \cdot O_j \cdot U \quad , \quad j \text{ even}. \tag{4.2.9}$$

This gives, for $j$ even

$$\partial_{x_j} U = 0, \tag{4.2.10}$$

and equation (4.2.7) now gives the first of the KdV hierarchy

$$4u_t - 6u_x^2 - u_{xxx} \quad = \quad 0. \tag{4.2.11}$$

Thus, we have shown that the integral equation (4.1.1) along with a suitable choice of the plane wave factor $\rho_k$ encodes the whole of the KP hierarchy through equation (4.2.6) and by imposing the additional symmetry (4.2.8) this can be reduced to the KdV hierarchy.

## 4.3 Discrete Case

If we now choose to work with discrete variables, the plane-wave factors $\rho_k$ and $\sigma_{k'}$ take the form

$$\rho_k = \prod_j (p_j + k)^{n_j} \quad , \quad \sigma_{k'} = \prod_j (p_j - k')^{-n_j} \ , \tag{4.3.1}$$

where $j = 1, 2, .., N$.

The system now consists of an N-dimensional lattice, with the dependent variable $\boldsymbol{u}_k$, a function now of $N$ discrete variables $n_1, n_2, .., n_N$ each of which has an associated lattice parameter $p_j$.

The discrete evolution is now given by

$$T_j \rho_k = (p_j + k)\rho_k \quad , \quad T_j \sigma_{k'} = \frac{1}{p_j - k'}\sigma_{k'},$$

which leads us to the following linear relation for the $C$

$$T_j C \cdot (p_j - {}^t\Lambda) = (p_j + \Lambda) \cdot C. \tag{4.3.2}$$

This then gives us the nonlinear relation for the $U$

$$T_j U \cdot (p_j - {}^t\Lambda) = (p + \Lambda) \cdot U - (T_j U) \cdot O \cdot U. \tag{4.3.3}$$

For the discrete case, dimensional reduction to the Gel'fand-Dikii hierarchy is achieved by iterating equation (4.3.3), this gives us

$$\left(\prod_{j=1}^{N} T_j\right) U \cdot \left[\prod_{j=1}^{N} (p_j - {}^t\Lambda)\right] = \left[\prod_{j=1}^{N} (p_j + \Lambda)\right] \cdot U - \left[\left(\prod_{j=1}^{N} T_j\right) U\right] \cdot O_{p_1,\ldots,p_N} \cdot U, \tag{4.3.4}$$

where

$$O_{p_1,\ldots,p_N} = \sum_{j=1}^{N} \left[\prod_{i=1}^{j-1} (p_j - {}^t\Lambda)\right] \cdot O \cdot \left[\prod_{i=j+1}^{N} (p_j + \Lambda)\right]. \tag{4.3.5}$$

The dimensional reduction to the lattice Gel'fand-Dikii hierarchy is obtained by imposing a constraint of the form

$$\left(\prod_{j=1}^{N} T_j\right) U = U. \tag{4.3.6}$$

For example, for the lattice KdV system, where $N = 2$, we have the constraint

$$T_p T_{-p} U = U. \tag{4.3.7}$$

And for the lattice BSQ system, where $N = 3$, we have the constraint

$$T_p T_{\omega p} T_{\omega^2 p} U = U. \tag{4.3.8}$$

This demonstrates how the lattice KP system is connected to the lattice KdV and BSQ systems. With this in mind we go on to derive the closed form lattice equations.

In order to derive closed for scalar equations we again need to introduce several objects which pick out certain entries of the infinite matrix $U$, however, as we no longer have the symmetry $U = {}^tU$ we require more than we did in the KdV case. The objects we now require are

$$\begin{aligned} u &= U_{0,0}, & s_{\alpha,\beta} &= \left(\tfrac{1}{\alpha+\Lambda} \cdot U \cdot \tfrac{1}{\beta + {}^t\Lambda}\right)_{0,0}, \\ v_\alpha &= 1 - \left(\tfrac{1}{\alpha+\Lambda} \cdot U\right)_{0,0}, & w_\beta &= 1 + \left(U \cdot \tfrac{1}{\beta+{}^t\Lambda}\right)_{0,0}, \\ s_\alpha &= \alpha - \left(\tfrac{1}{\alpha+\Lambda} \cdot U \cdot {}^t\Lambda\right)_{0,0}, & t_\beta &= \beta - \left(\Lambda \cdot U \cdot \tfrac{1}{\beta+{}^t\Lambda}\right)_{0,0}, \end{aligned} \tag{4.3.9a}$$

with $\alpha$ and $\beta$ again being arbitrary parameters.

We must also restrict the $N$-dimensional lattice to a three-dimensional lattice and to simplify the equations we make the following identifications

$$n_1 = n \ , \ n_2 = m \ , \ n_3 = h$$

$$p_1 = p \ , \ p_2 = q \ , \ p_3 = r$$

while for the discrete evolution we introduce the following notation

$$T_1\rho_k = \widetilde{\rho}_k \ , \ T_2\rho_k = \widehat{\rho}_k \ , \ T_3\rho_k = \bar{\rho}_k$$

each with their respective lattice parameter $p, q$ and $r$. Transformations of the objects in (4.3.9) are now interpreted as transformations of functions on a three-dimensional lattice with grid points $(n, m, h)$.

Equation (4.3.3) now gives the following relationships

$$\widetilde{s}_\alpha = (p+u)\widetilde{v}_\alpha - (p-\alpha)v_\alpha, \tag{4.3.10a}$$

$$t_\beta = (p+\beta)\widetilde{w}_\beta - (p-\widetilde{u})w_\beta, \tag{4.3.10b}$$

$$\widetilde{v}_\alpha w_\beta = 1-(p+\beta)\widetilde{s}_{\alpha,\beta} + (p-\alpha)s_{\alpha,\beta}, \tag{4.3.10c}$$

along with similar equations for the other two lattice directions.

Using equations (4.3.10a) and (4.3.10b) we can eliminate $s_\alpha$ and $t_\beta$ to derive the following Miura type relations which relate the variable $u$ to either $v_\alpha$ or $w_\beta$

$$p-q+\widehat{u}-\widetilde{u} = \frac{(p-\alpha)\widehat{v}_\alpha - (q-\alpha)\widetilde{v}_\alpha}{\widehat{\widetilde{v}}_\alpha} = \frac{(p+\beta)\widetilde{w}_\beta - (q+\beta)\widehat{w}_\beta}{w_\beta}, \tag{4.3.11}$$

again we have two other copies of this equation for each combination of lattice directions.

This Miura relation immediately gives us an equation relating the two objects $v_0$ and $w_0$

$$\frac{p\widehat{v}_0 - q\widetilde{v}_0}{\widehat{\widetilde{v}}} = \frac{p\widetilde{w}_0 - q\widehat{w}_0}{w_0}, \tag{4.3.12}$$

and this equation shall appear again as part of the Boussinesq system, see later.

Eliminating either a $v_p$ or a $w_p$ from equation (4.3.11) we obtain

$$\frac{p-q+\widehat{\bar{u}}-\widetilde{\bar{u}}}{p-q+\widehat{u}-\widetilde{u}} = \frac{p-r+\widehat{\bar{u}}-\widehat{\widetilde{u}}}{p-r+\bar{u}-\widetilde{u}}, \tag{4.3.13}$$

which is the lattice version of the (potential) KP equation.

We can alternatively eliminate the $u$ from equation (4.3.11) to give a closed form equation for the $v_\alpha$

$$\frac{(p-\alpha)\overline{v}_\alpha-(r-\alpha)\widetilde{v}_\alpha}{\overline{\widetilde{v}}_\alpha} = \frac{(p-\alpha)\widehat{v}_\alpha-(q-\alpha)\widetilde{v}_\alpha}{\widehat{\widetilde{v}}_\alpha}+\frac{(q-\alpha)\overline{v}_\alpha-(r-\alpha)\widehat{v}_\alpha}{\widehat{\overline{v}}_\alpha}, \tag{4.3.14}$$

Setting $\alpha = 0$ we get the following closed form lattice equation for $v_0$

$$p\left(\frac{\overline{v}_0}{\overline{\widetilde{v}}_0}-\frac{\widehat{v}_0}{\widehat{\widetilde{v}}_0}\right)+q\left(\frac{\widetilde{v}_0}{\widehat{\widetilde{v}}_0}-\frac{\overline{v}_0}{\widehat{\overline{v}}_0}\right)+r\left(\frac{\overline{v}_0}{\widehat{\overline{v}}_0}-\frac{\widetilde{v}_0}{\overline{\widetilde{v}}_0}\right) = 0, \tag{4.3.15}$$

which is the lattice version of the (potential) modified KP equation. It is also possible to derive similar equations for $w_\beta$ and $w_0$.

Turning now to equation (4.3.10c) and eliminating the $v_\alpha$ and $w_\beta$ we get the equation

$$\begin{aligned}&\frac{(1-(p+\beta)\widehat{\widetilde{s}}_{\alpha,\beta}+(p-\alpha)\widehat{s}_{\alpha,\beta})(1-(r+\beta)\widetilde{\overline{s}}_{\alpha,\beta}+(r-\alpha)\widetilde{s}_{\alpha,\beta})}{(1-(p+\beta)\widetilde{\overline{s}}_{\alpha,\beta}+(p-\alpha)\overline{s}_{\alpha,\beta})(1-(r+\beta)\widehat{\overline{s}}_{\alpha,\beta}+(r-\alpha)\widehat{s}_{\alpha,\beta})}\\ &\qquad\qquad = \frac{(1-(q+\beta)\widehat{\widetilde{s}}_{\alpha,\beta}+(q-\alpha)\widetilde{s}_{\alpha,\beta})}{(1-(q+\beta)\widehat{\overline{s}}_{\alpha,\beta}+(q-\alpha)\overline{s}_{\alpha,\beta})},\end{aligned} \tag{4.3.16}$$

which, if we define $z \equiv s_{0,0}-\frac{n}{p}-\frac{m}{q}-\frac{h}{r}$ gives us

$$\frac{(\widehat{\widetilde{z}}-\widehat{z})(\widetilde{\overline{z}}-\widetilde{z})(\widehat{\overline{z}}-\overline{z})}{(\widehat{\widetilde{z}}-\widetilde{z})(\widehat{\overline{z}}-\widehat{z})(\widetilde{\overline{z}}-\overline{z})} = 1, \tag{4.3.17}$$

which is the lattice version of the (potential) Schwarzian KP equation.

## 4.4 Similarity Reduction

For the similarity constraints we must again impose a scaling invariance on the integral equation. In the KP case such a constraint is given by

$$\int\int_D d\zeta(\ell,\ell')\left(\ell\frac{d}{d\ell}+\frac{d}{d\ell'}\ell'\right)f(\ell,\ell') = 0\,, \tag{4.4.1}$$

for functions $f(\ell,\ell')$ that are solutions of the integral equation (4.1.1), (or some objects constructed out of them). Again, we will not discuss at this point particular contours and measures that obey the constraint (4.4.1), but assume that a well-defined class of such integration measures and domains can be specified in connection with the solutions that we are interested in. Our purpose here is to investigate the algebraic structure underlying the similarity reductions on the lattice. We note that in equation (4.1.1) we have ignored the boundary conditions, leaving this aspect of the similarity constraint to future work. Thus, imposing (4.4.1), one aims to obtain a similarity reduction of the KP lattices along the same line as in [72].

Imposing (4.4.1) on the level of the infinite matrix structure, we obtain the following nonlinear relationship for the matrix $U$

$$\begin{aligned} U + I \cdot U + U \cdot {}^tI &= \sum_j p_j n_j \left( U \cdot \frac{1}{p_j + \Lambda} \right. \\ &\quad \left. - \frac{1}{p_j - {}^t\Lambda} \cdot U + U \cdot \frac{1}{p_j - {}^t\Lambda} \cdot O \cdot \frac{1}{p_j + \Lambda} \cdot U \right) . \end{aligned} \tag{4.4.2}$$

Using the relation

$$\frac{1}{p+\Lambda} \cdot \mathbf{I} + \frac{d}{dp} \frac{p}{p+\Lambda} = \mathbf{I} \cdot \frac{1}{p+\Lambda} ,$$

From the Miura transformation (1.1.2), one can now derive the following expression

$$\frac{1 - v_p}{1 - \tilde{v}_p} = 1 + \left( T_p^{-1} \frac{d}{dp} T_p \right) u. \tag{4.4.3}$$

Then the similarity constraint for the lattice KP equation can now be written in the following suggestive form

$$0 = u + \sum_p n \left( T_p^{-1} \frac{d}{dp} T_p \right) u \,. \tag{4.4.4}$$

The problem with this expression is that the discrete operators $\left( T_p^{-1} \frac{d}{dp} T_p \right) u$ in (4.4.4) are no longer the elementary translation operators that figure in the lattice KP itself. In order to use equation (4.4.4) we must first impose a dimensional reduction to the lattice

Gel'fand-Dikii hierarchy before imposing (4.4.4) . The full reduction of the KP lattice is therefore a two-fold approach.

We also note that recently there has been much work done on the subject of KP hierarchies. See, for example, Adler, Shiota and Van Moerbeke's work on vertex operators [93, 94] and Bogdanov and Konopelchenko's work on Calogero-Moser systems [95, 96].

# Chapter 5

# The Boussinesq System

## 5.1 Linearization of the Boussinesq System

The direct linearization of the Boussinesq (BSQ) system was first introduced in 1982 by Quispel, Nijhoff and Capel [41] where it was shown that, with the addition of an extra parameter in the kernel of the integral equation (2.1.1), one could obtain both the BSQ equation [97]

$$u_{tt} - u_{xx} - (u^2)_{xx} + u_{xxxx} = 0, \tag{5.1.1}$$

and the modified BSQ equation [41], (also given as a system earlier in[98])

$$3v_{tt} - 6v_t v_{xx} - 6v_x^2 v_{xx} + v_{xxxx} = 0, \tag{5.1.2}$$

along with Bäcklund transformation and reductions to the second and fourth Painlevé equations. In [41] they limited themselves to choosing the extra parameter to the cube root of unity, however, in 1991, along with Papageorgiou [99], they showed that in the more general case where the extra parameter is the $N^{\text{th}}$ root of unity, the integral equation then linearized the entire Gel'fand-Dikii Hierarchy. The integral equation in this case then reads,

$$u_k + \int_\Gamma d\lambda(l) \frac{\rho_k u_l}{k - \omega l} = \rho_k c_k, \tag{5.1.3}$$

where $c_k$, $\rho_k$, $\Gamma$ and $d\lambda$ are defined as in (2.1.1),

$\omega$ is the $N^{th}$ root of unity.

If $N = 2$ then we simply recover the KdV case and for $N = 3$ we have the BSQ system.

In [99] the lattice BSQ and lattice modified BSQ equations were presented along with their relevant Lax pairs and a gauge transformation between the two Lax pairs. In [87] this work was extended to include a lattice Schwarzian BSQ equation (but no Lax pair) and similarity constraints were also given for the BSQ and modified BSQ equations along with their associated monodromy problems. In this chapter we shall review the derivation of these previous results and extend the results for the BSQ system further.

## 5.2 Infinite Matrix Structure for the Boussinesq System

As before, we must now develop the infinite matrix structure. Most of the ingredients will be the same as for the KdV system given in section (2.2), but some amendments must be made in order to accommodate the addition of the $\omega$ in the integral equation (5.1.3).

The two main infinite matrices $C$ and $U$ are defined, as before by

$$C = \int_\Gamma d\lambda(l) \rho_l c\, {}^t c \quad , \quad U = \int_\Gamma d\lambda(l) u_l\, {}^t c.$$

As with the KdV system, the matrix $C$ provides linear relations while the $U$ provides nonlinear relations, specific entries of which shall give various closed form scalar equations. However, unlike the KdV system, we no longer have the symmetry in the matrix $U$ i.e. $U \neq {}^tU$.

Once again we require index raising matrices $\Lambda$ and ${}^t\Lambda$, index counting matrices $I$ and ${}^tI$ and a projection matrix $O$, all of which are defined as in equations (2.2.3a-2.2.3c). However, the matrix $\Omega$ must now satisfy the following relations

$$\Omega \cdot \Lambda^j - (-\omega\, {}^t\Lambda)^j \cdot \Omega = O_j, \tag{5.2.1a}$$

$${}^tI \cdot \Omega + \Omega \cdot I + \Omega = 0, \tag{5.2.1b}$$

$$\text{where} \qquad O_k = \sum_{j=0}^{k-1} (-\,{}^t\Lambda)^j \cdot O \cdot \Lambda^{k-1-j}. \tag{5.2.1c}$$

With all the ingredients in place we are again able to write the integral equation (5.1.3) in the following infinite matrix form

$$U = C \cdot (\mathbf{1} + \Omega \cdot C)^{-1}. \tag{5.2.2}$$

The form of this equation is the same as for equation (2.2.4) in the KdV case, but it should be noted that with the alterations in the infinite matrix scheme the above equation now represents the Boussinesq system. We now go on to derive the various closed form equations for this system.

## 5.3 Discrete Lattice Equations

For this section we shall limit ourselves to purely discrete variables. For this we choose a plane wave factor $\rho_k$ of the form

$$\rho_k \;=\; \prod_\nu \left(\frac{p_\nu + k}{p_\nu + \omega k}\right)^{n_\nu}, \tag{5.3.1}$$

giving us $N$ independent discrete variables $n_1, n_2 \ldots n_N$, each with an associated lattice parameter $p_\nu$ and a spectral parameter $k$.

The discrete evolution for the $\rho_k$ is now given by

$$T_\nu \rho_k \;=\; \left(\frac{p_\nu + k}{p_\nu + \omega k}\right) \rho_k, \tag{5.3.2}$$

which again leads to two linear relations for $C$

$$(T_\nu C) \cdot (p + \omega\, {}^t\Lambda) = (p + \Lambda) \cdot C, \tag{5.3.3a}$$

$$(p + \omega\Lambda) \cdot T_\nu C = C \cdot (p + {}^t\Lambda). \tag{5.3.3b}$$

However, unlike the KdV case, we now obtain two different nonlinear relations for the $U$ variable, one from each of the above equations,

$$(T_\nu U) \cdot (p + \omega\, {}^t\Lambda) \;=\; (p + \Lambda) \cdot U - (T_\nu U) \cdot O \cdot U, \tag{5.3.4a}$$

$$\begin{aligned}(p + \omega^2\Lambda) \cdot (p + \omega\Lambda) \cdot T_\nu U \;&=\; U \cdot (p + \omega^2\, {}^t\Lambda) \cdot (p + {}^t\Lambda) \\ &+\; U \cdot \left[(p + \omega^2\, {}^t\Lambda)\omega^2 \cdot O + \omega O \cdot (p + \omega^2\Lambda)\right] \cdot T_\nu U,\end{aligned} \tag{5.3.4b}$$

and, by eliminating the $\bar{U}$ from the above equations we obtain the following, purely algebraic expression:

$$\Lambda^3 \cdot U \;=\; U \cdot {}^t\Lambda^3 + U \cdot (O \cdot \Lambda^2 + \omega\, {}^t\Lambda \cdot O \cdot \Lambda + \omega^2\, {}^t\Lambda^2 \cdot O) \cdot U. \tag{5.3.5}$$

In order to derive the lattice equations of the Boussinesq system we must again introduce various scalar objects, however, due to the fact that $U \neq {}^tU$ we will require more than in the KdV case. The objects we now need are,

$$u \equiv U_{0,0} \quad , \quad s_{\alpha,\beta} \equiv \omega^2 \left(\frac{1}{\alpha+\Lambda} \cdot U \cdot \frac{1}{\beta + {}^t\Lambda}\right)_{0,0} \tag{5.3.6a}$$

$$v_\alpha \equiv 1 - \left(\frac{1}{\alpha+\Lambda} \cdot U\right)_{0,0} \quad , \quad w_\beta \equiv 1 + \omega^2 \left(U \cdot \frac{1}{\beta + {}^t\Lambda}\right)_{0,0} \tag{5.3.6b}$$

$$s_\alpha \equiv \alpha + \omega \left(\frac{1}{\alpha+\Lambda} \cdot U \cdot {}^t\Lambda\right)_{0,0} \quad , \quad t_\beta \equiv -\omega\beta + \omega^2 \left(\Lambda \cdot U \cdot \frac{1}{\beta + {}^t\Lambda}\right)_{0,0} \tag{5.3.6c}$$

$$a_\alpha \equiv -\alpha^2 + \omega^2 \left(\frac{1}{\alpha+\Lambda} \cdot U \cdot {}^t\Lambda^2\right)_{0,0} \quad , \quad b_\beta \equiv \omega^2\beta^2 + \omega^2 \left(\Lambda^2 \cdot U \cdot \frac{1}{\beta + {}^t\Lambda}\right)_{0,0} \tag{5.3.6d}$$

where, as in section 2.4, the subscript $(0,0)$ indicates the central element of the relevant infinite matrix and $\alpha$ and $\beta$ are free parameters which can take on any complex value. As well as the above scalars, we need the additional objects,

$$u^{(1,0)} \equiv U_{1,0} \quad , \quad u^{(0,1)} \equiv U_{0,1}. \tag{5.3.6e}$$

Closed form equations can be obtained by limiting the system to 2 dimensions and considering two different transformations

$$\rho_k \mapsto T_p(\rho_k) \equiv \widetilde{\rho}_k \quad , \quad \rho_k \mapsto T_p(\rho_k) \equiv \widehat{\rho}_k,$$

each of the same type, but for different lattice parameters, $p$ and $q$.

Equation (5.3.4a) then gives us the following relationships:

$$\widetilde{s}_\alpha = (p+u)\widetilde{v}_\alpha - (p-\alpha)v_\alpha, \tag{5.3.7a}$$

$$t_\beta = (p-\omega\beta)\widetilde{w}_\beta - (p-\widetilde{u})w_\beta, \tag{5.3.7b}$$

$$\widetilde{v}_\alpha w_\beta = 1 + (p-\omega\beta)\widetilde{s}_{\alpha,\beta} - (p-\alpha)s_{\alpha,\beta}, \tag{5.3.7c}$$

$$\widetilde{a}_\alpha = (p-\alpha)s_\alpha + p\widetilde{s}_\alpha + \omega\widetilde{v}_\alpha u^{(0,1)}, \tag{5.3.7d}$$

$$\widetilde{b}_\beta = (p-\omega\beta)\widetilde{t}_\beta - pt_\beta + \widetilde{u}^{(1,0)}w_\beta, \tag{5.3.7e}$$

$$\omega\widetilde{u}^{(0,1)} = u^{(1,0)} + p(u-\widetilde{u}) - \widetilde{u}u. \tag{5.3.7f}$$

While equation (5.3.4b) gives us these further relationships:

$$\begin{aligned} pv_\alpha\widetilde{w}_\beta - v_\alpha\widetilde{t}_\beta + s_\alpha\widetilde{w}_\beta &= (p+\alpha+\omega\beta) + (p^2+\alpha p+\alpha^2)\widetilde{s}_{\alpha,\beta} \\ &\quad -(p^2+\omega\beta p+\omega^2\beta^2)s_{\alpha,\beta}, \end{aligned} \tag{5.3.8a}$$

$$a_\alpha = (p(p-\widetilde{u}) + \widetilde{u}^{(1,0)})v_\alpha - (p^2+p\alpha+\alpha^2)\widetilde{v}_\alpha + (p-\widetilde{u})s_\alpha \tag{5.3.8b}$$

$$\widetilde{b}_\beta = (\omega u^{(1,0)} - p(p+u))\widetilde{w}_\beta + (p^2+\omega\beta p+\omega^2\beta^2)w_\beta + (p+u)\widetilde{t}_\beta. \tag{5.3.8c}$$

And finally, from equation (5.3.5) we get the relationship

$$v_\alpha b_\beta - a_\alpha w_\beta = s_\alpha t_\beta + (\alpha^2+\omega\alpha\beta+\omega^2\beta^2) + (\alpha^3-\beta^3)s_{\alpha,\beta}, \tag{5.3.9}$$

note : all of the equations (5.3.7a) - (5.3.9) have a dual obtained by replacing $p$ with $q$ and $\widetilde{\cdot}$ with $\widehat{\cdot}$.

Before we go on to derive the closed-form lattice equations we shall use the above equations to derive certain relationships for particular values of the parameters $\alpha$ and $\beta$.

Setting $\alpha = p$ and $\beta = \omega^2 p$ we obtain from equations (5.3.7b), (5.3.7c) and (5.3.7e)

$$v_p w_{\omega^2 p} = 1, \tag{5.3.10a}$$

$$t_{\omega^2 p} = -(p - \widetilde{u}) w_{\omega^2 p}, \tag{5.3.10b}$$

$$b_{\omega^2 p} = (p(p-u) + u^{(1,0)}) w_{\omega^2 p}. \tag{5.3.10c}$$

While the object $\frac{v_p}{\widetilde{v}_p}$ can be expressed in various ways for the parameter values $\alpha = \beta = 0$

$$3p^2 \frac{v_p}{\widetilde{v}_p} = \frac{p^3 - q^3}{p - q + \underset{\sim}{\widehat{u}} - u} + (p - q + \widehat{u} - \widetilde{u})(2p + q + \underset{\sim}{u} - \widehat{\widetilde{u}}), \tag{5.3.11a}$$

$$3p \frac{v_p}{\widetilde{v}_p} = \frac{\widehat{v}_0}{v_0} \frac{p^2 v_0 - q^2 \underset{\sim}{\widehat{v}_0}}{p \underset{\sim}{\widehat{v}_0} - q v_0} + \frac{\underset{\sim}{v_0}}{v_0} \frac{p\widehat{v}_0 - q\widetilde{v}_0}{\widehat{\widetilde{v}}_0} + p \frac{\widetilde{v}_0}{v_0}, \tag{5.3.11b}$$

$$3 \frac{v_p}{\widetilde{v}_p} = \frac{\widetilde{v}_0}{v_0} + \frac{\underset{\sim}{w_0}}{w_0} + \frac{\underset{\sim}{v_0} \widetilde{w}_0}{v_0 w_0}. \tag{5.3.11c}$$

Equations (5.3.10) and (5.3.11) shall be required throughout the rest of this section for deriving the various closed-form relations.

### 5.3.1 Closed Form Lattice Equations

We now use the relations from the previous section to derive closed-form equations for the variables $u, v_0$ and $s_{0,0}$.

From equation (5.3.7f) we get the following relationships

$$\omega \widehat{u}^{(0,1)} - \omega \widetilde{u}^{(0,1)} = p\widetilde{u} - q\widehat{u} - u(p - q + \widehat{u} - \widetilde{u}), \tag{5.3.12a}$$

$$\widehat{u}^{(1,0)} - \widetilde{u}^{(1,0)} = q\widetilde{u} - p\widehat{u} + \widehat{\widetilde{u}}(p - q + \widehat{u} - \widetilde{u}). \tag{5.3.12b}$$

While equations (5.3.7a), (5.3.7d) and (5.3.8b) gives us

$$\widehat{\widetilde{u}}^{(1,0)} - \omega u^{(0,1)} = pq - (p + q + u)(p + q - \widehat{\widetilde{u}}) + \frac{p^3 - q^3}{p - q + \widehat{u} - \widetilde{u}}. \tag{5.3.12c}$$

Now, by eliminating the variables $u^{(1,0)}$ and $u^{(0,1)}$ from the above three equations we obtain the following equation in terms of the variable $u$

$$\begin{aligned}\frac{p^3-q^3}{p-q+u_{n+1,m+1}-u_{n+2,m}} - \frac{p^3-q^3}{p-q+u_{n,m+2}-u_{n+1,m+1}} = \\ (p-q+-u_{n+1,m+2}-u_{n+2,m+1})(2p+q+u_{n,m+1}-u_{n+2,m+2}) \\ -(p-q+u_{n,m+1}-u_{n+1,m})(2p+q+u_{n,m}-u_{n+2,m+1}),\end{aligned} \tag{5.3.13}$$

which is a lattice version of the Boussinesq equation, [99].

Note: in the above equation, for clarity, we use the notation $\widetilde{u} = u_{n+1,m}$, $\widehat{u} = u_{n,m+1}$, etc.

A further set of relationships are obtained from equations (5.3.7a), (5.3.7f) and (5.3.8b)

$$(p-q+\widehat{u}-\widetilde{u})(p+q-\widehat{\widetilde{u}}+\frac{s_\alpha}{v_\alpha}) = (p^2+\alpha p+\alpha^2)\frac{\widetilde{v}_\alpha}{v_\alpha} -(q^2+\alpha q+\alpha^2)\frac{\widehat{v}_\alpha}{v_\alpha}, \tag{5.3.14}$$

$$p-q+\widehat{u}-\widetilde{u} = (p-\alpha)\frac{\widehat{v}_\alpha}{\widehat{\widetilde{v}}_\alpha} - (q-\alpha)\frac{\widetilde{v}_\alpha}{\widehat{\widetilde{v}}_\alpha}. \tag{5.3.15}$$

While, equations (5.3.7b), (5.3.7f) and (5.3.8c)

$$(p-q+\widehat{u}-\widetilde{u})(p+q+u-\frac{\widehat{\widetilde{t}}_\beta}{\widehat{\widetilde{w}}_\beta}) = (p^2+\omega\beta p+\omega^2\beta^2)\frac{\widehat{w}_\beta}{\widehat{\widetilde{w}}_\beta} -(q^2+\omega\beta q+\omega^2\beta^2)\frac{\widetilde{w}_\beta}{\widehat{\widetilde{w}}_\beta}, \tag{5.3.16}$$

$$p-q+\widehat{u}-\widetilde{u} = (p-\omega\beta)\frac{\widetilde{w}_\beta}{w_\beta} - (q-\omega\beta)\frac{\widehat{w}_\beta}{w_\beta}. \tag{5.3.17}$$

The first of these set of equations, along with equation (5.3.7a), upon elimination of the $s$ and $u$ variables give us the lattice version of the modified Boussinesq equation, [99]

$$\begin{aligned}(p-\alpha)(\frac{v_{n,m}}{v_{n+1,m}} - \frac{v_{n+1,m+2}}{v_{n+2,m+2}}) - (q-\alpha)(\frac{v_{n,m}}{v_{n,m+1}} - \frac{v_{n+2,m+1}}{v_{n+2,m+2}}) = \\ \frac{(p^2+\alpha p+\alpha^2)v_{n+1,m+1}-(q^2+\alpha q+\alpha^2)v_{n,m+2}}{(p-\alpha)v_{n,m+2}-(q-\alpha)v_{n+1,m+1}}\frac{v_{n+1,m+2}}{v_{n,m+1}} \\ -\frac{(p^2+\alpha p+\alpha^2)v_{n+2,m}-(q^2+\alpha q+\alpha^2)v_{n+1,m+1}}{(p-\alpha)v_{n+1,m+1}-(q-\alpha)v_{n+2,m}}\frac{v_{n+2,m+1}}{v_{n+1,m}},\end{aligned} \tag{5.3.18}$$

note: again, for clarity, we use the notation $\widetilde{v}_\alpha = v_{n+1,m}$, $\widehat{v}_\alpha = v_{n,m+1}$, etc.

And the second set, along with equation (5.3.7b) gives us a similar equation in terms of the $w$ variable.

Now, equations (5.3.12a), (5.3.12b), (5.3.14) and (5.3.7a) give us the relationship

$$\begin{aligned} 3p + \underset{\sim}{u} - \widehat{\widetilde{u}} &= \frac{(p^2+\alpha p+\alpha^2)\widetilde{v}_\alpha - (q^2+\alpha q+\alpha^2)\widehat{v}_\alpha}{(p-\alpha)\widehat{v}_\alpha - (q-\alpha)\widetilde{v}_\alpha}\frac{\widehat{\widetilde{v}}_\alpha}{v_\alpha} + (p-\omega\beta)\frac{\widetilde{\widetilde{w}}_\beta}{\widetilde{w}_\beta} \\ &\quad -(q-\omega\beta)\frac{\widehat{\widetilde{w}}_\beta}{\widetilde{w}_\beta} + (p-\alpha)\frac{\underset{\sim}{v}_\alpha}{v_\alpha}, \end{aligned}$$

and, equations (5.3.7a), (5.3.7b) and (5.3.8a) give us:

$$\begin{aligned} 3p + \underset{\sim}{u} - \widetilde{\widetilde{u}} &= \frac{(p+\alpha+\omega\beta) + (p^2+\alpha p+\alpha^2)\widetilde{s}_{\alpha,\beta} - (p^2+\omega\beta p+\omega^2\beta^2)s_{\alpha,\beta}}{v_\alpha \widetilde{w}_\beta} \\ &\quad +(p-\omega\beta)\frac{\widetilde{\widetilde{w}}_\beta}{\widetilde{w}_\beta} + (p-\alpha)\frac{\underset{\sim}{v}_\alpha}{v_\alpha}. \end{aligned}$$

By setting $\alpha = \beta = 0$ and defining $z = s_{0,0} + \frac{n}{p} + \frac{m}{q}$, these two relationships give us

$$v_0 \widehat{\widetilde{w}}_0 = \frac{[p^3(\widetilde{z}-z)(\widehat{\widetilde{z}}-\widehat{z}) - q^3(\widehat{z}-z)(\widehat{\widetilde{z}}-\widetilde{z})]}{pq(\widehat{z}-\widetilde{z})}, \tag{5.3.19}$$

and equation (5.3.7c) for $\alpha = \beta = 0$ can be rewritten as

$$p(\widetilde{z}-z) = \widetilde{v}_0 w_0, \tag{5.3.20}$$

noting that

$$\frac{(v_0\widehat{\widetilde{w}}_0)^{\widehat{}}}{(v_0\widehat{\widetilde{w}}_0)^{\widetilde{}}} = \frac{(\widehat{v}_0 w_0)(\widetilde{v}_0 w_0)^{\widehat{\widehat{\widetilde{}}}}}{(\widetilde{v}_0 w_0)(\widehat{v}_0 w_0)^{\widetilde{\widehat{\widetilde{}}}}},$$

we are able to write a closed-form equation in terms of the $z$ variable,

$$\begin{aligned} &\frac{(z_{n+2,m+2}-z_{n+1,m+2})(z_{n,m+2}-z_{n+1,m+1})(z_{n,m+1}-z_{n,m})}{(z_{n+2,m+2}-z_{n+2,m+1})(z_{n+2,m}-z_{n+1,m+1})(z_{n+1,m}-z_{n,m})} = \\ &\quad \frac{p^3(z_{n+1,m+2}-z_{n,m+2})(z_{n+1,m+1}-z_{n,m+1}) - q^3(z_{n+1,m+2}-z_{n+1,m+1})(z_{n,m+2}-z_{n,m+1})}{q^3(z_{n+2,m+1}-z_{n+2,m})(z_{n+1,m+1}-z_{n+1,m}) - p^3(z_{n+2,m+1}-z_{n+1,m+1})(z_{n+2,m}-z_{n+1,m})}, \end{aligned} \tag{5.3.21}$$

which is the lattice version of the Schwarzian Boussinesq equation, [87]

note: once again, for clarity, we use the notation $\widetilde{z} = z_{n+1,m}$, $\widehat{z} = z_{n,m+1}$, etc.

## 5.4 Lax Pairs for Discrete Equations

It is also possible to derive from the integral equation the following linear problem in terms of the infinite vectors $\boldsymbol{u}_k$ and $\widetilde{\boldsymbol{u}}_k$

$$(p+\omega k)\widetilde{\boldsymbol{u}}_k = (p+\Lambda-\bar{U}\cdot O)\cdot \boldsymbol{u}_k, \tag{5.4.1a}$$

$$\begin{aligned}(p+\omega^2 k)(p+k)\boldsymbol{u}_k &= (p+\omega^2\Lambda)\cdot(p+\omega\Lambda)\cdot\widetilde{\boldsymbol{u}}_k \\ &- U\cdot\left[(p+\omega^{2}\,{}^t\Lambda)\omega^2\cdot O+\omega O\cdot(p+\omega^2\Lambda)\right]\cdot\widetilde{\boldsymbol{u}}_k.\end{aligned} \tag{5.4.1b}$$

Thus, in terms of the vector

$$\phi_{n,m}(k) \equiv (p+\omega k)^n(q+\omega k)^m \begin{pmatrix} u_k^0(n,m) \\ u_k^1(n,m) \\ u_k^2(n,m) \end{pmatrix}, \tag{5.4.2}$$

we are able to derive the following 3x3 matrix Lax system, [99]

$$\widetilde{\phi}(k) = L_k^{(BSQ)}\cdot\phi(k), \tag{5.4.3a}$$

$$\widehat{\phi}(k) = M_k^{(BSQ)}\cdot\phi(k), \tag{5.4.3b}$$

where

$$L_k^{(BSQ)} = \begin{pmatrix} p-\widetilde{u} & 1 & 0 \\ -\widetilde{u}^(1,0) & p & 1 \\ k^3+p^3-(p^2+pu-\omega u^{(0,1)})(p-\widetilde{u})-(p+u)\widetilde{u}^{(1,0)} & \omega u^{(0,1)} & p+u \end{pmatrix},$$

and $M_k^{(BSQ)}$ is obtained from $L^{(BSQ)}$ by replacing $p$ with $q$ and $\widetilde{\cdot}$ with $\widehat{\cdot}$.

The compatibility condition of the Lax system

$$\widehat{L}_k^{(BSQ)}\cdot M_k^{(BSQ)} = \widetilde{M}_k^{(BSQ)}\cdot L_k^{(BSQ)}, \tag{5.4.4}$$

leads to equations (5.3.12a)-(5.3.12c) which in turn lead to the lattice Boussinesq equation (5.3.13).

We are also able to write down a Lax system for the vector

$$\psi_{n,m}(k) \equiv (p+\omega k)^n (q+\omega k)^m \begin{pmatrix} u_k^{(\alpha)}(n,m) \\ u_k^0(n,m) \\ u_k^1(n,m) \end{pmatrix}, \tag{5.4.5}$$

For this vector the Lax system is given by

$$\widetilde{\psi}(k) = L^{(mBSQ)} \cdot \psi(k), \tag{5.4.6a}$$

$$\widehat{\psi}(k) = M_k^{(mBSQ)} \cdot \psi(k), \tag{5.4.6b}$$

where

$$L^{(mBSQ)} = \begin{pmatrix} p-\alpha & \widetilde{v}_\alpha & 0 \\ 0 & p-\widetilde{u} & 1 \\ \frac{k^3+\alpha^3}{v_\alpha} & \frac{(pv+s_\alpha)(p-\widetilde{u})-(p^2+\alpha p+\alpha^2)\widetilde{v}_\alpha}{v_\alpha} & p+\frac{s_\alpha}{v_\alpha} \end{pmatrix},$$

and, again, $M_k^{(mBSQ)}$ is obtained from $L^{(mBSQ)}$ by replacing $p$ with $q$ and $\widetilde{\ }$ with $\widehat{\ }$. In this case the compatibility condition leads to equations (5.3.14), (5.3.15) and (5.3.7a) which give the lattice modified Boussinesq equation, [99].

The two vectors (5.4.2) and (5.4.5) are related to each other via the following gauge transformation:

$$\psi = V \cdot \phi, \tag{5.4.7}$$

where

$$V = \begin{pmatrix} \frac{-a_\alpha}{k^3+\alpha^3} & \frac{-s_\alpha}{k^3+\alpha^3} & \frac{v_\alpha}{k^3+\alpha^3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{5.4.8}$$

thus, the Lax matrices for the modified BSQ system $L_k^{(mBSQ)}$, $M_k^{(mBSQ)}$ are related to the Lax matrices for the BSQ system $L_k^{(BSQ)}$, $M_k^{(BSQ)}$ via

$$L_k^{(mBSQ)} = \widetilde{V} \cdot L_k^{(BSQ)} \cdot V^{-1}, \tag{5.4.9a}$$

$$M_k^{(mBSQ)} = \widehat{V} \cdot M_k^{(BSQ)} \cdot V^{-1}, \tag{5.4.9b}$$

To acquire the Lax representation of the Schwarzian Boussinesq equation we must apply a further gauge transformation

$$\chi = W \cdot \psi, \tag{5.4.10}$$

where

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{w_0} & 0 \\ 0 & -\frac{t_0}{w_0 \overline{w}_0} & \frac{1}{\overline{\overline{w}}_0} \end{pmatrix}. \tag{5.4.11}$$

Unlike the previous gauge transformation, this gauge contains a discrete shift. This is an arbitrary shift which can be associated with any of the independent variables and is hence represented with a "bar". We can either choose this arbitrary shift to be a $\widetilde{\cdot}$ or a $\widehat{\cdot}$ or we can leave it as a $\bar{\cdot}$ , both choices yield useful information as we shall see below.

This leads to the following Lax system,

$$\widetilde{\chi}(k) = L_k^{(SBSQ)} \cdot \chi(k), \tag{5.4.12a}$$

$$\widehat{\chi}(k) = M_k^{(SBSQ)} \cdot \chi(k), \tag{5.4.12b}$$

where

$$L_k^{(SBSQ)} = \begin{pmatrix} p & \widetilde{v}_0 w_0 & 0 \\ 0 & p & \frac{\overline{\widetilde{w}}_0}{\widetilde{w}_0} \\ \frac{k^3}{\overline{\overline{w}}_0} & 0 & p\frac{\widetilde{v}_0 w_0}{v_0 \widetilde{w}_0}\frac{\overline{w}_0}{w_0} \end{pmatrix},$$

again, $M_k^{(SBSQ)}$ is obtained from $L^{(SBSQ)}$ by replacing $p$ with $q$ and $\tilde{\cdot}$ with $\hat{\cdot}$ but without altering $\bar{\cdot}$ s.

If we leave the Lax matrices like this, the compatibility condition then yields the following relations

$$\frac{p\widehat{v}_0 - q\widetilde{v}_0}{\widehat{\widetilde{v}}_0} = \frac{p\widetilde{w}_0 - q\widehat{w}_0}{w_0} = \frac{p\widetilde{v}_0\widehat{w}_0 - q\widehat{v}_0\widetilde{w}_0}{v_0\widehat{\widetilde{w}}_0}, \tag{5.4.13}$$

with all the $\bar{\cdot}$ shifts canceling out.

Further shifts of the above equation can be used to eliminate all the $w$'s, giving the Modified Boussinesq Equation (5.3.18), however there are some advantages to focusing on (5.4.13) as a coupled system between variables $v_0$ and $w_0$.

Alternatively, if we choose the arbitrary $\bar{\cdot}$ shift to be a $\tilde{\cdot}$ shift we can use equations (5.3.19) and (5.3.20) to express the Lax matrices purely in terms of $z$ variables, giving

$$L_k^{(SBSQ)} = \begin{pmatrix} p & p(\widetilde{z} - z) & 0 \\ 0 & p & 1 \\ \frac{k^3p^2(\widehat{\widetilde{\widetilde{z}}} - \widehat{\widetilde{z}})*}{(\widetilde{\widetilde{\widetilde{z}}} - \widetilde{\widetilde{z}})} & 0 & \frac{p^4(\widetilde{z} - z)(\widehat{\widetilde{\widetilde{z}}} - \widehat{\widetilde{z}})*}{(\widetilde{\widetilde{\widetilde{z}}} - \widetilde{\widetilde{z}})} \end{pmatrix},$$

$$M_k^{(SBSQ)} = \begin{pmatrix} q & q(\widehat{z} - z) & 0 \\ 0 & q & 1 \\ k^3pq* & 0 & \frac{q^4(\widehat{z} - z)(\widehat{\widetilde{z}} - \widetilde{z})}{(\widehat{\widehat{z}} - \widehat{z})*} \end{pmatrix},$$

$$\text{where} \qquad * = \frac{\widehat{z} - \widetilde{z}}{p^3(\widetilde{z} - z)(\widehat{\widetilde{z}} - \widehat{z}) - q^3(\widehat{z} - z)(\widehat{\widetilde{z}} - \widetilde{z})}.$$

With the Lax matrices in this form, the compatibility condition immediately yields the Schwarzian Boussinesq equation (5.3.22).

## 5.5 Differential-Difference Equations

We now go on to derive the differential-difference relations for the Boussinesq system. For these we require the dependence of the wave factor $\rho_k$ on the lattice parameters $p_\nu$, which is given by

$$\frac{\partial}{\partial p_\nu}\rho_k = n_\nu\left(\frac{1}{p_\nu+k}-\frac{1}{p_\nu+\omega k}\right)\rho_k. \tag{5.5.1}$$

This leads to two possible linear relations for the matrix $C$

$$\frac{\partial}{\partial p_\nu}C = n_\nu\left(\frac{1}{p_\nu+\Lambda}\cdot C - C\cdot\frac{1}{p_\nu+\omega\,{}^t\Lambda}\right), \tag{5.5.2a}$$

$$\frac{\partial}{\partial p_\nu}C = n_\nu\left(C\cdot\frac{1}{p_\nu+{}^t\Lambda}-\frac{1}{p_\nu+\omega\Lambda}\cdot C\right), \tag{5.5.2b}$$

which, due to the fact that $C$ and $\Lambda$ are both symmetric, are equivalent. However, as $\Omega$ is not symmetric, only the first of these equations (5.5.2a) yields a nonlinear relation for the $U$ variable, namely

$$\frac{\partial}{\partial p_\nu}U = n_\nu\left(\frac{1}{p_\nu+\Lambda}\cdot U - U\cdot\frac{1}{p_\nu+\omega\,{}^t\Lambda}+U\cdot\frac{1}{p_\nu+\omega\,{}^t\Lambda}\cdot O\frac{1}{p_\nu+\Lambda}\cdot U\right). \tag{5.5.3}$$

Now, using the objects in (5.3.6), equation (5.5.3) gives us the following scalar differential relations for each of the main variables in the Miura chain, $u$, $v_\alpha$, $w_\alpha$ and $s_{\alpha,\beta}$

$$\frac{\partial u}{\partial p} = n\left(1-v_p w_{\omega^2 p}\right), \tag{5.5.4a}$$

$$\frac{\partial v_\alpha}{\partial p} = n\left(\frac{v_p-v_\alpha}{p-\alpha}-v_p s_{\alpha,\omega^2 p}\right), \tag{5.5.4b}$$

$$\frac{\partial w_\alpha}{\partial p} = n\left(w_{\omega^2 p}s_{p,\beta}-\frac{\omega^2(w_\beta-w_{\omega^2 p})}{\omega^2 p-\beta}\right), \tag{5.5.4c}$$

$$\frac{\partial s_{\alpha,\beta}}{\partial p} = n\left(\frac{s_{\alpha,\beta}-s_{p,\beta}}{p-\alpha}-\frac{\omega^2(s_{\alpha,\beta}-s_{\alpha,\omega^2 p})}{\omega^2 p-\beta}+s_{\alpha,\omega^2 p}s_{p,\beta}\right). \tag{5.5.4d}$$

Using various relations it is possible write these in the following way

$$\frac{\partial u}{\partial p} = n\left(1-\frac{3p^2(p-q+\underset{\sim}{\widehat{u}}-u)}{p^3-q^3+(p-q+\widehat{u}-\widetilde{u})(2p+q+\underset{\sim}{u}-\widehat{\widetilde{u}})(p-q+\underset{\sim}{\widehat{u}}-u)}\right), \tag{5.5.5a}$$

$$\frac{\partial v_0}{\partial p} = \frac{n}{p}\left(\frac{3pv_0\widetilde{v}_0\widehat{\widetilde{v}}_0(p\underset{\sim}{\widehat{v}}_0-qv_0)}{\widehat{v}_0\widehat{\widetilde{v}}_0(p^2v_0-q^2\underset{\sim}{\widehat{v}}_0)+\underset{\sim}{v}(p\widehat{v}_0-q\widetilde{v}_0)(p\underset{\sim}{\widehat{v}}_0-qv_0)+p\widetilde{v}_0\widehat{\widetilde{v}}_0(p\underset{\sim}{\widehat{v}}_0-qv_0)}-v_0\right), \tag{5.5.5b}$$

$$\frac{\partial z}{\partial p} = -\frac{n}{p^2}\left(\frac{3p^4(\widetilde{z}-z)(z-\underset{\sim}{z})(\widehat{\widetilde{z}}-\widehat{z})(\underset{\sim}{\widehat{z}}-z)}{p^3(\widetilde{z}-\underset{\sim}{z})(\widehat{\widetilde{z}}-\widehat{z})(\underset{\sim}{\widehat{z}}-z)+p^3(\widehat{\widetilde{z}}-\widetilde{z})(\widehat{z}-\underset{\sim}{\widehat{z}})(z-\underset{\sim}{z})-q^3(\widehat{\widetilde{z}}-\widetilde{z})(\widehat{z}-z)(\underset{\sim}{\widehat{z}}-\underset{\sim}{z})}\right). \tag{5.5.5c}$$

However, these equations are not strictly closed form as they involve shifts not associated with the lattice parameter $p$. For the variables $v_0$ and $w_0$ it is possible to give a closed form coupled system, namely

$$\frac{\partial}{\partial p}\log v_0 = \frac{n}{p}\left(\frac{2\widetilde{v}_0w_0-v_0\underset{\sim}{w}_0-\underset{\sim}{v}_0\widetilde{w}_0}{\widetilde{v}_0w_0+v_0\underset{\sim}{w}_0+\underset{\sim}{v}_0\widetilde{w}_0}\right), \tag{5.5.6a}$$

$$\frac{\partial}{\partial p}\log w_0 = \frac{n}{p}\left(\frac{2v_0\underset{\sim}{w}_0-\widetilde{v}_0w_0-\underset{\sim}{v}_0\widetilde{w}_0}{\widetilde{v}_0w_0+v_0\underset{\sim}{w}_0+\underset{\sim}{v}_0\widetilde{w}_0}\right). \tag{5.5.6b}$$

For the variable $z$ we can give the following closed-form expression for the differential-difference equation

$$\frac{\partial}{\partial p}\log\left(p(\underset{\sim}{z}-\widetilde{z})-\frac{3n(\widetilde{z}-z)(z-\underset{\sim}{z})}{z_p}\right) = \frac{\underset{\sim}{z}_p}{\underset{\approx}{z}-\underset{\sim}{z}}+\frac{\widetilde{z}_p}{\widetilde{z}-\widetilde{\widetilde{z}}}-\frac{2n}{p}, \tag{5.5.7}$$

where $z_p \equiv \frac{\partial z}{\partial p}$. We shall now proceed to derive their associated linear problems and, in the spirit of [85], investigate the compatibility condition of the linear system for the variable $z$ in order to derive the generating PDE for the Schwarzian BSQ hierarchy.

## 5.6 Linear Problems for Differential-Difference Equations

In this section we develop the associated linear problems for the differential-difference relations and attempt to use them to derive the generating equation for the hierarchy of

Schwarzian Boussinesq equations.

For this we require the linear differential-difference relation in terms of the infinite vector $\boldsymbol{u}_k$

$$\frac{\partial}{\partial p}\boldsymbol{u}_k = n\left(\frac{1}{p+\Lambda} - \frac{1}{p+\omega k} + U \cdot \frac{1}{p+\omega\,{}^t\Lambda} \cdot O \cdot \frac{1}{p+\Lambda}\right) \cdot \boldsymbol{u}_k, \tag{5.6.1}$$

we now pick out certain entries of the vector $\boldsymbol{u}_k$

$$\frac{\partial u_k^0}{\partial p} = n\left(w_{\omega^2 p} u_k^{(p)} - \frac{1}{p+\omega k} u_k^0\right), \tag{5.6.2a}$$

$$\frac{\partial u_k^1}{\partial p} = n\left(t_{\omega^2 p} u_k^{(p)} + u_k^0 - \frac{1}{p+\omega k} u_k^1\right), \tag{5.6.2b}$$

$$\frac{\partial u_k^2}{\partial p} = n\left(b_{\omega^2 p} u_k^{(p)} - p u_k^0 + u_k^1 - \frac{1}{p+\omega k} u_k^2\right), \tag{5.6.2c}$$

where

$$u_k^{(\alpha)} = \left(\frac{1}{\alpha+\Lambda} \cdot \boldsymbol{u}_k\right)_0, \tag{5.6.3}$$

Eliminating the $u_k^{(p)}$ from the above equations using

$$(p+\omega k)\widetilde{u}_k^{(\alpha)} = (p-\alpha)u_k^{(\alpha)} + \widetilde{v}_\alpha u_k^0, \tag{5.6.4}$$

which is obtained from equation (5.4.1a), we obtain the following linear problem for the differential of the vector $\phi(k)$, (5.4.2), with respect to the lattice parameter $p$

$$\frac{\partial \phi}{\partial p} = n\,A_1 \cdot \phi + n\,A_2 \cdot \underset{\sim}{\phi}, \tag{5.6.5}$$

where

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -p & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} w_{\omega^2 p} v_p & 0 & 0 \\ t_{\omega^2 p} v_p & 0 & 0 \\ b_{\omega^2 p} v_p & 0 & 0 \end{pmatrix},$$

along with a dual equation for the differential with respect to $q$, obtained via the usual replacements. Utilizing equations (5.3.10) we can eliminate the auxiliary variables and express it purely in terms of the variables $u$, $u^{(1,0)}$ and $u^{(0,1)}$ and using the inverse of the Lax matrix $L_k^{(BSQ)}$ from equation (5.4.3a) we obtain

$$\frac{\partial \phi}{\partial p} = n\left[A_3 + \frac{v_p w_{\omega^2 p}}{p^3+k^3} A_4 \cdot A_5\right] \cdot \phi, \tag{5.6.6}$$

where

$$A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -p & 1 & 0 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 1 & 0 & 0 \\ (p-\widetilde{u}) & 0 & 0 \\ (p(p-u)+u^{(1,0)}) & 0 & 0 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} (p^2 + p\underset{\sim}{u} - \omega \underset{\sim}{u}^{(0,1)}) & (p+\underset{\sim}{u}) & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$v_p w_{\omega^2 p} = \frac{3p^2(p-q+\widehat{\underset{\sim}{u}}-u)}{p^3+q^3+(p-q+\widehat{u}-\widetilde{u})(2p+q+\underset{\sim}{u}-\widehat{\widetilde{u}})(p-q+\widehat{\underset{\sim}{u}}-u)},$$

again with a dual obtained via the usual replacements.

In order to derive the linear problem for the modified vector $\psi_k$, (5.4.5), we need the additional relation

$$\frac{\partial u_k^{(\alpha)}}{\partial p} = n\left(\frac{1}{p-\alpha}(u_k^{(\alpha)} - u_k^{(p)}) - \frac{1}{p+\omega k}u_k^{(\alpha)} + \frac{s_{\alpha,\omega^2 p} v_p \underset{\sim}{u}_k^0}{p+\omega k}\right), \tag{5.6.7}$$

which, again having eliminated the $u_k^{(p)}$ using equation (5.6.4) we get

$$\frac{\partial \psi}{\partial p} = n\, B_1 \cdot \psi + n\, B_2 \cdot \underset{\sim}{\psi}, \tag{5.6.8}$$

where

$$B_1 = \begin{pmatrix} \frac{1}{p-\alpha} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & \left(s_{\alpha,\omega^2 p} - \frac{1}{p-\alpha}\right) & 0 \\ 0 & w_{\omega^2 p} v_p & 0 \\ 0 & t_{\omega^2 p} v_p & 0 \end{pmatrix},$$

Again, there is a dual equation for the differential of $\psi_k$ with respect to $q$ obtained with the usual replacements. As with the linear problem for $\phi$ we can eliminate as many of the auxiliary variables as possible and use the inverse of the Lax matrix $L_k^{(mBSQ)}$ from equation (5.4.6a) to eliminate the undershift in the $\psi$ we obtain

$$\frac{\partial\psi}{\partial p} = n\left[B_3 + \frac{v_p w_{\omega^2 p}}{(p^3+k^3)v_\alpha} B_4 \cdot B_5\right]\cdot\psi, \qquad (5.6.9)$$

where

$$B_3 = \begin{pmatrix} \frac{1}{p-\alpha} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad B_4 = \begin{pmatrix} -\frac{\tilde{v}_\alpha}{p-\alpha} & 0 & 0 \\ 1 & 0 & 0 \\ -(p-\tilde{u} & 0 & 0 \end{pmatrix},$$

$$B_5 = \begin{pmatrix} k^3+\alpha^3 & (p-\alpha)(p\underset{\sim}{v}_\alpha + s_\alpha) & -(p-\alpha)\underset{\sim}{v}_\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$v_p w_{\omega^2 p} = \frac{3pv_0\hat{\hat{v}}_0(p\underset{\sim}{\hat{v}}_0 - qv_0)}{\hat{v}_0\hat{\hat{v}}_0(p^2v_0 - q^2\underset{\sim}{\hat{v}}_0) + \underset{\sim}{v}(p\hat{v}_0 - q\tilde{v}_0)(p\underset{\sim}{\hat{v}}_0 - qv_0) + p\tilde{v}_0\hat{\hat{v}}_0(p\underset{\sim}{\hat{v}}_0 - qv_0)},$$

again with dual for the differential with respect to $q$.

As with the Lax pairs for the lattice equations, the linear differential relations (5.6.5) and (5.6.8), (for parameter value $\alpha = 0$), are related via the gauge transformation $V$, (5.4.8), by

$$V_p\cdot V^{-1} + V\cdot A_1\cdot V^{-1} + V\cdot A_2\cdot(\underset{\sim}{L}^{(BSQ)})^{-1}\cdot V^{-1} = B_1 + B_2\cdot(\underset{\sim}{L}^{(mBSQ)})^{-1}. \qquad (5.6.10)$$

Carrying out a similar gauge transformation on the linear differential relation for $\psi_k$, this time using (5.4.11) we obtain the following linear problem for the vector $\chi_k$

$$\frac{\partial\chi}{\partial p} = n\left[C_1 + \frac{v_p w_{\omega^2 p}}{(p^3+k^3)} C_2\right]\cdot\chi, \qquad (5.6.11)$$

where

$$C_1 = \begin{pmatrix} \frac{1}{p} & 0 & 0 \\ 0 & \frac{\partial}{\partial p}\log w_0 & 0 \\ 0 & \frac{\widetilde{w}_0}{\overline{w}_0}\left(\frac{p}{n}\frac{\partial}{\partial p}\log w_0 + 1\right) & -\frac{\partial}{\partial p}\log \overline{w}_0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} \frac{k^3}{p}\frac{\widetilde{v}_0}{v_0} & -\widetilde{v}_0\underset{\sim}{w_0} & \frac{\widetilde{v}_0}{v_0}\underset{\sim}{v_0}\overline{w}_0 \\ k^3\frac{1}{v_0 w_0} & p^2\frac{\underset{\sim}{w_0}}{w_0} & -p\frac{v_0\overline{w}_0}{v_0 w_0} \\ -k^3 p\frac{\widetilde{w}_0}{v_0 w_0 \overline{w}_0} & -p^3\frac{\widetilde{w}_0 \underset{\sim}{w_0}}{w_0 \overline{w}_0} & p^2\frac{v_0\widetilde{w}_0}{v_o w_0} \end{pmatrix},$$

with a dual obtained from the usual replacements.

We now make the following definitions

$$a = \frac{\widetilde{v}_0}{v_0}v_p w_{\omega^2 p} \quad , \quad b = \frac{\underset{\sim}{w_0}}{w_0}v_p w_{\omega^2 p} \quad , \quad c = \frac{\underset{\sim}{v_0}\widetilde{w}_0}{v_0 w_0}v_p w_{\omega^2 p} \quad , \quad X = \frac{\widetilde{w}_0}{\overline{w}_0},$$
$$e = \frac{\widehat{v}_0}{v_0}v_q w_{\omega^2 q} \quad , \quad f = \frac{\underset{\wedge}{w_0}}{w_0}v_q w_{\omega^2 q} \quad , \quad d = \frac{\underset{\wedge}{v_0}\widehat{w}_0}{v_0 w_0}v_q w_{\omega^2 q} \quad , \quad Y = \frac{\widehat{w}_0}{\overline{w}_0},$$

these variables are related, via equation (5.3.11c), by

$$3 = a + b + c \qquad , \qquad 3 = e + f + d, \tag{5.6.13}$$

and using these new variables the linear problem can be rewritten as

$$\frac{\partial \chi}{\partial p} = (C_3 + C_4)\cdot\chi, \tag{5.6.14a}$$

$$C_3 = \begin{pmatrix} \frac{n}{p} & 0 & 0 \\ 0 & -n\left(b - \frac{1}{p}\right) & 0 \\ 0 & npbX & -\overline{n}\left(\overline{b} - \frac{1}{p}\right) \end{pmatrix},$$

$$C_4 = \frac{1}{p^3 + k^3}\begin{pmatrix} \frac{nk^3}{p}a & p^3 z_p & -\frac{p^2 c z_p}{bX} \\ -\frac{n^2 k^2}{p^2}\frac{ab}{z_p} & np^2 b & \frac{npc}{X} \\ \frac{n^2 k^3}{p}\frac{abX}{z_p} & -np^3 bX & np^2 c \end{pmatrix},$$

$$\frac{\partial \chi}{\partial q} = (D_3 + D_4) \cdot \chi, \tag{5.6.14b}$$

$$D_3 = \begin{pmatrix} \frac{m}{q} & 0 & 0 \\ 0 & -m\left(f - \frac{1}{q}\right) & 0 \\ 0 & mqfY & -\overline{m}\left(\overline{f} - \frac{1}{q}\right) \end{pmatrix},$$

$$D_4 = \frac{1}{p^3 + k^3} \begin{pmatrix} \frac{mk^3}{q}e & q^3 z_q & -\frac{q^2 d z_q}{fY} \\ -\frac{m^2k^2}{q^2}\frac{ef}{z_q} & mq^2 f & \frac{mqd}{Y} \\ \frac{m^2k^3}{q}\frac{efY}{z_q} & -mq^3 fY & mq^2 d \end{pmatrix},$$

(where $z_p \equiv \frac{\partial z}{\partial p}$ and $z_q \equiv \frac{\partial z}{\partial q}$).

If we now calculate the compatibility condition for (5.6.14a) and (5.6.14b)

$$(C_3 + C_4)_q - (D_3 + D_4)_p - [(C_3 + C_4), (D_3 + D_4)] = 0 \tag{5.6.15}$$

we obtain the following relations

$$\begin{aligned} na_q = me_p = \frac{1}{q^3 - p^3} &\left[ n^2 \frac{z_q}{z_p} \left( pq \frac{abd}{f} \frac{X}{Y} + q^3 ab \right) \right. \\ &\left. -n^2 \frac{z_p}{z_q} \left( pq \frac{cef}{b} \frac{Y}{X} + p^3 ef \right) \right], \end{aligned} \tag{5.6.16a}$$

$$\begin{aligned} nb_q = mf_p = \frac{1}{q^3 - p^3} &\left[ n^2 \frac{z_q}{z_p} q^3 ab - m^2 \frac{z_p}{z_q} p^3 ef \right. \\ &\left. +nm \left( cf\frac{Y}{X} - bd\frac{X}{Y} \right) pq \right], \end{aligned} \tag{5.6.16b}$$

$$\begin{aligned} z_{pq} = \frac{1}{q^3 - p^3} &\left[ mpz_p \left( p^2(e - f) - q\frac{cf}{b}\frac{Y}{X} \right) \right. \\ &\left. -nqz_q \left( q^2(a - b) - p\frac{db}{f}\frac{X}{Y} \right) \right], \end{aligned} \tag{5.6.16c}$$

$$\begin{aligned} (\log(Y))_p = \frac{1}{q^3 - p^3} &\left( q - p\frac{X}{Y} \right) \left( \frac{n^2}{m} q^2 \frac{z_q}{z_p} \frac{ab}{f} + n\left( q^2 b + pc\frac{Y}{X} \right) \right) \\ &-\frac{1}{p}\left( n - \overline{n}(1 - p\overline{b}) \right), \end{aligned} \tag{5.6.16d}$$

$$(\log(X))_q = \frac{1}{q^3-p^3}\left(q\frac{Y}{X}-p\right)\left(\frac{m^2}{n}p^2\frac{z_p}{z_q}\frac{ef}{b}+m\left(p^2f+qd\frac{X}{Y}\right)\right)$$
$$-\frac{1}{q}\left(m-\overline{m}(1-q\overline{f})\right). \quad (5.6.16e)$$

This gives us a coupled system for the variables $z, a, b, c, e, f, d, X, Y$, but unfortunately there is still one discrete shift left in the system so we are unable to derive a closed form PDE in $z$. The derivation of the above system follows the same procedure as that given in section 2.6 for the Schwarzian KdV hierarchy. Indeed, comparing equations (5.6.16) with equations (2.6.2) we see that there is a close correlation between the two sets of PDEs, but as expected the BSQ system is more complicated than the KdV system and as such the derivation of a closed form PDE in $z$ will also be much more complicated. However, this correlation leads us to believe that the system (5.6.16) will eventually lead to the generating PDE for the Schwarian BSQ hierarchy.

## 5.7 Integrability of the Boussinesq Lattice Equations

In section 5.3 we derived two-dimensional closed form lattice equations for the Boussinesq system, namely the BSQ equation (5.3.13), the mBSQ equation (5.3.18) and the SBSQ equation (5.3.22), see also the recent papers by Bobenko et al. [100, 101]. In this section we wish to investigate whether these equations, as is the case in the KdV system, represent *compatible parameter-families of partial difference equations.* It turns out that in the Boussinesq the process of embedding these equations in a multi-dimensional lattice is somewhat more complicated and hence we shall deal with each equation separately. We shall however, throughout this section, adopt the notation of section 3.2 in which we have a vector $\boldsymbol{n}$ of discrete independent variables $n_i$, and an associated vector $\boldsymbol{p}$ of lattice parameters $p_i$. Forward shifts of a dependent variable $f(\boldsymbol{n};\boldsymbol{p})$ are now denoted by a right superscript and backward shifts with a left subscript, i.e.

$$f^j = T_j f(\mathbf{n};\mathbf{p}) = f(\mathbf{n}+\mathbf{e}_j;\mathbf{p}) \quad , \quad {}_jf = T_j^{-1} f(\mathbf{n};\mathbf{p}) = f(\mathbf{n}-\mathbf{e}_j;\mathbf{p}) \; ,$$

where $\mathbf{e}_j$ is a vector with single nonzero entry equal to unity in its $j^{\text{th}}$ component.

### 5.7.1 The BSQ Equation

For the lattice BSQ equation (5.3.13) we must first consider the coupled system of equations (5.3.7f), (5.3.12b) and (5.3.12c) from which the closed form lattice equation is derived. On any two-dimensional sublattice of the multi-dimensional lattice we can embed a copy of equations (5.3.7f) and (5.3.12b) giving us

$$u^{ij} = \frac{c^j - c^i + p_i u^j - p_j u^i}{p_i - p_j + u^j - u^i}, \tag{5.7.1a}$$

$$c^{ij} = \omega d + p_i p_j - (p_i + p_j + u)(p_i + p_j - u^{ij}) + \frac{p_i^3 - p_j^3}{p_i - p_j + u^j - u^i}, \tag{5.7.1b}$$

while on any any one-dimensional sublattice we can embed a copy of equation (5.3.12c), giving

$$\omega d^i = c + p_i(u - u^i) - u u^i. \tag{5.7.1c}$$

(where, for simplicity, we have defined $c \equiv u^{(1,0)}$ and $d \equiv u^{(0,1)}$)

These equations are represented schematically in figure 5.1.

Now, given the initial data points as indicated in figure 5.2 we can, using equations (5.7.1a) and (5.7.1b), uniquely calculate the values of $u^{12}$, $c^{12}$, $u^{13}$, $c^{13}$, $u^{23}$ and $c^{23}$, while equation (5.7.1c) uniquely determines $d^1$, $d^2$ and $d^3$. However, for the points $d^{12}$, $d^{13}$ and $d^{23}$, we have already reached a point of possible inconsistency with, for

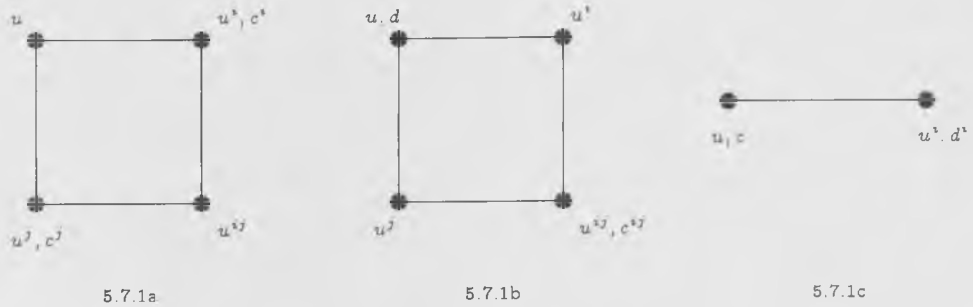



Figure 5.1: *Symbolic representation of equations (5.7.1).*

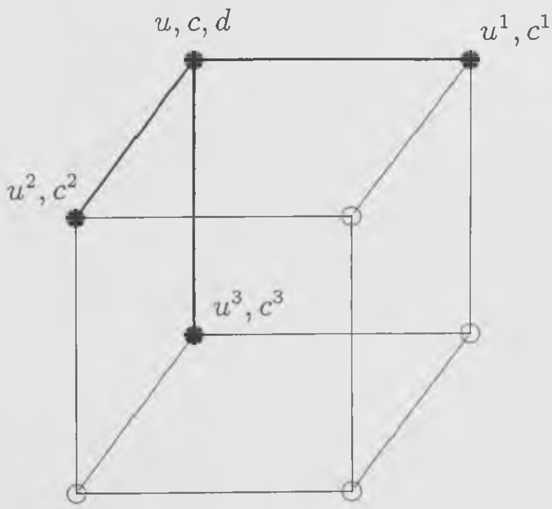



Figure 5.2: *Initial data points for equations (5.7.1).*

example, two ways of calculating $d^{12}$. Shifting equation (5.7.1c) and eliminating the $u^{ij}$ with equation (5.7.1a) we find that

$$\omega d^{ij} = \frac{p_i p_j(u^i - u^j) + p_i(c^i - c^j) + u^i u^j(p_j - p_i) + c^i u^j - c^j u^i}{p_i - p_j + u^j - u^i}. \qquad (5.7.2)$$

This expression is invariant on interchanging $i$ and $j$, thus showing that any potential inconsistency is avoided, hence allowing us to calculate the points $d$ uniquely for all successive points in the multi-dimensional lattice. Continuing to iterate through the cube, we find that there are three different ways to calculate either $u^{123}$ or $c^{123}$. Using equations (5.7.1) to eliminate all intermediary points in favour of only the given initial data points we can show that we still do not obtain any inconsistencies, i.e. that

$$
\begin{aligned}
u^{ijk} &= u^{jki} = u^{kij}, \\
c^{ijk} &= c^{jki} = c^{kij}, \\
d^{ijk} = d^{jki} &= d^{kij} = d^{ikj} = d^{kji} = d^{jik}.
\end{aligned}
$$

(the explicit formula are too large to reproduce here)

Equations (5.7.1) therefore form a compatible parameter-family of coupled partial difference equations and as a result can be described as a *holonomic system of coupled partial difference equations.*

Turning now to the closed form lattice equation for the variable $u$, (5.3.13), on any two-dimensional sublattice we impose a copy of this equation giving

$$
\begin{aligned}
\frac{p_i^3 - p_j^3}{p_i - p_j + u^{ij} - u^{ii}} - \frac{p_i^3 - p_j^3}{p - q + u^{jj} - u^{ij}} &= (p_i - p_j + u^{ijj} - u^{iij})(2p_i + p_j + u^{j} - u^{iijj}) \\
&- (p_i - p_j + u^{j} - u^{i})(2_i + p_j + u - u^{iij}).
\end{aligned}
\tag{5.7.3}
$$

The consistency for this equation is illustrated by figure 5.3. For the three-dimensional sublattice we have nine copies of equation (5.7.3), three in each direction. Imposing a given set of initial data points, marked with a $\bullet$, we can calculate uniquely from one copy of the lattice equation all the points marked with a $\circ$. However, for the point marked with a $\square$ there are three possible ways to calculate it, and it is at this point that the consistency must verified. If we impose the initial data points arbitrarily then we obtain different values for the function at this point and hence equation (5.7.3), in itself does form a compatible parameter family of equations. If, however, we impose as initial data points the variables shown symbolically on figure 5.3 and use equations

(5.7.1) to calculate the remainder of the points marked with a $\bullet$ then we find that there is no inconsistency. Although the calculations involved are enormous and extend even MAPLE's symbolic capabilities, as a result it is only possible to test this consistency numerically. We therefore consider equation (5.7.3) to be a compatible parameter-family of equations modulo equations (5.7.1). It should also be noted that the KP lattice equation (4.3.13) can be consistently imposed on any three-dimensional sub-lattice.

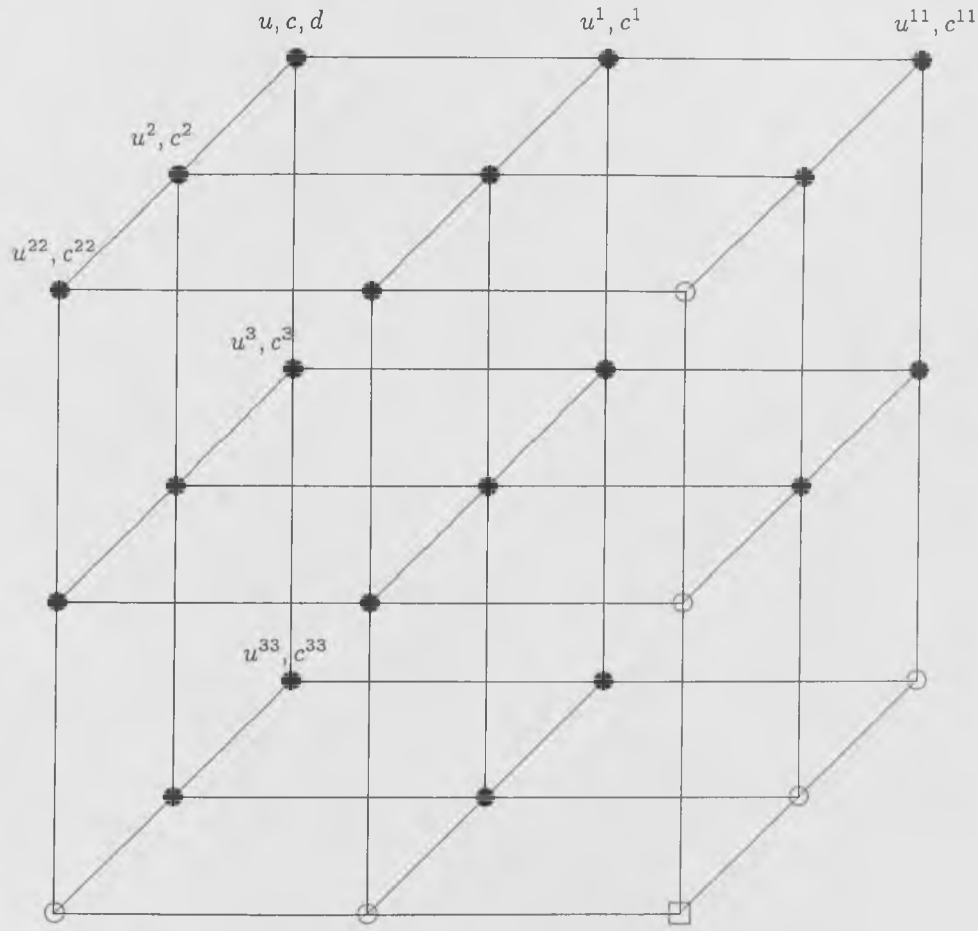



Figure 5.3: *Compatibility of the BSQ lattice equation.*

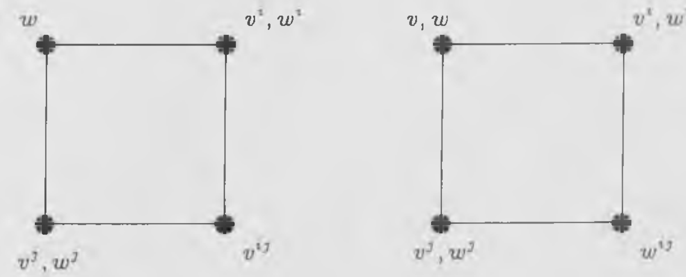
### 5.7.2 The Modified BSQ Equation

For the modified equation we must deal with the coupled system (5.4.13). Embedding on each two-dimensional sublattice a copy of these equations we get

$$v^{ij} = \frac{(p_i v^j - p_j v^i)w}{p_i w^i - p_j w^j}, \tag{5.7.4a}$$

$$w^{ij} = \frac{(p_i v^i w^j - p_j v^j w^i)w}{(p_i w^i - p_j w^j)v}, \tag{5.7.4b}$$

which are schematically represented by figure 5.4




5.7.4a 5.7.4b

Figure 5.4: *Symbolic representation of equations (5.7.4).*

The consistency is now illustrated by figure 5.5 with $v^{12}$, $w^{12}$, $v^{13}$, $w^{13}$, $v^{23}$ and $w^{23}$ calculated uniquely by equations (5.7.4) but three ways of calculating $v^{123}$ and $w^{123}$. Using equations (5.7.4) to eliminate all but the given initial points we find,

$$v^{ijk} = \frac{\left[p_i^2 w^i(p_j v^k - p_k v^j) + p_j^2 w^j(p_k v^i - p_i v^k) + p_k^2 w^k(p_i v^j - p_j v^i)\right] v}{p_i^2 v^i(p_j w^j - p_k w^k) + p_j^2 v^j(p_k w^k - p_i w^i) + p_k^2 v^k(p_i w^i - p_j w^j)}, \tag{5.7.5a}$$

$$w^{ijk} = \frac{\left[p_i^2(p_j v^j w^k - p_k v^k w^j) + p_j^2(p_k v^k w^i - p_i v^i w^k) + p_k^2(p_i v^i w^j - p_j v^j w^i)\right] w}{p_i^2 v^i(p_j w^j - p_k w^k) + p_j^2 v^j(p_k w^k - p_i w^i) + p_k^2 v^k(p_i w^i - p_j w^j)}. \tag{5.7.5b}$$

Both of these expressions are invariant for any permutation of $i, j, k$ thus, any consistency is avoided and equations (5.7.4) form a compatible parameter-family of coupled partial difference equations.

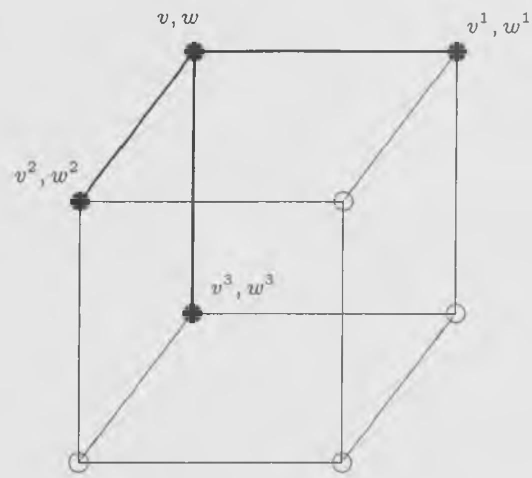



Figure 5.5: *Consistency of equations (5.7.4).*

For the closed-form equation for the variable $v$, on each two-dimensional sublattice we now impose a copy of equation (5.3.18) giving

$$p_i\left(\frac{v}{v^i}-\frac{v^{ijj}}{v^{iijj}}\right)-p_j\left(\frac{v}{v^j}-\frac{v^{iij}}{v^{iijj}}\right)=\frac{p_i^2v^{ij}-p_j^2v^{jj}}{p_iv^{jj}-p_jv^{ij}}\frac{v^{ijj}}{v^j}-\frac{p_i^2v^{ii}-p_j^2v^{ij}}{p_iv^{ij}-p_jv^{ii}}\frac{v^{iij}}{v^i}. \quad (5.7.6)$$

The consistency of this equation is illustrated by figure 5.6 and again we find that if we impose the initial data points, marked $\bullet$, completely arbitrarily then at the point marked with a $\square$ there is an inconsistency. In this case we must give the initial points indicated symbolically in figure 5.6 and use equations (5.7.4) to complete the required set of initial points. Given this set of initial data we can again show numerically that there is no inconsistency. Hence, equation (5.7.6) is a compatible parameter-family of equations modulo equations (5.7.4). Again, it can also be shown that the modified KP equation (4.3.15) can be consistently imposed on any three-dimensional sub-lattice.

### 5.7.3 The Schwarzian BSQ Equation

For the Schwarzian BSQ equation we simply supplement the coupled system for the modified BSQ (5.7.4) with equation (5.3.20). Embedding a copy of this equation on

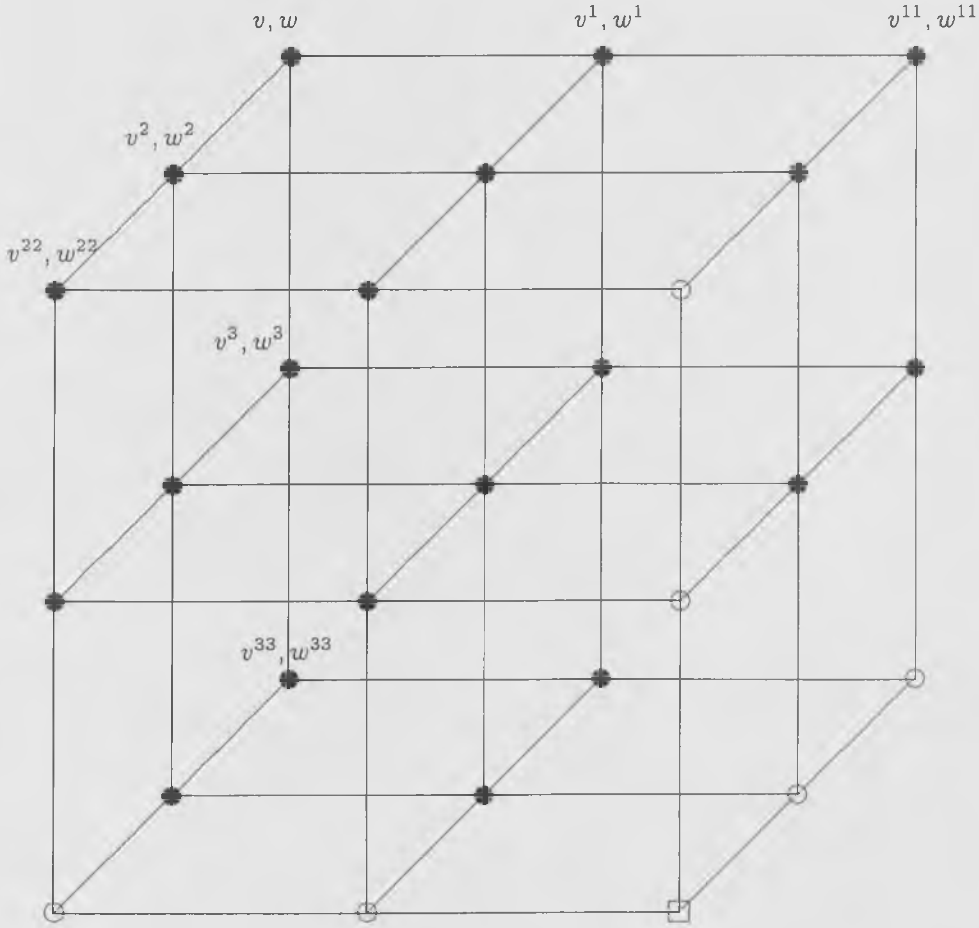



Figure 5.6: *Compatibility of the modified BSQ lattice equation.*

each one-dimensional sublattice we get

$$z^i = \frac{v^i w}{p_i} + z. \tag{5.7.7}$$

The consistency is again illustrated by figure 5.5 to which we must add one extra initial data point, namely $z$. With this point added we can use equation (5.7.7) to uniquely calculate $z^1, z^2$ and $z^3$. We reach a possible inconsistency at the points $z^{12}, z^{13}$ and $z^{23}$. Shifting equation (5.7.7) and eliminating all points except the initial data with equations (5.7.4a) and (5.7.7) we find that,

$$z^{ij} = \frac{p_i^2(p_j w^i z + v^j w^i w) - p_j^2(p_i w^j z + v^i w^j w)}{p_i p_j (p_i w^i - p_j w^j)}. \tag{5.7.8}$$

This expression is invariant on interchanging $i$ and $j$, hence equation (5.7.7) along with equations (5.7.4) also forms a compatible parameter-family of equations.

For the closed-form equation for the variable $z$ we impose a copy of equation (5.3.22) on each two-dimensional sublattice, giving

$$\frac{(z^{iijj}-z^{ijj})(z^{jj}-z^{ij})(z^{j}-z)}{(z^{iijj}-z^{iij})(z^{ii}-z^{i})(z^{i}-z)}=\frac{p^3(z^{ijj}-z^{jj})(z^{i}-z^{j})-q^3(z^{ijj}-z^{ij})(z^{jj}-z^{j})}{q^3(z^{iij}-z^{ii})(z^{ij}-z^{i})-p^3(z^{iij}-z^{ij})(z^{ii}-z^{i})}. \tag{5.7.9}$$

In this case, to avoid any inconsistencies, we use the same initial points, given symbolically in figure 5.6, for the modified BSQ equation. To this we again add the extra initial data point $z$ and then use equations (5.7.7) and (5.7.4) to calculate the values of $z, v$ and $w$ for each point marked with a $\bullet$. With this set of initial data we can then use equation (5.7.9) to calculate the remaining points, finding no inconsistency at the point marked with a $\square$. Hence, equation (5.7.9) is a compatible parameter-family of equations modulo equations (5.7.7) and (5.7.4).

## 5.8 Similarity Constraints for the Boussinesq System

In this section show how to derive the similarity constraints for the Boussinesq system, along with their associated monodromy problems. We shall then go on to demonstrate that, In the case of the modified system, the constraint are compatible with the lattice equation derived in section 5.3.

For the Boussinesq system, the dependence of the plane-wave factor $\rho_k$ on the spectral parameter $k$ is given by

$$k\frac{\partial}{\partial k}\rho_k = \sum_{\nu} n_\nu p_\nu \left(\frac{1}{p_\nu+\omega k}-\frac{1}{p_\nu+k}\right)\rho_k. \tag{5.8.1}$$

This leads us to impose the following linear relationship for the matrix $C$

$$\begin{aligned} C + I \cdot C + C \cdot {}^tI &= [l\rho_l \boldsymbol{c}_l {}^t\boldsymbol{c}_l]_{\partial\Gamma} + \\ & \sum_\nu n_\nu p_\nu \left( \frac{1}{p_\nu + \Lambda} \cdot C - C \cdot \frac{1}{p_\nu + \omega\, {}^t\Lambda} \right), \end{aligned} \tag{5.8.2}$$

which gives the following nonlinear relationship for the matrix $U$

$$\begin{aligned} U + I \cdot U + U \cdot {}^tI &= [l\rho_l^{-1} \boldsymbol{u}_j {}^t\boldsymbol{u}_l]_{\partial\Gamma} + \\ & \sum_\nu n_\nu p_\nu \left( \frac{1}{p_\nu + \Lambda} \cdot U - U \cdot \frac{1}{p_\nu + \omega\, {}^t\Lambda} \right. \\ & \left. + U \cdot \frac{1}{p_\nu + \omega\, {}^t\Lambda} \cdot O \cdot \frac{1}{p_\nu + \Lambda} U \right). \end{aligned} \tag{5.8.3}$$

Limiting ourselves to two dimensions, with independent discrete variables $n$ and $m$ and lattice parameters $p$ and $q$, we obtain closed-form similarity constraints for the objects defined in (5.3.6). The similarity constraint of the lattice BSQ equation reads

$$\begin{aligned} &(\lambda\omega^{-n-m}(\omega^2 - 1) - 1)u = \\ &\quad 3np^3 \left[ \frac{p^3 - q^3}{p - q + \underset{\sim}{\widehat{u}} - u} + (p - q + \widehat{u} - \widetilde{u})(2p + q + \underset{\sim}{u} - \widehat{\widetilde{u}}) \right]^{-1} \\ &\quad + 3mq^3 \left[ \frac{p^3 - q^3}{p - q + u - \underset{\frown}{\widetilde{u}}} - (p - q + \widehat{u} - \widetilde{u})(2p + q + \underset{\sim}{u} - \widehat{\widetilde{u}}) \right]^{-1}. \end{aligned} \tag{5.8.4}$$

The similarity constraint of the lattice mBSQ equation reads

$$\begin{aligned} & np \left[ p + \frac{\widehat{v}_0}{\widetilde{v}_0} \frac{p^2 v_0 - q^2 \underset{\sim}{\widehat{v}}_0}{p \underset{\sim}{\widehat{v}}_0 - q v_0} + \frac{\underset{\sim}{v}_0}{\widetilde{v}_0} \frac{p\widehat{v}_0 - q\widetilde{v}_0}{\widehat{\widetilde{v}}_0} \right]^{-1} + \\ & + mq \left[ q + \frac{\widetilde{v}_0}{\widehat{v}_0} \frac{q^2 v_0 - p^2 \underset{\frown}{\widetilde{v}}_0}{q \underset{\frown}{\widetilde{v}}_0 - p v_0} + \frac{\underset{\frown}{v}_0}{\widehat{v}_0} \frac{q\widetilde{v}_0 - p\widehat{v}_0}{\widehat{\widetilde{v}}_0} \right]^{-1} = \frac{1 - 2\mu}{1 - \omega^2} + \lambda\omega^{-n-m} + n + m. \end{aligned} \tag{5.8.5}$$

Finally, the similarity constraint for the Schwarzian lattice BSQ equation reads

$$2\mu z = 3np \frac{(\widetilde{z} - z)(z - \underset{\sim}{z})}{p(\widetilde{z} - z) + \underset{\sim}{v}_0 \widetilde{w}_0} + 3mq \frac{(\widehat{z} - z)(z - \underset{\frown}{z})}{p(\widehat{z} - z) + \underset{\frown}{v}_0 \widehat{w}_0}, \tag{5.8.6}$$

where $$\underset{\sim}{v}_0\tilde{w}_0 = \frac{(\hat{\tilde{z}} - \tilde{z})}{p^2(\hat{\tilde{z}} - \hat{z})(\hat{\underset{\sim}{z}} - z)}[p^3(\hat{z} - \hat{\underset{\sim}{z}})(z - \underset{\sim}{z}) - q^3(\hat{z} - z)(\hat{\underset{\sim}{z}} - \underset{\sim}{z})],$$

with a similar expression for $\underset{\wedge}{v}_0\hat{w}_0$.

Alternatively, for the variables $v_0$ and $w_0$ we obtain the following coupled set of similarity constraints

$$\frac{1-2\mu}{1-\omega^2} + \lambda\omega^{-n-m} = n\left(\frac{2\tilde{v}_0w_0 - v_0\underset{\sim}{w}_0 - \underset{\sim}{v}_0\tilde{w}_0}{\tilde{v}_0w_0 + v_0\underset{\sim}{w}_0 + \underset{\sim}{v}_0\tilde{w}_0}\right) + m\left(\frac{2\hat{v}_0w_0 - v_0\underset{\wedge}{w}_0 - \underset{\wedge}{v}_0\hat{w}_0}{\hat{v}_0w_0 + v_0\underset{\wedge}{w}_0 + \underset{\wedge}{v}_0\hat{w}_0}\right), \quad (5.8.7)$$

$$\frac{1-2\mu}{1-\omega^2} + \lambda\omega^{-n-m} = -\omega n\left(\frac{2v_0\underset{\wedge}{w}_0 - \tilde{v}_0w_0 - \underset{\sim}{v}_0\tilde{w}_0}{\tilde{v}_0w_0 + v_0\underset{\sim}{w}_0 + \underset{\sim}{v}_0\tilde{w}_0}\right) - \omega m\left(\frac{2v_0\underset{\wedge}{w}_0 - \hat{v}_0w_0 - \underset{\wedge}{v}_0\hat{w}_0}{\hat{v}_0w_0 + v_0\underset{\wedge}{w}_0 + \underset{\wedge}{v}_0\hat{w}_0}\right). \quad (5.8.8)$$

For this coupled system we now investigate the compatibility of these constraints with the lattice equation (5.3.18). This will follow closely the scheme described in section (3.4.1) for the 2-dimensional KdV system, however, in this case the situation is more complicated as we are now dealing with a coupled system for two dependent variables $v_0$ and $w_0$.

So, for this case we have two lattice equations (5.4.13) and two similarity constraints (5.8.7). Schematically these are represented by figure 5.7. The compatibility of these equations is also illustrated by figure 3.3, however, for each initial data point (marked with a $\bullet$) we must now assign values of both $v_0$ and $w_0$. Again we proceed by calculating each successive point using either the lattice equation (5.3.18), points marked with a $\circ$, or the similarity constraints (5.8.7), points marked with a $\times$. For the points calculated using the similarity constraint we must solve the system of equations (5.8.7) simultaneously. At a certain point in the iteration we reach a point, marked with a $\otimes$, which can be calculated using either the lattice equations or the similarity constraints and it is at this point that the compatibility must be confirmed.

Again the calculations involved are very large and cannot be reproduced here, but have been verified by MAPLE.

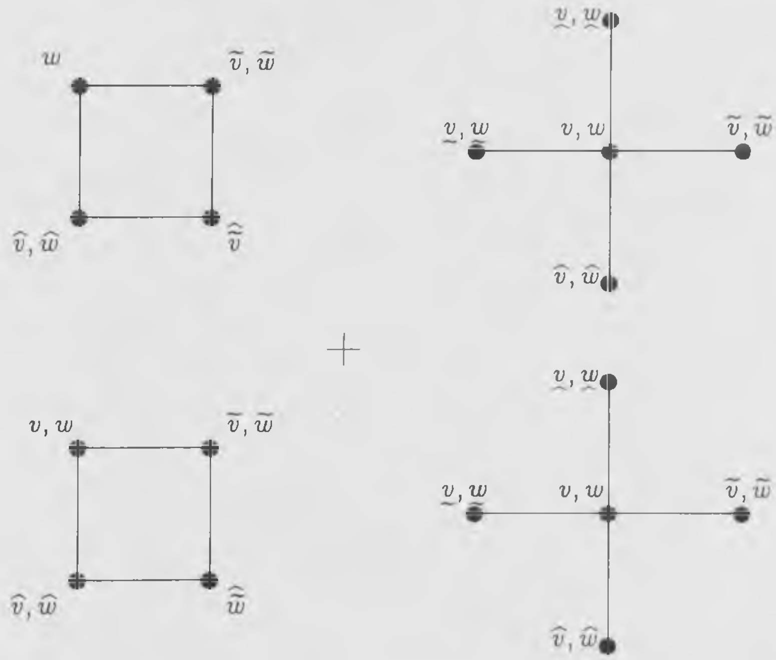


Figure 5.7: *Symbolic representation of lattice equation and similarity constraint.*

Thus, we have a coupled system of lattice equations and a compatible system of similarity constraints for which we can give a well-posed IVP. In theory we should therefore be able to use the similarity constraint to reduce the lattice equation to a O$\Delta$E of Painlevé type. However, to date we have been unable to determine the relevant variables which would allow us to do this explicitly.

## 5.9 Semicontinuous Limits

In this section we show why we consider the lattice equations and similarity constraints, derived in sections (5.3.1) and (5.8), to be of BSQ type. By investigating what happens under a continuum limit compatible with the integrability structure, we shall show that we recover the continuum situation. As it was established in [41] that the similarity reduction for the continuum BSQ equation leads to Painlevé IV, the discrete similarity reduction must necessarily lead to a discrete analogue of $P_{IV}$. Thus, even if we can only obtain on the discrete level the reduction in the form of a system containing the lattice equation and a similarity constraint, this system reduces in a continuum limit to the Painlevé equation. Since we have two discrete variables in the lattice equation, namely $n$ and $m$, we have to perform the continuum limit in two steps: one letting the variable $m$ become continuous, reducing our equation to a differential-difference equation, i.e. an equation with one discrete and one continuous variable, and a second step in which the remaining discrete variable will become continuous. Both steps are achieved by shrinking the corresponding lattice step (encoded in the parameters $p$ and $q$) to zero. In this section, we shall perform the first continuum limit in two different ways, one which we call the "skew" limit because it involves a change of variables on the lattice, and one which we will call the "straight" limit which can be obtained directly. The results in both cases are different, but from both the full continuum limit can be obtained and leads to the same result. It should also be noted that the differential-difference equations obtained either by either continuum limit are not the same as those obtained in section (5.5).

### 5.9.1 Skew Continuum Limit

The most convenient way of doing this is first, on the lattice, to do a change of discrete variables, namely $u_{n,m} = u_{n'}(m)$, and then doing the limit by taking

$$\delta \equiv p - q \mapsto 0 \quad , \quad m \mapsto \infty \quad , \quad \delta m \mapsto \tau \ , \tag{5.9.1}$$

where $n' = n + m$ is to remain fixed. This limit is motivated from the behaviour of discrete plane-wave factors

$$\rho_k(n,m) = \left(\frac{p+k}{p+\omega k}\right)^n \left(\frac{q+k}{q+\omega k}\right)^m , \tag{5.9.2}$$

which under (5.9.1) behave as

$$\left(\frac{p+k}{p+\omega k}\right)^n \left(\frac{q+k}{q+\omega k}\right)^m \mapsto \left(\frac{p+k}{p+\omega k}\right)^{n'} \exp\left[\frac{(1-\omega)k\tau}{(p+k)(p+\omega k)}\right] , \tag{5.9.3}$$

cf. [65, 66]. By this limit, the lattice BSQ (5.3.13) goes over into the following differential-difference equation (we omit the prime of the $n'$ variable)

$$\begin{aligned} 3p^2\partial_\tau \log(1+\dot{u}_n) &= (3p + u_{n-1} - u_{n+2})(1+\dot{u}_n)(1+\dot{u}_{n+1}) \\ &\quad - (3p + u_{n-2} - u_{n+1})(1+\dot{u}_{n-1})(1+\dot{u}_n) , \end{aligned} \tag{5.9.4}$$

($\dot{u}_n = \partial_\tau u_n$). In fact, as $n' = n + m$ remains fixed in the limit, we need to do first a change of variables on the lattice, namely $u(n,m) \equiv u_{n+m}(m)$ , and then use a systematic expansion of the form

$$u_n(m) \mapsto u_n(\tau) + \delta \dot{u}_n(\tau) + \frac{1}{2}\ddot{u}_n(\tau) + \cdots ,$$

in which one retains the dominant term in the small parameter $\delta$ to obtain (5.9.4). Performing the same limit on the similarity constraint (5.8.4) we obtain

$$\begin{aligned} 0 &= \frac{u_n + \tau - np}{1+\dot{u}_n} + (np - 3\tau) \\ &+ \frac{p\tau}{3p^2}\left[(3p + u_{n-1} - u_{n+2})(1+\dot{u}_n)(1+\dot{u}_{n+1}) + (3p + u_{n-2} - u_{n+1})(1+\dot{u}_{n-1})(1+\dot{u}_n)\right] , \end{aligned} \tag{5.9.5}$$

which imposes scaling-invariance on the solutions of (5.9.4).

Next, we can apply the same continuum limit to the lattice mBSQ equation (5.3.18), leading to

$$\partial_\tau\left(\frac{v_{n+1}}{v_{n-1}}\frac{(p^2+p\alpha+\alpha^2)\dot{v}_n-(2p+\alpha)v_n}{(p-\alpha)\dot{v}_n+v_n}\right)=$$
$$=\frac{v_{n+1}}{v_{n+2}}\left((p-\alpha)\frac{\dot{v}_{n+1}}{v_{n+1}}+1\right)-\frac{v_{n-2}}{v_{n-1}}\left((p-\alpha)\frac{\dot{v}_{n-1}}{v_{n-1}}+1\right). \tag{5.9.6}$$

Taking $\alpha=0$ and introducing the variable $Q_n=1+p\partial_\tau(\log v_n)$ , this equation can be cast into the form

$$\begin{aligned}3p\partial_\tau\log Q_n &= \left(1-\frac{v_{n-2}}{v_{n+1}}\right)Q_{n-1}Q_n-3(Q_n+Q_{n-1})\\ &\quad -\left(1-\frac{v_{n-1}}{v_{n+2}}\right)Q_nQ_{n+1}+3(Q_{n+1}+Q_n).\end{aligned} \tag{5.9.7}$$

The similarity constraint in this limit takes the form

$$\begin{aligned}n(Q_n-1) &= \frac{\tau}{p}Q_n\left[3-(Q_{n+1}+Q_n+Q_{n-1})+\right.\\ &\quad \left.+\frac{1}{3}\left(1-\frac{v_{n-1}}{v_{n+2}}\right)Q_nQ_{n+1}+\frac{1}{3}\left(1-\frac{v_{n-2}}{v_{n+1}}\right)Q_{n-1}Q_n\right].\end{aligned} \tag{5.9.8}$$

Finally, the continuum limit of the lattice Schwarzian BSQ (5.3.22) is given by

$$\partial_\tau\log\left(3\frac{(z_{n+1}-z_n)(z_n-z_{n-1})}{\dot{z}_n}-p(z_{n+1}-z_{n-1})\right)=\frac{\dot{z}_{n-1}}{z_{n-1}-z_{n-2}}-\frac{\dot{z}_{n+1}}{z_{n+2}-z_{n+1}}, \tag{5.9.9}$$

whilst the similarity constraint for the variable $z$ reduces to

$$\begin{aligned}\frac{z_n}{\dot{z}_n} &= np-3\tau+\\ &+p\tau\left[\frac{\dot{z}_{n+1}(z_{n+2}-z_n)}{(z_{n+2}-z_{n+1})(z_{n+1}-z_n)}+\frac{\dot{z}_n(z_{n+1}-z_{n-1})}{(z_{n+1}-z_n)(z_n-z_{n-1})}+\frac{\dot{z}_{n-1}(z_n-z_{n-2})}{(z_n-z_{n-1})(z_{n-1}-z_{n-2})}\right.\\ &\left.+\frac{p}{3}\frac{\dot{z}_{n+1}\dot{z}_n(z_{n-1}-z_{n+2})}{(z_{n+2}-z_{n+1})(z_{n+1}-z_n)(z_n-z_{n-1})}+\frac{p}{3}\frac{\dot{z}_n\dot{z}_{n-1}(z_{n-2}-z_{n+1})}{(z_{n+1}-z_n)(z_n-z_{n-1})(z_{n-1}-z_{n-2})}\right].\end{aligned} \tag{5.9.10}$$

### 5.9.2 Straight Continuum Limit

The limit (5.9.1) is not the only way to obtain semi-continuous versions of the lattice BSQ equations. We could equally well apply a "straight" limit, i.e. one that doesn't involve a change of variables on the initial lattice. In that case we can consider simply

$$q \mapsto \infty \quad , \quad m \mapsto \infty \quad , \quad \frac{m}{q} \mapsto x \; , \tag{5.9.11}$$

corresponding to the following limiting behaviour of the discrete plane-wave factors

$$\left(\frac{p+k}{p+\omega k}\right)^n \left(\frac{q+k}{q+\omega k}\right)^m \mapsto \left(\frac{p+k}{p+\omega k}\right)^n e^{(1-\omega)kx} \; . \tag{5.9.12}$$

By this limit, using the expansion

$$u_{n,m} \mapsto u_n(x) + \frac{1}{q} u_n'(x) + \frac{1}{2q^2} u_n''(x) + \dots \; ,$$

in which $u_n' \equiv \partial_x u_n$, the lattice BSQ equations (5.3.13) go over into another set of differential-difference equations, namely

$$\begin{aligned} \partial_x^2(u_{n+1}+u_n+u_{n-1}) &= 3u_{n+1}'(p+u_n-u_{n+1}) - 3u_{n-1}'(p+u_{n-1}-u_n) \\ &\quad + (p+u_n-u_{n+1})^3 - (p+u_{n-1}-u_n)^3 \; , \end{aligned} \tag{5.9.13}$$

for the BSQ equation, and

$$\begin{aligned} &\partial_x^2 \log(v_{n+1}v_nv_{n-1}) = \\ &= \left(\frac{v'^{\,2}_{n+1}}{v_{n+1}^2} - \frac{v'^{\,2}_{n-1}}{v_{n-1}^2} + \frac{v_{n+1}'}{v_{n+1}}\frac{v_n'}{v_n} - \frac{v_n'}{v_n}\frac{v_{n-1}'}{v_{n-1}}\right) + 3p\left(\frac{v_{n+1}'v_n}{v_{n+1}^2} - \frac{v_{n-1}'}{v_n}\right) \\ &\quad + p^2\left(\frac{v_n}{v_{n-1}} - \frac{v_{n-1}^2}{v_n^2} + \frac{v_n^2}{v_{n+1}^2} - \frac{v_{n+1}}{v_n}\right) . \end{aligned} \tag{5.9.14}$$

for the mBSQ equation (5.3.18). Finally the continuum limit of (5.3.22) is given by

$$\partial_x \log\left[p^3(z_{n+1}-z_{n-1}) + \frac{z_{n+1}'z_n'z_{n-1}'}{(z_{n+1}-z_n)(z_n-z_{n-1})}\right] = \frac{z_{n+1}'}{z_{n+1}-z_n} - \frac{z_{n-1}'}{z_n-z_{n-1}} . \tag{5.9.15}$$

The continuum limit of the similarity constraints are the following

$$u_n + xu'_n = np\left[1 - 3p^2(2u_n - u_{n+1} - u_{n-1})\times\right.$$
$$\left.\times\left[(2u_n - u_{n+1} - u_{n-1})(u'_{n+1} + u'_n + u'_{n-1}) + (p + u_n - u_{n+1})^3 - (p + u_{n-1} - u_n)^3\right]^{-1}\right], \quad (5.9.16)$$

for the differential-difference BSQ (5.9.13), and

$$\begin{aligned} 0 &= x\frac{v'_n}{v_n} + \\ &+n\left[-1 + 3p\frac{v_{n+1}}{v_{n-1}}\left[\frac{v'_{n+1}}{v_{n+1}} + \frac{v'_n}{v_n} + \frac{v'_{n-1}}{v_{n-1}} + p\left(\frac{v_{n+1}}{v_{n-1}} + \frac{v_{n-1}}{v_n} + \frac{v_n}{v_{n+1}}\right)\right]^{-1}\right], \end{aligned} \quad (5.9.17)$$

for the differential-difference mBSQ equation (5.9.14), and

$$z_n = xz'_n + 3np^3\frac{(z_{n+1} - z_n)^2(z_n - z_{n-1})^2}{z'_{n+1}z'_n z'_{n-1} + p^3(z_n - z_{n+1})(z_{n+1} - z_{n-1})(z_{n-1} - z_n)}, \quad (5.9.18)$$

for the differential-difference Schwarzian BSQ equation (5.9.15). If we compare the skew and straight continuum BSQ systems, one immediately observes that, whereas the former are generally of higher-order in the discrete variable, the latter ones involve derivatives with respect to the shifted variables.

### 5.9.3 Full Continuum Limit

We shall concentrate on the semi-continuous BSQ equation obtained by taking the continuum limits under the "skew limit". To recover the fully continuous BSQ, together with its similarity reduction, a second continuum limit is performed by taking

$$p \to \infty\ , \quad n \to \infty\ , \quad \tau \to \infty\ , \quad (5.9.19a)$$

such that

$$\frac{n}{p} + \frac{\tau}{p^2} \mapsto x\ , \quad \frac{n}{2p^2} + \frac{\tau}{p^3} \mapsto t\ . \quad (5.9.19b)$$

in which case the plane-wave factor $\rho_k$ takes the from

$$\rho_k \quad \mapsto \quad \exp\left[(1-\omega)kx - (1-\omega^2)k^2 t\right] \ , \tag{5.9.20}$$

and (5.9.4) goes over into the potential BSQ equation

$$u_{tt} + \frac{1}{3}u_{xxxx} + 4u_x u_{xx} = 0 \ , \tag{5.9.21}$$

which is the integrated version of the BSQ equation.

The full continuum limit of (5.9.6) gives

$$v_{tt} + \frac{1}{3}v_{xxxx} + 2v_t v_{xx} - 2v_x^2 v_{xx} = 0 \ . \tag{5.9.22}$$

Which is is the the potential modified BSQ equation.

Finally, for the Schwarzian BSQ equation the full continuum limit of the equation for $z$, gives

$$3\left(\frac{z_t}{z_x}\right)_t + \left(\frac{z_{xxx}}{z_x} + \frac{3}{2}\frac{z_t^2 - z_{xx}^2}{z_x^2}\right)_x = 0 \ , \tag{5.9.23}$$

which is the Schwarzian BSQ equation.

Thus we see that continuum limits of the lattice equations of section 5.3.1 yield continuous BSQ equations and can rightly be referred to as lattice BSQ equations.

# Chapter 6

# The AKNS System

## 6.1 Introduction

In this section, following on from [80], we shall extend the direct linearization scheme to a system of two coupled integral equations in terms of two dependent variables $\varphi_k$ and $\psi_k$. Thus we now have the following set of linear integral equations

$$\varphi_k + \int_{C_1} d\lambda_1(l') \frac{\rho_k \psi_{l'}}{k - l'} = \rho_k \boldsymbol{c}_k, \tag{6.1.1a}$$

$$\psi_{k'} - \int_{C_2} d\lambda_2(l) \frac{\sigma_{k'} \varphi_l}{k' - l} = 0. \tag{6.1.1b}$$

In the above system, $\boldsymbol{c}_k$ is the infinite vector defined in (2.1.1) for the KdV system. However we now have two arbitrary contours $C_1$ and $C_2$, two arbitrary measures $d\lambda_1$ and $d\lambda_2$ and two plane-wave factors $\rho_k$ and $\sigma_k$. As with the previous systems, it is the plane-wave factors which shall determine the number of dimensions of the system and whether they are discrete or continuous.

If we integrate equation (6.1.1a) over the second contour $C_2$ with respect to $d\lambda_2$ and integrate equation (6.1.1b) over the first contour $C_1$ with respect to $d\lambda_1$ we find that $\varphi_k$ and $\psi_{k'}$ both satisfy integral equations of the same type but with different inhomogeneous terms, i.e.

$$\varphi_k + \int_{C_1} d\lambda_1(l') \int_{C_2} d\lambda_2(l) \frac{\rho_k \sigma_{l'} \varphi_{l'}}{(k-l')(l'-l)} = \rho_k c_k, \tag{6.1.2a}$$

$$\psi_{k'} + \int_{C_1} d\lambda_1(l') \int_{C_2} d\lambda_2(l) \frac{\rho_l \sigma_{k'} \psi_{l'}}{(k'-l)(l-l')} = \int_{C_2} d\lambda_2(l) \frac{\sigma_{k'} \rho_l}{(k'-l)} c_l. \tag{6.1.2b}$$

To complete the system we must also define the following adjoint coupled system in terms of two further independent variables $\chi_k$ and $\theta_k$

$$\chi_{k'} + \int_{C_2} d\lambda_2(l) \frac{\sigma_{k'} \theta_l}{k'-l} = \sigma_{k'} c_{k'}, \tag{6.1.3a}$$

$$\theta_k - \int_{C_1} d\lambda_1(l') \frac{\rho_k \chi_{l'}}{k-l'} = 0. \tag{6.1.3b}$$

Again, integrating the first of these (6.1.3a) over the first contour $C_1$ with respect to $d\lambda_1$ and the second over the second contour $C_2$ with respect to $d\lambda_2$ we find that $\chi_{k'}$ and $\theta_k$ also both satisfy integral equations of the same type as in (6.1.2) but with different inhomogeneous terms, i.e.

$$\chi_{k'} + \int_{C_2} d\lambda_2(l) \int_{C_1} d\lambda_1(l') \frac{\rho_l \sigma_{k'} \chi_{l'}}{(k'-l)(l-l')} = \sigma_{k'} c_{k'}, \tag{6.1.4a}$$

$$\theta_k + \int_{C_1} d\lambda_1(l') \int_{C_2} d\lambda_2(l) \frac{\rho_k \sigma_{l'} \theta_l}{(k-l')(l'-l)} = \int_{C_1} d\lambda_1(l') \frac{\sigma_{l'} \rho_k}{(k-l')} c_{l'}. \tag{6.1.4b}$$

Thus we have a coupled system of integral equations, (6.1.1) and (6.1.3), in terms of four dependent variables $\varphi_k$, $\psi_k$, $\chi_k$ and $\theta_k$ each of which satisfy second order linear integral equations of the same type. As before, in order to simplify the equations and investigate the underlying algebraic structure, we must develop an infinite matrix structure for this system.

## 6.2 Infinite Matrix Formalization

Firstly we require two infinite $(\mathbb{Z} \times \mathbb{Z})$ matrices defined by

$$\varphi^o = \int_{C_2} d\lambda_2(l) \rho_l \boldsymbol{c}_l \, {}^t\boldsymbol{c}_l \quad , \quad \psi^o = \int_{C_1} d\lambda_1(l') \sigma_{l'} \boldsymbol{c}_{l'} \, {}^t\boldsymbol{c}_{l'}, \tag{6.2.1}$$

which we incorporate into the following $(2 \times 2)$ matrix

$$\boldsymbol{C} = \begin{pmatrix} 0 & \varphi^o \\ -\psi^o & 0 \end{pmatrix}. \tag{6.2.2}$$

This $(2 \times 2)$ matrix will be used to derive the linear relationships for the system.

For the nonlinear relations we must introduce the following four infinite matrices

$$\varphi = \int_{C_2} d\lambda_2(l) \boldsymbol{\varphi}_l \, {}^t\boldsymbol{c}_l \quad , \quad \psi = \int_{C_1} d\lambda_1(l') \boldsymbol{\psi}_{l'} \, {}^t\boldsymbol{c}_{l'}, \tag{6.2.3a}$$

$$\chi = \int_{C_1} d\lambda_1(l') \boldsymbol{\chi}_{l'} \, {}^t\boldsymbol{c}_{l'} \quad , \quad \theta = \int_{C_2} d\lambda_2(l) \boldsymbol{\psi}_l \, {}^t\boldsymbol{c}_l, \tag{6.2.3b}$$

which we again incorporate into a $(2 \times 2)$ matrix

$$\boldsymbol{H} = \begin{pmatrix} \psi & \varphi \\ -\chi & \theta \end{pmatrix}. \tag{6.2.4}$$

Apart from these main objects, we again require several other infinite matrices

i) Index raising matrices $\Lambda$ and ${}^t\Lambda$.

ii) Projection matrix $O$.

iii) Index counting matrices $I$ and ${}^tI$.

iv) Matrix $\Omega$.

All of which are as defined in (2.2.3) for the KdV system. However, we also require similar objects which we can consistently use along with the matrices $\boldsymbol{C}$ and $\boldsymbol{H}$. For this purpose we define the following $(2 \times 2)$ matrices,

$$\boldsymbol{\Lambda} = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \quad , \quad {}^t\boldsymbol{\Lambda} = \begin{pmatrix} {}^t\Lambda & 0 \\ 0 & {}^t\Lambda \end{pmatrix} \quad , \quad \boldsymbol{O} = \begin{pmatrix} O & 0 \\ 0 & O \end{pmatrix}, \tag{6.2.5a}$$

$$\boldsymbol{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad , \quad {}^t\boldsymbol{I} = \begin{pmatrix} {}^tI & 0 \\ 0 & {}^tI \end{pmatrix} \quad , \quad \boldsymbol{\Omega} = \begin{pmatrix} \Omega & 0 \\ 0 & \Omega \end{pmatrix}. \tag{6.2.5b}$$

As each of these is a multiple of the identity matrix, equations (2.2.3d)-(2.2.3e) still hold for these $2 \times 2$ matrices, i.e.

$$\boldsymbol{\Omega} \cdot \boldsymbol{\Lambda}^j - (-{}^t\boldsymbol{\Lambda})^j \cdot \boldsymbol{\Omega} = \boldsymbol{O}_j, \tag{6.2.6a}$$

$$\text{where} \qquad \boldsymbol{O}_k = \sum_{j=0}^{k-1} (-{}^t\boldsymbol{\Lambda})^j \cdot \boldsymbol{O} \cdot \boldsymbol{\Lambda}^{k-1-j},$$

$${}^t\boldsymbol{I} \cdot \boldsymbol{\Omega} + \boldsymbol{\Omega} \cdot \boldsymbol{I} + \boldsymbol{\Omega} = 0. \tag{6.2.6b}$$

We now have a set of objects which will allow us to describe our system in terms of either infinite matrices or in terms of $(2 \times 2)$ matrices, each entry of which is an infinite matrix (these $(2 \times 2)$ matrices shall be written in boldface). For example, the system of integral equations (6.1.1) and (6.1.3) can be written in terms of infinite matrices as follow

$$\varphi + \psi \cdot \Omega \cdot \varphi^o = \varphi^o, \tag{6.2.7a}$$

$$\psi - \varphi \cdot \Omega \cdot \psi^o = 0, \tag{6.2.7b}$$

$$\chi + \theta \cdot \Omega \cdot \psi^o = \psi^o, \tag{6.2.7c}$$

$$\theta - \chi \cdot \Omega \cdot \varphi = 0. \tag{6.2.7d}$$

While, in terms of $(2 \times 2)$ matrices, this becomes

$$\boldsymbol{H} = \boldsymbol{C} \cdot (1 + \boldsymbol{\Omega} \cdot \boldsymbol{C})^{-1}. \tag{6.2.8}$$

The form of the above equation is the same as (2.2.4) in the KdV case, however, now it describes a system of coupled equations.

### 6.2.1 Symmetries

The infinite matrices in (6.2.3) can be shown to have certain symmetries which shall lead us to an important symmetry for the object $\boldsymbol{H}$

i) Symmetry of $\varphi$ and $\chi$

From the definition of $\varphi$, if we eliminate the ${}^t c_l$ using the transpose of equation (6.1.1a) we obtain

$$\varphi = \int_{C_2} d\lambda_2(l)\boldsymbol{\varphi}_l\frac{1}{\rho_l}\left[{}^t\boldsymbol{\varphi}_l + \int_{C_1} d\lambda_1(l')\frac{\rho_l\,{}^t\boldsymbol{\psi}_{l'}}{l-l'}\right].$$

Expanding this and using equation (6.1.1b) we find that

$$\varphi = \int_{C_2} d\lambda_2(l)\frac{\boldsymbol{\varphi}_l\,{}^t\boldsymbol{\varphi}_l}{\rho_l} - \int_{C_1} d\lambda_1(l')\frac{\boldsymbol{\psi}_{l'}\,{}^t\boldsymbol{\psi}_l}{\rho_l}.$$

This expression is clearly invariant under transposition, and hence

$$\varphi = {}^t\varphi. \tag{6.2.9}$$

Starting with the definition of $\chi$, a similar argument using equations (6.1.3a) and (6.1.3b), yields

$$\chi = {}^t\chi. \tag{6.2.10}$$

ii) Symmetry of $\psi$ and $\theta$

Starting now with the definitions of $\psi$ and $\theta$ a similar calculation gives us the following two results

$$\begin{aligned}\varphi &= \int_{C_1} d\lambda_1(l')\int_{C_2} d\lambda_2(l)\frac{\boldsymbol{\varphi}_l\,{}^t\boldsymbol{\chi}_{l'} + \boldsymbol{\psi}_{l'}\,{}^t\boldsymbol{\theta}_l}{l'-l},\\ \theta &= \int_{C_1} d\lambda_1(l')\int_{C_2} d\lambda_2(l)\frac{\boldsymbol{\chi}_{l'}\,{}^t\boldsymbol{\varphi}_l + \boldsymbol{\theta}_l\,{}^t\boldsymbol{\psi}_{l'}}{l-l'}.\end{aligned}$$

From these results we can clearly see that

$$\theta = -{}^t\psi. \tag{6.2.11}$$

Applying (6.2.9)-(6.2.11) to the definition of $\boldsymbol{H}$ we find

$${}^t\boldsymbol{H} = \begin{pmatrix} {}^t\psi & {}^t\varphi \\ -{}^t\chi & {}^t\theta \end{pmatrix} = -\begin{pmatrix} \theta & -\varphi \\ \chi & \psi \end{pmatrix},$$

which gives us the following symmetry for $\boldsymbol{H}$

$${}^t\boldsymbol{H} = -cof(\boldsymbol{H}), \tag{6.2.12}$$

where we use $cof$ to indicate the matrix of cofactors of a $2 \times 2$ matrix

### 6.2.2 Useful Identities

Throughout the rest of this section we shall be dealing mainly with $2 \times 2$ matrices, each entry of which is an infinite ($\mathbb{Z} \times \mathbb{Z}$) matrix, and it will be useful at this point to clarify the notation and give some of the identities these matrices obey that shall be used in the calculations to come. We have already defined most of the $2 \times 2$ matrices we shall require in (6.2.2), (6.2.4) and (6.2.5), but along with these we shall also require the following $2 \times 2$ matrices each entry of which is a scalar

$$\boldsymbol{\eta} = \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{P} = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix},$$

although these matrices have scalar entries we write them in boldface as they shall be used in calculations along with the $2 \times 2$ matrices with infinite matrix entries.

Given that we have matrices within matrices, these are two possible ways of transposing these objects. We can either transpose each entry of the $2 \times 2$ matrix or we can transpose the the whole matrix, we use a left superscript $t$ for the former and a right superscript $T$ for the later, i.e. for

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \qquad a, b, c, d \text{ infinite } \mathbb{Z} \times \mathbb{Z} \text{ matrices,}$$

$${}^{t}\boldsymbol{A} = \begin{pmatrix} {}^{t}a & {}^{t}b \\ {}^{t}c & {}^{t}d \end{pmatrix} \quad , \quad \boldsymbol{A}^{T} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Using this notation we can give the following identities

$${}^{t}(\boldsymbol{A} \cdot \boldsymbol{B})^{T} = ({}^{t}\boldsymbol{B}^{T} \cdot {}^{t}\boldsymbol{A}^{T}), \tag{6.2.13a}$$

$$(cof(\boldsymbol{A}))^{T} = \sigma_2 \cdot \boldsymbol{A} \cdot \sigma_2, \tag{6.2.13b}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are $2 \times 2$ matrices, each entry of which is a $\mathbf{Z} \times \mathbf{Z}$ matrix and $\sigma_2$ is the Pauli matrix

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

For diagonal matrices we use a "bar" to indicate that the entries of the diagonal have been swapped, this should not be confused with the previous chapters in which a "bar" was used as a shift in one of the lattice directions. i.e.

$$\boldsymbol{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad , \quad \overline{\boldsymbol{A}} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}.$$

Thus, for diagonal matrices we have the relation

$$\sigma_2 \cdot \boldsymbol{A} = \overline{\boldsymbol{A}} \cdot \sigma_2. \tag{6.2.14}$$

With these definitions and identities in place we can proceed to derive the lattice equations for the AKNS section.

## 6.3 Algebraic Relations

In order to derive a purely algebraic relation for the system we return to the original integral equations. Multiplying through equation (6.1.2a) by $k^p$ with $p \in \mathbb{N}$ we obtain the following

$$\begin{aligned} k^p\boldsymbol{\varphi}_k + \int_{C_1} d\lambda_1(l')\int_{C_2} d\lambda_2(l)\frac{\rho_k\sigma_{l'}}{(k-l')(l'-l)}l^p\boldsymbol{\varphi}_l &= k^p\rho_k\boldsymbol{c}_k \\ -\int_{C_1} d\lambda_1(l')\int_{C_2} d\lambda_2(l)\frac{\rho_k\sigma_{l'}(k^p-l'^p)}{(k-l')(l'-l)}\boldsymbol{\varphi}_l &- \int_{C_1} d\lambda_1(l')\int_{C_2} d\lambda_2(l)\frac{\rho_k\sigma_{l'}(l'^p-l^p)}{(k-l')(l'-l)}\boldsymbol{\varphi}_l. \end{aligned} \tag{6.3.1}$$

Using the following identity

$$(k^p - l^p) = (k-l)\sum_{j=0}^{p-1} k^{p-1-j}l^j,$$

and the integral equations (6.1.1b), equation (6.3.1) can be expressed as

$$\begin{aligned} k^p\boldsymbol{\varphi}_k + \int_{C_1} d\lambda_1(l')\int_{C_2} d\lambda_2(l)\frac{\rho_k\sigma_{l'}l^p\boldsymbol{\varphi}_l}{(k-l')(l'-l)} &= k^p\rho_k\boldsymbol{c}_k - \psi\cdot O_p\cdot\boldsymbol{c}_k\rho_k \\ &\quad -\varphi\cdot O_p\int_{C_1} d\lambda_1(l')\frac{\rho_k\sigma_{l'}}{(k-l')}\boldsymbol{c}_{l'}. \end{aligned}$$

Thus, using the integral equations (6.1.2a) and (6.1.4b) this gives us

$$k^p\cdot\boldsymbol{\varphi}_k = \Lambda^p\cdot\boldsymbol{\varphi}_k - \psi\cdot O_p\cdot\boldsymbol{\varphi}_k - \varphi\cdot O_p\cdot\boldsymbol{\theta}_k. \tag{6.3.2a}$$

Similar arguments yield

$$k'^p\cdot\boldsymbol{\psi}_{k'} = \Lambda^p\cdot\boldsymbol{\psi}_{k'} - \psi\cdot O_p\cdot\boldsymbol{\psi}_{k'} + \varphi\cdot O_p\cdot\boldsymbol{\chi}_{k'}, \tag{6.3.2b}$$

$$k'^p\cdot\boldsymbol{\chi}_{k'} = \Lambda\cdot\boldsymbol{\chi}_{k'} - \theta\cdot O_p\cdot\boldsymbol{\chi}_{k'} - \chi\cdot O_p\cdot\boldsymbol{\psi}_k, \tag{6.3.2c}$$

$$k^p\cdot\boldsymbol{\theta}_k = \Lambda\cdot\boldsymbol{\theta}_k - \theta\cdot O_p\cdot\boldsymbol{\theta}_k + \chi\cdot O_p\cdot\boldsymbol{\varphi}_k. \tag{6.3.2d}$$

Integrating equations (6.3.2a) and (6.3.2d) over $C_2$ with respect to $d\lambda_2$ and integrating equations (6.3.2b) and (6.3.2c) over $C_1$ with respect to $d\lambda_1$, equations (6.3.2) can be expressed in term of the $(2 \times 2)$ matrix $\boldsymbol{H}$, giving us

$$\boldsymbol{H} \cdot {}^t\boldsymbol{\Lambda}^p \;=\; (\boldsymbol{\Lambda}^p - \boldsymbol{H} \cdot \boldsymbol{O}_p) \cdot \boldsymbol{H}. \tag{6.3.3}$$

This is our purely algebraic relation for $\boldsymbol{H}$ in which $p$ is an arbitrary parameter which can take on any integer value.

## 6.4 Discrete Lattice Equations

For this section we shall concentrate on discrete equations and immediately limit ourselves to the two-dimensional case, Hence we define the plane-wave factors $\rho_k$ and $\sigma_{k'}$ by

$$\rho_k = \left(\eta\frac{p_1 - k}{p_2 - k}\right)^n \left(\eta'\frac{q_1 - k}{q_2 - k}\right)^m \quad , \quad \sigma_{k'} = \left(\frac{1}{\eta}\frac{p_2 - k'}{p_1 - k'}\right)^n \left(\frac{1}{\eta'}\frac{q_2 - k'}{q_1 - k'}\right)^m . \tag{6.4.1}$$

The factors $\rho_k$ and $\sigma_{k'}$ now depend on two independent discrete variables $n$ and $m$ each of which now has two lattice parameters associated with it, $p_1, p_2$ and $q_1, q_2$ respectively, as well as an extra parameter for each independent variable, $\eta$ and $\eta'$ respectively.

For the discrete evolution we can impose on $\rho_k$ and $\sigma_{k'}$ two different transformations of the same type but for each of the discrete dimensions, namely

$$\rho_k \mapsto \widetilde{\rho}_k \equiv \rho_k(n+1, m) \qquad , \qquad \rho_k \mapsto \widehat{\rho}_k \equiv \rho_k(n, m+1),$$

with similar expressions for $\sigma_{k'}$. The discrete evolution of $\rho_k$ and $\sigma_{k'}$ is therefore given by,

$$\widetilde{\rho}_k = \eta \left(\frac{p_1 - k}{p_2 - k}\right) \rho_k \quad , \quad \widetilde{\sigma}_{k'} = \frac{1}{\eta}\left(\frac{p_2 - k'}{p_1 - k'}\right) \sigma_{k'}, \tag{6.4.2}$$

with similar expressions for the $\widehat{\ }$ shifts given by replacing $p_1, p_2, \eta$ with $q_1, q_2, \eta'$.

This leads us to impose the following linear relationships for $\varphi^\circ$ and $\psi^\circ$

$$\widetilde{\varphi}^\circ \cdot (p_2 - {}^t\Lambda) = \eta(p_1 - \Lambda) \cdot \varphi^\circ \quad , \quad \widetilde{\psi}^\circ \cdot (p_1 - {}^t\Lambda)\eta = (p_2 - \Lambda) \cdot \psi^\circ,$$

which, in terms of $2 \times 2$ matrices can be expressed as

$$\widetilde{\boldsymbol{C}} \cdot (\boldsymbol{P} - {}^t\boldsymbol{\Lambda}) \cdot \boldsymbol{\eta} = \boldsymbol{\eta} \cdot (\boldsymbol{P} - \boldsymbol{\Lambda}) \cdot \boldsymbol{C}. \tag{6.4.3}$$

Equations (6.2.8) and (6.2.6a) can now be used to give the following nonlinear relation for $\boldsymbol{H}$

$$\widetilde{\boldsymbol{H}} \cdot (\boldsymbol{P} - {}^t\boldsymbol{\Lambda}) \cdot \boldsymbol{\eta} \;=\; \boldsymbol{\eta} \cdot (\boldsymbol{P} - \boldsymbol{\Lambda}) \cdot \boldsymbol{H} + \widetilde{\boldsymbol{H}} \cdot \boldsymbol{\eta} \cdot \boldsymbol{O} \cdot \boldsymbol{H}, \tag{6.4.4}$$

from which the lattice equations will be derived.

## 6.5 Similarity Reduction

In order to derive the similarity constraints we need to see how the factors $\rho_k$ and $\sigma_k$ behave under the action of the differential operator $k\frac{\partial}{\partial k}$. This is given by

$$k\frac{\partial}{\partial k}\rho_k = n\left(\frac{p_2}{p_2 - k} - \frac{p_1}{p_1 - k}\right)\rho_k \quad , \quad k\frac{\partial}{\partial k}\sigma_k = n\left(\frac{p_1}{p_1 - k} - \frac{p_2}{p_2 - k}\right)\sigma_k, \tag{6.5.1}$$

now consider the object

$$\begin{aligned}\int_{C_2} d\lambda_2(l)(l\frac{\partial}{\partial l}\rho_l)c_l\,{}^tc_l \;&=\; \int_{C_2} d\lambda_2(l)\frac{\partial}{\partial l}(l\rho_l c_l\,{}^tc_l) - \int_{C_2} d\lambda_2(l)\rho_l c_l\,{}^tc_l \\ &-\; \int_{C_2} d\lambda_2(l)\rho_l(l\frac{\partial}{\partial l}c_l)\,{}^tc_l - \int_{C_2} d\lambda_2(l)\rho_l c_l(l\frac{\partial}{\partial l}\,{}^tc_l),\end{aligned}$$

hence

$$\begin{aligned}\int_{C_2} d\lambda_2(l)(l\frac{\partial}{\partial l}\rho_l)c_l\,{}^tc_l \;&=\; [l\rho_l c_l\,{}^tc_l]_{\partial C_2} - \int_{C_2} d\lambda_2(l)\rho_l)c_l\,{}^tc_l \\ &-\; \int_{C_2} d\lambda_2(l)\rho_l(l\frac{\partial}{\partial l}c_l)\,{}^tc_l - \int_{C_2} d\lambda_2(l)\rho_l c_l(l\frac{\partial}{\partial l}\,{}^tc_l),\end{aligned}$$

now, ignoring boundary conditions and noting that

$$l\frac{\partial}{\partial l}c_l = I \cdot c_l \quad \text{and} \quad l\frac{\partial}{\partial l}\,{}^tc_l = {}^tc_l \cdot {}^tI,$$

we have

$$\int_{C_2} d\lambda_2(l)(l\frac{\partial}{\partial l}\rho_l)c_l\,{}^tc_l \quad = \quad -\boldsymbol{\varphi}^o - \boldsymbol{I}\cdot\boldsymbol{\varphi}^o - \boldsymbol{\varphi}^o\cdot{}^t\boldsymbol{I}.$$

Similarly

$$\int_{C_1} d\lambda_1(l')(l\frac{\partial}{\partial l}\sigma_l)c_l\,{}^tc_l \quad = \quad -\boldsymbol{\psi}^o - \boldsymbol{I}\cdot\boldsymbol{\psi}^o - \boldsymbol{\psi}^o\cdot{}^t\boldsymbol{I}.$$

Now, by inserting equation(6.5.1) into the above equations we get

$$n\left(\frac{p_1}{p_1-\Lambda}\cdot\boldsymbol{\varphi}^o - \boldsymbol{\varphi}^o\cdot\frac{p_2}{p_2-{}^t\Lambda}\right) \quad = \quad \boldsymbol{\varphi}^o + \boldsymbol{I}\cdot\boldsymbol{\varphi}^o + \boldsymbol{\varphi}^o\cdot{}^t\boldsymbol{I}, \tag{6.5.2}$$
$$n\left(\frac{p_1}{p_1-\Lambda}\cdot\boldsymbol{\psi}^o - \boldsymbol{\psi}^c\cdot\frac{p_2}{p_2-{}^t\Lambda}\right) \quad = \quad -\boldsymbol{\psi}^o + \boldsymbol{I}\cdot\boldsymbol{\psi}^o + \boldsymbol{\psi}^o\cdot{}^t\boldsymbol{I},$$

transposing the second of these equations gives

$$n\left(\frac{p_2}{p_2-\Lambda}\cdot(-\boldsymbol{\psi}^o) - (-\boldsymbol{\psi}^o)\cdot\frac{p_1}{p_1-{}^t\Lambda}\right) \quad = \quad -\boldsymbol{\psi}^o + \boldsymbol{I}\cdot\boldsymbol{\psi}^o + \boldsymbol{\psi}^o\cdot{}^t\boldsymbol{I}, \tag{6.5.3}$$

equations (6.5.2) and (6.5.3) can now be expressed in terms of $2\times 2$ matrices, giving us the following linear relation for $\boldsymbol{C}$

$$\boldsymbol{C} + \boldsymbol{I}\cdot\boldsymbol{C} + \boldsymbol{C}\cdot{}^t\boldsymbol{I} \quad = \quad n\left(\frac{\boldsymbol{P}}{\boldsymbol{P}-\boldsymbol{\Lambda}}\cdot\boldsymbol{C} - \boldsymbol{C}\cdot\frac{\boldsymbol{P}}{\boldsymbol{P}-{}^t\boldsymbol{\Lambda}}\right), \tag{6.5.4}$$

where

$$\frac{\boldsymbol{P}}{\boldsymbol{P}-\boldsymbol{\Lambda}} \equiv \begin{pmatrix} \frac{p_1}{p_1-\Lambda} & 0 \\ 0 & \frac{p_2}{p_2-\Lambda} \end{pmatrix}, \quad \frac{\boldsymbol{P}}{\boldsymbol{P}-{}^t\boldsymbol{\Lambda}} \equiv \begin{pmatrix} \frac{p_1}{p_1-{}^t\Lambda} & 0 \\ 0 & \frac{p_2}{p_2-{}^t\Lambda} \end{pmatrix}.$$

Having obtained an equation for $C$ we now wish to obtain the corresponding nonlinear equation for $H$. Which is given by,

$$\boldsymbol{H} + \boldsymbol{I}\cdot\boldsymbol{H} + \boldsymbol{H}\cdot{}^t\boldsymbol{I} =$$
$$n\left(\frac{\boldsymbol{P}}{\boldsymbol{P}-\boldsymbol{\Lambda}}\cdot\boldsymbol{H} - \boldsymbol{H}\cdot\frac{\boldsymbol{P}}{\boldsymbol{P}-{}^t\boldsymbol{\Lambda}} - \boldsymbol{H}\cdot\frac{\boldsymbol{P}}{\boldsymbol{P}-{}^t\boldsymbol{\Lambda}}\cdot\boldsymbol{O}\cdot\frac{\boldsymbol{P}}{\boldsymbol{P}-\boldsymbol{\Lambda}}\cdot\boldsymbol{H}\right). \tag{6.5.5}$$

It is particular entries of this equation which will give us the similarity constraints for the AKNS system.

## 6.6 Lattice Equations

In order to derive the lattice equations of the AKNS system we must introduce the following objects

$$h \equiv \boldsymbol{H}_{0,0} \quad , \quad s_{\alpha,\beta} \equiv \left(\frac{1}{\alpha - \boldsymbol{\Lambda}} \cdot \boldsymbol{H} \cdot \frac{1}{\beta - {}^t\boldsymbol{\Lambda}}\right)_{0,0}, \tag{6.6.1a}$$

$$v_\alpha \equiv 1 + \left(\frac{1}{\alpha - \boldsymbol{\Lambda}} \cdot \boldsymbol{H}\right)_{0,0} \quad , \quad w_\beta \equiv 1 - \left(\boldsymbol{H} \cdot \frac{1}{\beta - {}^t\boldsymbol{\Lambda}}\right)_{0,0}, \tag{6.6.1b}$$

$$t_\alpha \equiv \alpha + \left(\frac{1}{\alpha - \boldsymbol{\Lambda}} \cdot \boldsymbol{H} \cdot {}^t\boldsymbol{\Lambda}\right)_{0,0} \quad , \quad s_\beta \equiv \beta - \left(\boldsymbol{\Lambda} \cdot \boldsymbol{H} \cdot \frac{1}{\beta - {}^t\boldsymbol{\Lambda}}\right)_{0,0}, \tag{6.6.1c}$$

where the subscript $(0,0)$ now means to pick out the central element of each of the infinite matrices in the relevant $2 \times 2$ matrix. This means that each of the objects in (6.6.1) is a $2 \times 2$ matrix with scalar entries.

Using the relations in section (6.2.2) and the symmetry of $\boldsymbol{H}$, it is possible to show that these objects possess the following symmetries

$$s_{\alpha,\beta} = -\sigma_2 \cdot (s_{\overline{\beta},\overline{\alpha}})^T \cdot \sigma_2, \tag{6.6.2a}$$

$$v_\alpha = \sigma_2 \cdot (w_{\overline{\alpha}})^T \cdot \sigma_2 \quad , \quad w_\beta = \sigma_2 \cdot (v_{\overline{\beta}})^T \cdot \sigma_2, \tag{6.6.2b}$$

$$t_\alpha = \sigma_2 \cdot (s_{\overline{\alpha}})^T \cdot \sigma_2 \quad , \quad s_\beta = \sigma_2 \cdot (t_{\overline{\beta}})^T \cdot \sigma_2. \tag{6.6.2c}$$

We now proceed to derive the lattice equations and similarity constraints.

### 6.6.1 Algebraic Lattice Equations

Starting with the algebraic relation (6.3.3), taking $p = 1$ we get

$$\boldsymbol{H} \cdot {}^t\boldsymbol{\Lambda} = \boldsymbol{\Lambda} \cdot \boldsymbol{H} - \boldsymbol{H} \cdot \boldsymbol{O} \cdot \boldsymbol{H}. \tag{6.6.3}$$

This equation now gives us the following algebraic relations for the objects in (6.6.1)

$$\boldsymbol{v}_0 \cdot \boldsymbol{w}_0 = 1, \tag{6.6.4a}$$

$$1 - \boldsymbol{\alpha} \boldsymbol{s}_{\alpha,0} = \boldsymbol{v}_\alpha \cdot \boldsymbol{w}_0, \tag{6.6.4b}$$

$$\boldsymbol{v}_0 \cdot \boldsymbol{w}_\beta = 1 + \boldsymbol{\beta} \boldsymbol{s}_{0,\beta}, \tag{6.6.4c}$$

$$\boldsymbol{v}_\alpha \cdot \boldsymbol{w}_\beta = 1 + \boldsymbol{\beta} \boldsymbol{s}_{\alpha,\beta} - \boldsymbol{\alpha} \boldsymbol{s}_{\alpha,\beta}. \tag{6.6.4d}$$

Turning now to the shifted equation (6.4.4) we obtain the following Miura type equation

$$\boldsymbol{\eta} - \widetilde{\boldsymbol{s}}_{\alpha,\beta} \cdot (\boldsymbol{P} - \boldsymbol{\beta}) \cdot \boldsymbol{\eta} + \boldsymbol{\eta} \cdot (\boldsymbol{P} - \boldsymbol{\alpha}) \cdot \boldsymbol{s}_{\alpha,\beta} = \widetilde{\boldsymbol{v}}_\alpha \cdot \boldsymbol{\eta} \cdot \boldsymbol{w}_\beta, \tag{6.6.5a}$$

and by considering a second transformation of the same type but with different parameters $Q$ and $\eta'$ we obtain the similar relation

$$\boldsymbol{\eta}' - \widehat{\boldsymbol{s}}_{\alpha,\beta} \cdot (\boldsymbol{Q} - \boldsymbol{\beta}) \cdot \boldsymbol{\eta}' + \boldsymbol{\eta}' \cdot (\boldsymbol{Q} - \boldsymbol{\alpha}) \cdot \boldsymbol{s}_{\alpha,\beta} = \widehat{\boldsymbol{v}}_\alpha \cdot \boldsymbol{\eta}' \cdot \boldsymbol{w}_\beta, \tag{6.6.5b}$$

now, by applying the symmetries (6.6.2) to equations (6.6.5b) and (6.6.5b) we get

$$\begin{aligned} \overline{\boldsymbol{\eta}} + \overline{\boldsymbol{\eta}} \cdot (\overline{\boldsymbol{P}} - \boldsymbol{\alpha}) \cdot \widetilde{\boldsymbol{s}}_{\alpha,\beta} - \boldsymbol{s}_{\alpha,\beta} \cdot (\overline{\boldsymbol{P}} - \boldsymbol{\beta}) \cdot \overline{\boldsymbol{\eta}} &= \boldsymbol{v}_\alpha \cdot \overline{\boldsymbol{\eta}} \cdot \widetilde{\boldsymbol{w}}_\beta, \\ \overline{\boldsymbol{\eta}}' + \overline{\boldsymbol{\eta}}' \cdot (\overline{\boldsymbol{Q}} - \boldsymbol{\alpha}) \cdot \widehat{\boldsymbol{s}}_{\alpha,\beta} - \boldsymbol{s}_{\alpha,\beta} \cdot (\overline{\boldsymbol{Q}} - \boldsymbol{\beta}) \cdot \overline{\boldsymbol{\eta}}' &= \boldsymbol{v}_\alpha \cdot \overline{\boldsymbol{\eta}}' \cdot \widehat{\boldsymbol{w}}_\beta, \end{aligned}$$

applying extra shifts to these equations gives

$$\overline{\boldsymbol{\eta}} + \overline{\boldsymbol{\eta}} \cdot (\overline{\boldsymbol{P}} - \boldsymbol{\alpha}) \cdot \widehat{\widetilde{\boldsymbol{s}}}_{\alpha,\beta} - \widehat{\boldsymbol{s}}_{\alpha,\beta} \cdot (\overline{\boldsymbol{P}} - \boldsymbol{\beta}) \cdot \overline{\boldsymbol{\eta}} = \widehat{\boldsymbol{v}}_\alpha \cdot \overline{\boldsymbol{\eta}} \cdot \widehat{\widetilde{\boldsymbol{w}}}_\beta, \tag{6.6.5c}$$

$$\overline{\boldsymbol{\eta}}' + \overline{\boldsymbol{\eta}}' \cdot (\overline{\boldsymbol{Q}} - \boldsymbol{\alpha}) \cdot \widehat{\widetilde{\boldsymbol{s}}}_{\alpha,\beta} - \widetilde{\boldsymbol{s}}_{\alpha,\beta} \cdot (\overline{\boldsymbol{Q}} - \boldsymbol{\beta}) \cdot \overline{\boldsymbol{\eta}}' = \widetilde{\boldsymbol{v}}_\alpha \cdot \overline{\boldsymbol{\eta}}' \cdot \widetilde{\widehat{\boldsymbol{w}}}_\beta, \tag{6.6.5d}$$

eliminating $\boldsymbol{w}_\beta$ from equations (6.6.5a) and (6.6.5b) gives

$$\begin{aligned} \widetilde{\boldsymbol{v}}_\alpha \cdot \boldsymbol{\eta} \cdot \frac{1}{\boldsymbol{\eta}'} \cdot \widehat{\boldsymbol{v}}_\alpha^{-1} &= [\boldsymbol{\eta} - \widetilde{\boldsymbol{s}}_{\alpha,\beta} \cdot (\boldsymbol{P} - \boldsymbol{\beta}) \cdot \boldsymbol{\eta} + \boldsymbol{\eta} \cdot (\boldsymbol{P} - \boldsymbol{\alpha}) \cdot \boldsymbol{s}_{\alpha,\beta}] \times \\ &\quad \times [\boldsymbol{\eta}' - \widehat{\boldsymbol{s}}_{\alpha,\beta} \cdot (\boldsymbol{Q} - \boldsymbol{\beta}) \cdot \boldsymbol{\eta}' + \boldsymbol{\eta}' \cdot (\boldsymbol{Q} - \boldsymbol{\alpha}) \cdot \boldsymbol{s}_{\alpha,\beta}]^{-1}, \end{aligned}$$

while eliminating $\widehat{\widetilde{\boldsymbol{w}}}_{\beta}$ from equations (6.6.5c) and (6.6.5d) gives

$$\begin{aligned}\widetilde{\boldsymbol{v}}_{\alpha}\cdot\overline{\boldsymbol{\eta}}'\cdot\frac{1}{\overline{\overline{\boldsymbol{\eta}}}}\cdot\widehat{\boldsymbol{v}}_{\alpha}^{-1} &= [\overline{\boldsymbol{\eta}}'+\overline{\boldsymbol{\eta}}'\cdot(\overline{\boldsymbol{Q}}-\boldsymbol{\alpha})\cdot\widetilde{\widehat{\boldsymbol{s}}}_{\alpha,\beta}-\widetilde{\boldsymbol{s}}_{\alpha,\beta}\cdot(\overline{\boldsymbol{Q}}-\boldsymbol{\beta})\cdot\overline{\boldsymbol{\eta}}']\times\\ &\quad\times[\overline{\boldsymbol{\eta}}+\overline{\boldsymbol{\eta}}\cdot(\overline{\boldsymbol{P}}-\boldsymbol{\alpha})\cdot\widetilde{\widehat{\boldsymbol{s}}}_{\alpha,\beta}-\widehat{\boldsymbol{s}}_{\alpha,\beta}\cdot(\overline{\boldsymbol{P}}-\boldsymbol{\beta})\cdot\overline{\boldsymbol{\eta}}]^{-1}.\end{aligned}$$

In order to derive a closed-form equation from the above equations the LHS of each equation must be equivalent. This is the case only if we apply the extra condition $\eta = \eta'$, in this case we get

$$\begin{aligned}&[\boldsymbol{\eta}-\widetilde{\boldsymbol{s}}_{\alpha,\beta}\cdot(\boldsymbol{P}-\boldsymbol{\beta})\cdot\boldsymbol{\eta}+\boldsymbol{\eta}\cdot(\boldsymbol{P}-\boldsymbol{\alpha})\cdot\boldsymbol{s}_{\alpha,\beta}]\times\\ &\qquad\times[\boldsymbol{\eta}'-\widehat{\boldsymbol{s}}_{\alpha,\beta}\cdot(\boldsymbol{Q}-\boldsymbol{\beta})\cdot\boldsymbol{\eta}'+\boldsymbol{\eta}'\cdot(\boldsymbol{Q}-\boldsymbol{\alpha})\cdot\boldsymbol{s}_{\alpha,\beta}]^{-1}=\\ &[\overline{\boldsymbol{\eta}}'+\overline{\boldsymbol{\eta}}'\cdot(\overline{\boldsymbol{Q}}-\boldsymbol{\alpha})\cdot\widehat{\widetilde{\boldsymbol{s}}}_{\alpha,\beta}-\widetilde{\boldsymbol{s}}_{\alpha,\beta}\cdot(\overline{\boldsymbol{Q}}-\boldsymbol{\beta})\cdot\overline{\boldsymbol{\eta}}']\times\\ &\qquad\times[\overline{\boldsymbol{\eta}}+\overline{\boldsymbol{\eta}}\cdot(\overline{\boldsymbol{P}}-\boldsymbol{\alpha})\cdot\widehat{\widetilde{\boldsymbol{s}}}_{\alpha,\beta}-\widehat{\boldsymbol{s}}_{\alpha,\beta}\cdot(\overline{\boldsymbol{P}}-\boldsymbol{\beta})\cdot\overline{\boldsymbol{\eta}}]^{-1}.\end{aligned}\tag{6.6.6}$$

If we now introduce the following object

$$\boldsymbol{z}=s_{0,0}-n\boldsymbol{P}^{-1}-m\boldsymbol{Q}^{-1}\quad,\quad cof(\boldsymbol{z})=\boldsymbol{z}'=-s_{0,0}-n\overline{\boldsymbol{P}}^{-1}-m\boldsymbol{Q}^{-1}.\tag{6.6.7}$$

Equation (6.6.6) gives

$$\begin{aligned}&(\boldsymbol{\eta}\cdot\boldsymbol{P}\cdot\boldsymbol{z}-\widetilde{\boldsymbol{z}}\cdot\boldsymbol{P}\cdot\boldsymbol{\eta})\times(\boldsymbol{\eta}'\cdot\boldsymbol{Q}\cdot\boldsymbol{z}-\widehat{\boldsymbol{z}}\cdot\boldsymbol{Q}\cdot\boldsymbol{\eta}')^{-1}=\\ &\qquad(\widetilde{\boldsymbol{z}}'\cdot\overline{\boldsymbol{Q}}\cdot\overline{\boldsymbol{\eta}}'-\overline{\boldsymbol{\eta}}'\cdot\overline{\boldsymbol{Q}}\cdot\widehat{\widetilde{\boldsymbol{z}}}')\times(\widehat{\boldsymbol{z}}'\cdot\overline{\boldsymbol{P}}\cdot\overline{\boldsymbol{\eta}}-\overline{\boldsymbol{\eta}}\cdot\overline{\boldsymbol{P}}\cdot\widehat{\widetilde{\boldsymbol{z}}}')^{-1}.\end{aligned}\tag{6.6.8}$$

We can also obtain from equation (6.4.4) the following relations

$$\widetilde{\boldsymbol{t}}_{\alpha}\cdot\boldsymbol{\eta} = \widetilde{\boldsymbol{v}}_{\alpha}\cdot\boldsymbol{\eta}\cdot(\boldsymbol{P}-\boldsymbol{h})-\boldsymbol{\eta}\cdot(\boldsymbol{P}-\boldsymbol{\alpha})\cdot\boldsymbol{v}_{\alpha},\tag{6.6.9a}$$

$$\boldsymbol{t}_{\alpha}\cdot\overline{\boldsymbol{\eta}} = \boldsymbol{v}_{\alpha}\cdot\overline{\boldsymbol{\eta}}\cdot(\overline{\boldsymbol{P}}-\widetilde{\boldsymbol{h}})-(\overline{\boldsymbol{P}}-\boldsymbol{\alpha})\cdot\overline{\boldsymbol{\eta}}\cdot\overline{\boldsymbol{v}}_{\alpha},\tag{6.6.9b}$$

now, by applying different shifts to these equations we we obtain another set of Miura type relations, namely

$$
\begin{aligned}
&\boldsymbol{\eta}' \cdot \boldsymbol{Q} \cdot \boldsymbol{\eta} - \boldsymbol{\eta} \cdot \boldsymbol{P} \cdot \boldsymbol{\eta}' - \boldsymbol{\eta}' \cdot \widehat{\widetilde{\boldsymbol{h}}} \cdot \boldsymbol{\eta} + \boldsymbol{\eta} \cdot \widehat{\boldsymbol{h}} \cdot \boldsymbol{\eta}' = \\
&\qquad (\widehat{\widetilde{\boldsymbol{v}}}_\alpha)^{-1} \cdot (\boldsymbol{\eta}' \cdot (\boldsymbol{Q} - \boldsymbol{\alpha}) \cdot \widetilde{\boldsymbol{v}}_\alpha \cdot \boldsymbol{\eta} - \boldsymbol{\eta} \cdot (\boldsymbol{P} - \boldsymbol{\alpha}) \cdot \widehat{\boldsymbol{v}}_\alpha \cdot \boldsymbol{\eta}'), \qquad (6.6.10a) \\
&\overline{\boldsymbol{\eta}}' \cdot \overline{\boldsymbol{Q}} \cdot \boldsymbol{\eta} - \boldsymbol{\eta} \cdot \boldsymbol{P} \cdot \overline{\boldsymbol{\eta}}' + \boldsymbol{\eta} \cdot \boldsymbol{h} \cdot \overline{\boldsymbol{\eta}}' - \overline{\boldsymbol{\eta}}' \cdot \widehat{\widehat{\boldsymbol{h}}} \cdot \boldsymbol{\eta} = \\
&\qquad (\widetilde{\boldsymbol{v}}_\alpha)^{-1} \cdot (\overline{\boldsymbol{\eta}}' \cdot (\overline{\boldsymbol{Q}} - \boldsymbol{\alpha}) \cdot \widehat{\widehat{\boldsymbol{v}}}_\alpha \cdot \boldsymbol{\eta} - \boldsymbol{\eta} \cdot (\boldsymbol{P} - \boldsymbol{\alpha}) \cdot \boldsymbol{v}_\alpha \cdot \overline{\boldsymbol{\eta}}'), \qquad (6.6.10b)
\end{aligned}
$$

for which, certain parameter values will yield further closed form equations.

### 6.6.2 Similarity Lattice Equations

In the case of the similarity constraint we obtain the following equations

$$
\begin{aligned}
\boldsymbol{h} &= n\left(\frac{1}{\boldsymbol{\eta}} \cdot \frac{1}{\widetilde{\widetilde{\boldsymbol{v}}}_P} \cdot \boldsymbol{P} \cdot \boldsymbol{\eta} \cdot \boldsymbol{v}_P - \boldsymbol{P}\right) + m\left(\frac{1}{\boldsymbol{\eta}'} \cdot \frac{1}{\widehat{\widehat{\boldsymbol{v}}}_Q} \cdot \boldsymbol{Q} \cdot \boldsymbol{\eta}' \cdot \boldsymbol{v}_Q - \boldsymbol{Q}\right) \qquad (6.6.11a) \\
0 &= n\left(\frac{1}{\boldsymbol{P}} \cdot \frac{1}{\boldsymbol{\eta}} \cdot \widetilde{\boldsymbol{v}}_0 \cdot \frac{1}{\widetilde{\widetilde{\boldsymbol{v}}}_P} \cdot \boldsymbol{\eta} \cdot \boldsymbol{P} \cdot \boldsymbol{v}_P \cdot \frac{1}{\boldsymbol{v}_0} - 1\right) \\
&\quad + m\left(\frac{1}{\boldsymbol{Q}} \cdot \frac{1}{\boldsymbol{\eta}'} \cdot \widehat{\boldsymbol{v}}_0 \cdot \frac{1}{\widehat{\widehat{\boldsymbol{v}}}_Q} \cdot \boldsymbol{\eta}' \cdot \boldsymbol{Q} \cdot \boldsymbol{v}_Q \cdot \frac{1}{\boldsymbol{v}_0} - 1\right), \qquad (6.6.11b) \\
-z &= n\left(\frac{1}{\boldsymbol{P}} \cdot \frac{1}{\boldsymbol{\eta}} \cdot \widetilde{\boldsymbol{v}}_0 \cdot \frac{1}{\widetilde{\widetilde{\boldsymbol{v}}}_P} \cdot \boldsymbol{\eta} \cdot \boldsymbol{P} \cdot \boldsymbol{v}_P \cdot \boldsymbol{\eta} \cdot \frac{1}{\underset{\sim}{\boldsymbol{v}}_0} \cdot \frac{1}{\boldsymbol{\eta}} \cdot \frac{1}{\boldsymbol{P}}\right) \\
&\quad + m\left(\frac{1}{\boldsymbol{Q}} \cdot \frac{1}{\boldsymbol{\eta}'} \cdot \widehat{\boldsymbol{v}}_0 \cdot \frac{1}{\widehat{\widehat{\boldsymbol{v}}}_Q} \cdot \boldsymbol{\eta}' \cdot \boldsymbol{Q} \cdot \boldsymbol{v}_Q \cdot \boldsymbol{\eta}' \cdot \frac{1}{\underset{\frown}{\boldsymbol{v}}_0} \cdot \frac{1}{\boldsymbol{\eta}'} \cdot \frac{1}{\boldsymbol{Q}}\right), \qquad (6.6.11c)
\end{aligned}
$$

where the fractions in the above equations should be treated as the inverse of the relevant $2 \times 2$ matrix.

These equations may be used as the basis for deriving closed-form similarity constraints.

In this section we have succeeded in laying down the foundations of the direct linearization method for the lattice AKNS system. We have formulated the infinite matrix structure and defined the basic equations needed to derive the relevant lattice equations and similarity constraints. The derivation of these equations is however left for the subject of future work.

# Chapter 7

# Conclusions

This thesis deals with the subject of integrable lattice equations and their similarity reductions, in particular to equations of Painlevé type.

In Chapter 1 some key notions of continuous integrable systems are highlighted. In particular we review the Inverse Scattering Transform, Lax pairs, Bäcklund and Miura transformations and conservation laws. We also recall ingredients from the classical theory of Painlevé equations and the Painlevé property and showed how this is related to integrability via the ARS-conjecture. Having summarized the main features of continuous integrable systems, we consider their discrete analogues, giving examples of both integrable P$\Delta$Es and discrete Painlevé equations. Emphasis was given here to the the linearization of the KdV equation by Fokas and Ablowitz as this is the basis of the direct linearization method employed throughout the thesis.

In chapter 2 we review the direct linearisation of the KdV system. Whilst most of the results in the chapter have already appeared in the literature we need to establish the notations and to develop the tools for the subsequent chapters. In particular this entails the setting up of an infinite matrix structure, which, albeit being a strictly formal tool, has proven to be extremely powerful in establishing the algebraic properties of the system. It is demonstrated how this infinite matrix structure can be used to derive a host of results including closed-form lattice equations, associated differential-difference equations and similarity constraints that govern their symmetry reductions. Furthermore the linear systems associated with each of these equations are derived. In this chapter also, a new method for obtaining Lax pairs for the lattice equations directly from the equations themselves is presented. This relatively simple technique is expected to become a useful way of obtaining Lax pairs in cases were they are unknown. Finally in this section we recalled the nonlinear PDE which are the generating equation for the entire hierarchy of Schwarzian KdV equations.

The next chapters in the thesis contain new material. In chapter 3 we deal with extending the reduction of the lattice KdV equations to Painlevé type equations of higher order. This reduction is achieved by means of a similarity constraint, compatible with the lattice equation, linking the various dimensions of the multi-dimensional lattice. Thus, we show that both the lattice equations and the differential-difference equations for the KdV system can be consistently embedded in the multidimensional lattice and hence form compatible parameter-families of equations. Extending the similarity constraint in a similar way, we derive a coupled set of second-order nonlinear ODEs which make up the continuous $P_{VI}$ hierarchy along with a coupled set of first order nonlinear O$\Delta$Es which give the discrete analogy of this hierarchy. With these systems in place we give explicit examples of these reductions for the two and three dimensional cases. The results for the two-dimensional case, although already known, were reproduced by means of the general multi-dimensional systems. For the three dimensional case, it was shown that an IVP could be well-posed and that this does indeed lead to higher order Painlevé type equations. These systems were then compared to the Garnier systems.

In chapter 4 we introduce the higher dimensional KP system. We indicate how this system can be dimensionally reduced to the lattice Gel'fand-Dikii hierarchy. We also develop the direct linearisation method for the lattice KP system and give closed form lattice equations. For the similarity reductions we shown that the system must first be dimensionally reduced to the Gel'fand-Dikii hierarchy before the similarity can be implemented.

In chapter 5 we extend the direct linearisation method to the Boussinesq system, deriving the lattice equations and their associated Lax pairs as well as new differential-difference equations along with their linear systems. Using these linear systems we derive a coupled system of nonlinear PDEs from which we expect the analogue of the generating PDE for the Schwarzian BSQ hierarchy can be extracted. However, so far, we have not been successful in obtaining a closed-form equation of this type. We then proceed to show how the lattice BSQ equations can be consistently embedded in a multi-dimensional lattice and hence can also be considered as compatible parameter-families of equations. It is seen that this is best achieved by considering these lattice equations as coupled systems rather than closed-form equations. Finally we rederive the similarity constraints for the BSQ system along with the associated monodromy problems. The compatibility of these constraints with the lattice equations is confirmed for the case of the modified system. Again this is best achieved via coupled systems for both the lattice equation and the constraint. In theory the constraint can therefore be used to reduce the lattice equation to an O$\Delta$E and as this O$\Delta$E will have been derived as a self-similar reduction of an integrable lattice equation we would expect it to be of Painlevé type, however this reduction is not done explicitly as we are unable to determine the necessary similarity variables.

In the final chapter we develop the direct linearisation method for the lattice AKNS system. We introduce the notation and derive the matrix equations from which the lattice equations and similarity constraints can be found. The derivation of the explicit form of these equations is left to future work.

In conclusion, in this thesis we have presented new examples of similarity reduced lattice equations and demonstrated that there is an intimate interplay between continuous and discrete systems. We are confident that these these results will prove to be of importance in the development of a coherent theory of integrable partial difference equations. In first instance, further analysis of the AKNS system is required as well as a full picture for the entire lattice Gel'fand-Dikii hierarchy. In future extensions to lattice systems of elliptic type, e.g. the lattice Krichever-Novikov system, need to be investigated.

# Appendix: Coefficients for equation (3.4.10)

In section 3.4.2 we gave the following coupled system of second-order nonlinear ODE's, equation (3.4.10)

$$\begin{aligned}
\frac{\partial^2 X}{\partial p^2} &= A_1\left(\frac{\partial X}{\partial p}\right)^2 + A_2\left(\frac{\partial Y}{\partial p}\right)^2 + A_3\left(\frac{\partial X}{\partial p}\right)\left(\frac{\partial Y}{\partial p}\right) + A_4\left(\frac{\partial X}{\partial p}\right) + A_5\left(\frac{\partial Y}{\partial p}\right) + A_6 \\
\frac{\partial^2 Y}{\partial p^2} &= B_1\left(\frac{\partial X}{\partial p}\right)^2 + B_2\left(\frac{\partial Y}{\partial p}\right)^2 + B_3\left(\frac{\partial X}{\partial p}\right)\left(\frac{\partial Y}{\partial p}\right) + B_4\left(\frac{\partial X}{\partial p}\right) + B_5\left(\frac{\partial Y}{\partial p}\right) + B_6
\end{aligned}$$

where the coefficients, as calculated by MAPLE, are given by:

$$A_1 = -2\left(-\frac{q\,(p\,X-q)\,(q-\frac{p}{X})}{p^2-q^2}+p\right)\%2\left(2\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,p\,(p^2-q^2)^2}{(p^2-r^2)\,\%3\,(p\,X-q)^2\,(q-\frac{p}{X})^2}+\right.$$

$$2(p\,Y-r)\,(r-\frac{p}{\mathrm{y}(p)})\,p\,(p^2-q^2)\left(-2\,\frac{\%2\,p\,(p^2-q^2)\,(p\,\mathrm{y}(p)-r)\,(r-\frac{p}{Y})}{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p^2-r^2)}\right.$$

$$\left.\left.+2\,\frac{\%2\,\%1^2\,p}{(p^2-q^2)^2}-2\,\frac{p\,\%1\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)}+2\,\frac{\%2^2\,\%1\,p}{(p\,X-q)^2\,(q-\frac{p}{X})^2}\right)\right/($$

$$(p^2-r^2)\,\%3^2\,(p\,X-q)\,(q-\frac{p}{X}))-q\left(\right.$$

$$2\,\frac{(-\%6+2\,\%4)\,(p\,Y-r)\,(r-\frac{p}{Y})\,p\,(p^2-q^2)^2}{(p^2-r^2)\,\%3\,(p\,X-q)^2\,(q-\frac{p}{X})^2}$$

$$+2\,\frac{(p\,Y-r)\,(r-\dfrac{p}{Y})\,p\,(p^2-q^2)\,(\%6-2\,\%4)\,Y}{(p^2-r^2)\,\%3\,(p\,X-q)\,(q-\dfrac{p}{X})}\Bigg)\Big/(q^2-p^2)\Bigg)\Big/(p\,(p^2-q^2))-\Bigg($$

$$-\frac{p^3\,(p\,X-q)^2\,(q-\dfrac{p}{X})^2\,q\,Y}{(p^2-q^2)^2}+\frac{p\,(p\,X-q)^2\,(q-\dfrac{p}{X})^2\,q\,r^2\,Y}{(p^2-q^2)^2}$$

$$+2\,\frac{p^4\,(p\,X-q)\,(q-\dfrac{p}{X})\,Y}{p^2-q^2}-2\,\frac{q^2\,(p\,X-q)\,(q-\dfrac{p}{X})\,r^2\,Y}{p^2-q^2}-q\,p^3\,Y$$

$$+q\,p\,r^2\,Y-\frac{p^3\,(p\,X-q)\,(q-\dfrac{p}{X})\,Y^2\,r}{p^2-q^2}+\frac{q^2\,(p\,X-q)\,(q-\dfrac{p}{X})\,p\,Y^2\,r}{p^2-q^2}$$

$$-\frac{p^3\,(p\,X-q)\,(q-\dfrac{p}{X})\,r}{p^2-q^2}+\frac{q^2\,(p\,X-q)\,(q-\dfrac{p}{X})\,r\,p}{p^2-q^2}\Bigg)\Bigg(-\%5\,p\,(p^2-q^2)\Bigg($$

$$-2\,\frac{\%2\,p\,(p^2-q^2)\,(p\,Y-r)\,(r-\dfrac{p}{Y})}{(p\,X-q)^2\,(q-\dfrac{p}{X})^2\,(p^2-r^2)}+2\,\frac{\%2\,\%1^2\,p}{(p^2-q^2)^2}$$

$$-2\,\frac{p\,\%1\,(p\,\mathrm{y}(p)-r)\,(r-\dfrac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)}+2\,\frac{\%2^2\,\%1\,p}{(p\,X-q)^2\,(q-\dfrac{p}{\mathrm{x}(p)})^2}\Bigg)\Big/(\%3^2\,(p\,X-q)$$

$$(q-\frac{p}{X}))-\frac{\%5\,p\,(p^2-q^2)^2}{\%3\,(p\,X-q)^2\,(q-\dfrac{p}{X})^2}$$

$$-\frac{r\left(-\dfrac{(-\%6+2\,\%4)\,\%5\,p\,(p^2-q^2)}{\%3\,(p\,X-q)\,(q-\dfrac{p}{X})\,Y}-\dfrac{\%5\,p\,(\%6-2\,\%4)}{\%3}\right)}{r^2-p^2}$$

$$+\frac{\left(\dfrac{p\,\%1}{p^2-q^2}+\dfrac{\%2\,p\,(p^2-q^2)}{(p\,X-q)^2\,(q-\dfrac{p}{X})^2}\right)p\,(p^2-q^2)}{\%3\,(p\,X-q)\,(q-\dfrac{p}{X})}\Bigg)\Big/(p\,Y\,(p^2-r^2)\,(p^2-q^2))$$

$$\%1:=q-\frac{p\,(p^2-q^2)}{(p\,X-q)\,(q-\dfrac{p}{X})}$$

$$\%2:=\frac{p\,(p\,X-q)\,(q-\dfrac{p}{X})}{p^2-q^2}-q$$

$$\%3:=-2\,\frac{\%2\,\%1\,(p\,Y-r)\,(r-\dfrac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)}+\frac{\%2^2\,\%1^2}{(p^2-q^2)^2}+\frac{(p\,Y-r)^2\,(r-\dfrac{p}{Y})^2}{(p^2-r^2)^2}$$

$$\%4:=\frac{(p\,Y-r)\,(r-\dfrac{p}{Y})\,p\,(p^2-q^2)}{(p^2-r^2)\,\%3\,(p\,X-q)\,(q-\dfrac{p}{X})}$$

$$\%5:=\frac{\%2\,\%1}{p^2-q^2}+\frac{(p\,Y-r)\,(r-\dfrac{p}{Y})}{p^2-r^2}$$

$$\%6:=\frac{\%5\,p\,(p^2-q^2)}{\%3\,(p\,X-q)\,(q-\dfrac{p}{X})}$$

$$A_2 = -2 \left( -\frac{q\,(p\,X - q)\,(q - \frac{p}{X})}{p^2 - q^2} + p \right) \%4 \Bigg($$

$$-\frac{q \left( -\frac{(2\,\%8 - \%7)\,\%6\,p\,(p^2 - r^2)\,(p^2 - q^2)}{\%5\,(p\,Y - r)\,(r - \frac{p}{Y})\,(p\,X - q)\,(q - \frac{p}{X})} - \frac{\%6\,p\,(-2\,\%8 + \%7)}{\%5} \right)}{q^2 - p^2}$$

$$-\frac{\%6\,p\,(p^2 - r^2)^2}{\%5\,(p\,Y - r)^2\,(r - \frac{p}{Y})^2} - \%6\,p\,(p^2 - r^2) \Bigg( -2\,\frac{\%4\,\%3\,\%2\,p\,(p^2 - r^2)}{(p^2 - q^2)\,(p\,Y - r)^2\,(r - \frac{p}{Y})^2}$$

$$-2\,\frac{\%4\,\%3\,p\,\%1}{(p^2 - q^2)\,(p^2 - r^2)} + 2\,\frac{\%2\,\%1^2\,p}{(p^2 - r^2)^2} + 2\,\frac{\%2^2\,\%1\,p}{(p\,Y - r)^2\,(r - \frac{p}{Y})^2} \Bigg) \Big/ (\%5^2$$

$$(p\,Y - r)\,(r - \frac{p}{\mathrm{y}(p)})) + \frac{\left( \frac{p\,\%1}{p^2 - r^2} + \frac{\%2\,p\,(p^2 - r^2)}{(p\,Y - r)^2\,(r - \frac{p}{Y})^2} \right) p\,(p^2 - r^2)}{\%5\,(p\,Y - r)\,(r - \frac{p}{\mathrm{y}(p)})} \Bigg) / (p\,(p^2 - q^2)) - \Bigg($$

$$-\frac{p^3\,(p\,X - q)^2\,(q - \frac{p}{X})^2\,q\,(p\,Y - r)\,(r - \frac{p}{Y})}{(p^2 - q^2)^2\,(p^2 - r^2)}$$

$$+\frac{p\,(p\,X - q)^2\,(q - \frac{p}{X})^2\,q\,r^2\,(p\,Y - r)\,(r - \frac{p}{Y})}{(p^2 - q^2)^2\,(p^2 - r^2)}$$

$$+2\,\frac{p^4\,(p\,X - q)\,(q - \frac{p}{X})\,(p\,Y - r)\,(r - \frac{p}{Y})}{(p^2 - q^2)\,(p^2 - r^2)}$$

$$-2\,\frac{q^2\,(p\,X - q)\,(q - \frac{p}{X})\,r^2\,(p\,\mathrm{y}(p) - r)\,(r - \frac{p}{Y})}{(p^2 - q^2)\,(p^2 - r^2)} - \frac{q\,p^3\,(p\,Y - r)\,(r - \frac{p}{Y})}{p^2 - r^2}$$

$$+\frac{q\,p\,r^2\,(p\,Y - r)\,(r - \frac{p}{Y})}{p^2 - r^2} - \frac{p^3\,(p\,X - q)\,(q - \frac{p}{X})\,(p\,Y - r)^2\,(r - \frac{p}{Y})^2\,r}{(p^2 - q^2)\,(p^2 - r^2)^2}$$

$$+\frac{q^2\,(p\,X - q)\,(q - \frac{p}{X})\,p\,(p\,Y - r)^2\,(r - \frac{p}{Y})^2\,r}{(p^2 - q^2)\,(p^2 - r^2)^2} - \frac{p^3\,(p\,X - q)\,(q - \frac{p}{X})\,r}{p^2 - q^2}$$

$$+\frac{q^2\,(p\,X - q)\,(q - \frac{p}{X})\,r\,p}{p^2 - q^2} \Bigg) \Bigg( -r \Bigg( 2\,\frac{(2\,\%8 - \%7)\,\%4\,\%3\,p\,(p^2 - r^2)^2}{(p^2 - q^2)\,\%5\,(p\,Y - r)^2\,(r - \frac{p}{Y})^2}$$

$$+2\,\frac{\%4\,\%3\,p\,(p^2 - r^2)\,(-2\,\%8 + \%7)\,(p\,X - q)\,(q - \frac{p}{X})}{(p^2 - q^2)^2\,\%5\,(p\,Y - r)\,(r - \frac{p}{Y})} \Bigg) / (r^2 - p^2)$$

$$+2\,\frac{\%4\,\%3\,p\,(p^2 - r^2)^2}{(p^2 - q^2)\,\%5\,(p\,Y - r)^2\,(r - \frac{p}{Y})^2} + 2\%4\,\%3\,p\,(p^2 - r^2) \Bigg($$

$$-2\,\frac{\%4\,\%3\,\%2\,p\,(p^2 - r^2)}{(p^2 - q^2)\,(p\,Y - r)^2\,(r - \frac{p}{Y})^2} - 2\,\frac{\%4\,\%3\,p\,\%1}{(p^2 - q^2)\,(p^2 - r^2)} + 2\,\frac{\%2\,\%1^2\,p}{(p^2 - r^2)^2}$$

$$+ 2\,\frac{\%2^2\,\%1\,p}{(p\,Y-r)^2\,(r-\frac{p}{Y})^2}\Bigg) \Bigg/ ((p^2-q^2)\,\%5^2\,(p\,Y-r)\,(r-\frac{p}{Y}))\Bigg) \Bigg/ (p$$

$$(p\,Y-r)\,(r-\frac{p}{\mathrm{y}(p)})\,(p^2-q^2))$$

$$\%1 := r - \frac{p\,(p^2-r^2)}{(p\,Y-r)\,(r-\frac{p}{Y})}$$

$$\%2 := \frac{p\,(p\,Y-r)\,(r-\frac{p}{Y})}{p^2-r^2} - r$$

$$\%3 := q - \frac{p\,(p^2-q^2)}{(p\,X-q)\,(q-\frac{p}{X})}$$

$$\%4 := \frac{p\,(p\,X-q)\,(q-\frac{p}{X})}{p^2-q^2} - q$$

$$\%5 := -2\,\frac{\%4\,\%3\,\%2\,\%1}{(p^2-q^2)\,(p^2-r^2)} + \frac{\%4^2\,\%3^2}{(p^2-q^2)^2} + \frac{\%2^2\,\%1^2}{(p^2-r^2)^2}$$

$$\%6 := \frac{\%4\,\%3}{p^2-q^2} + \frac{\%2\,\%1}{p^2-r^2}$$

$$\%7 := \frac{\%6\,p\,(p^2-r^2)}{\%5\,(p\,Y-r)\,(r-\frac{p}{Y})}$$

$$\%8 := \frac{\%4\,\%3\,p\,(p^2-r^2)}{(p^2-q^2)\,\%5\,(p\,Y-r)\,(r-\frac{p}{Y})}$$

$$A_3 = -2\left(-\frac{q\,(p\,X-q)\,(q-\frac{p}{X})}{p^2-q^2} + p\right)\%4\,\Bigg(2\%2\,\%1\,p\,(p^2-q^2)\,\Bigg($$

$$-2\,\frac{\%4\,\%3\,\%2\,p\,(p^2-r^2)}{(p^2-q^2)\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2} - 2\,\frac{\%4\,\%3\,p\,\%1}{(p^2-q^2)\,(p^2-r^2)} + 2\,\frac{\%2\,\%1^2\,p}{(p^2-r^2)^2}$$

$$+ 2\,\frac{\%2^2\,\%1\,p}{(p\,Y-r)^2\,(r-\frac{p}{\mathrm{y}(p)})^2}\Bigg) \Bigg/ ((p^2-r^2)\,\%5^2\,(p\,X-q)\,(q-\frac{p}{X})) - \%6\,p$$

$$(p^2-r^2)\Bigg(-2\,\frac{\%4\,p\,(p^2-q^2)\,\%2\,\%1}{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p^2-r^2)} + 2\,\frac{\%4\,\%3^2\,p}{(p^2-q^2)^2} - 2\,\frac{p\,\%3\,\%2\,\%1}{(p^2-q^2)\,(p^2-r^2)}$$

$$+ 2\,\frac{\%4^2\,\%3\,p}{(p\,X-q)^2\,(q-\frac{p}{\mathrm{x}(p)})^2}\Bigg) \Bigg/ (\%5^2\,(p\,Y-r)\,(r-\frac{p}{Y}))$$

$$-2\,\frac{\%2\,p^2\,(p^2-r^2)\,(p^2-q^2)}{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,\%5\,(p\,X-q)\,(q-\frac{p}{X})}$$

$$
+\frac{\left(\frac{p\,\%3}{p^2-q^2}+\frac{\%4\,p\,(p^2-q^2)}{(p\,X-q)^2\,(q-\frac{p}{X})^2}\right)p\,(p^2-r^2)}{\%5\,(p\,Y-r)\,(r-\frac{p}{Y})}
$$

$$
-2\,\frac{p^2\,\%1\,(p^2-q^2)}{(p^2-r^2)\,\%5\,(p\,X-q)\,(q-\frac{p}{X})}-q\Bigg(
$$

$$
\left(2\,\frac{(2\,\%8-\%7)\,\%2\,\%1\,p\,(p^2-q^2)}{(p^2-r^2)\,\%5\,(p\,X-q)\,(q-\frac{p}{X})}-\frac{(-\%10+2\,\%9)\,\%6\,p\,(p^2-r^2)}{\%5\,(p\,Y-r)\,(r-\frac{p}{Y})}\right)(p^2-q^2)
$$

$$
\Big/((p\,X-q)\,(q-\frac{p}{X}))-
$$

$$
\left(\frac{\%6\,p\,(p^2-r^2)\,(\%10-2\,\%9)}{\%5\,(p\,Y-r)\,(r-\frac{p}{Y})}-2\,\frac{\%2\,\%1\,p\,(p^2-q^2)\,(-2\,\%8+\%7)}{(p^2-r^2)\,\%5\,(p\,X-q)\,(q-\frac{p}{X})}\right)(p\,Y-r)
$$

$$
\left.\left.(r-\frac{p}{Y})/(p^2-r^2)\right)/(q^2-p^2)\right)/(p\,(p^2-q^2))-\Bigg(
$$

$$
-\frac{p^3\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)^2\,(p^2-r^2)}
$$

$$
+\frac{p\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q\,r^2\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)^2\,(p^2-r^2)}
$$

$$
+2\,\frac{p^4\,(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)}
$$

$$
-2\,\frac{q^2\,(p\,X-q)\,(q-\frac{p}{X})\,r^2\,(p\,\mathrm{y}(p)-r)\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)}-\frac{q\,p^3\,(p\,Y-r)\,(r-\frac{p}{Y})}{p^2-r^2}
$$

$$
+\frac{q\,p\,r^2\,(p\,Y-r)\,(r-\frac{p}{Y})}{p^2-r^2}-\frac{p^3\,(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,r}{(p^2-q^2)\,(p^2-r^2)^2}
$$

$$
+\frac{q^2\,(p\,X-q)\,(q-\frac{p}{X})\,p\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,r}{(p^2-q^2)\,(p^2-r^2)^2}-\frac{p^3\,(p\,X-q)\,(q-\frac{p}{X})\,r}{p^2-q^2}
$$

$$
\left.+\frac{q^2\,(p\,X-q)\,(q-\frac{p}{X})\,r\,p}{p^2-q^2}\right)\Bigg(-\%6\,p\,(p^2-q^2)\left(-2\,\frac{\%4\,\%3\,\%2\,p\,(p^2-r^2)}{(p^2-q^2)\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2}\right.
$$

$$
\left.-2\,\frac{\%4\,\%3\,p\,\%1}{(p^2-q^2)\,(p^2-r^2)}+2\,\frac{\%2\,\%1^2\,p}{(p^2-r^2)^2}+2\,\frac{\%2^2\,\%1\,p}{(p\,Y-r)^2\,(r-\frac{p}{Y})^2}\right)\Big/(\%5^2
$$

$$
(p\,X-q)\,(q-\frac{p}{\mathrm{x}(p)}))+2\%4\,\%3\,p\,(p^2-r^2)\left(-2\,\frac{\%4\,p\,(p^2-q^2)\,\%2\,\%1}{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p^2-r^2)}\right.
$$

$$
\left.+2\,\frac{\%4\,\%3^2\,p}{(p^2-q^2)^2}-2\,\frac{p\,\%3\,\%2\,\%1}{(p^2-q^2)\,(p^2-r^2)}+2\,\frac{\%4^2\,\%3\,p}{(p\,X-q)^2\,(q-\frac{p}{X})^2}\right)\Big/((p^2-q^2)
$$

$$\%5^2\,(p\,Y-r)\,(r-\frac{p}{Y}))-r\Bigg($$

$$\frac{\left(-\frac{(2\,\%8-\%7)\,\%6\,p\,(p^2-q^2)}{\%5\,(p\,X-q)\,(q-\frac{p}{X})}+2\,\frac{(-\%10+2\,\%9)\,\%4\,\%3\,p\,(p^2-r^2)}{(p^2-q^2)\,\%5\,(p\,Y-r)\,(r-\frac{p}{Y})}\right)(p^2-r^2)}{(p\,Y-r)\,(r-\frac{p}{Y})}-$$

$$\left(-2\,\frac{\%4\,\%3\,p\,(p^2-r^2)\,(\%10-2\,\%9)}{(p^2-q^2)\,\%5\,(p\,Y-r)\,(r-\frac{p}{Y})}+\frac{\%6\,p\,(p^2-q^2)\,(-2\,\%8+\%7)}{\%5\,(p\,X-q)\,(q-\frac{p}{X})}\right)(p\,X-q)$$

$$(q-\frac{p}{X})/(p^2-q^2)\Bigg)/(r^2-p^2)-2\,\frac{\%4\,p^2\,(p^2-q^2)\,(p^2-r^2)}{(p\,X-q)^2\,(q-\frac{p}{X})^2\,\%5\,(p\,Y-r)\,(r-\frac{p}{Y})}$$

$$-2\,\frac{p^2\,\%3\,(p^2-r^2)}{(p^2-q^2)\,\%5\,(p\,Y-r)\,(r-\frac{p}{Y})}$$

$$+\frac{\left(\frac{p\,\%1}{p^2-r^2}+\frac{\%2\,p\,(p^2-r^2)}{(p\,Y-r)^2\,(r-\frac{p}{Y})^2}\right)p\,(p^2-q^2)}{\%5\,(p\,X-q)\,(q-\frac{p}{X})}\Bigg)\Big/(p\,(p\,\mathrm{y}(p)-r)\,(r-\frac{p}{Y})$$

$$(p^2-q^2))$$

$$\%1:=r-\frac{p\,(p^2-r^2)}{(p\,Y-r)\,(r-\frac{p}{Y})}$$

$$\%2:=\frac{p\,(p\,Y-r)\,(r-\frac{p}{Y})}{p^2-r^2}-r$$

$$\%3:=q-\frac{p\,(p^2-q^2)}{(p\,X-q)\,(q-\frac{p}{X})}$$

$$\%4:=\frac{p\,(p\,X-q)\,(q-\frac{p}{X})}{p^2-q^2}-q$$

$$\%5:=-2\,\frac{\%4\,\%3\,\%2\,\%1}{(p^2-q^2)\,(p^2-r^2)}+\frac{\%4^2\,\%3^2}{(p^2-q^2)^2}+\frac{\%2^2\,\%1^2}{(p^2-r^2)^2}$$

$$\%6:=\frac{\%4\,\%3}{p^2-q^2}+\frac{\%2\,\%1}{p^2-r^2}$$

$$\%7:=\frac{\%6\,p\,(p^2-r^2)}{\%5\,(p\,Y-r)\,(r-\frac{p}{Y})}$$

$$\%8:=\frac{\%4\,\%3\,p\,(p^2-r^2)}{(p^2-q^2)\,\%5\,(p\,Y-r)\,(r-\frac{p}{Y})}$$

$$\%9:=\frac{\%2\,\%1\,p\,(p^2-q^2)}{(p^2-r^2)\,\%5\,(p\,X-q)\,(q-\frac{p}{X})}$$

$$\%10:=\frac{\%6\,p\,(p^2-q^2)}{\%5\,(p\,X-q)\,(q-\frac{p}{X})}$$

$$A_4 = -2\left(-\frac{q\,(p\,X-q)\,(q-\frac{p}{X})}{p^2-q^2}+p\right)\%6\left(2\,\frac{\%9\,\%8\,\%7\,\%13}{(p^2-r^2)\,\%10^2}+2\%9\,\%8\,p\,(p^2-q^2)\,\left(\right.\right.$$

$$-2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,\%5\,\%9\,\%8}{(p^2-q^2)^2\,(p^2-r^2)}-2\,\frac{\%9^2\,\%8}{(p^2-r^2)\,(p\,Y-r)\,(r-\frac{p}{Y})}$$

$$-4\,\frac{\%6^2\,\%5^2\,p}{(p^2-q^2)^3}+2\,\frac{\%6\,\%9\,\%8}{(p\,X-q)\,(q-\frac{p}{X})\,(p^2-r^2)}+4\,\frac{\%6\,\%5\,\%9\,\%8\,p}{(p^2-q^2)\,(p^2-r^2)^2}$$

$$+4\,\frac{\%6\,\%5\,\%9\,\%8\,p}{(p^2-q^2)^2\,(p^2-r^2)}-2\,\frac{\%6\,\%5\,(p\,Y-r)\,(r-\frac{p}{Y})\,\%8}{(p^2-q^2)\,(p^2-r^2)^2}$$

$$+2\,\frac{\%6\,\%5^2\,(p\,X-q)\,(q-\frac{p}{X})}{(p^2-q^2)^3}+2\,\frac{\%6\,\%5\,\%9}{(p^2-q^2)\,(p\,Y-r)\,(r-\frac{p}{Y})}$$

$$-4\,\frac{\%9^2\,\%8^2\,p}{(p^2-r^2)^3}+2\,\frac{\%9\,\%8^2\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-r^2)^3}$$

$$\left.-2\,\frac{\%6^2\,\%5}{(p^2-q^2)\,(p\,X-q)\,(q-\frac{p}{X})}\right)\Big/((p^2-r^2)\,\%10^2\,(p\,X-q)\,(q-\frac{p}{X}))$$

$$+4\,\frac{\%9\,\%8\,p^2\,(p^2-q^2)}{(p^2-r^2)^2\,\%10\,(p\,\mathrm{x}(p)-q)\,(q-\frac{p}{X})}$$

$$-2\,\frac{\%9\,\%8\left(\frac{(\mu-\nu)\,\%6\,p\,(p^2-q^2)}{(p\,X-q)^2\,(q-\frac{p}{X})^2}+\frac{(n+m)\,p\,q\,\%14}{p^2-q^2}+\frac{(\mu-\nu)\,p\,\%5}{p^2-q^2}+\%3\right)}{(p^2-r^2)\,\%10}$$

$$-\frac{\%15\,\%12\,\%13}{\%10^2}+2\,\frac{\%9\,p\,(p^2-q^2)}{(p\,Y-r)\,(r-\frac{p}{\mathrm{y}(p)})\,\%10\,(p\,X-q)\,(q-\frac{p}{X})}$$

$$+\frac{\left(\frac{p\,\%5}{p^2-q^2}+\frac{\%6\,p\,(p^2-q^2)}{(p\,X-q)^2\,(q-\frac{p}{X})^2}\right)\%12}{\%10}-2\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,\%8\,p\,(p^2-q^2)}{(p^2-r^2)^2\,\%10\,(p\,X-q)\,(q-\frac{p}{X})}$$

$$+\frac{\%15\,m\,p\,q\,\%14}{\%10\,(p^2-q^2)}-q((2$$

$$(n+\mu-\nu+2\,\frac{\%9\,\%8\,\%7}{(p^2-r^2)\,\%10}+2\,\frac{\%6\,\%5\,\%12}{(p^2-q^2)\,\%10}-\frac{\%15\,\%12}{\%10}-\frac{\%15\,\%7}{\%10})\,\%9\,\%8\,p$$

$$(p^2-q^2)\Big/((p^2-r^2)\,\%10\,(p\,X-q)\,(q-\frac{p}{X}))$$

$$+(-\%17+2\,\%16)\,(m+2\,\frac{\%9\,\%8\,\%7}{(p^2-r^2)\,\%10}-\frac{\%15\,\%12}{\%10}))(p^2-q^2)\Big/((p\,X-q)$$

$$(q-\frac{p}{X}))-((m-2\,\frac{\%9\,\%8\,\%7}{(p^2-r^2)\,\%10}+\frac{\%15\,\%12}{\%10})\,(\%17-2\,\%16)-2\%9\,\%8\,p$$

$$(p^2-q^2)\,(n-\mu+\nu-2\,\frac{\%9\,\%8\,\%7}{(p^2-r^2)\,\%10}-2\,\frac{\%6\,\%5\,\%12}{(p^2-q^2)\,\%10}+\frac{\%15\,\%12}{\%10}+\frac{\%15\,\%7}{\%10})$$

$$\Big/((p^2-r^2)\,\%10\,(p\,X-q)\,(q-\frac{p}{X})))(p\,Y-r)\,(r-\frac{p}{Y})/(p^2-r^2))/(q^2-p^2$$

$$)\Bigg)\Big/(p\,(p^2-q^2))-\Bigg(-\frac{p^3\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)^2\,(p^2-r^2)}$$

$$+\frac{p\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q\,r^2\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)^2\,(p^2-r^2)}$$

$$+2\,\frac{p^4\,(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)}$$

$$-2\,\frac{q^2\,(p\,X-q)\,(q-\frac{p}{X})\,r^2\,(p\,\mathrm{y}(p)-r)\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)}-\frac{q\,p^3\,(p\,Y-r)\,(r-\frac{p}{Y})}{p^2-r^2}$$

$$+\frac{q\,p\,r^2\,(p\,Y-r)\,(r-\frac{p}{Y})}{p^2-r^2}-\frac{p^3\,(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,r}{(p^2-q^2)\,(p^2-r^2)^2}$$

$$+\frac{q^2\,(p\,X-q)\,(q-\frac{p}{X})\,p\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,r}{(p^2-q^2)\,(p^2-r^2)^2}-\frac{p^3\,(p\,X-q)\,(q-\frac{p}{X})\,r}{p^2-q^2}$$

$$+\frac{q^2\,(p\,X-q)\,(q-\frac{p}{X})\,r\,p}{p^2-q^2}\Bigg)\Bigg($$

$$\frac{\%15\left(\frac{(\mu-\nu)\,\%6\,p\,(p^2-q^2)}{(p\,X-q)^2\,(q-\frac{p}{X})^2}+\frac{(n+m)\,p\,q\,\%14}{p^2-q^2}+\frac{(\mu-\nu)\,p\,\%5}{p^2-q^2}+\%3\right)}{\%10}$$

$$-\frac{\%15\,\%7\,\%13}{\%10^2}-\%15\,p\,(p^2-q^2)\Bigg(-2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,\%5\,\%9\,\%8}{(p^2-q^2)^2\,(p^2-r^2)}$$

$$-2\,\frac{\%9^2\,\%8}{(p^2-r^2)\,(p\,Y-r)\,(r-\frac{p}{\mathrm{y}(p)})}-4\,\frac{\%6^2\,\%5^2\,p}{(p^2-q^2)^3}$$

$$+2\,\frac{\%6\,\%9\,\%8}{(p\,X-q)\,(q-\frac{p}{X})\,(p^2-r^2)}+4\,\frac{\%6\,\%5\,\%9\,\%8\,p}{(p^2-q^2)\,(p^2-r^2)^2}+4\,\frac{\%6\,\%5\,\%9\,\%8\,p}{(p^2-q^2)^2\,(p^2-r^2)}$$

$$-2\,\frac{\%6\,\%5\,(p\,Y-r)\,(r-\frac{p}{Y})\,\%8}{(p^2-q^2)\,(p^2-r^2)^2}+2\,\frac{\%6\,\%5^2\,(p\,X-q)\,(q-\frac{p}{X})}{(p^2-q^2)^3}$$

$$+2\,\frac{\%6\,\%5\,\%9}{(p^2-q^2)\,(p\,Y-r)\,(r-\frac{p}{Y})}-4\,\frac{\%9^2\,\%8^2\,p}{(p^2-r^2)^3}$$

$$+2\,\frac{\%9\,\%8^2\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-r^2)^3}-2\,\frac{\%6^2\,\%5}{(p^2-q^2)\,(p\,X-q)\,(q-\frac{p}{X})}\Bigg)\Big/(\%10^2$$

$$(p\,X-q)\,(q-\frac{p}{X}))-r((-$$

$$(n+\mu-\nu+2\,\frac{\%9\,\%8\,\%7}{(p^2-r^2)\,\%10}+2\,\frac{\%6\,\%5\,\%12}{(p^2-q^2)\,\%10}-\frac{\%15\,\%12}{\%10}-\frac{\%15\,\%7}{\%10})\,\%15\,p$$

$$(p^2-q^2)\Big/(\%10\,(p\,X-q)\,(q-\frac{p}{X}))$$

$$+(-\%17+2\,\%16)\,(-\%11-\frac{\%15\,\%7}{\%10}+2\,\frac{\%6\,\%5\,\%12}{(p^2-q^2)\,\%10}))(p^2-r^2)\Big/((p\,Y-r)$$

$$(r-\frac{p}{Y}))-((-\%11+\frac{\%15\,\%7}{\%10}-2\,\frac{\%6\,\%5\,\%12}{(p^2-q^2)\,\%10})\,(\%17-2\,\%16)+\%15\,p\,(p^2-q^2)$$

$$(n-\mu+\nu-2\,\frac{\%9\,\%8\,\%7}{(p^2-r^2)\,\%10}-2\,\frac{\%6\,\%5\,\%12}{(p^2-q^2)\,\%10}+\frac{\%15\,\%12}{\%10}+\frac{\%15\,\%7}{\%10})\Big/(\%10$$

$$(p\,X-q)\,(q-\frac{p}{X})))(p\,X-q)\,(q-\frac{p}{X})/(p^2-q^2))/(r^2-p^2)$$

$$-2\,\frac{\%6\,\%5\,m\,p\,q\,\%14}{(p^2-q^2)^2\,\%10}+2\,\frac{\%6\,\%5\,\%12\,\%13}{(p^2-q^2)\,\%10^2}+\Bigg(\frac{(p\,X-q)\,(q-\frac{p}{X})\,\%5}{(p^2-q^2)^2}$$

$$+\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,\%8}{(p^2-r^2)^2}-\frac{\%9}{(p\,Y-r)\,(r-\frac{p}{Y})}-\frac{\%6}{(p\,X-q)\,(q-\frac{p}{X})}$$

$$-2\,\frac{\%9\,\%8\,p}{(p^2-r^2)^2}-2\,\frac{\%6\,\%5\,p}{(p^2-q^2)^2}\Bigg)\,p\,(p^2-q^2)\Big/(\%10\,(p\,X-q)\,(q-\frac{p}{X}))$$

$$-2\,\frac{\%6\,p\,(p^2-q^2)\,\%12}{(p\,X-q)^2\,(q-\frac{p}{X})^2\,\%10}-2\,\frac{p\,\%5\,\%12}{(p^2-q^2)\,\%10}$$

$$+\frac{\left(\frac{p\,\%5}{p^2-q^2}+\frac{\%6\,p\,(p^2-q^2)}{(p\,X-q)^2\,(q-\frac{p}{X})^2}\right)\%7}{\%10}\Bigg)\Big/(p\,(p\,\mathrm{y}(p)-r)\,(r-\frac{p}{Y})\,(p^2-q^2))$$

$$\%1:=\frac{p^2-r^2}{(p\,\mathrm{y}(p)-r)\,(r-\frac{p}{Y})}-\frac{(p\,Y-r)\,(r-\frac{p}{Y})}{p^2-r^2}$$

$$\%2:=-4\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,\mathrm{y}(p)-r)^2\,(r-\frac{p}{Y})^2\,p\,q\,r^4}{(p^2-q^2)^3\,(p^2-r^2)^2}$$

$$+2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,q\,p^2\,r^3}{(p^2-q^2)\,(p^2-r^2)}$$

$$+2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,q^3\,p^2\,r}{(p^2-q^2)\,(p^2-r^2)}$$

$$-2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,p^4\,q\,r}{(p^2-q^2)^3\,(p^2-r^2)^3}$$

$$+2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,p^2\,q\,r^3}{(p^2-q^2)^3\,(p^2-r^2)^3}$$

$$-4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)\,(r-\frac{p}{Y})\,p^3\,r^3}{(p^2-q^2)^2\,(p^2-r^2)}$$

$$-2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,q\,p^4\,r}{(p^2-q^2)\,(p^2-r^2)}$$

$$+2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,p^2\,q^3\,r}{(p^2-q^2)^3\,(p^2-r^2)^3}$$

$$
\begin{aligned}
&- 2 \frac{(p\,X - q)^3 \,(q - \frac{p}{X})^3 \,(p\,Y - r)\,(r - \frac{p}{Y})\,p^4\,q\,r}{(p^2 - q^2)^3\,(p^2 - r^2)} \\
&+ 4 \frac{(p\,X - q)^2 \,(q - \frac{p}{X})^2 \,(p\,Y - r)^2\,(r - \frac{p}{Y})^2\,p^2\,r^4}{(p^2 - q^2)^2\,(p^2 - r^2)^2} \\
&- 4 \frac{(p\,X - q)^2 \,(q - \frac{p}{X})^2 \,(p\,Y - r)^3\,(r - \frac{p}{Y})^3\,q^4\,p\,r}{(p^2 - q^2)^2\,(p^2 - r^2)^3} \\
&+ 4 \frac{(p\,X - q)^2 \,(q - \frac{p}{X})^2 \,(p\,Y - r)^2\,(r - \frac{p}{Y})^2\,q^4\,p^2}{(p^2 - q^2)^2\,(p^2 - r^2)^2} \\
&- 4 \frac{(p\,X - q)^2 \,(q - \frac{p}{X})^2 \,(p\,Y - r)\,(r - \frac{p}{Y})\,q^4\,r\,p}{(p^2 - q^2)^2\,(p^2 - r^2)} \\
&+ \frac{q^4\,(p\,X - q)^2 \,(q - \frac{p}{X})^2\,r^2}{(p^2 - q^2)^2} \\
&+ 4 \frac{(p\,Y - r)^2\,(r - \frac{p}{Y})^2\,p^3\,(p\,X - q)\,(q - \frac{p}{\mathrm{x}(p)})\,q\,r^2}{(p^2 - r^2)^2\,(p^2 - q^2)} \\
&+ 2 \frac{(p\,Y - r)^2\,(r - \frac{p}{Y})^2\,q^4\,(p\,X - q)^2\,(q - \frac{p}{X})^2\,r^2}{(p^2 - r^2)^2\,(p^2 - q^2)^2} \\
&+ 4 \frac{(p\,Y - r)^2\,(r - \frac{p}{Y})^2\,q^3\,p\,(p\,X - q)\,(q - \frac{p}{X})\,r^2}{(p^2 - r^2)^2\,(p^2 - q^2)} \\
&+ \frac{(p\,X - q)^2 \,(q - \frac{p}{X})^2\,p^4\,(p\,\mathrm{y}(p) - r)^4\,(r - \frac{p}{Y})^4\,r^2}{(p^2 - q^2)^2\,(p^2 - r^2)^4} \\
&- 2 \frac{(p\,X - q)^2 \,(q - \frac{p}{X})^2\,p^2\,(p\,Y - r)^4\,(r - \frac{p}{Y})^4\,r^2\,q^2}{(p^2 - q^2)^2\,(p^2 - r^2)^4} \\
&+ 2 \frac{(p\,X - q)^3 \,(q - \frac{p}{X})^3 \,(p\,Y - r)\,(r - \frac{p}{Y})\,p^2\,q\,r^3}{(p^2 - q^2)^3\,(p^2 - r^2)} \\
&+ 2 \frac{(p\,X - q)^3 \,(q - \frac{p}{X})^3 \,(p\,Y - r)\,(r - \frac{p}{Y})\,p^2\,q^3\,r}{(p^2 - q^2)^3\,(p^2 - r^2)} \\
&- 2 \frac{(p\,X - q)^3 \,(q - \frac{p}{X})^3 \,(p\,Y - r)\,(r - \frac{p}{Y})\,q^3\,r^3}{(p^2 - q^2)^3\,(p^2 - r^2)} \\
&- 4 \frac{(p\,X - q)^2 \,(q - \frac{p}{X})^2 \,(p\,Y - r)^3\,(r - \frac{p}{Y})^3\,p^3\,r^3}{(p^2 - q^2)^2\,(p^2 - r^2)^3} \\
&- 2 \frac{(p\,X - q)^3 \,(q - \frac{p}{X})^3 \,(p\,Y - r)^3\,(r - \frac{p}{Y})^3\,q^3\,r^3}{(p^2 - q^2)^3\,(p^2 - r^2)^3} \\
&+ 2 \frac{(p\,Y - r)^2\,(r - \frac{p}{Y})^2\,p^4\,(p\,X - q)^2\,(q - \frac{p}{X})^2\,q^2}{(p^2 - r^2)^2\,(p^2 - q^2)^2} \\
&- 2 \frac{(p\,Y - r)^2\,(r - \frac{p}{Y})^2\,p^2\,(p\,X - q)^4\,(q - \frac{p}{X})^4\,q^2\,r^2}{(p^2 - r^2)^2\,(p^2 - q^2)^4} \\
&+ 4 \frac{(p\,Y - r)^2\,(r - \frac{p}{Y})^2\,p^3\,(p\,X - q)^3\,(q - \frac{p}{X})^3\,q\,r^2}{(p^2 - r^2)^2\,(p^2 - q^2)^3}
\end{aligned}
$$

$$
\begin{aligned}
&- 2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q\,p^4\,r}{(p^2-q^2)\,(p^2-r^2)^3} \\
&+ \frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,(p\,X-q)^4\,(q-\frac{p}{X})^4\,q^2\,r^4}{(p^2-r^2)^2\,(p^2-q^2)^4} \\
&+ 4\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p\,(p\,X-q)^3\,(q-\frac{p}{X})^3\,q^3\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^3} \\
&- 16\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^2\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q^2\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^2} \\
&+ 4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,p^3\,(p\,\mathrm{y}(p)-r)\,(r-\frac{p}{Y})\,r\,q^2}{(p^2-q^2)^2\,(p^2-r^2)} \\
&+ \frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)^4\,(r-\frac{p}{Y})^4\,r^2\,q^4}{(p^2-q^2)^2\,(p^2-r^2)^4} \\
&+ 4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,r^3\,p\,(p\,Y-r)\,(r-\frac{p}{Y})\,q^2}{(p^2-q^2)^2\,(p^2-r^2)} \\
&+ 2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q\,p^2\,r^3}{(p^2-q^2)\,(p^2-r^2)^3} \\
&+ 2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q^3\,p^2\,r}{(p^2-q^2)\,(p^2-r^2)^3} \\
&- 2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q^3\,r^3}{(p^2-q^2)\,(p^2-r^2)^3} \\
&- 4\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q\,p\,r^4}{(p^2-q^2)\,(p^2-r^2)^2} \\
&- 2\,\frac{p^2\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q^2\,r^2}{(p^2-q^2)^2} \\
&+ \frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^4\,(p\,X-q)^4\,(q-\frac{p}{X})^4\,q^2}{(p^2-r^2)^2\,(p^2-q^2)^4} \\
&- 2\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q^2\,p^2\,r^2}{(p^2-r^2)^2} + \frac{p^4\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,r^2}{(p^2-q^2)^2} \\
&- 4\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q^3\,p^3}{(p^2-q^2)\,(p^2-r^2)^2} \\
&- 2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,q^3\,r^3}{(p^2-q^2)\,(p^2-r^2)} \\
&+ \frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q^2\,p^4}{(p^2-r^2)^2} + \frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q^2\,r^4}{(p^2-r^2)^2} \\
&+ 4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,p^3\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,r\,q^2}{(p^2-q^2)^2\,(p^2-r^2)^3} \\
&+ 4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,p\,(p\,\mathrm{y}(p)-r)^3\,(r-\frac{p}{Y})^3\,r^3\,q^2}{(p^2-q^2)^2\,(p^2-r^2)^3}
\end{aligned}
$$

$$+\,2\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^4\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^2}$$

$$+\,2\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q^2\,r^4}{(p^2-r^2)^2\,(p^2-q^2)^2}$$

$$-\,4\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^3\,q^3}{(p^2-q^2)^3\,(p^2-r^2)^2}$$

$$\%3 := \frac{p^2-q^2}{(p\,\mathrm{x}(p)-q)\,(q-\frac{p}{X})}$$

$$\%4 := \%3 - \frac{(p\,X-q)\,(q-\frac{p}{X})}{p^2-q^2}$$

$$\%5 := q - \frac{p\,(p^2-q^2)}{(p\,X-q)\,(q-\frac{p}{X})}$$

$$\%6 := \frac{p\,(p\,X-q)\,(q-\frac{p}{X})}{p^2-q^2} - q$$

$$\%7 := \mu+\nu+\frac{(\mu-\nu)\,\%6\,\%5}{p^2-q^2}+\frac{(n+m)\,p\,q\,\%4}{p^2-q^2}$$

$$-\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,(p^2-q^2)\,(p\,X-q)\,(q-\frac{p}{X})\,p\,r\,\%1}{\%2}$$

$$\%8 := r - \frac{p\,(p^2-r^2)}{(p\,Y-r)\,(r-\frac{p}{Y})}$$

$$\%9 := \frac{p\,(p\,Y-r)\,(r-\frac{p}{Y})}{p^2-r^2} - r$$

$$\%10 := -2\,\frac{\%6\,\%5\,\%9\,\%8}{(p^2-q^2)\,(p^2-r^2)}+\frac{\%6^2\,\%5^2}{(p^2-q^2)^2}+\frac{\%9^2\,\%8^2}{(p^2-r^2)^2}$$

$$\%11 := \frac{(p^2-r^2)\,(p\,Y-r)\,(r-\frac{p}{Y})\,(p^2-q^2)\,(p\,X-q)\,(q-\frac{p}{\mathrm{x}(p)})}{\%2}$$

$$\%12 := \mu+\nu+\frac{(\mu-\nu)\,\%9\,\%8}{p^2-r^2}+\frac{(n-\%11)\,p\,r\,\%1}{p^2-r^2}+\frac{m\,p\,q\,\%4}{p^2-q^2}$$

$$\%13 := -2\,\frac{\%6\,p\,(p^2-q^2)\,\%9\,\%8}{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p^2-r^2)}+2\,\frac{\%6\,\%5^2\,p}{(p^2-q^2)^2}-2\,\frac{p\,\%5\,\%9\,\%8}{(p^2-q^2)\,(p^2-r^2)}$$

$$+\,2\,\frac{\%6^2\,\%5\,p}{(p\,X-q)^2\,(q-\frac{p}{X})^2}$$

$$\%14 := -\frac{(p^2-q^2)^2}{(p\,X-q)^2\,(q-\frac{p}{X})^2} - 1$$

$$\%15 := \frac{\%6\,\%5}{p^2-q^2}+\frac{\%9\,\%8}{p^2-r^2}$$

$$\%16 := \frac{\%9\,\%8\,p\,(p^2-q^2)}{(p^2-r^2)\,\%10\,(p\,X-q)\,(q-\frac{p}{X})}$$

$$\%17 := \frac{\%15\,p\,(p^2-q^2)}{\%10\,(p\,X-q)\,(q-\frac{p}{X})}$$

$$A_5 = -2 \left( - \frac{q\,(p\,X - q)\,(q - \frac{p}{X})}{p^2 - q^2} + p \right) \%4 \Bigg( - q((-$$

$$(n + \mu - \nu + 2\,\frac{\%2\,\%1\,\%10}{(p^2 - r^2)\,\%5} + 2\,\frac{\%4\,\%3\,\%13}{(p^2 - q^2)\,\%5} - \frac{\%11\,\%13}{\%5} - \frac{\%11\,\%10}{\%5})\,\%11\,p$$

$$(p^2 - r^2) \Big/ (\%5\,(p\,Y - r)\,(r - \frac{p}{Y}))$$

$$+ (2\,\%15 - \%14)\,(m + 2\,\frac{\%2\,\%1\,\%10}{(p^2 - r^2)\,\%5} - \frac{\%11\,\%13}{\%5}))(p^2 - q^2) \Big/ ((p\,X - q)$$

$$(q - \frac{p}{X})) - ((m - 2\,\frac{\%2\,\%1\,\%10}{(p^2 - r^2)\,\%5} + \frac{\%11\,\%13}{\%5})\,(-2\,\%15 + \%14) + \%11\,p\,(p^2 - r^2)$$

$$(n - \mu + \nu - 2\,\frac{\%2\,\%1\,\%10}{(p^2 - r^2)\,\%5} - 2\,\frac{\%4\,\%3\,\%13}{(p^2 - q^2)\,\%5} + \frac{\%11\,\%13}{\%5} + \frac{\%11\,\%10}{\%5}) \Big/ (\%5$$

$$(p\,Y - r)\,(r - \frac{p}{Y})))(p\,Y - r)\,(r - \frac{p}{Y})/(p^2 - r^2))/(q^2 - p^2)$$

$$+ \frac{\%11 \left( \frac{(\mu - \nu)\,\%2\,p\,(p^2 - r^2)}{(p\,Y - r)^2\,(r - \frac{p}{Y})^2} + \frac{(n - \%12)\,p\,r\,\%16}{p^2 - r^2} + \%6 + \frac{(\mu - \nu)\,p\,\%1}{p^2 - r^2} \right)}{\%5}$$

$$+ 2\,\frac{\%2\,\%1\,(p\,Y - r)\,(r - \frac{p}{Y})\,(p^2 - q^2)\,(p\,X - q)\,(q - \frac{p}{X})\,p\,r\,\%16}{\%5\,(p^2 - r^2)\,\%8}$$

$$- 2\,\frac{p\,\%1\,\%10}{(p^2 - r^2)\,\%5} - 2\,\frac{\%2\,p\,(p^2 - r^2)\,\%10}{(p\,Y - r)^2\,(r - \frac{p}{Y})^2\,\%5} + \Bigg( \frac{(p\,X - q)\,(q - \frac{p}{X})\,\%3}{(p^2 - q^2)^2}$$

$$+ \frac{(p\,Y - r)\,(r - \frac{p}{Y})\,\%1}{(p^2 - r^2)^2} - \frac{\%2}{(p\,Y - r)\,(r - \frac{p}{Y})} - \frac{\%4}{(p\,X - q)\,(q - \frac{p}{X})}$$

$$- 2\,\frac{\%2\,\%1\,p}{(p^2 - r^2)^2} - 2\,\frac{\%4\,\%3\,p}{(p^2 - q^2)^2} \Bigg)\,p\,(p^2 - r^2) \Big/ (\%5\,(p\,Y - r)\,(r - \frac{p}{Y}))$$

$$+ \frac{\left( \frac{p\,\%1}{p^2 - r^2} + \frac{\%2\,p\,(p^2 - r^2)}{(p\,Y - r)^2\,(r - \frac{p}{Y})^2} \right) \%13}{\%5} + 2\,\frac{\%2\,\%1\,\%10\,\%17}{(p^2 - r^2)\,\%5^2} - \frac{\%11\,\%13\,\%17}{\%5^2} -$$

$$\%11\,p\,(p^2 - r^2) \Bigg( - 2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,\%3\,\%2\,\%1}{(p^2 - q^2)^2\,(p^2 - r^2)}$$

$$- 2\,\frac{\%2^2\,\%1}{(p^2 - r^2)\,(p\,Y - r)\,(r - \frac{p}{Y})} - 4\,\frac{\%4^2\,\%3^2\,p}{(p^2 - q^2)^3}$$

$$+ 2\,\frac{\%4\,\%2\,\%1}{(p\,X - q)\,(q - \frac{p}{X})\,(p^2 - r^2)} + 4\,\frac{\%4\,\%3\,\%2\,\%1\,p}{(p^2 - q^2)\,(p^2 - r^2)^2} + 4\,\frac{\%4\,\%3\,\%2\,\%1\,p}{(p^2 - q^2)^2\,(p^2 - r^2)}$$

$$
-2\,\frac{\%4\,\%3\,(p\,Y-r)\,(r-\frac{p}{Y})\,\%1}{(p^2-q^2)\,(p^2-r^2)^2}+2\,\frac{\%4\,\%3^2\,(p\,X-q)\,(q-\frac{p}{X})}{(p^2-q^2)^3}
$$

$$
+2\,\frac{\%4\,\%3\,\%2}{(p^2-q^2)\,(p\,Y-r)\,(r-\frac{p}{Y})}-4\,\frac{\%2^2\,\%1^2\,p}{(p^2-r^2)^3}
$$

$$
\left.+2\,\frac{\%2\,\%1^2\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-r^2)^3}-2\,\frac{\%4^2\,\%3}{(p^2-q^2)\,(p\,X-q)\,(q-\frac{p}{X})}\right)\Bigg/(\%5^2
$$

$$
\left.(p\,Y-r)\,(r-\frac{p}{\mathrm{y}(p)})\right)\Bigg/(p\,(p^2-q^2))-\Bigg(
$$

$$
-\frac{p^3\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)^2\,(p^2-r^2)}
$$

$$
+\frac{p\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q\,r^2\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)^2\,(p^2-r^2)}
$$

$$
+2\,\frac{p^4\,(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)}
$$

$$
-2\,\frac{q^2\,(p\,X-q)\,(q-\frac{p}{X})\,r^2\,(p\,\mathrm{y}(p)-r)\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)}-\frac{q\,p^3\,(p\,Y-r)\,(r-\frac{p}{Y})}{p^2-r^2}
$$

$$
+\frac{q\,p\,r^2\,(p\,Y-r)\,(r-\frac{p}{Y})}{p^2-r^2}-\frac{p^3\,(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,r}{(p^2-q^2)\,(p^2-r^2)^2}
$$

$$
+\frac{q^2\,(p\,X-q)\,(q-\frac{p}{X})\,p\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,r}{(p^2-q^2)\,(p^2-r^2)^2}-\frac{p^3\,(p\,X-q)\,(q-\frac{p}{X})\,r}{p^2-q^2}
$$

$$
\left.+\frac{q^2\,(p\,X-q)\,(q-\frac{p}{X})\,r\,p}{p^2-q^2}\right)\Bigg(-\frac{\%11\,\%10\,\%17}{\%5^2}
$$

$$
+\frac{\left(\frac{p\,\%1}{p^2-r^2}+\frac{\%2\,p\,(p^2-r^2)}{(p\,Y-r)^2\,(r-\frac{p}{Y})^2}\right)\%10}{\%5}+2\,\frac{\%4\,\%3\,\%13\,\%17}{(p^2-q^2)\,\%5^2}+2\%4\,\%3\,p
$$

$$
(p^2-r^2)\Bigg(-2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,\%3\,\%2\,\%1}{(p^2-q^2)^2\,(p^2-r^2)}-2\,\frac{\%2^2\,\%1}{(p^2-r^2)\,(p\,Y-r)\,(r-\frac{p}{Y})}
$$

$$
-4\,\frac{\%4^2\,\%3^2\,p}{(p^2-q^2)^3}+2\,\frac{\%4\,\%2\,\%1}{(p\,X-q)\,(q-\frac{p}{X})\,(p^2-r^2)}+4\,\frac{\%4\,\%3\,\%2\,\%1\,p}{(p^2-q^2)\,(p^2-r^2)^2}
$$

$$
+4\,\frac{\%4\,\%3\,\%2\,\%1\,p}{(p^2-q^2)^2\,(p^2-r^2)}-2\,\frac{\%4\,\%3\,(p\,Y-r)\,(r-\frac{p}{Y})\,\%1}{(p^2-q^2)\,(p^2-r^2)^2}
$$

$$
+2\,\frac{\%4\,\%3^2\,(p\,X-q)\,(q-\frac{p}{X})}{(p^2-q^2)^3}+2\,\frac{\%4\,\%3\,\%2}{(p^2-q^2)\,(p\,Y-r)\,(r-\frac{p}{Y})}
$$

$$
-4\,\frac{\%2^2\,\%1^2\,p}{(p^2-r^2)^3}+2\,\frac{\%2\,\%1^2\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-r^2)^3}
$$

$$- 2 \frac{\%4^2\,\%3}{(p^2 - q^2)\,(p\,X - q)\,(q - \frac{p}{X})}\Bigg) \Bigg/ ((p^2 - q^2)\,\%5^2\,(p\,Y - r)\,(r - \frac{p}{Y}))$$

$$- 2 \frac{\%4\,\%3 \left( \frac{(\mu - \nu)\,\%2\,p\,(p^2 - r^2)}{(p\,Y - r)^2\,(r - \frac{p}{Y})^2} + \frac{(n - \%12)\,p\,r\,\%16}{p^2 - r^2} + \%6 + \frac{(\mu - \nu)\,p\,\%1}{p^2 - r^2} \right)}{(p^2 - q^2)\,\%5}$$

$$- \frac{\%11\,(p\,Y - r)\,(r - \frac{p}{Y})\,(p^2 - q^2)\,(p\,X - q)\,(q - \frac{p}{\mathrm{x}(p)})\,p\,r\,\%16}{\%5\,\%8}$$

$$- 2 \frac{(p\,X - q)\,(q - \frac{p}{X})\,\%3\,p\,(p^2 - r^2)}{(p^2 - q^2)^2\,\%5\,(p\,Y - r)\,(r - \frac{p}{Y})} - r((2$$

$$(n + \mu - \nu + 2 \frac{\%2\,\%1\,\%10}{(p^2 - r^2)\,\%5} + 2 \frac{\%4\,\%3\,\%13}{(p^2 - q^2)\,\%5} - \frac{\%11\,\%13}{\%5} - \frac{\%11\,\%10}{\%5})\,\%4\,\%3\,p$$

$$(p^2 - r^2) \Big/ ((p^2 - q^2)\,\%5\,(p\,Y - r)\,(r - \frac{p}{Y}))$$

$$+ (2\,\%15 - \%14)\,(-\%12 - \frac{\%11\,\%10}{\%5} + 2 \frac{\%4\,\%3\,\%13}{(p^2 - q^2)\,\%5}))(p^2 - r^2) \Big/ ((p\,Y - r)$$

$$(r - \frac{p}{Y})) - ((-\%12 + \frac{\%11\,\%10}{\%5} - 2 \frac{\%4\,\%3\,\%13}{(p^2 - q^2)\,\%5})\,(-2\,\%15 + \%14) - 2\%4\,\%3\,p$$

$$(p^2 - r^2)\,(n - \mu + \nu - 2 \frac{\%2\,\%1\,\%10}{(p^2 - r^2)\,\%5} - 2 \frac{\%4\,\%3\,\%13}{(p^2 - q^2)\,\%5} + \frac{\%11\,\%13}{\%5} + \frac{\%11\,\%10}{\%5})$$

$$\Big/ ((p^2 - q^2)\,\%5\,(p\,Y - r)\,(r - \frac{p}{Y})))(p\,X - q)\,(q - \frac{p}{X})/(p^2 - q^2))/(r^2 - p^2)$$

$$+ 2 \frac{\%4\,p\,(p^2 - r^2)}{(p\,X - q)\,(q - \frac{p}{\mathrm{x}(p)})\,\%5\,(p\,Y - r)\,(r - \frac{p}{Y})}$$

$$+ 4 \frac{\%4\,\%3\,p^2\,(p^2 - r^2)}{(p^2 - q^2)^2\,\%5\,(p\,\mathrm{y}(p) - r)\,(r - \frac{p}{Y})}\Bigg) \Bigg/ (p\,(p\,Y - r)\,(r - \frac{p}{Y})\,(p^2 - q^2))$$

$$\%1 := r - \frac{p\,(p^2 - r^2)}{(p\,Y - r)\,(r - \frac{p}{Y})}$$

$$\%2 := \frac{p\,(p\,Y - r)\,(r - \frac{p}{Y})}{p^2 - r^2} - r$$

$$\%3 := q - \frac{p\,(p^2 - q^2)}{(p\,X - q)\,(q - \frac{p}{X})}$$

$$\%4 := \frac{p\,(p\,X - q)\,(q - \frac{p}{X})}{p^2 - q^2} - q$$

$$\%5 := -2 \frac{\%4\,\%3\,\%2\,\%1}{(p^2 - q^2)\,(p^2 - r^2)} + \frac{\%4^2\,\%3^2}{(p^2 - q^2)^2} + \frac{\%2^2\,\%1^2}{(p^2 - r^2)^2}$$

$$\%6 := \frac{p^2 - r^2}{(p\,Y - r)\,(r - \frac{p}{Y})}$$

$$\%7 := \%6 - \frac{(p\,Y - r)\,(r - \frac{p}{Y})}{p^2 - r^2}$$

$$
\begin{aligned}
\%8 := & -4\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p\,q\,r^4}{(p^2-q^2)^3\,(p^2-r^2)^2}\\
&+2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,q\,p^2\,r^3}{(p^2-q^2)\,(p^2-r^2)}\\
&+2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,q^3\,p^2\,r}{(p^2-q^2)\,(p^2-r^2)}\\
&-2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,p^4\,q\,r}{(p^2-q^2)^3\,(p^2-r^2)^3}\\
&+2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,p^2\,q\,r^3}{(p^2-q^2)^3\,(p^2-r^2)^3}\\
&-4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)\,(r-\frac{p}{Y})\,p^3\,r^3}{(p^2-q^2)^2\,(p^2-r^2)}\\
&-2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,q\,p^4\,r}{(p^2-q^2)\,(p^2-r^2)}\\
&+2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,p^2\,q^3\,r}{(p^2-q^2)^3\,(p^2-r^2)^3}\\
&-2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)\,(r-\frac{p}{Y})\,p^4\,q\,r}{(p^2-q^2)^3\,(p^2-r^2)}\\
&+4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^2\,r^4}{(p^2-q^2)^2\,(p^2-r^2)^2}\\
&-4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q^4\,p\,r}{(p^2-q^2)^2\,(p^2-r^2)^3}\\
&+4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q^4\,p^2}{(p^2-q^2)^2\,(p^2-r^2)^2}\\
&-4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)\,(r-\frac{p}{Y})\,q^4\,r\,p}{(p^2-q^2)^2\,(p^2-r^2)}\\
&+\frac{q^4\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,r^2}{(p^2-q^2)^2}\\
&+4\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^3\,(p\,X-q)\,(q-\frac{p}{\mathrm{x}(p)})\,q\,r^2}{(p^2-r^2)^2\,(p^2-q^2)}\\
&+2\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q^4\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^2}\\
&+4\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q^3\,p\,(p\,X-q)\,(q-\frac{p}{X})\,r^2}{(p^2-r^2)^2\,(p^2-q^2)}\\
&+\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,p^4\,(p\,\mathrm{y}(p)-r)^4\,(r-\frac{p}{Y})^4\,r^2}{(p^2-q^2)^2\,(p^2-r^2)^4}\\
&-2\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,p^2\,(p\,Y-r)^4\,(r-\frac{p}{Y})^4\,r^2\,q^2}{(p^2-q^2)^2\,(p^2-r^2)^4}
\end{aligned}
$$

$$
\begin{aligned}
&+2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)\,(r-\frac{p}{Y})\,p^2\,q\,r^3}{(p^2-q^2)^3\,(p^2-r^2)}\\
&+2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)\,(r-\frac{p}{Y})\,p^2\,q^3\,r}{(p^2-q^2)^3\,(p^2-r^2)}\\
&-2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)\,(r-\frac{p}{Y})\,q^3\,r^3}{(p^2-q^2)^3\,(p^2-r^2)}\\
&-4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,p^3\,r^3}{(p^2-q^2)^2\,(p^2-r^2)^3}\\
&-2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q^3\,r^3}{(p^2-q^2)^3\,(p^2-r^2)^3}\\
&+2\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^4\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q^2}{(p^2-r^2)^2\,(p^2-q^2)^2}\\
&-2\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^2\,(p\,X-q)^4\,(q-\frac{p}{X})^4\,q^2\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^4}\\
&+4\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^3\,(p\,X-q)^3\,(q-\frac{p}{X})^3\,q\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^3}\\
&-2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q\,p^4\,r}{(p^2-q^2)\,(p^2-r^2)^3}\\
&+\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,(p\,X-q)^4\,(q-\frac{p}{X})^4\,q^2\,r^4}{(p^2-r^2)^2\,(p^2-q^2)^4}\\
&+4\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p\,(p\,X-q)^3\,(q-\frac{p}{X})^3\,q^3\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^3}\\
&-16\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^2\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q^2\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^2}\\
&+4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,p^3\,(p\,\mathrm{y}(p)-r)\,(r-\frac{p}{Y})\,r\,q^2}{(p^2-q^2)^2\,(p^2-r^2)}\\
&+\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)^4\,(r-\frac{p}{Y})^4\,r^2\,q^4}{(p^2-q^2)^2\,(p^2-r^2)^4}\\
&+4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,r^3\,p\,(p\,Y-r)\,(r-\frac{p}{Y})\,q^2}{(p^2-q^2)^2\,(p^2-r^2)}\\
&+2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q\,p^2\,r^3}{(p^2-q^2)\,(p^2-r^2)^3}\\
&+2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q^3\,p^2\,r}{(p^2-q^2)\,(p^2-r^2)^3}\\
&-2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q^3\,r^3}{(p^2-q^2)\,(p^2-r^2)^3}\\
&-4\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q\,p\,r^4}{(p^2-q^2)\,(p^2-r^2)^2}
\end{aligned}
$$

$$- 2\,\frac{p^2\,(p\,X-q)^2\,(q-\dfrac{p}{X})^2\,q^2\,r^2}{(p^2-q^2)^2}$$
$$+ \frac{(p\,Y-r)^2\,(r-\dfrac{p}{Y})^2\,p^4\,(p\,X-q)^4\,(q-\dfrac{p}{X})^4\,q^2}{(p^2-r^2)^2\,(p^2-q^2)^4}$$
$$- 2\,\frac{(p\,Y-r)^2\,(r-\dfrac{p}{Y})^2\,q^2\,p^2\,r^2}{(p^2-r^2)^2} + \frac{p^4\,(p\,X-q)^2\,(q-\dfrac{p}{X})^2\,r^2}{(p^2-q^2)^2}$$
$$- 4\,\frac{(p\,X-q)\,(q-\dfrac{p}{X})\,(p\,Y-r)^2\,(r-\dfrac{p}{Y})^2\,q^3\,p^3}{(p^2-q^2)\,(p^2-r^2)^2}$$
$$- 2\,\frac{(p\,X-q)\,(q-\dfrac{p}{X})\,(p\,Y-r)\,(r-\dfrac{p}{Y})\,q^3\,r^3}{(p^2-q^2)\,(p^2-r^2)}$$
$$+ \frac{(p\,Y-r)^2\,(r-\dfrac{p}{Y})^2\,q^2\,p^4}{(p^2-r^2)^2} + \frac{(p\,Y-r)^2\,(r-\dfrac{p}{Y})^2\,q^2\,r^4}{(p^2-r^2)^2}$$
$$+ 4\,\frac{(p\,X-q)^2\,(q-\dfrac{p}{X})^2\,p^3\,(p\,Y-r)^3\,(r-\dfrac{p}{Y})^3\,r\,q^2}{(p^2-q^2)^2\,(p^2-r^2)^3}$$
$$+ 4\,\frac{(p\,X-q)^2\,(q-\dfrac{p}{X})^2\,p\,(p\,\mathrm{y}(p)-r)^3\,(r-\dfrac{p}{Y})^3\,r^3\,q^2}{(p^2-q^2)^2\,(p^2-r^2)^3}$$
$$+ 2\,\frac{(p\,Y-r)^2\,(r-\dfrac{p}{Y})^2\,p^4\,(p\,X-q)^2\,(q-\dfrac{p}{X})^2\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^2}$$
$$+ 2\,\frac{(p\,Y-r)^2\,(r-\dfrac{p}{Y})^2\,(p\,X-q)^2\,(q-\dfrac{p}{X})^2\,q^2\,r^4}{(p^2-r^2)^2\,(p^2-q^2)^2}$$
$$- 4\,\frac{(p\,X-q)^3\,(q-\dfrac{p}{X})^3\,(p\,Y-r)^2\,(r-\dfrac{p}{Y})^2\,p^3\,q^3}{(p^2-q^2)^3\,(p^2-r^2)^2}$$

$$\%9 := \frac{p^2-q^2}{(p\,\mathrm{x}(p)-q)\,(q-\dfrac{p}{X})} - \frac{(p\,X-q)\,(q-\dfrac{p}{X})}{p^2-q^2}$$

$$\%10 := \mu+\nu+\frac{(\mu-\nu)\,\%4\,\%3}{p^2-q^2} + \frac{(n+m)\,p\,q\,\%9}{p^2-q^2}$$
$$- \frac{(p\,Y-r)\,(r-\dfrac{p}{Y})\,(p^2-q^2)\,(p\,X-q)\,(q-\dfrac{p}{X})\,p\,r\,\%7}{\%8}$$

$$\%11 := \frac{\%4\,\%3}{p^2-q^2} + \frac{\%2\,\%1}{p^2-r^2}$$

$$\%12 := \frac{(p^2-r^2)\,(p\,Y-r)\,(r-\dfrac{p}{Y})\,(p^2-q^2)\,(p\,X-q)\,(q-\dfrac{p}{X})}{\%8}$$

$$\%13 := \mu+\nu+\frac{(\mu-\nu)\,\%2\,\%1}{p^2-r^2} + \frac{(n-\%12)\,p\,r\,\%7}{p^2-r^2} + \frac{m\,p\,q\,\%9}{p^2-q^2}$$

$$\%14 := \frac{\%11\,p\,(p^2-r^2)}{\%5\,(p\,Y-r)\,(r-\dfrac{p}{Y})}$$

$$\%15 := \frac{\%4\,\%3\,p\,(p^2-r^2)}{(p^2-q^2)\,\%5\,(p\,Y-r)\,(r-\dfrac{p}{Y})}$$

$$\%16 := -\frac{(p^2-r^2)^2}{(p\,Y-r)^2\,(r-\dfrac{p}{Y})^2} - 1$$

$$\%17 := -2\,\frac{\%4\,\%3\,\%2\,p\,(p^2-r^2)}{(p^2-q^2)\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2} - 2\,\frac{\%4\,\%3\,p\,\%1}{(p^2-q^2)\,(p^2-r^2)} + 2\,\frac{\%2\,\%1^2\,p}{(p^2-r^2)^2}$$
$$+ 2\,\frac{\%2^2\,\%1\,p}{(p\,Y-r)^2\,(r-\frac{p}{Y})^2}$$

$$A_6 = -\left(-\frac{p^3\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)^2\,(p^2-r^2)}\right.$$
$$+\frac{p\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q\,r^2\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)^2\,(p^2-r^2)}$$
$$+2\,\frac{p^4\,(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)}$$
$$-2\,\frac{q^2\,(p\,X-q)\,(q-\frac{p}{X})\,r^2\,(p\,\mathrm{y}(p)-r)\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)} - \frac{q\,p^3\,(p\,Y-r)\,(r-\frac{p}{Y})}{p^2-r^2}$$
$$+\frac{q\,p\,r^2\,(p\,Y-r)\,(r-\frac{p}{Y})}{p^2-r^2} - \frac{p^3\,(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,r}{(p^2-q^2)\,(p^2-r^2)^2}$$
$$+\frac{q^2\,(p\,X-q)\,(q-\frac{p}{X})\,p\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,r}{(p^2-q^2)\,(p^2-r^2)^2} - \frac{p^3\,(p\,X-q)\,(q-\frac{p}{X})\,r}{p^2-q^2}$$
$$\left.+\frac{q^2\,(p\,X-q)\,(q-\frac{p}{X})\,r\,p}{p^2-q^2}\right)\left(4\,\frac{\%5\,\%4\,\%12\,p}{(p^2-q^2)^2\,\%9} + \left(\frac{(p\,X-q)\,(q-\frac{p}{X})\,\%4}{(p^2-q^2)^2}\right.\right.$$
$$+\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,\%7}{(p^2-r^2)^2} - \frac{\%8}{(p\,Y-r)\,(r-\frac{p}{Y})} - \frac{\%5}{(p\,X-q)\,(q-\frac{p}{X})}$$
$$\left.-2\,\frac{\%8\,\%7\,p}{(p^2-r^2)^2} - 2\,\frac{\%5\,\%4\,p}{(p^2-q^2)^2}\right)\%6/(\%9) + \%10\left(-\frac{(\mu-\nu)\,\%5}{(p\,X-q)\,(q-\frac{p}{X})}\right.$$
$$-2\,\frac{(\mu-\nu)\,\%5\,\%4\,p}{(p^2-q^2)^2} + \frac{(\mu-\nu)\,(p\,X-q)\,(q-\frac{p}{X})\,\%4}{(p^2-q^2)^2} + \frac{(n+m)\,q\,\%3}{p^2-q^2}$$
$$-\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,(p^2-q^2)\,(p\,X-q)\,(q-\frac{p}{X})\,r\,\%1}{\%2}$$
$$+2\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,(p^2-q^2)\,(p\,X-q)\,(q-\frac{p}{X})\,p^2\,r\,\%1}{(p^2-r^2)\,\%2}$$
$$\left.-2\,\frac{(n+m)\,p^2\,q\,\%3}{(p^2-q^2)^2}\right)/(\%9) + 2\,\frac{\%5\,\%12}{(p\,X-q)\,(q-\frac{p}{X})\,\%9} + 2\,\frac{\%5\,\%4\,\%12\,\%13}{(p^2-q^2)\,\%9^2}$$
$$-\frac{\%10\,\%6\,\%13}{\%9^2} - r($$

$$(n+\mu-\nu+2\,\frac{\%8\,\%7\,\%6}{(p^2-r^2)\,\%9}+2\,\frac{\%5\,\%4\,\%12}{(p^2-q^2)\,\%9}-\frac{\%10\,\%12}{\%9}-\frac{\%10\,\%6}{\%9})$$

$$(-\%11-\frac{\%10\,\%6}{\%9}+2\,\frac{\%5\,\%4\,\%12}{(p^2-q^2)\,\%9})\,(p^2-r^2)\Big/((p\,Y-r)\,(r-\frac{p}{Y}))-$$

$$(-\%11+\frac{\%10\,\%6}{\%9}-2\,\frac{\%5\,\%4\,\%12}{(p^2-q^2)\,\%9})$$

$$(n-\mu+\nu-2\,\frac{\%8\,\%7\,\%6}{(p^2-r^2)\,\%9}-2\,\frac{\%5\,\%4\,\%12}{(p^2-q^2)\,\%9}+\frac{\%10\,\%12}{\%9}+\frac{\%10\,\%6}{\%9})\,(p\,X-q)$$

$$(q-\frac{p}{X})/(p^2-q^2))/(r^2-p^2)-2\,\frac{(p\,\mathrm{x}(p)-q)\,(q-\frac{p}{X})\,\%4\,\%12}{(p^2-q^2)^2\,\%9}-2\%5\,\%4\,\Bigg($$

$$\frac{(n-\%11)\,r\,\%1}{p^2-r^2}-2\,\frac{(\mu-\nu)\,\%8\,\%7\,p}{(p^2-r^2)^2}+\frac{(\mu-\nu)\,(p\,Y-r)\,(r-\frac{p}{Y})\,\%7}{(p^2-r^2)^2}$$

$$-2\,\frac{m\,p^2\,q\,\%3}{(p^2-q^2)^2}-\frac{(\mu-\nu)\,\%8}{(p\,Y-r)\,(r-\frac{p}{Y})}-2\,\frac{(n-\%11)\,p^2\,r\,\%1}{(p^2-r^2)^2}+\frac{m\,q\,\%3}{p^2-q^2}\Bigg)/((p^2-q^2)$$

$$\%9)\Bigg)\Big/(p\,(p\,\mathrm{y}(p)-r)\,(r-\frac{p}{Y})\,(p^2-q^2))-2\left(-\frac{q\,(p\,X-q)\,(q-\frac{p}{X})}{p^2-q^2}+p\right)\%5\,\Bigg(-$$

$$2\%8\,\%7\left(-\frac{(\mu-\nu)\,\%5}{(p\,X-q)\,(q-\frac{p}{X})}-2\,\frac{(\mu-\nu)\,\%5\,\%4\,p}{(p^2-q^2)^2}\right.$$

$$+\frac{(\mu-\nu)\,(p\,X-q)\,(q-\frac{p}{X})\,\%4}{(p^2-q^2)^2}+\frac{(n+m)\,q\,\%3}{p^2-q^2}$$

$$-\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,(p^2-q^2)\,(p\,X-q)\,(q-\frac{p}{X})\,r\,\%1}{\%2}$$

$$+2\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,(p^2-q^2)\,(p\,X-q)\,(q-\frac{p}{X})\,p^2\,r\,\%1}{(p^2-r^2)\,\%2}$$

$$\left.-2\,\frac{(n+m)\,p^2\,q\,\%3}{(p^2-q^2)^2}\right)/((p^2-r^2)\,\%9)+4\,\frac{\%8\,\%7\,\%6\,p}{(p^2-r^2)^2\,\%9}+\%10\,\Bigg(\frac{(n-\%11)\,r\,\%1}{p^2-r^2}$$

$$-2\,\frac{(\mu-\nu)\,\%8\,\%7\,p}{(p^2-r^2)^2}+\frac{(\mu-\nu)\,(p\,Y-r)\,(r-\frac{p}{Y})\,\%7}{(p^2-r^2)^2}-2\,\frac{m\,p^2\,q\,\%3}{(p^2-q^2)^2}$$

$$-\frac{(\mu-\nu)\,\%8}{(p\,Y-r)\,(r-\frac{p}{Y})}-2\,\frac{(n-\%11)\,p^2\,r\,\%1}{(p^2-r^2)^2}+\frac{m\,q\,\%3}{p^2-q^2}\Bigg)/(\%9)$$

$$+2\,\frac{\%8\,\%6}{(p\,Y-r)\,(r-\frac{p}{Y})\,\%9}+\Bigg(\frac{(p\,X-q)\,(q-\frac{p}{X})\,\%4}{(p^2-q^2)^2}$$

$$+\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,\%7}{(p^2-r^2)^2}-\frac{\%8}{(p\,Y-r)\,(r-\frac{p}{Y})}-\frac{\%5}{(p\,X-q)\,(q-\frac{p}{X})}$$

$$-2\,\frac{\%8\,\%7\,p}{(p^2-r^2)^2}-2\,\frac{\%5\,\%4\,p}{(p^2-q^2)^2}\Bigg)\,\%12/(\%9)-2\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,\%7\,\%6}{(p^2-r^2)^2\,\%9}$$

$$+2\,\frac{\%8\,\%7\,\%6\,\%13}{(p^2-r^2)\,\%9^2}-\frac{\%10\,\%12\,\%13}{\%9^2}-q($$
$$(n+\mu-\nu+2\,\frac{\%8\,\%7\,\%6}{(p^2-r^2)\,\%9}+2\,\frac{\%5\,\%4\,\%12}{(p^2-q^2)\,\%9}-\frac{\%10\,\%12}{\%9}-\frac{\%10\,\%6}{\%9})$$
$$(m+2\,\frac{\%8\,\%7\,\%6}{(p^2-r^2)\,\%9}-\frac{\%10\,\%12}{\%9})\,(p^2-q^2)\left/((p\,X-q)\,(q-\frac{p}{X}))-\right.$$
$$(m-2\,\frac{\%8\,\%7\,\%6}{(p^2-r^2)\,\%9}+\frac{\%10\,\%12}{\%9})$$
$$(n-\mu+\nu-2\,\frac{\%8\,\%7\,\%6}{(p^2-r^2)\,\%9}-2\,\frac{\%5\,\%4\,\%12}{(p^2-q^2)\,\%9}+\frac{\%10\,\%12}{\%9}+\frac{\%10\,\%6}{\%9})\,(p\,Y-r)$$
$$\left.(r-\frac{p}{Y})/(p^2-r^2))/(q^2-p^2)\right)\,/(p\,(p^2-q^2))$$

$$\%1 := \frac{p^2-r^2}{(p\,\mathrm{y}(p)-r)\,(r-\frac{p}{Y})}-\frac{(p\,Y-r)\,(r-\frac{p}{Y})}{p^2-r^2}$$

$$\%2 := -4\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,\mathrm{y}(p)-r)^2\,(r-\frac{p}{Y})^2\,p\,q\,r^4}{(p^2-q^2)^3\,(p^2-r^2)^2}$$
$$+2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,q\,p^2\,r^3}{(p^2-q^2)\,(p^2-r^2)}$$
$$+2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,q^3\,p^2\,r}{(p^2-q^2)\,(p^2-r^2)}$$
$$-2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,p^4\,q\,r}{(p^2-q^2)^3\,(p^2-r^2)^3}$$
$$+2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,p^2\,q\,r^3}{(p^2-q^2)^3\,(p^2-r^2)^3}$$
$$-4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)\,(r-\frac{p}{Y})\,p^3\,r^3}{(p^2-q^2)^2\,(p^2-r^2)}$$
$$-2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,q\,p^4\,r}{(p^2-q^2)\,(p^2-r^2)}$$
$$+2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,p^2\,q^3\,r}{(p^2-q^2)^3\,(p^2-r^2)^3}$$
$$-2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)\,(r-\frac{p}{Y})\,p^4\,q\,r}{(p^2-q^2)^3\,(p^2-r^2)}$$
$$+4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^2\,r^4}{(p^2-q^2)^2\,(p^2-r^2)^2}$$
$$-4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q^4\,p\,r}{(p^2-q^2)^2\,(p^2-r^2)^3}$$
$$+4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q^4\,p^2}{(p^2-q^2)^2\,(p^2-r^2)^2}$$
$$-4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)\,(r-\frac{p}{Y})\,q^4\,r\,p}{(p^2-q^2)^2\,(p^2-r^2)}$$

$$
\begin{aligned}
&+ \frac{q^4\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,r^2}{(p^2-q^2)^2} \\
&+ 4\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^3\,(p\,X-q)\,(q-\frac{p}{\mathrm{x}(p)})\,q\,r^2}{(p^2-r^2)^2\,(p^2-q^2)} \\
&+ 2\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q^4\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^2} \\
&+ 4\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q^3\,p\,(p\,X-q)\,(q-\frac{p}{X})\,r^2}{(p^2-r^2)^2\,(p^2-q^2)} \\
&+ \frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,p^4\,(p\,\mathrm{y}(p)-r)^4\,(r-\frac{p}{Y})^4\,r^2}{(p^2-q^2)^2\,(p^2-r^2)^4} \\
&- 2\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,p^2\,(p\,Y-r)^4\,(r-\frac{p}{Y})^4\,r^2\,q^2}{(p^2-q^2)^2\,(p^2-r^2)^4} \\
&+ 2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)\,(r-\frac{p}{Y})\,p^2\,q\,r^3}{(p^2-q^2)^3\,(p^2-r^2)} \\
&+ 2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)\,(r-\frac{p}{Y})\,p^2\,q^3\,r}{(p^2-q^2)^3\,(p^2-r^2)} \\
&- 2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)\,(r-\frac{p}{Y})\,q^3\,r^3}{(p^2-q^2)^3\,(p^2-r^2)} \\
&- 4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,p^3\,r^3}{(p^2-q^2)^2\,(p^2-r^2)^3} \\
&- 2\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q^3\,r^3}{(p^2-q^2)^3\,(p^2-r^2)^3} \\
&+ 2\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^4\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q^2}{(p^2-r^2)^2\,(p^2-q^2)^2} \\
&- 2\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^2\,(p\,X-q)^4\,(q-\frac{p}{X})^4\,q^2\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^4} \\
&+ 4\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^3\,(p\,X-q)^3\,(q-\frac{p}{X})^3\,q\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^3} \\
&- 2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q\,p^4\,r}{(p^2-q^2)\,(p^2-r^2)^3} \\
&+ \frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,(p\,X-q)^4\,(q-\frac{p}{X})^4\,q^2\,r^4}{(p^2-r^2)^2\,(p^2-q^2)^4} \\
&+ 4\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p\,(p\,X-q)^3\,(q-\frac{p}{X})^3\,q^3\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^3} \\
&- 16\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^2\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q^2\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^2} \\
&+ 4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,p^3\,(p\,\mathrm{y}(p)-r)\,(r-\frac{p}{Y})\,r\,q^2}{(p^2-q^2)^2\,(p^2-r^2)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,(p\,Y-r)^4\,(r-\frac{p}{Y})^4\,r^2\,q^4}{(p^2-q^2)^2\,(p^2-r^2)^4}\\
&+ 4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,r^3\,p\,(p\,Y-r)\,(r-\frac{p}{Y})\,q^2}{(p^2-q^2)^2\,(p^2-r^2)}\\
&+ 2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q\,p^2\,r^3}{(p^2-q^2)\,(p^2-r^2)^3}\\
&+ 2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q^3\,p^2\,r}{(p^2-q^2)\,(p^2-r^2)^3}\\
&- 2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,q^3\,r^3}{(p^2-q^2)\,(p^2-r^2)^3}\\
&- 4\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q\,p\,r^4}{(p^2-q^2)\,(p^2-r^2)^2}\\
&- 2\,\frac{p^2\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q^2\,r^2}{(p^2-q^2)^2}\\
&+ \frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^4\,(p\,X-q)^4\,(q-\frac{p}{X})^4\,q^2}{(p^2-r^2)^2\,(p^2-q^2)^4}\\
&- 2\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q^2\,p^2\,r^2}{(p^2-r^2)^2} + \frac{p^4\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,r^2}{(p^2-q^2)^2}\\
&- 4\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q^3\,p^3}{(p^2-q^2)\,(p^2-r^2)^2}\\
&- 2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,q^3\,r^3}{(p^2-q^2)\,(p^2-r^2)}\\
&+ \frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q^2\,p^4}{(p^2-r^2)^2} + \frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,q^2\,r^4}{(p^2-r^2)^2}\\
&+ 4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,p^3\,(p\,Y-r)^3\,(r-\frac{p}{Y})^3\,r\,q^2}{(p^2-q^2)^2\,(p^2-r^2)^3}\\
&+ 4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,p\,(p\,\mathrm{y}(p)-r)^3\,(r-\frac{p}{Y})^3\,r^3\,q^2}{(p^2-q^2)^2\,(p^2-r^2)^3}\\
&+ 2\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^4\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,r^2}{(p^2-r^2)^2\,(p^2-q^2)^2}\\
&+ 2\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,(p\,X-q)^2\,(q-\frac{p}{X})^2\,q^2\,r^4}{(p^2-r^2)^2\,(p^2-q^2)^2}\\
&- 4\,\frac{(p\,X-q)^3\,(q-\frac{p}{X})^3\,(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p^3\,q^3}{(p^2-q^2)^3\,(p^2-r^2)^2}
\end{aligned}
$$

$$\%3 := \frac{p^2-q^2}{(p\,\mathrm{x}(p)-q)\,(q-\frac{p}{X})} - \frac{(p\,X-q)\,(q-\frac{p}{X})}{p^2-q^2}$$

$$\%4 := q - \frac{p\,(p^2-q^2)}{(p\,X-q)\,(q-\frac{p}{X})}$$

$$\%5 := \frac{p\,(p\,X - q)\,(q - \frac{p}{X})}{p^2 - q^2} - q$$

$$\%6 := \mu + \nu + \frac{(\mu - \nu)\,\%5\,\%4}{p^2 - q^2} + \frac{(n + m)\,p\,q\,\%3}{p^2 - q^2}$$
$$- \frac{(p\,Y - r)\,(r - \frac{p}{Y})\,(p^2 - q^2)\,(p\,X - q)\,(q - \frac{p}{X})\,p\,r\,\%1}{\%2}$$

$$\%7 := r - \frac{p\,(p^2 - r^2)}{(p\,Y - r)\,(r - \frac{p}{Y})}$$

$$\%8 := \frac{p\,(p\,Y - r)\,(r - \frac{p}{Y})}{p^2 - r^2} - r$$

$$\%9 := -2\,\frac{\%5\,\%4\,\%8\,\%7}{(p^2 - q^2)\,(p^2 - r^2)} + \frac{\%5^2\,\%4^2}{(p^2 - q^2)^2} + \frac{\%8^2\,\%7^2}{(p^2 - r^2)^2}$$

$$\%10 := \frac{\%5\,\%4}{p^2 - q^2} + \frac{\%8\,\%7}{p^2 - r^2}$$

$$\%11 := \frac{(p^2 - r^2)\,(p\,Y - r)\,(r - \frac{p}{Y})\,(p^2 - q^2)\,(p\,X - q)\,(q - \frac{p}{X})}{\%2}$$

$$\%12 := \mu + \nu + \frac{(\mu - \nu)\,\%8\,\%7}{p^2 - r^2} + \frac{(n - \%11)\,p\,r\,\%1}{p^2 - r^2} + \frac{m\,p\,q\,\%3}{p^2 - q^2}$$

$$\%13 := -2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,\%4\,\%8\,\%7}{(p^2 - q^2)^2\,(p^2 - r^2)} - 2\,\frac{\%8^2\,\%7}{(p^2 - r^2)\,(p\,Y - r)\,(r - \frac{p}{Y})}$$
$$- 4\,\frac{\%5^2\,\%4^2\,p}{(p^2 - q^2)^3} + 2\,\frac{\%5\,\%8\,\%7}{(p\,X - q)\,(q - \frac{p}{X})\,(p^2 - r^2)} + 4\,\frac{\%5\,\%4\,\%8\,\%7\,p}{(p^2 - q^2)\,(p^2 - r^2)^2}$$
$$+ 4\,\frac{\%5\,\%4\,\%8\,\%7\,p}{(p^2 - q^2)^2\,(p^2 - r^2)} - 2\,\frac{\%5\,\%4\,(p\,Y - r)\,(r - \frac{p}{Y})\,\%7}{(p^2 - q^2)\,(p^2 - r^2)^2}$$
$$+ 2\,\frac{\%5\,\%4^2\,(p\,X - q)\,(q - \frac{p}{X})}{(p^2 - q^2)^3} + 2\,\frac{\%5\,\%4\,\%8}{(p^2 - q^2)\,(p\,Y - r)\,(r - \frac{p}{Y})}$$
$$- 4\,\frac{\%8^2\,\%7^2\,p}{(p^2 - r^2)^3} + 2\,\frac{\%8\,\%7^2\,(p\,Y - r)\,(r - \frac{p}{Y})}{(p^2 - r^2)^3}$$
$$- 2\,\frac{\%5^2\,\%4}{(p^2 - q^2)\,(p\,X - q)\,(q - \frac{p}{X})}$$

$$
\begin{aligned}
B_1 \;=\; & -(-p^3\,X^2\,q\,Y + p\,\mathrm{x}(p)^2\,q\,r^2\,Y + 2\,p^4\,X\,Y - 2\,q^2\,X\,r^2\,Y - q\,p^3\,Y \\
& + q\,p\,r^2\,Y - p^3\,X\,\mathrm{y}(p)^2\,r + q^2\,X\,p\,Y^2\,r - p^3\,X\,r + q^2\,X\,r\,p)\,\Bigg( \\
& 2\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,p}{(p^2-r^2)\,\%1\,X^2} + 2(p\,Y-r)\,(r-\frac{p}{Y})\,p\,\Bigg( \\
& -2\,\frac{(p\,X-q)\,p\,(p\,Y-r)\,(r-\frac{p}{Y})}{X^2\,(p^2-q^2)\,(p^2-r^2)} + 2\,\frac{(p\,X-q)\,(q-\frac{p}{X})^2\,p}{(p^2-q^2)^2} \\
& -2\,\frac{p\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)} + 2\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})\,p}{(p^2-q^2)^2\,X^2}\Bigg)\,/((p^2-r^2)\,\%1^2 \\
& X) - q\,\Bigg(2\,\frac{(-\frac{\%3\,p}{\%1\,X} + 2\,\%2)\,(p\,Y-r)\,(r-\frac{p}{Y})\,p}{(p^2-r^2)\,\%1\,X^2} \\
& +2\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,p\,(\frac{\%3\,p}{\%1\,X} - 2\,\%2)\,Y}{(p^2-r^2)\,\%1\,X}\Bigg)\,/(q^2-p^2)\Bigg)\,/(p\,X\,(p^2-r^2) \\
& (p^2-q^2)) - 2(-r\,Y+p)\,(p\,\mathrm{y}(p)-r)\,\Bigg(-\%3\,p\,\Bigg(-2\,\frac{(p\,X-q)\,p\,(p\,Y-r)\,(r-\frac{p}{Y})}{X^2\,(p^2-q^2)\,(p^2-r^2)} \\
& +2\,\frac{(p\,X-q)\,(q-\frac{p}{X})^2\,p}{(p^2-q^2)^2} - 2\,\frac{p\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)} \\
& +2\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})\,p}{(p^2-q^2)^2\,X^2}\Bigg)\,/(\%1^2\,\mathrm{x}(p)) - \frac{\%3\,p}{\%1\,X^2} \\
& -\frac{r\left(-\frac{(-\frac{\%3\,p}{\%1\,X} + 2\,\%2)\,\%3\,p}{\%1\,X\,Y} - \frac{\%3\,p\,(\frac{\%3\,p}{\%1\,X} - 2\,\%2)}{\%1}\right)}{r^2-p^2} \\
& +\frac{\left(\frac{p\,(q-\frac{p}{X})}{p^2-q^2} + \frac{(p\,X-q)\,p}{\mathrm{x}(p)^2\,(p^2-q^2)}\right)\,p}{\%1\,X}\Bigg)\,/(p\,(p^2-r^2)) \\
& \%1 := -2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)} + \frac{(p\,X-q)^2\,(q-\frac{p}{X})^2}{(p^2-q^2)^2} \\
& + \frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2}{(p^2-r^2)^2} \\
& \%2 := \frac{(p\,Y-r)\,(r-\frac{p}{Y})\,p}{(p^2-r^2)\,\%1\,X} \\
& \%3 := \frac{(p\,X-q)\,(q-\frac{p}{X})}{p^2-q^2} + \frac{(p\,Y-r)\,(r-\frac{p}{Y})}{p^2-r^2}
\end{aligned}
$$

$$
\begin{aligned}
B_2 \;=\; & -(-p^3 X^2 q Y + p\,\mathrm{x}(p)^2 q r^2 Y + 2 p^4 X Y - 2 q^2 X r^2 Y - q p^3 Y \\
& + q p r^2 Y - p^3 X\,\mathrm{y}(p)^2 r + q^2 X p Y^2 r - p^3 X r + q^2 X r p) \Bigg( \\
& - \frac{q \left( - \frac{(2\,\%3 - \frac{\%2\,p}{\%1\,Y})\,\%2\,p}{\%1\,Y\,\mathrm{x}(p)} - \frac{\%2\,p\,(-2\,\%3 + \frac{\%2\,p}{\%1\,Y})}{\%1} \right)}{q^2 - p^2} - \frac{\%2\,p}{\%1\,Y^2} - \%2\,p \Bigg( \\
& -2\,\frac{(p X - q)(q - \frac{p}{X})(p Y - r)\,p}{(p^2 - q^2)\,Y^2\,(p^2 - r^2)} - 2\,\frac{(p X - q)(q - \frac{p}{X})\,p\,(r - \frac{p}{\mathrm{y}(p)})}{(p^2 - q^2)(p^2 - r^2)} \\
& + 2\,\frac{(p Y - r)(r - \frac{p}{Y})^2 p}{(p^2 - r^2)^2} + 2\,\frac{(p\,\mathrm{y}(p) - r)^2 (r - \frac{p}{Y})\,p}{(p^2 - r^2)^2\,Y^2} \Bigg) / (\%1^2\,\mathrm{y}(p)) \\
& + \frac{\left( \frac{p\,(r - \frac{p}{Y})}{p^2 - r^2} + \frac{(p Y - r)\,p}{\mathrm{y}(p)^2\,(p^2 - r^2)} \right) p}{\%1\,Y} \Bigg) / (p X (p^2 - r^2)(p^2 - q^2)) - 2(-r Y + p) \\
& (p Y - r) \Bigg( - r \Bigg( 2\,\frac{(2\,\%3 - \frac{\%2\,p}{\%1\,Y})(p X - q)(q - \frac{p}{X})\,p}{(p^2 - q^2)\,\%1\,Y^2} \\
& + 2\,\frac{(p X - q)(q - \frac{p}{X})\,p\,(-2\,\%3 + \frac{\%2\,p}{\%1\,Y})\,X}{(p^2 - q^2)\,\%1\,Y} \Bigg) / (r^2 - p^2) \\
& + 2\,\frac{(p X - q)(q - \frac{p}{X})\,p}{(p^2 - q^2)\,\%1\,Y^2} + 2(p X - q)(q - \frac{p}{X})\,p \Bigg( \\
& -2\,\frac{(p X - q)(q - \frac{p}{X})(p Y - r)\,p}{(p^2 - q^2)\,Y^2\,(p^2 - r^2)} - 2\,\frac{(p X - q)(q - \frac{p}{X})\,p\,(r - \frac{p}{\mathrm{y}(p)})}{(p^2 - q^2)(p^2 - r^2)} \\
& + 2\,\frac{(p Y - r)(r - \frac{p}{Y})^2 p}{(p^2 - r^2)^2} + 2\,\frac{(p Y - r)^2 (r - \frac{p}{Y})\,p}{(p^2 - r^2)^2\,Y^2} \Bigg) / ((p^2 - q^2)\,\%1^2\,Y) \Bigg) / (p \\
& (p^2 - r^2))
\end{aligned}
$$

$$
\%1 := -2\,\frac{(p X - q)(q - \frac{p}{X})(p Y - r)(r - \frac{p}{Y})}{(p^2 - q^2)(p^2 - r^2)} + \frac{(p X - q)^2 (q - \frac{p}{X})^2}{(p^2 - q^2)^2} + \frac{(p Y - r)^2 (r - \frac{p}{Y})^2}{(p^2 - r^2)^2}
$$

$$
\%2 := \frac{(p X - q)(q - \frac{p}{X})}{p^2 - q^2} + \frac{(p Y - r)(r - \frac{p}{Y})}{p^2 - r^2}
$$

$$
\%3 := \frac{(p X - q)(q - \frac{p}{X})\,p}{(p^2 - q^2)\,\%1\,Y}
$$

$$
\begin{aligned}
B_3 \;=\; & -(-p^3\,X^2\,q\,Y + p\,\mathrm{x}(p)^2\,q\,r^2\,Y + 2\,p^4\,X\,Y - 2\,q^2\,X\,r^2\,Y - q\,p^3\,Y \\
& + q\,p\,r^2\,Y - p^3\,X\,\mathrm{y}(p)^2\,r + q^2\,X\,p\,Y^2\,r - p^3\,X\,r + q^2\,X\,r\,p)\left(2(p\,Y - r)\right. \\
& (r - \frac{p}{Y})\,p\left(-2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,(p\,Y - r)\,p}{(p^2 - q^2)\,Y^2\,(p^2 - r^2)}\right. \\
& - 2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,p\,(r - \frac{p}{Y})}{(p^2 - q^2)\,(p^2 - r^2)} + 2\,\frac{(p\,Y - r)\,(r - \frac{p}{Y})^2\,p}{(p^2 - r^2)^2} \\
& \left. + 2\,\frac{(p\,Y - r)^2\,(r - \frac{p}{Y})\,p}{(p^2 - r^2)^2\,Y^2}\right)/((p^2 - r^2)\,\%1^2\,X) - \%2\,p\left(\right. \\
& -2\,\frac{(p\,X - q)\,p\,(p\,Y - r)\,(r - \frac{p}{Y})}{X^2\,(p^2 - q^2)\,(p^2 - r^2)} + 2\,\frac{(p\,X - q)\,(q - \frac{p}{X})^2\,p}{(p^2 - q^2)^2} \\
& \left. - 2\,\frac{p\,(q - \frac{p}{X})\,(p\,Y - r)\,(r - \frac{p}{Y})}{(p^2 - q^2)\,(p^2 - r^2)} + 2\,\frac{(p\,X - q)^2\,(q - \frac{p}{X})\,p}{(p^2 - q^2)^2\,X^2}\right)/(\%1^2\,\mathrm{y}(p)) \\
& - 2\,\frac{(p\,Y - r)\,p^2}{Y^2\,(p^2 - r^2)\,\%1\,X} + \frac{\left(\frac{p\,(q - \frac{p}{X})}{p^2 - q^2} + \frac{(p\,X - q)\,p}{\mathrm{x}(p)^2\,(p^2 - q^2)}\right)p}{\%1\,Y} - 2\,\frac{p^2\,(r - \frac{p}{Y})}{(p^2 - r^2)\,\%1\,X} \\
& - q\left(\frac{2\,\frac{(2\,\%3 - \frac{\%2\,p}{\%1\,Y})\,(p\,Y - r)\,(r - \frac{p}{Y})\,p}{(p^2 - r^2)\,\%1\,X} - \frac{(-\frac{\%2\,p}{\%1\,X} + 2\,\%4)\,\%2\,p}{\%1\,Y}}{X}\right. \\
& \left. - \left(\frac{\%2\,p\,(\frac{\%2\,p}{\%1\,X} - 2\,\%4)}{\%1\,Y} - 2\,\frac{(p\,Y - r)\,(r - \frac{p}{Y})\,p\,(-2\,\%3 + \frac{\%2\,p}{\%1\,Y})}{(p^2 - r^2)\,\%1\,X}\right)Y\right)/( \\
& \left. q^2 - p^2)\right)/(p\,X\,(p^2 - r^2)\,(p^2 - q^2)) - 2(-r\,Y + p)\,(p\,Y - r)\left(-\%2\,p\left(\right.\right. \\
& -2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,(p\,Y - r)\,p}{(p^2 - q^2)\,Y^2\,(p^2 - r^2)} - 2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,p\,(r - \frac{p}{\mathrm{y}(p)})}{(p^2 - q^2)\,(p^2 - r^2)} \\
& \left. + 2\,\frac{(p\,Y - r)\,(r - \frac{p}{Y})^2\,p}{(p^2 - r^2)^2} + 2\,\frac{(p\,\mathrm{y}(p) - r)^2\,(r - \frac{p}{Y})\,p}{(p^2 - r^2)^2\,Y^2}\right)/(\%1^2\,\mathrm{x}(p)) + 2(p\,X - q) \\
& (q - \frac{p}{X})\,p\left(-2\,\frac{(p\,X - q)\,p\,(p\,\mathrm{y}(p) - r)\,(r - \frac{p}{Y})}{\mathrm{x}(p)^2\,(p^2 - q^2)\,(p^2 - r^2)} + 2\,\frac{(p\,X - q)\,(q - \frac{p}{\mathrm{x}(p)})^2\,p}{(p^2 - q^2)^2}\right. \\
& \left. - 2\,\frac{p\,(q - \frac{p}{X})\,(p\,Y - r)\,(r - \frac{p}{Y})}{(p^2 - q^2)\,(p^2 - r^2)} + 2\,\frac{(p\,X - q)^2\,(q - \frac{p}{X})\,p}{(p^2 - q^2)^2\,X^2}\right)/((p^2 - q^2)\,\%1^2
\end{aligned}
$$

$$Y) - r\left(\frac{-\dfrac{(2\,\%3 - \dfrac{\%2\,p}{\%1\,Y})\,\%2\,p}{\%1\,X} + 2\,\dfrac{(-\dfrac{\%2\,p}{\%1\,X} + 2\,\%4)\,(p\,X - q)\,(q - \dfrac{p}{X})\,p}{(p^2 - q^2)\,\%1\,Y}}{Y}\right.$$

$$\left. - \left(-2\,\frac{(p\,\mathrm{x}(p) - q)\,(q - \dfrac{p}{X})\,p\,(\dfrac{\%2\,p}{\%1\,X} - 2\,\%4)}{(p^2 - q^2)\,\%1\,Y} + \frac{\%2\,p\,(-2\,\%3 + \dfrac{\%2\,p}{\%1\,Y})}{\%1\,X}\right)\,X\right)/($$

$$r^2 - p^2) - 2\,\frac{(p\,\mathrm{x}(p) - q)\,p^2}{X^2\,(p^2 - q^2)\,\%1\,Y} - 2\,\frac{p^2\,(q - \dfrac{p}{X})}{(p^2 - q^2)\,\%1\,Y}$$

$$\left. + \frac{\left(\dfrac{p\,(r - \dfrac{p}{Y})}{p^2 - r^2} + \dfrac{(p\,Y - r)\,p}{Y^2\,(p^2 - r^2)}\right)\,p}{\%1\,X}\right)/(p\,(p^2 - r^2))$$

$$\%1 := -2\,\frac{(p\,X - q)\,(q - \dfrac{p}{X})\,(p\,Y - r)\,(r - \dfrac{p}{Y})}{(p^2 - q^2)\,(p^2 - r^2)} + \frac{(p\,X - q)^2\,(q - \dfrac{p}{X})^2}{(p^2 - q^2)^2}$$

$$+ \frac{(p\,Y - r)^2\,(r - \dfrac{p}{Y})^2}{(p^2 - r^2)^2}$$

$$\%2 := \frac{(p\,X - q)\,(q - \dfrac{p}{X})}{p^2 - q^2} + \frac{(p\,Y - r)\,(r - \dfrac{p}{Y})}{p^2 - r^2}$$

$$\%3 := \frac{(p\,X - q)\,(q - \dfrac{p}{X})\,p}{(p^2 - q^2)\,\%1\,Y}$$

$$\%4 := \frac{(p\,Y - r)\,(r - \dfrac{p}{Y})\,p}{(p^2 - r^2)\,\%1\,X}$$

$$B_4 = -(-p^3\,X^2\,q\,Y + p\,\mathrm{x}(p)^2\,q\,r^2\,Y + 2\,p^4\,X\,Y - 2\,q^2\,X\,r^2\,Y - q\,p^3\,Y$$

$$+ q\,p\,r^2\,Y - p^3\,X\,\mathrm{y}(p)^2\,r + q^2\,X\,p\,Y^2\,r - p^3\,X\,r + q^2\,X\,r\,p)\Bigg($$

$$2\,\frac{(p\,Y - r)\,(r - \dfrac{p}{Y})\,\%2\,\%6}{(p^2 - r^2)\,\%3^2} + 2(p\,Y - r)\,(r - \frac{p}{Y})\,p\,\Bigg($$

$$-2\,\frac{X\,(q - \dfrac{p}{X})\,(p\,Y - r)\,(r - \dfrac{p}{Y})}{(p^2 - q^2)\,(p^2 - r^2)} - 2\,\frac{(p\,Y - r)^2\,(r - \dfrac{p}{Y})}{(p^2 - r^2)^2\,Y}$$

$$-4\,\frac{(p\,X - q)^2\,(q - \dfrac{p}{X})^2\,p}{(p^2 - q^2)^3} + 2\,\frac{(p\,X - q)\,(p\,Y - r)\,(r - \dfrac{p}{Y})}{X\,(p^2 - q^2)\,(p^2 - r^2)}$$

$$+4\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,p}{(p^2-q^2)\,(p^2-r^2)^2}$$
$$+4\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,p}{(p^2-q^2)^2\,(p^2-r^2)}$$
$$-2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,Y\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)}+2\,\frac{(p\,X-q)\,(q-\frac{p}{X})^2\,X}{(p^2-q^2)^2}$$
$$+2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)}{(p^2-q^2)\,Y\,(p^2-r^2)}-4\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p}{(p^2-r^2)^3}$$
$$\left.+2\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})^2\,Y}{(p^2-r^2)^2}-2\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})}{(p^2-q^2)^2\,X}\right)/((p^2-r^2)\,\%3^2\,X)$$
$$+4\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,p^2}{(p^2-r^2)^2\,\%3\,X}-2(p\,Y-r)\,(r-\frac{p}{Y})$$
$$\left(\frac{(\mu-\nu)\,(p\,X-q)\,p}{X^2\,(p^2-q^2)}+\frac{(n+m)\,p\,q\,(-\frac{1}{X^2}-1)}{p^2-q^2}+\frac{(\mu-\nu)\,p\,(q-\frac{p}{X})}{p^2-q^2}+\frac{1}{X}\right)/($$
$$(p^2-r^2)\,\%3)-\frac{\%7\,\%5\,\%6}{\%3^2}+2\,\frac{(p\,Y-r)\,p}{Y\,(p^2-r^2)\,\%3\,X}$$
$$+\frac{\left(\frac{p\,(q-\frac{p}{X})}{p^2-q^2}+\frac{(p\,X-q)\,p}{\mathrm{x}(p)^2\,(p^2-q^2)}\right)\%5}{\%3}-2\,\frac{Y\,(r-\frac{p}{Y})\,p}{(p^2-r^2)\,\%3\,\mathrm{x}(p)}+\frac{\%7\,m\,p\,q\,(-\frac{1}{X^2}-1)}{\%3\,(p^2-q^2)}$$
$$-q\left(\left(2\,\frac{(n+\mu-\nu+2\,\%9+2\,\%8-\frac{\%7\,\%5}{\%3}-\frac{\%7\,\%2}{\%3})\,(p\,Y-r)\,(r-\frac{p}{Y})\,p}{(p^2-r^2)\,\%3\,X}\right.\right.$$
$$\left.\left.+(-\frac{\%7\,p}{\%3\,X}+2\,\%10)\,(m+2\,\%9-\frac{\%7\,\%5}{\%3})\right)\right/X-\Bigg($$
$$(m-2\,\%9+\frac{\%7\,\%5}{\%3})\,(\frac{\%7\,p}{\%3\,X}-2\,\%10)$$
$$\left.\left.-2\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,p\,(n-\mu+\nu-2\,\%9-2\,\%8+\frac{\%7\,\%5}{\%3}+\frac{\%7\,\%2}{\%3})}{(p^2-r^2)\,\%3\,X}\right)Y\right)/($$
$$\left.q^2-p^2)\right)/(p\,X\,(p^2-r^2)\,(p^2-q^2))-2(-r\,Y+p)\,(p\,Y-r)\,\Bigg(\%7$$
$$\left(\frac{(\mu-\nu)\,(p\,X-q)\,p}{X^2\,(p^2-q^2)}+\frac{(n+m)\,p\,q\,(-\frac{1}{X^2}-1)}{p^2-q^2}+\frac{(\mu-\nu)\,p\,(q-\frac{p}{X})}{p^2-q^2}+\frac{1}{X}\right)/(\%3)$$
$$-\frac{\%7\,\%2\,\%6}{\%3^2}-\%7\,p\,\Bigg(-2\,\frac{X\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)}$$
$$-2\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})}{(p^2-r^2)^2\,Y}-4\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})^2\,p}{(p^2-q^2)^3}$$

$$+ 2 \frac{(p\,X - q)\,(p\,Y - r)\,(r - \frac{p}{Y})}{X\,(p^2 - q^2)\,(p^2 - r^2)}$$
$$+ 4 \frac{(p\,X - q)\,(q - \frac{p}{X})\,(p\,Y - r)\,(r - \frac{p}{Y})\,p}{(p^2 - q^2)\,(p^2 - r^2)^2}$$
$$+ 4 \frac{(p\,X - q)\,(q - \frac{p}{X})\,(p\,Y - r)\,(r - \frac{p}{Y})\,p}{(p^2 - q^2)^2\,(p^2 - r^2)}$$
$$- 2 \frac{(p\,X - q)\,(q - \frac{p}{X})\,Y\,(r - \frac{p}{Y})}{(p^2 - q^2)\,(p^2 - r^2)} + 2 \frac{(p\,X - q)\,(q - \frac{p}{X})^2\,X}{(p^2 - q^2)^2}$$
$$+ 2 \frac{(p\,X - q)\,(q - \frac{p}{X})\,(p\,Y - r)}{(p^2 - q^2)\,Y\,(p^2 - r^2)} - 4 \frac{(p\,Y - r)^2\,(r - \frac{p}{Y})^2\,p}{(p^2 - r^2)^3}$$
$$+ 2 \frac{(p\,Y - r)\,(r - \frac{p}{Y})^2\,Y}{(p^2 - r^2)^2} - 2 \frac{(p\,X - q)^2\,(q - \frac{p}{X})}{(p^2 - q^2)^2\,X} \Bigg) /(\%3^2\,X) - r \Bigg(\Bigg($$
$$- \frac{(n + \mu - \nu + 2\,\%9 + 2\,\%8 - \frac{\%7\,\%5}{\%3} - \frac{\%7\,\%2}{\%3})\,\%7\,p}{\%3\,X}$$
$$+ (- \frac{\%7\,p}{\%3\,X} + 2\,\%10)\,(-\%4 - \frac{\%7\,\%2}{\%3} + 2\,\%8)\Bigg) /Y - \Bigg($$
$$(-\%4 + \frac{\%7\,\%2}{\%3} - 2\,\%8)\,(\frac{\%7\,p}{\%3\,X} - 2\,\%10)$$
$$+ \frac{\%7\,p\,(n - \mu + \nu - 2\,\%9 - 2\,\%8 + \frac{\%7\,\%5}{\%3} + \frac{\%7\,\%2}{\%3})}{\%3\,X}\Bigg)\,X\Bigg) /(r^2 - p^2)$$
$$- 2 \frac{(p\,X - q)\,(q - \frac{p}{X})\,m\,p\,q\,(- \frac{1}{X^2} - 1)}{(p^2 - q^2)^2\,\%3} + 2 \frac{(p\,X - q)\,(q - \frac{p}{X})\,\%5\,\%6}{(p^2 - q^2)\,\%3^2} + \Bigg($$
$$\frac{X\,(q - \frac{p}{X})}{p^2 - q^2} + \frac{Y\,(r - \frac{p}{Y})}{p^2 - r^2} - \frac{p\,Y - r}{Y\,(p^2 - r^2)} - \frac{p\,X - q}{X\,(p^2 - q^2)}$$
$$- 2 \frac{(p\,Y - r)\,(r - \frac{p}{Y})\,p}{(p^2 - r^2)^2} - 2 \frac{(p\,X - q)\,(q - \frac{p}{X})\,p}{(p^2 - q^2)^2}\Bigg)\,p/(\%3\,\mathrm{x}(p))$$
$$- 2 \frac{(p\,X - q)\,p\,\%5}{X^2\,(p^2 - q^2)\,\%3} - 2 \frac{p\,(q - \frac{p}{X})\,\%5}{(p^2 - q^2)\,\%3} + \frac{\left(\frac{p\,(q - \frac{p}{X})}{p^2 - q^2} + \frac{(p\,X - q)\,p}{X^2\,(p^2 - q^2)}\right)\,\%2}{\%3}\Bigg) /(p$$
$$(p^2 - r^2))$$

$$\%1 := p^4\,X^2\,r^2 - 4\,X\,Y^2\,q^3\,p^3 - 2\,X\,Y\,q^3\,r^3 + Y^2\,q^2\,p^4 + Y^2\,q^2\,r^4$$
$$+ 4\,X^2\,p^3\,Y^3\,r\,q^2 + 4\,X^2\,p\,Y^3\,r^3\,q^2 + 4\,\mathrm{x}(p)^2\,p^3\,Y\,r\,q^2 + X^2\,Y^4\,r^2\,q^4$$
$$+ 4\,X^2\,r^3\,p\,Y\,q^2 + 2\,X\,Y^3\,q\,p^2\,r^3 + 2\,X\,Y^3\,q^3\,p^2\,r - 2\,X\,Y^3\,q^3\,r^3$$
$$- 4\,X\,Y^2\,q\,p\,r^4 - 2\,p^2\,X^2\,q^2\,r^2 + Y^2\,p^4\,X^4\,q^2 + 2\,Y^2\,p^4\,X^2\,r^2$$
$$+ 2\,Y^2\,p^4\,X^2\,q^2 - 2\,Y^2\,p^2\,X^4\,q^2\,r^2 + 4\,Y^2\,p^3\,X^3\,q\,r^2 - 2\,X\,Y^3\,q\,p^4\,r$$

$$+ Y^2 X^4 q^2 r^4 + 4 Y^2 p X^3 q^3 r^2 - 16 Y^2 p^2 X^2 q^2 r^2 + 2 Y^2 X^2 q^2 r^4$$

$$+ 4 Y^2 p^3 X q r^2 + 2 Y^2 q^4 X^2 r^2 - 2 Y^2 q^2 p^2 r^2 + 4 Y^2 q^3 p X r^2$$

$$+ X^2 p^4 Y^4 r^2 - 2 X^2 p^2 Y^4 r^2 q^2 + 2 X^3 Y p^2 q r^3 + 2 X^3 Y p^2 q^3 r$$

$$- 2 X^3 Y q^3 r^3 - 4 X^2 Y^3 p^3 r^3 - 2 X^3 Y^3 q^3 r^3 - 4 X^3 Y^2 p^3 q^3$$

$$- 4 X^3 Y^2 p q r^4 + 2 X Y q p^2 r^3 + 2 X Y q^3 p^2 r - 2 X^3 Y^3 p^4 q r$$

$$+ 2 X^3 Y^3 p^2 q r^3 - 4 X^2 Y p^3 r^3 - 2 X Y q p^4 r + 2 X^3 Y^3 p^2 q^3 r$$

$$- 2 X^3 Y p^4 q r + 4 X^2 Y^2 p^2 r^4 - 4 X^2 Y^3 q^4 p r + 4 X^2 Y^2 q^4 p^2$$

$$- 4 X^2 Y q^4 r p + q^4 X^2 r^2$$

$$\%2 := \mu + \nu + \frac{(\mu - \nu)(p X - q)(q - \frac{p}{X})}{p^2 - q^2} + \frac{(n + m) p q (\frac{1}{X} - X)}{p^2 - q^2}$$

$$- \frac{(p^2 - r^2) Y (p^2 - q^2)^2 X p r (\frac{1}{Y} - Y)}{\%1}$$

$$\%3 := -2 \frac{(p X - q)(q - \frac{p}{X})(p Y - r)(r - \frac{p}{Y})}{(p^2 - q^2)(p^2 - r^2)} + \frac{(p X - q)^2 (q - \frac{p}{X})^2}{(p^2 - q^2)^2}$$

$$+ \frac{(p Y - r)^2 (r - \frac{p}{Y})^2}{(p^2 - r^2)^2}$$

$$\%4 := \frac{(p^2 - r^2)^2 Y (p^2 - q^2)^2 X}{\%1}$$

$$\%5 := \mu + \nu + \frac{(\mu - \nu)(p Y - r)(r - \frac{p}{Y})}{p^2 - r^2} + \frac{(n - \%4) p r (\frac{1}{Y} - Y)}{p^2 - r^2}$$

$$+ \frac{m p q (\frac{1}{X} - X)}{p^2 - q^2}$$

$$\%6 := -2 \frac{(p X - q) p (p Y - r)(r - \frac{p}{Y})}{X^2 (p^2 - q^2)(p^2 - r^2)} + 2 \frac{(p X - q)(q - \frac{p}{X})^2 p}{(p^2 - q^2)^2}$$

$$- 2 \frac{p (q - \frac{p}{X})(p Y - r)(r - \frac{p}{Y})}{(p^2 - q^2)(p^2 - r^2)} + 2 \frac{(p X - q)^2 (q - \frac{p}{X}) p}{(p^2 - q^2)^2 X^2}$$

$$\%7 := \frac{(p X - q)(q - \frac{p}{X})}{p^2 - q^2} + \frac{(p Y - r)(r - \frac{p}{Y})}{p^2 - r^2}$$

$$\%8 := \frac{(p X - q)(q - \frac{p}{X}) \%5}{(p^2 - q^2) \%3}$$

$$\%9 := \frac{(p Y - r)(r - \frac{p}{Y}) \%2}{(p^2 - r^2) \%3}$$

$$\%10 := \frac{(p Y - r)(r - \frac{p}{Y}) p}{(p^2 - r^2) \%3 X}$$

$$
\begin{aligned}
B_5 \;=\; & -(-p^3\,X^2\,q\,Y + p\,\mathrm{x}(p)^2\,q\,r^2\,Y + 2\,p^4\,X\,Y - 2\,q^2\,X\,r^2\,Y - q\,p^3\,Y \\
& + q\,p\,r^2\,Y - p^3\,X\,\mathrm{y}(p)^2\,r + q^2\,X\,p\,Y^2\,r - p^3\,X\,r + q^2\,X\,r\,p)\Bigg(-q\,\Bigg(\Bigg(\Bigg( \\
& -\frac{(n+\mu-\nu+2\,\%8+2\,\%7-\dfrac{\%4\,\%6}{\%1}-\dfrac{\%4\,\%3}{\%1})\,\%4\,p}{\%1\,Y} \\
& + (2\,\%9-\frac{\%4\,p}{\%1\,Y})\,(m+2\,\%8-\frac{\%4\,\%6}{\%1})\Bigg)\Bigg/X - \Bigg( \\
& (m-2\,\%8+\frac{\%4\,\%6}{\%1})\,(-2\,\%9+\frac{\%4\,p}{\%1\,Y}) \\
& + \frac{\%4\,p\,(n-\mu+\nu-2\,\%8-2\,\%7+\dfrac{\%4\,\%6}{\%1}+\dfrac{\%4\,\%3}{\%1})}{\%1\,Y}\Bigg)\,Y\Bigg)\Bigg/(q^2-p^2) + \%4 \\
& \left(\frac{(\mu-\nu)\,(p\,Y-r)\,p}{Y^2\,(p^2-r^2)} + \frac{(n-\%5)\,p\,r\,(-\dfrac{1}{\mathrm{y}(p)^2}-1)}{p^2-r^2} + \frac{1}{Y} + \frac{(\mu-\nu)\,p\,(r-\dfrac{p}{Y})}{p^2-r^2}\right)\Bigg/(\%1) \\
& + 2\,\frac{(p\,Y-r)\,(r-\dfrac{p}{Y})\,Y\,(p^2-q^2)^2\,X\,p\,r\,(-\dfrac{1}{\mathrm{y}(p)^2}-1)}{\%1\,\%2} - 2\,\frac{p\,(r-\dfrac{p}{Y})\,\%3}{(p^2-r^2)\,\%1} \\
& - 2\,\frac{(p\,Y-r)\,p\,\%3}{Y^2\,(p^2-r^2)\,\%1} + \left(\frac{X\,(q-\dfrac{p}{X})}{p^2-q^2} + \frac{\mathrm{y}(p)\,(r-\dfrac{p}{Y})}{p^2-r^2} - \frac{p\,Y-r}{Y\,(p^2-r^2)}\right. \\
& \left. - \frac{p\,X-q}{\mathrm{x}(p)\,(p^2-q^2)} - 2\,\frac{(p\,Y-r)\,(r-\dfrac{p}{Y})\,p}{(p^2-r^2)^2} - 2\,\frac{(p\,X-q)\,(q-\dfrac{p}{X})\,p}{(p^2-q^2)^2}\right)\,p/(\%1\,Y \\
& ) + \frac{\left(\dfrac{p\,(r-\dfrac{p}{Y})}{p^2-r^2} + \dfrac{(p\,Y-r)\,p}{\mathrm{y}(p)^2\,(p^2-r^2)}\right)\,\%6}{\%1} + 2\,\frac{(p\,Y-r)\,(r-\dfrac{p}{Y})\,\%3\,\%10}{(p^2-r^2)\,\%1^2} \\
& - \frac{\%4\,\%6\,\%10}{\%1^2} - \%4\,p\,\Bigg(-2\,\frac{X\,(q-\dfrac{p}{X})\,(p\,Y-r)\,(r-\dfrac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)} \\
& - 2\,\frac{(p\,Y-r)^2\,(r-\dfrac{p}{Y})}{(p^2-r^2)^2\,Y} - 4\,\frac{(p\,X-q)^2\,(q-\dfrac{p}{X})^2\,p}{(p^2-q^2)^3} \\
& + 2\,\frac{(p\,X-q)\,(p\,Y-r)\,(r-\dfrac{p}{Y})}{X\,(p^2-q^2)\,(p^2-r^2)} \\
& + 4\,\frac{(p\,X-q)\,(q-\dfrac{p}{X})\,(p\,Y-r)\,(r-\dfrac{p}{Y})\,p}{(p^2-q^2)\,(p^2-r^2)^2} \\
& + 4\,\frac{(p\,X-q)\,(q-\dfrac{p}{X})\,(p\,Y-r)\,(r-\dfrac{p}{Y})\,p}{(p^2-q^2)^2\,(p^2-r^2)} \\
& - 2\,\frac{(p\,X-q)\,(q-\dfrac{p}{X})\,Y\,(r-\dfrac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)} + 2\,\frac{(p\,X-q)\,(q-\dfrac{p}{X})^2\,X}{(p^2-q^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
&+2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)}{(p^2-q^2)\,Y\,(p^2-r^2)}-4\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p}{(p^2-r^2)^3}\\
&\left.\left.+2\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})^2\,Y}{(p^2-r^2)^2}-2\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})}{(p^2-q^2)^2\,X}\right)/(\%1^2\,Y)\right)/(p\,X\\
&(p^2-r^2)\,(p^2-q^2))-2(-r\,\mathrm{y}(p)+p)\,(p\,Y-r)\left(-\frac{\%4\,\%3\,\%10}{\%1^2}\right.\\
&+\frac{\left(\frac{p\,(r-\frac{p}{Y})}{p^2-r^2}+\frac{(p\,Y-r)\,p}{\mathrm{y}(p)^2\,(p^2-r^2)}\right)\%3}{\%1}+2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,\%6\,\%10}{(p^2-q^2)\,\%1^2}+2\\
&(p\,X-q)\,(q-\frac{p}{X})\,p\left(-2\,\frac{X\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)}\right.\\
&-2\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})}{(p^2-r^2)^2\,Y}-4\,\frac{(p\,\mathrm{x}(p)-q)^2\,(q-\frac{p}{X})^2\,p}{(p^2-q^2)^3}\\
&+2\,\frac{(p\,X-q)\,(p\,Y-r)\,(r-\frac{p}{Y})}{X\,(p^2-q^2)\,(p^2-r^2)}\\
&+4\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,p}{(p^2-q^2)\,(p^2-r^2)^2}\\
&+4\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)\,(r-\frac{p}{Y})\,p}{(p^2-q^2)^2\,(p^2-r^2)}\\
&-2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,Y\,(r-\frac{p}{Y})}{(p^2-q^2)\,(p^2-r^2)}+2\,\frac{(p\,X-q)\,(q-\frac{p}{X})^2\,X}{(p^2-q^2)^2}\\
&+2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,(p\,Y-r)}{(p^2-q^2)\,Y\,(p^2-r^2)}-4\,\frac{(p\,Y-r)^2\,(r-\frac{p}{Y})^2\,p}{(p^2-r^2)^3}\\
&\left.+2\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})^2\,Y}{(p^2-r^2)^2}-2\,\frac{(p\,X-q)^2\,(q-\frac{p}{X})}{(p^2-q^2)^2\,X}\right)/((p^2-q^2)\,\%1^2\,Y)-2\\
&(p\,X-q)\,(q-\frac{p}{\mathrm{x}(p)})\\
&\left(\frac{(\mu-\nu)\,(p\,Y-r)\,p}{Y^2\,(p^2-r^2)}+\frac{(n-\%5)\,p\,r\,(-\frac{1}{Y^2}-1)}{p^2-r^2}+\frac{1}{Y}+\frac{(\mu-\nu)\,p\,(r-\frac{p}{Y})}{p^2-r^2}\right)/(\\
&(p^2-q^2)\,\%1)-\frac{\%4\,(p^2-r^2)\,Y\,(p^2-q^2)^2\,X\,p\,r\,(-\frac{1}{Y^2}-1)}{\%1\,\%2}-2\,\frac{X\,(q-\frac{p}{X})\,p}{(p^2-q^2)\,\%1\,Y}\\
&-r\left(\left(2\,\frac{(n+\mu-\nu+2\,\%8+2\,\%7-\frac{\%4\,\%6}{\%1}-\frac{\%4\,\%3}{\%1})\,(p\,X-q)\,(q-\frac{p}{X})\,p}{(p^2-q^2)\,\%1\,Y}\right.\right.\\
&\left.+(2\,\%9-\frac{\%4\,p}{\%1\,Y})\,(-\%5-\frac{\%4\,\%3}{\%1}+2\,\%7)\right)/Y-\Bigg(
\end{aligned}
$$

$$(-\%5 + \frac{\%4\,\%3}{\%1} - 2\,\%7)\,(-2\,\%9 + \frac{\%4\,p}{\%1\,Y})$$

$$-2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,p\,(n - \mu + \nu - 2\,\%8 - 2\,\%7 + \frac{\%4\,\%6}{\%1} + \frac{\%4\,\%3}{\%1})}{(p^2 - q^2)\,\%1\,Y}\Bigg)\,X\Bigg)\Big/($$

$$r^2 - p^2) + 2\,\frac{(p\,\mathrm{x}(p) - q)\,p}{X\,(p^2 - q^2)\,\%1\,Y} + 4\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,p^2}{(p^2 - q^2)^2\,\%1\,Y}\Bigg)\Big/(p\,(p^2 - r^2))$$

$$\%1 := -2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,(p\,Y - r)\,(r - \frac{p}{Y})}{(p^2 - q^2)\,(p^2 - r^2)} + \frac{(p\,X - q)^2\,(q - \frac{p}{X})^2}{(p^2 - q^2)^2}$$

$$+ \frac{(p\,Y - r)^2\,(r - \frac{p}{Y})^2}{(p^2 - r^2)^2}$$

$$\%2 := p^4\,X^2\,r^2 - 4\,X\,Y^2\,q^3\,p^3 - 2\,X\,Y\,q^3\,r^3 + Y^2\,q^2\,p^4 + Y^2\,q^2\,r^4$$

$$+ 4\,X^2\,p^3\,Y^3\,r\,q^2 + 4\,X^2\,p\,Y^3\,r^3\,q^2 + 4\,\mathrm{x}(p)^2\,p^3\,Y\,r\,q^2 + X^2\,Y^4\,r^2\,q^4$$

$$+ 4\,X^2\,r^3\,p\,Y\,q^2 + 2\,X\,Y^3\,q\,p^2\,r^3 + 2\,X\,Y^3\,q^3\,p^2\,r - 2\,X\,Y^3\,q^3\,r^3$$

$$- 4\,X\,Y^2\,q\,p\,r^4 - 2\,p^2\,X^2\,q^2\,r^2 + Y^2\,p^4\,X^4\,q^2 + 2\,Y^2\,p^4\,X^2\,r^2$$

$$+ 2\,Y^2\,p^4\,X^2\,q^2 - 2\,Y^2\,p^2\,X^4\,q^2\,r^2 + 4\,Y^2\,p^3\,X^3\,q\,r^2 - 2\,X\,Y^3\,q\,p^4\,r$$

$$+ Y^2\,X^4\,q^2\,r^4 + 4\,Y^2\,p\,X^3\,q^3\,r^2 - 16\,Y^2\,p^2\,X^2\,q^2\,r^2 + 2\,Y^2\,X^2\,q^2\,r^4$$

$$+ 4\,Y^2\,p^3\,X\,q\,r^2 + 2\,Y^2\,q^4\,X^2\,r^2 - 2\,Y^2\,q^2\,p^2\,r^2 + 4\,Y^2\,q^3\,p\,X\,r^2$$

$$+ X^2\,p^4\,Y^4\,r^2 - 2\,X^2\,p^2\,Y^4\,r^2\,q^2 + 2\,X^3\,Y\,p^2\,q\,r^3 + 2\,X^3\,Y\,p^2\,q^3\,r$$

$$- 2\,X^3\,Y\,q^3\,r^3 - 4\,X^2\,Y^3\,p^3\,r^3 - 2\,X^3\,Y^3\,q^3\,r^3 - 4\,X^3\,Y^2\,p^3\,q^3$$

$$- 4\,X^3\,Y^2\,p\,q\,r^4 + 2\,X\,Y\,q\,p^2\,r^3 + 2\,X\,Y\,q^3\,p^2\,r - 2\,X^3\,Y^3\,p^4\,q\,r$$

$$+ 2\,X^3\,Y^3\,p^2\,q\,r^3 - 4\,X^2\,Y\,p^3\,r^3 - 2\,X\,Y\,q\,p^4\,r + 2\,X^3\,Y^3\,p^2\,q^3\,r$$

$$- 2\,X^3\,Y\,p^4\,q\,r + 4\,X^2\,Y^2\,p^2\,r^4 - 4\,X^2\,Y^3\,q^4\,p\,r + 4\,X^2\,Y^2\,q^4\,p^2$$

$$- 4\,X^2\,Y\,q^4\,r\,p + q^4\,X^2\,r^2$$

$$\%3 := \mu + \nu + \frac{(\mu - \nu)\,(p\,X - q)\,(q - \frac{p}{X})}{p^2 - q^2} + \frac{(n + m)\,p\,q\,(\frac{1}{X} - X)}{p^2 - q^2}$$

$$- \frac{(p^2 - r^2)\,Y\,(p^2 - q^2)^2\,X\,p\,r\,(\frac{1}{Y} - Y)}{\%2}$$

$$\%4 := \frac{(p\,X - q)\,(q - \frac{p}{X})}{p^2 - q^2} + \frac{(p\,Y - r)\,(r - \frac{p}{Y})}{p^2 - r^2}$$

$$\%5 := \frac{(p^2 - r^2)^2\,Y\,(p^2 - q^2)^2\,X}{\%2}$$

$$\%6 := \mu + \nu + \frac{(\mu - \nu)\,(p\,Y - r)\,(r - \frac{p}{Y})}{p^2 - r^2} + \frac{(n - \%5)\,p\,r\,(\frac{1}{Y} - Y)}{p^2 - r^2}$$
$$+ \frac{m\,p\,q\,(\frac{1}{X} - X)}{p^2 - q^2}$$
$$\%7 := \frac{(p\,X - q)\,(q - \frac{p}{X})\,\%6}{(p^2 - q^2)\,\%1}$$
$$\%8 := \frac{(p\,Y - r)\,(r - \frac{p}{Y})\,\%3}{(p^2 - r^2)\,\%1}$$
$$\%9 := \frac{(p\,X - q)\,(q - \frac{p}{X})\,p}{(p^2 - q^2)\,\%1\,Y}$$
$$\%10 := -2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,(p\,Y - r)\,p}{(p^2 - q^2)\,Y^2\,(p^2 - r^2)}$$
$$-2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,p\,(r - \frac{p}{Y})}{(p^2 - q^2)\,(p^2 - r^2)} + 2\,\frac{(p\,Y - r)\,(r - \frac{p}{Y})^2\,p}{(p^2 - r^2)^2}$$
$$+2\,\frac{(p\,Y - r)^2\,(r - \frac{p}{Y})\,p}{(p^2 - r^2)^2\,Y^2}$$

$$B_6 = -2(-r\,Y + p)\,(p\,Y - r)\left(4\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,\%6\,p}{(p^2 - q^2)^2\,\%3} + \left(\frac{X\,(q - \frac{p}{X})}{p^2 - q^2}\right.\right.$$
$$+ \frac{Y\,(r - \frac{p}{Y})}{p^2 - r^2} - \frac{p\,Y - r}{Y\,(p^2 - r^2)} - \frac{p\,X - q}{X\,(p^2 - q^2)} - 2\,\frac{(p\,Y - r)\,(r - \frac{p}{Y})\,p}{(p^2 - r^2)^2}$$
$$\left. - 2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,p}{(p^2 - q^2)^2}\right)\%2/(\%3) + \%4\left(-\frac{(\mu - \nu)\,(p\,X - q)}{X\,(p^2 - q^2)}\right.$$
$$-2\,\frac{(\mu - \nu)\,(p\,X - q)\,(q - \frac{p}{X})\,p}{(p^2 - q^2)^2} + \frac{(\mu - \nu)\,X\,(q - \frac{p}{X})}{p^2 - q^2}$$
$$+ \frac{(n + m)\,q\,(\frac{1}{X} - X)}{p^2 - q^2} - \frac{(p^2 - r^2)\,Y\,(p^2 - q^2)^2\,X\,r\,(\frac{1}{Y} - Y)}{\%1}$$
$$\left. + 2\,\frac{Y\,(p^2 - q^2)^2\,X\,p^2\,r\,(\frac{1}{\mathrm{y}(p)} - Y)}{\%1} - 2\,\frac{(n + m)\,p^2\,q\,(\frac{1}{X} - \mathrm{x}(p))}{(p^2 - q^2)^2}\right)/(\%3)$$
$$+ 2\,\frac{(p\,X - q)\,\%6}{X\,(p^2 - q^2)\,\%3} + 2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,\%6\,\%9}{(p^2 - q^2)\,\%3^2} - \frac{\%4\,\%2\,\%9}{\%3^2} - r\,\Bigg($$

$$\frac{(n+\mu-\nu+2\,\%8+2\,\%7-\frac{\%4\,\%6}{\%3}-\frac{\%4\,\%2}{\%3})(-\%5-\frac{\%4\,\%2}{\%3}+2\,\%7)}{Y}$$

$$-(-\%5+\frac{\%4\,\%2}{\%3}-2\,\%7)(n-\mu+\nu-2\,\%8-2\,\%7+\frac{\%4\,\%6}{\%3}+\frac{\%4\,\%2}{\%3})\,X\Bigg)\Big/($$

$$r^2-p^2)-2\,\frac{X\,(q-\frac{p}{X})\,\%6}{(p^2-q^2)\,\%3}-2(p\,X-q)\,(q-\frac{p}{X})\Bigg(\frac{(n-\%5)\,r\,(\frac{1}{Y}-Y)}{p^2-r^2}$$

$$-2\,\frac{(\mu-\nu)\,(p\,Y-r)\,(r-\frac{p}{Y})\,p}{(p^2-r^2)^2}+\frac{(\mu-\nu)\,Y\,(r-\frac{p}{Y})}{p^2-r^2}-2\,\frac{m\,p^2\,q\,(\frac{1}{X}-X)}{(p^2-q^2)^2}$$

$$-\frac{(\mu-\nu)\,(p\,Y-r)}{Y\,(p^2-r^2)}-2\,\frac{(n-\%5)\,p^2\,r\,(\frac{1}{Y}-Y)}{(p^2-r^2)^2}+\frac{m\,q\,(\frac{1}{X}-X)}{p^2-q^2}\Bigg)\Big/((p^2-q^2)$$

$$\%3)\Bigg)\Big/(p\,(p^2-r^2))-(-p^3\,X^2\,q\,Y+p\,\mathrm{x}(p)^2\,q\,r^2\,Y+2\,p^4\,X\,Y$$

$$-2\,q^2\,X\,r^2\,Y-q\,p^3\,Y+q\,p\,r^2\,Y-p^3\,X\,\mathrm{y}(p)^2\,r+q^2\,X\,p\,Y^2\,r$$

$$-p^3\,X\,r+q^2\,X\,r\,p)\Bigg(-2(p\,Y-r)\,(r-\frac{p}{Y})\Bigg(-\frac{(\mu-\nu)\,(p\,X-q)}{X\,(p^2-q^2)}$$

$$-2\,\frac{(\mu-\nu)\,(p\,X-q)\,(q-\frac{p}{X})\,p}{(p^2-q^2)^2}+\frac{(\mu-\nu)\,X\,(q-\frac{p}{X})}{p^2-q^2}$$

$$+\frac{(n+m)\,q\,(\frac{1}{X}-X)}{p^2-q^2}-\frac{(p^2-r^2)\,Y\,(p^2-q^2)^2\,X\,r\,(\frac{1}{Y}-Y)}{\%1}$$

$$+2\,\frac{Y\,(p^2-q^2)^2\,X\,p^2\,r\,(\frac{1}{\mathrm{y}(p)}-Y)}{\%1}-2\,\frac{(n+m)\,p^2\,q\,(\frac{1}{X}-\mathrm{x}(p))}{(p^2-q^2)^2}\Bigg)\Big/((p^2-r^2)\,\%3$$

$$)+4\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,\%2\,p}{(p^2-r^2)^2\,\%3}+\%4\Bigg(\frac{(n-\%5)\,r\,(\frac{1}{Y}-Y)}{p^2-r^2}$$

$$-2\,\frac{(\mu-\nu)\,(p\,Y-r)\,(r-\frac{p}{Y})\,p}{(p^2-r^2)^2}+\frac{(\mu-\nu)\,Y\,(r-\frac{p}{Y})}{p^2-r^2}-2\,\frac{m\,p^2\,q\,(\frac{1}{X}-X)}{(p^2-q^2)^2}$$

$$-\frac{(\mu-\nu)\,(p\,Y-r)}{Y\,(p^2-r^2)}-2\,\frac{(n-\%5)\,p^2\,r\,(\frac{1}{Y}-Y)}{(p^2-r^2)^2}+\frac{m\,q\,(\frac{1}{X}-X)}{p^2-q^2}\Bigg)\Big/(\%3)$$

$$+2\,\frac{(p\,Y-r)\,\%2}{Y\,(p^2-r^2)\,\%3}+\Bigg(\frac{X\,(q-\frac{p}{X})}{p^2-q^2}+\frac{\mathrm{y}(p)\,(r-\frac{p}{Y})}{p^2-r^2}-\frac{p\,Y-r}{Y\,(p^2-r^2)}$$

$$-\frac{p\,X-q}{X\,(p^2-q^2)}-2\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,p}{(p^2-r^2)^2}-2\,\frac{(p\,X-q)\,(q-\frac{p}{X})\,p}{(p^2-q^2)^2}\Bigg)\,\%6/(\%3)$$

$$-2\,\frac{Y\,(r-\frac{p}{Y})\,\%2}{(p^2-r^2)\,\%3}+2\,\frac{(p\,Y-r)\,(r-\frac{p}{Y})\,\%2\,\%9}{(p^2-r^2)\,\%3^2}-\frac{\%4\,\%6\,\%9}{\%3^2}-q\Bigg($$

$$\frac{(n+\mu-\nu+2\,\%8+2\,\%7-\frac{\%4\,\%6}{\%3}-\frac{\%4\,\%2}{\%3})\,(m+2\,\%8-\frac{\%4\,\%6}{\%3})}{X}$$

$$- (m - 2\,\%8 + \frac{\%4\,\%6}{\%3})\,(n - \mu + \nu - 2\,\%8 - 2\,\%7 + \frac{\%4\,\%6}{\%3} + \frac{\%4\,\%2}{\%3})\,Y\Big)\,/(q^2 - p^2)\Big)\,/$$

$$(p\,X\,(p^2 - r^2)\,(p^2 - q^2))$$

$$\%1 := p^4\,X^2\,r^2 - 4\,X\,Y^2\,q^3\,p^3 - 2\,X\,Y\,q^3\,r^3 + Y^2\,q^2\,p^4 + Y^2\,q^2\,r^4$$
$$+ 4\,X^2\,p^3\,Y^3\,r\,q^2 + 4\,X^2\,p\,Y^3\,r^3\,q^2 + 4\,\mathrm{x}(p)^2\,p^3\,Y\,r\,q^2 + X^2\,Y^4\,r^2\,q^4$$
$$+ 4\,X^2\,r^3\,p\,Y\,q^2 + 2\,X\,Y^3\,q\,p^2\,r^3 + 2\,X\,Y^3\,q^3\,p^2\,r - 2\,X\,Y^3\,q^3\,r^3$$
$$- 4\,X\,Y^2\,q\,p\,r^4 - 2\,p^2\,X^2\,q^2\,r^2 + Y^2\,p^4\,X^4\,q^2 + 2\,Y^2\,p^4\,X^2\,r^2$$
$$+ 2\,Y^2\,p^4\,X^2\,q^2 - 2\,Y^2\,p^2\,X^4\,q^2\,r^2 + 4\,Y^2\,p^3\,X^3\,q\,r^2 - 2\,X\,Y^3\,q\,p^4\,r$$
$$+ Y^2\,X^4\,q^2\,r^4 + 4\,Y^2\,p\,X^3\,q^3\,r^2 - 16\,Y^2\,p^2\,X^2\,q^2\,r^2 + 2\,Y^2\,X^2\,q^2\,r^4$$
$$+ 4\,Y^2\,p^3\,X\,q\,r^2 + 2\,Y^2\,q^4\,X^2\,r^2 - 2\,Y^2\,q^2\,p^2\,r^2 + 4\,Y^2\,q^3\,p\,X\,r^2$$
$$+ X^2\,p^4\,Y^4\,r^2 - 2\,X^2\,p^2\,Y^4\,r^2\,q^2 + 2\,X^3\,Y\,p^2\,q\,r^3 + 2\,X^3\,Y\,p^2\,q^3\,r$$
$$- 2\,X^3\,Y\,q^3\,r^3 - 4\,X^2\,Y^3\,p^3\,r^3 - 2\,X^3\,Y^3\,q^3\,r^3 - 4\,X^3\,Y^2\,p^3\,q^3$$
$$- 4\,X^3\,Y^2\,p\,q\,r^4 + 2\,X\,Y\,q\,p^2\,r^3 + 2\,X\,Y\,q^3\,p^2\,r - 2\,X^3\,Y^3\,p^4\,q\,r$$
$$+ 2\,X^3\,Y^3\,p^2\,q\,r^3 - 4\,X^2\,Y\,p^3\,r^3 - 2\,X\,Y\,q\,p^4\,r + 2\,X^3\,Y^3\,p^2\,q^3\,r$$
$$- 2\,X^3\,Y\,p^4\,q\,r + 4\,X^2\,Y^2\,p^2\,r^4 - 4\,X^2\,Y^3\,q^4\,p\,r + 4\,X^2\,Y^2\,q^4\,p^2$$
$$- 4\,X^2\,Y\,q^4\,r\,p + q^4\,X^2\,r^2$$

$$\%2 := \mu + \nu + \frac{(\mu - \nu)\,(p\,X - q)\,(q - \frac{p}{X})}{p^2 - q^2} + \frac{(n + m)\,p\,q\,(\frac{1}{X} - X)}{p^2 - q^2}$$
$$- \frac{(p^2 - r^2)\,Y\,(p^2 - q^2)^2\,X\,p\,r\,(\frac{1}{Y} - Y)}{\%1}$$

$$\%3 := -2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,(p\,Y - r)\,(r - \frac{p}{Y})}{(p^2 - q^2)\,(p^2 - r^2)} + \frac{(p\,X - q)^2\,(q - \frac{p}{X})^2}{(p^2 - q^2)^2}$$
$$+ \frac{(p\,Y - r)^2\,(r - \frac{p}{Y})^2}{(p^2 - r^2)^2}$$

$$\%4 := \frac{(p\,X - q)\,(q - \frac{p}{X})}{p^2 - q^2} + \frac{(p\,Y - r)\,(r - \frac{p}{Y})}{p^2 - r^2}$$

$$\%5 := \frac{(p^2 - r^2)^2\,Y\,(p^2 - q^2)^2\,X}{\%1}$$

$$\%6 := \mu + \nu + \frac{(\mu - \nu)\,(p\,Y - r)\,(r - \frac{p}{Y})}{p^2 - r^2} + \frac{(n - \%5)\,p\,r\,(\frac{1}{Y} - Y)}{p^2 - r^2}$$
$$+ \frac{m\,p\,q\,(\frac{1}{X} - X)}{p^2 - q^2}$$

$$\%7 := \frac{(p\,X - q)\,(q - \frac{p}{X})\,\%6}{(p^2 - q^2)\,\%3}$$

$$\%8 := \frac{(p\,Y - r)\,(r - \frac{p}{Y})\,\%2}{(p^2 - r^2)\,\%3}$$

$$\%9 := -2\,\frac{X\,(q - \frac{p}{X})\,(p\,Y - r)\,(r - \frac{p}{Y})}{(p^2 - q^2)\,(p^2 - r^2)} - 2\,\frac{(p\,Y - r)^2\,(r - \frac{p}{Y})}{(p^2 - r^2)^2\,Y}$$
$$- 4\,\frac{(p\,X - q)^2\,(q - \frac{p}{X})^2\,p}{(p^2 - q^2)^3} + 2\,\frac{(p\,X - q)\,(p\,Y - r)\,(r - \frac{p}{Y})}{X\,(p^2 - q^2)\,(p^2 - r^2)}$$
$$+ 4\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,(p\,Y - r)\,(r - \frac{p}{Y})\,p}{(p^2 - q^2)\,(p^2 - r^2)^2}$$
$$+ 4\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,(p\,Y - r)\,(r - \frac{p}{Y})\,p}{(p^2 - q^2)^2\,(p^2 - r^2)}$$
$$- 2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,Y\,(r - \frac{p}{Y})}{(p^2 - q^2)\,(p^2 - r^2)} + 2\,\frac{(p\,X - q)\,(q - \frac{p}{X})^2\,X}{(p^2 - q^2)^2}$$
$$+ 2\,\frac{(p\,X - q)\,(q - \frac{p}{X})\,(p\,Y - r)}{(p^2 - q^2)\,Y\,(p^2 - r^2)} - 4\,\frac{(p\,Y - r)^2\,(r - \frac{p}{Y})^2\,p}{(p^2 - r^2)^3}$$
$$+ 2\,\frac{(p\,Y - r)\,(r - \frac{p}{Y})^2\,Y}{(p^2 - r^2)^2} - 2\,\frac{(p\,X - q)^2\,(q - \frac{p}{X})}{(p^2 - q^2)^2\,X}$$

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