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# Internal resonance between in-plane and out-of-plane modes of vibration of inclined cables subjected to vertical support excitation.

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**ABSTRACT:** Inclined cables are important structural elements of cable-stayed bridges. When the bridge deck oscillates, large amplitude cable vibrations can arise in various modes as a result of the low cable damping, parametric excitation or non-linear modal coupling. The resulting vibrations are undesirable and potentially damaging to the long-term performance of the bridge. The phenomena can be modelled considering internal resonances between in-plane and out-of-plane modes of vibration of the cable. Here they have been studied using a four-mode model that represents the response of an inclined cable vertically excited at the lower end (i.e. from the deck) with a frequency that is close to the natural frequency of the second cable mode in each plane and twice the frequency of the first mode in each plane. The modal equations of the model are investigated using the software package AUTO for the numerical continuation of solutions of a system of ODEs. This allows the identification of the important solution branches in the cable model responsible for unwanted vibration behaviour. The result of our analysis is that we identify amplitudes of excitation above which modes other than the directly excited mode (the second in-plane mode) start contributing to the response of the cable. In addition, we show that the response amplitudes in these additional modes is of similar magnitude to the amplitudes in the directly excited mode, which could be considered an issue in the design of cable-stayed bridges. In summary, by using a numerical continuation technique we predict when the response of the cable will change from a single in-plane mode to coupled responses in two or more modes, in-plane or in both planes, and the modal amplitudes involved in these coupled responses.

**KEY WORDS:** Cable vibration; Internal resonance; Sway motion; Modal interaction; Bifurcation analysis.

## 1 INTRODUCTION

The inclined cables of cable-stayed bridges are typically lightly damped. When the bridge deck oscillates (typically from wind or traffic excitation) it provides a support motion input to the cable. This type of excitation can lead to large amplitude vibrations of the cable [1]. A case of particular interest is when the deck motion is at two times a natural frequency of the cable, which can lead to large amplitude parametrically excited vibrations. This phenomenon was studied in [2] by using a nonlinear Mathieu-type equation to model the parametric resonance [3]. However, since the cables are taut, the natural frequencies are almost at pure harmonics of the fundamental mode, so two times one natural frequency closely coincides with another mode, which is directly excited by the same input [4], [5]. Furthermore internal resonance occurs between the in-plane and out-of-plane modes, so non-linear coupling leads to out-of-plane responses for in-plane excitation. The deck motions are generally larger at lower frequencies, so the most significant responses often occur for deck motions around the second natural frequency of the cable, where small deck inputs can cause large responses in the first and second modes in both planes.

In this paper the Warnitchai equations [6] (see also [7] for a detailed derivation) are used to model the vibration of the cable. It is assumed that the longitudinal vibrations of the cable can be neglected, so that the planes of interest are vertical (in-plane) and sway (out-of-plane).

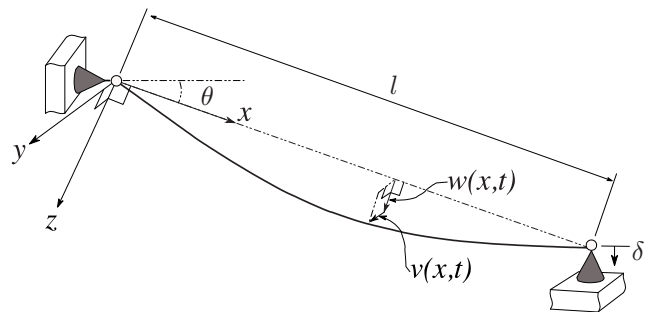


Figure 1. Schematic representation of an inclined cable with vertical input motion at the lower attachment point.

Four modes are included in the model, two in-plane and two out-of-plane, which enables the important low frequency dynamic behaviour to be modelled [8], [9]. The Warnitchai equations are scaled and averaged using the same procedure as previously presented in [4], [5]. Then a bifurcation study is carried out by means of numerical continuation [10] with the software package AUTO [11]. In this way, a series of bifurcation diagrams are produced for different values of the detuning parameter of the excitation frequency (with respect to the second in-plane natural frequency) with the aim of studying the behaviour of modal amplitudes of the response of the cable close to the resonance frequency. The result is a general picture of the stable and unstable branches and all interactions between the bifurcation

$$\begin{aligned}
y'_{1c} &= \left( \frac{W_{12}z_{1c}z_{1s}}{16\omega_1} - \xi\omega_1 \right) y_{1c} - \left( \mu\omega_1 + \frac{N_1\Delta}{4\omega_1} - \frac{W_{12}}{32\omega_1} (C_1 - 2z_{1c}^2) \right) y_{1s}, \\
y'_{1s} &= \left( \mu\omega_1 - \frac{N_1\Delta}{4\omega_1} - \frac{W_{12}}{32\omega_1} (C_1 - 2z_{1s}^2) \right) y_{1c} - \left( \xi\omega_1 + \frac{W_{12}z_{1c}z_{1s}}{16\omega_1} \right) y_{1s}, \\
y'_{2c} &= \left( \frac{W_{12}z_{2c}z_{2s}}{2\omega_1} - 2\xi\omega_1 \right) y_{2c} - \left( 2\mu\omega_1 - \frac{W_{12}}{8\omega_1} (C_2 - 4z_{2c}^2) \right) y_{2s}, \\
y'_{2s} &= \left( 2\mu\omega_1 - \frac{W_{12}}{8\omega_1} (C_2 - 4z_{2s}^2) \right) y_{2c} - \left( 2\xi\omega_1 + \frac{W_{12}z_{2c}z_{2s}}{2\omega_1} \right) y_{2s}, \\
z'_{1c} &= \left( \frac{W_{12}y_{1c}y_{1s}}{16\omega_1} - \xi\omega_1 \right) z_{1c} - \left( (\mu - \kappa)\omega_1 + \frac{N_1\Delta}{4\omega_1} - \frac{W_{12}}{32\omega_1} (C_1 - 2y_{1c}^2) \right) z_{1s}, \\
z'_{1s} &= \left( (\mu - \kappa)\omega_1 - \frac{N_1\Delta}{4\omega_1} - \frac{W_{12}}{32\omega_1} (C_1 - 2y_{1s}^2) \right) z_{1c} - \left( \xi\omega_1 + \frac{W_{12}y_{1c}y_{1s}}{16\omega_1} \right) z_{1s}, \\
z'_{2c} &= \left( \frac{W_{12}y_{2c}y_{2s}}{2\omega_1} - 2\xi\omega_1 \right) z_{2c} - \left( 2\mu\omega_1 - \frac{W_{12}}{8\omega_1} (C_2 - 4y_{2c}^2) \right) z_{2s}, \\
z'_{2s} &= \left( 2\mu\omega_1 - \frac{W_{12}}{8\omega_1} (C_2 - 4y_{2s}^2) \right) z_{2c} - \left( 2\xi\omega_1 + \frac{W_{12}y_{2c}y_{2s}}{2\omega_1} \right) z_{2s} - B\Delta\omega_1. \tag{1}
\end{aligned}$$

loci curves in the space defined by the norm of the solutions and both main parameters (input amplitude and frequency). In addition, one of these solution branches is discussed in more detail regarding the response amplitudes in all modes that our model includes.

The remainder of this paper is structured as follows. In Section 2 the equations of motion for the inclined cable model are presented as derived via scaling and first-order averaging. Section 3 describes the bifurcation study and numerical continuation analysis. Conclusions are drawn in Section 4.

## 2 MODAL AMPLITUDE EQUATIONS

In this paper we use a mathematical model of an inclined cable, see Figure 1, which is excited vertically at the bottom anchorage with amplitude  $\Delta$  and angular frequency  $\Omega$  where its upper anchorage stays in the rest position [4]. Our model is based on the Warnitchai equations used in [4], [5], [6], [7]. In the Warnitchai derivation, the out-of-plane and in-plane modal response of the cable are given by

$$\begin{aligned}
v(x,t) &= \sum_{n=1}^{\infty} \phi_n(x)y_n(t), \\
w(x,t) &= \sum_{n=1}^{\infty} \psi_n(x)z_n(t),
\end{aligned}$$

respectively, where  $x \in [0, \ell]$ ,  $\ell$  is the support separation distance, the spatial functions  $\phi(x)$  and  $\psi(x)$  are the out-of-plane and in-plane linear modes of a cable with fixed ends, and  $y_n(t)$  and  $z_n(t)$  their corresponding time-dependent generalised coordinates. The mode shapes for the out-of-plane and even in-plane modes

of the linearised system are assumed to be:

$$\begin{aligned}
\phi_n &= \sin\left(n\pi\frac{x}{\ell}\right) & \text{for } n = 1, 2, 3, \dots, \\
\psi_n &= \sin\left(n\pi\frac{x}{\ell}\right) & \text{for } n = 2, 4, 6, \dots
\end{aligned}$$

For the odd in-plane modes the mode shapes are more complex; however, as is discussed in [7], for taut cables they may be approximated by sine functions of the same form as the out-of-plane modes. To derive the time-dependent generalised coordinates we can use the non-autonomous system with eight ODEs, see [5] where the original Warnitchai equations of motion of the cable are scaled and averaged to become Equations (1). Here,  $C_1 = 3M_1 + 8M_2$ ,  $C_2 = M_1 + 6M_2$ , where  $M_1 = y_{1c}^2 + y_{1s}^2 + z_{1c}^2 + z_{1s}^2$  and  $M_2 = y_{2c}^2 + y_{2s}^2 + z_{2c}^2 + z_{2s}^2$ . Furthermore,  $\kappa = k_1/2$ ,  $W_{nk} = v_{nk}/m$ ,  $N_n = 2\eta_n \sin\theta/m$ ,  $B = \zeta_2 \cos\theta/m$  and  $\delta = \Delta \cos(\Omega t)$ ; numerical values for the cable parameters are given in Table 1 and Table 2 based on the scaled-model cable tested in [5]. In the expressions for  $W_{nk}$ ,  $N_n$  and  $B$ :

Table 1. Dimensional cable parameters

$N_1$ [Hz <sup>2</sup> /m]	$W_{12}$ [1/(s.m) <sup>2</sup> ]	$\omega_1$ [rad/s]
$1.04 \times 10^{-4}$	$5.19 \times 10^{-4}$	20.4852

Table 2. Nondimensional cable parameters.

$B$	$\xi$	$\kappa$
0.2939	0.002	0.0234

$m = \rho A \ell / 2$  is the modal mass of the cable (for all modes),  $\theta$  is the angle between the cable chord and the horizontal plane,  $\sigma_s$  is the static stress,  $\lambda^2$  is Irvine's parameter,  $A$  is the cross sectional area,  $\rho$  is the density,  $g$  is gravity and  $E$  is Young's modulus;

$\omega_n$  is the  $n$ -th natural frequency of the cable. The equivalent modulus of the cable  $E_q$ , the distributed weight perpendicular to the cable chord  $\gamma$ , and the parameters  $\nu_{nk}, \eta_n, \zeta_n$  are given by:

$$E_q = \frac{1}{1 + \lambda^2/12} E, \quad \lambda^2 = \frac{E}{\sigma_s} \left( \frac{\gamma \ell}{\sigma_s} \right)^2, \quad \gamma = \rho g \cos \theta,$$

$$\nu_{nk} = \frac{EA\pi^4 n^2 k^2}{8\ell^3}, \quad \eta_n = \frac{E_q A \pi^2 n^2}{4\ell^2}, \quad \zeta_n = \frac{2m}{n\pi}.$$

The out-of-plane and in-plane natural frequencies,  $\omega_{yn}$  and  $\omega_{zn}$  respectively, are given by

$$\omega_{yn} = \frac{n\pi}{\ell} \sqrt{\frac{\sigma_s}{\rho}}, \quad \omega_{zn} = \frac{n\pi}{\ell} \sqrt{\frac{\sigma_s}{\rho} (1 + k_n)},$$

where  $k_n$  is due to the effect of sag and is given by

$$k_n = \left( \frac{2\lambda^2}{\pi^4 n^4} \right) (1 + (-1)^{n+1})^2.$$

The vibration response of the cable presented by Equations (1) can be expressed by the amplitude contributions of two in-plane and two out-of-plane modal amplitudes:

- $Z_2 = \sqrt{z_{2c}^2 + z_{2s}^2}$  for the second in-plane modal amplitude;
- $Z_1 = \sqrt{z_{1c}^2 + z_{1s}^2}$  for the first in-plane modal amplitude;
- $Y_2 = \sqrt{y_{2c}^2 + y_{2s}^2}$  for the second out-of-plane modal amplitude;
- $Y_1 = \sqrt{y_{1c}^2 + y_{1s}^2}$  for the first out-of-plane modal amplitude.

The amplitude of excitation  $\Delta$  and the detuning  $\mu = \Omega/\omega_2 - 1$  between the frequency of the actuator and the second natural frequency of the cable are the main parameters of Equations (1). The non-dimensional values that are used when we vary the parameters of the system are  $\Delta/L$  and  $\mu$ , where  $L = 5.4m$  is the untensioned cable length measured from the top of the cable to its bottom, and  $L \approx \ell$ . The norm  $\|N\| = \sqrt{Z_2^2 + Z_1^2 + Y_2^2 + Y_1^2}/L$  of the solutions of Equations (1) is a function of  $\Delta/L$  and  $\mu$ , which we use to measure the changes of the behaviour of the cable response.

### 3 BIFURCATION STUDY

Our system in the form of ODEs (1) can be examined with the continuation software AUTO [10], [11]. In our study we follow the stable solutions of Equations (1) from zero forcing amplitude, hence zero response amplitude, and different values of forcing frequency close to the second natural frequency of the cable. Therefore we fix the forcing frequency and vary the excitation amplitude. This leads to increasing amplitude of the response of the cable. Because the averaging technique is based on modal decomposition, we are analysing how the modal amplitudes of the response change as the excitation amplitude increases from zero to positive values. In the resulting bifurcation diagrams [12], [13], stable and unstable solution branches can be identified. Since the stable solutions define the possible modes of behaviour of the physical cable system we can calculate each modal contribution to the overall vibration response. By combining multiple continuations of this type, each one for a different frequency close to 2:1 resonance, we obtain a set of solution curves of Equations (1).

#### 3.1 Stability boundaries in $(\Delta/L, \mu, \|N\|)$ -space

Series of solution curves of Equations (1) of the following types ( $Z_2 \neq 0, Z_1 = Y_2 = Y_1 = 0$ ) for  $(\Delta/L, \mu) \in [0, 6.2 \times 10^{-3}] \times [-0.03, 0.07]$  give us a set of solution branches  $\Gamma$  in  $(\Delta/L, \mu, \|N\|)$ -space, see Figure 2. The solution curve for  $\mu = 0.01$  is shown colourless to identify the intersection of  $\Gamma$  with the  $(\Delta/L, 0.01, \|N\|)$ -plane; it is the bold white curve that is labelled  $l_1$ . Later in this paper, the bifurcation curve  $l_1$  will be explained in detail.  $\Gamma$  could be constructed if we derive the equation of the surface that contains it by numerically solving only the last couple of equations responsible for the  $Z_2$ -mode [5], the directly excited mode, while taking into account that all other modes have no contribution.

In Figure 2 dark grey represents the stable parts of the solution curves and light the unstable ones. The curves  $B_{Y_1}, B_{Y_2}, B_{Z_1}$  and  $F_{Z_2}$ , drawn black, are respectively the loci of the branch points of  $Y_1, Y_2$  and  $Z_1$ -modal amplitudes and the fold points of the  $Z_2$ -modal amplitudes. When it forms a boundary between dark and light parts then the respective bifurcation concerns a stable solution. This is the case for  $B_{Y_1}$  on the left of the crossing point  $N$ , for  $B_{Y_2}$  between the crossing points  $N$  and  $M$ , for  $F_{Z_2}$  between the crossing points  $M$  and the crossing point  $K$ , and for  $B_{Z_1}$  between crossing point  $K$  and the right end of the region of  $\mu$ . Otherwise, the respective bifurcation concerns an unstable solution. In Figure 2 we can see the interactions of the bifurcation curves at crossing points  $N, M, K$  and the cusp point  $C$  [12], [13] therefore, we have a general picture of how the dynamics of the cable changes when we vary  $\mu$  or  $\Delta/L$  by the already shown stability boundaries of the pure  $Z_2$ -modal amplitudes response region, the dark grey region of  $\Gamma$  in Figure 2.

The approximate coordinates of the points  $N, M, C$  and  $K$  in  $(\Delta/L, \mu, \|N\|)$ -space, for the chosen cable parameters, are  $(0.8 \times 10^{-4}, 0.0036, 2.4 \times 10^{-3})$  for  $N$ ,  $(0.9 \times 10^{-4}, 0.062, 6.8 \times 10^{-3})$  for  $M$ ,  $(0.2 \times 10^{-4}, 0.0035, 1.3 \times 10^{-3})$  for  $C$ , and  $(2.2 \times 10^{-4}, 0.021, 2.3 \times 10^{-3})$  for  $K$ .

Hence if the  $B_{Y_1}$ -curve is crossed for  $\mu \in [-0.03, 0.0036]$  as  $\Delta/L$  is increased for a fixed  $\mu$ , we would experience out-of-plane motion due to the presence of stable branches that include the  $Y_1$  mode. The parts  $B_{Y_1}$  and  $F_{Z_2}$  which concern stable solutions, approach each other, and they enclose a region that corresponds to pure  $Z_2$ -modal amplitude. This region is crossed by the  $B_{Z_1}$ -curve. There are two crossings, one is between  $B_{Y_1}$  and  $B_{Z_1}$  and another is between  $F_{Z_2}$  and  $B_{Z_1}$  but, because they are close enough, we only indicate the point  $M$  in between to denote these crossings. Thus if the  $B_{Y_2}$ -curve is crossed for  $\mu \in [0.0036, 0.062]$ , that is, between points  $N$  and  $M$ , as  $\Delta/L$  is increased for a fixed  $\mu$ , we would experience out-of-plane motion due to presence of stable branches that include  $Y_2$  mode. If the  $F_{Z_2}$ -curve is crossed for  $\mu \in [0.0035, 0.0062]$ , that is, between points  $C$  and  $M$ , as  $\Delta/L$  is decreased for a fixed  $\mu$ , we would experience a hysteresis jump down to another pure  $Z_2$ -modal amplitudes that correspond to solutions with lower norm  $\|N\|$ . If the  $F_{Z_2}$ -curve is crossed for  $\mu \in [0.0035, 0.021]$ , that is, between points  $C$  and  $K$ , as  $\Delta/L$  is increased for a fixed  $\mu$ , we would experience out-of-plane motion due to a hysteresis

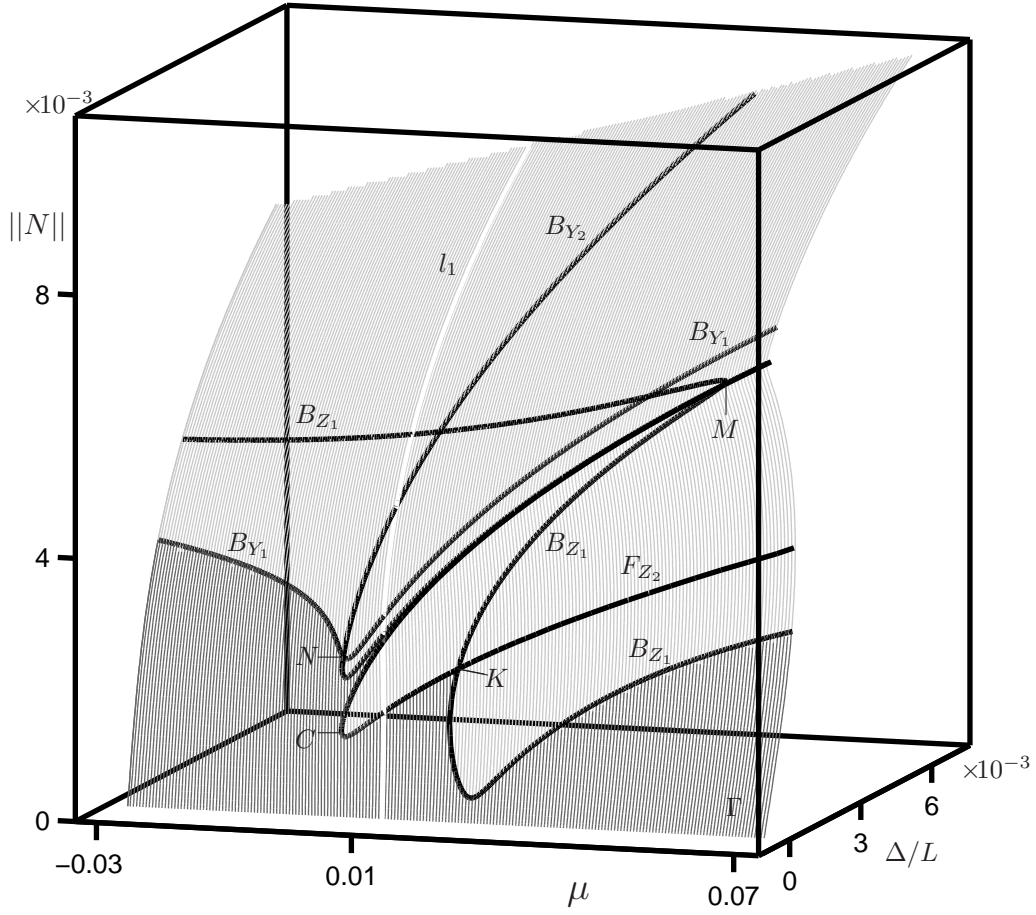


Figure 2. The set of solution branches  $\Gamma$  in  $(\Delta/L, \mu, \|N\|)$ -space, obtained by a series of continuations of Equation (1); its intersection with the  $(\Delta/L, \|N\|)$ -plane for  $\mu = 0.01$  is the bold white curve denoted  $l_1$ . Solution branches including a pure  $Z_2$ -mode are gray; stable regions are shown dark and unstable region light. Bifurcation curves  $B_{Y_1}, B_{Y_2}, B_{Z_1}$  and  $F_{Z_2}$ , are black, and the bifurcation points  $C, K, N$  and  $M$  are labeled.

jump to another stable solution branches that include  $Y_1$  and  $Y_2$  modes. As an example of this case we will discuss in details how the cable response changes when  $\Delta/L$  is increased for fixed  $\mu = 0.01$ . Finally if the  $B_{Z_1}$ -curve is crossed for  $\mu \geq 0.021$  as  $\Delta/L$  is increased for a fixed  $\mu$ , where  $B_{Z_1}$  concerns stable solution, we would experience vibration amplitudes that are not only in a pure  $Z_2$ -mode. In this case our theoretical study is not complete yet, the possible presence of stable limit cycles in this region makes the prediction of the dynamics beyond the  $Z_1$  branch point curve hard to achieve. As can be seen in [5], most of the stability boundaries of  $Z_2$ -amplitude that are shown in Figure 2 are experimentally confirmed.

### 3.2 One-parameter bifurcation diagram

The set  $\Gamma$  of solutions in the  $(\Delta/L, \mu, \|N\|)$ -space shown in Fig. 2 features four bifurcation curves. As we explained, these curves may give rise to additional branches of stable solutions or to some kind of hysteresis jump to other branches of stable solutions. To become familiar with one of these situations we pick the value  $\mu = 0.01$  and perform a detailed continuation analysis by variation of the amplitude of excitation  $\Delta/L \in [0, 7 \times$

$10^{-4}]$ . Thus, Figure 3 presents the one-parameter bifurcation diagram of Equations (1) for fixed  $\mu = 0.01$ ; it shows the properties of the solution branches  $l_1 \in \Gamma_1, l_2, l_3$  and  $l_4$ , and how their stable (denoted by black) and unstable (denoted by grey) parts meet in bifurcation points. The bifurcation points that lie on  $l_1$  also lie on  $\Gamma$  so they are the crossing points between  $l_1$  and the bifurcation curves shown in Figure 2. More precisely, branch point bifurcations  $B_1, B_3 \in B_{Y_2}$  and  $B_2 \in B_{Y_1}$  are denoted by  $\blacksquare$ . Fold bifurcation points  $F_1, F_2 \in F_{Z_2}$  are denoted by  $\bullet$ .

The continuation starts at zero amplitude and, as the excitation increases, we follow  $l_1$  on which only the  $Z_2$ -amplitudes exist. Then at  $\Delta/L$  approximately  $0.74 \times 10^{-4}$  a fold bifurcation point  $F_1$  appears. The curve  $l_1$  loses stability at  $F_1$  and becomes unstable until it reaches a second bifurcation point  $F_2$  at  $\Delta/L \approx 0.37 \times 10^{-4}$ . There it becomes stable again for a very small interval, see Figure 4 (enlarged view of Figure 3), until a branch point  $B_1$  for  $\Delta/L \approx 0.4 \times 10^{-4}$ . Beyond the point  $B_1$ ,  $l_1$  stays unstable to the end of the region of interest of  $\Delta/L$ . According to the instability of  $l_1$  beyond  $F_1$ , if we increase  $\Delta/L$  above  $0.74 \times 10^{-4}$ , a transition to another stable solution branch  $l_4$  takes place. On  $l_4$  the modal amplitudes  $Z_2, Y_1$ , and  $Y_2$  are

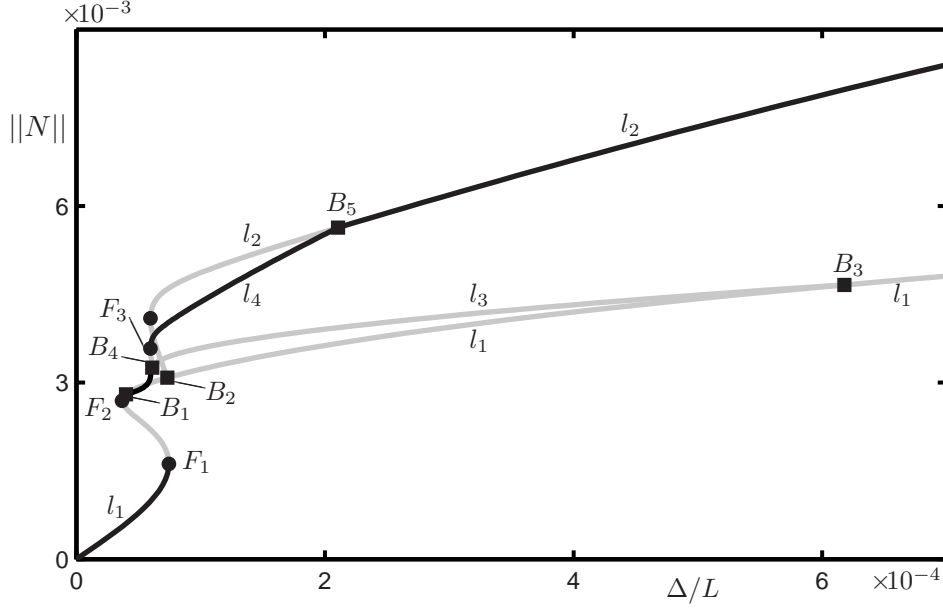


Figure 3. One-parameter bifurcation diagram of Equation (1) for  $\mu = 0.01$ , showing solution branches represented by their norm  $\|N\| = \sqrt{Z_2^2 + Z_1^2 + Y_2^2 + Y_1^2}/L$  for  $\Delta/L \in [0, 7 \times 10^{-4}]$ . The stable parts of the solution branches  $l_1$ – $l_4$  (black curves) are connected via their unstable parts (grey curves), which meet at bifurcation points. Specifically,  $B_1$  –  $B_5$  are branch points (denoted by  $\blacksquare$ ) and  $F_1$  –  $F_3$  are fold points (denoted by  $\bullet$ ).

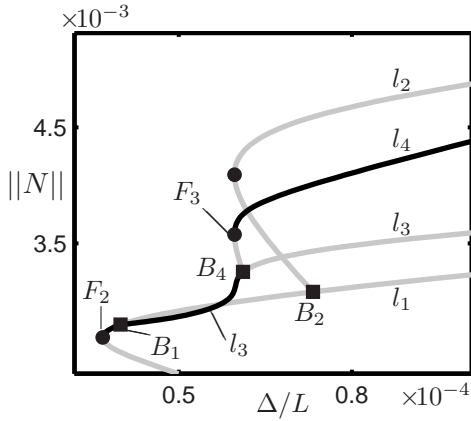


Figure 4. Scaled view of Figure 3, we look in the domain  $(\Delta/L, \|N\|) \in [0.3 \times 10^{-4}, 1 \times 10^{-4}] \times [2.3 \times 10^{-3}, 5.3 \times 10^{-3}]$  where we can distinguish how  $F_2$  and  $B_1$  are separated by a stable part of the  $l_1$  solution branch, and the connections between the  $l_1$  –  $l_4$  branches.

non-zero, hence for  $\Delta/L$  above  $0.74 \times 10^{-4}$ , strong out-of-plane amplitudes start contributing to the response of the cable. If we further increase  $\Delta/L$  we will reach the branch point  $B_5$  at  $\Delta/L \approx 2.1 \times 10^{-4}$  that connects  $l_4$  to the stable part of  $l_2$  on which  $Y_1$  and  $Z_2$  amplitudes exist. Until the end of the region of continuation,  $l_2$  remains stable and  $Y_1$  is the only modal amplitude apart from  $Z_2$  that contributes to the overall response of the cable. Here we have to explain how we found the stable part of  $l_2$  and  $l_4$ , which are significant to our study because they give us the information on the dynamics of the cable past the point  $F_1$  where we find an out-of-plane response. From the branch point  $B_1$ , see Figure 4, we start to compute solution curve  $l_3$ , which represents the  $Y_2$  contributions to the cable response. In the beginning of

this curve between  $B_1$  and  $B_4$ , there exists a region where  $l_3$  is stable. Beyond the bifurcation point  $B_4$  at  $\Delta/L \approx 0.61 \times 10^{-4}$ ,  $l_3$  becomes unstable. It joins again  $l_1$  at  $\Delta/L \approx 6.2 \times 10^{-4}$ . The point  $B_3$  where  $l_3$  joins  $l_1$  again can be seen in Figure 2 as the intersection between the  $B_{Y_2}$  branch point curve and  $l_1$ . At  $\Delta/L \approx 0.61 \times 10^{-4}$  there is a branch point  $B_4$  on  $l_3$ , this branch point is the place where  $l_3$  joins  $l_4$ , which is the solution curve on which the modal amplitudes  $Z_2$ ,  $Y_1$  and  $Y_2$  exist. Until the fold point  $F_3$  at  $\Delta/L \approx 0.6 \times 10^{-4}$   $l_4$  is unstable; beyond this point the stable part of  $l_4$  begins and it becomes physically observable as the solution branch that is responsible for the out-of-plane cable response that arises when  $\Delta/L$  becomes larger than  $0.74 \times 10^{-4}$ , beyond the  $F_1$  fold point. The onset of a contribution of the  $Y_1$ -amplitude to the  $Z_2$ -amplitude, i.e. the solution curve  $l_2$  on which we remain for values of  $\Delta/L$  higher than  $2.1 \times 10^{-4}$ , beyond branch point  $B_5$  in Figure 3, goes back to the branch point  $B_2$  on  $l_1$  at  $\Delta/L \approx 0.73 \times 10^{-4}$ , see Figure 4. Between  $B_2$  and  $B_5$ , the branch  $l_2$  is unstable and does not effect the response of the cable.

The stable parts of the solution branches  $l_1$ ,  $l_2$ ,  $l_3$ , and  $l_4$  are shown in Figures 5(a), (b), and (c) in order to show how the  $Z_2$ ,  $Y_1$ , and  $Y_2$ -amplitudes contribute to the overall response of the cable for  $\mu = 0.01$ . Figure 5(a) shows that the  $Z_2$ -amplitude contributes to all the stable branches  $l_1$  to  $l_4$ . The only modal amplitude that is involved in  $l_1$  is the  $Z_2$ -amplitude; note that in Figures 5(b) and (c)  $l_1$  is shown as zero response. Then, if  $\Delta/L$  becomes larger than  $0.74 \times 10^{-4}$ , we jump from the point  $F_1$  and the cable response vibrates with out-of-plane contributions from the  $Y_1$ -amplitude and from the  $Y_2$ -amplitude. As can be seen in Figures 5(b) and (c),  $l_4$  appears as a non-zero curve. At  $B_5$ ,  $l_4$  joins  $l_2$ , which is the point where the  $Y_2$ -amplitude disappears and the cable response shows  $(Z_2, Y_1)$ -amplitudes only, until the end of the region of continuation of  $\Delta/L$ . The contribution

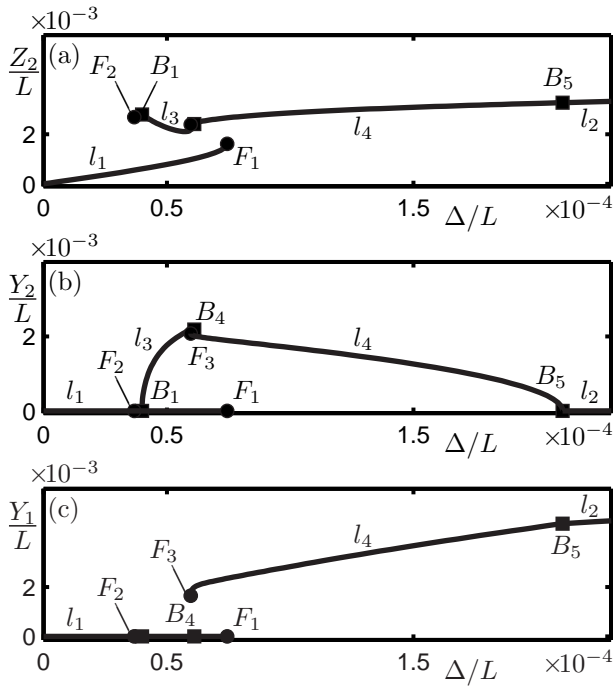


Figure 5. Modal amplitude contributions to the stable branches  $l_1 - l_4$  for  $\mu = 0.01$  from Figures 3 and 4, showing the contributions of (a)  $Z_2$ , (b)  $Y_2$ , and (c)  $Y_1$  modes.

of the  $Z_2$ -amplitude and of the  $Y_2$ -amplitude to  $l_3$ , shown in Figure 5(a),(b), begins at the branch point  $B_1$  and ends at the branch point  $B_4$ . The point  $B_1$  lies on the small stable part of  $l_1$ , preceded by the unstable one beyond the fold point  $F_1$ , see Figures 3 and 4. Hence due to the unstable part of  $l_1$  between  $F_1$  and  $F_2$ , where  $\Delta/L$  decreases, in Figure 5(a),(b) and (c), the points  $F_2$  and  $B_1$  come before  $F_1$ . Also, because the stable parts of both  $l_1$  and  $l_3$  appear as zero solutions in Figure 5(c), they overlap each other. It is important to note that the amplitudes at vibration modes  $Y_1$  and  $Y_2$  on solution branch  $l_4$ , and  $Y_1$  on  $l_2$  are both compatible with the amplitude of the directly excited mode  $Z_2$ .

#### 4 CONCLUSIONS

In this paper we presented a theoretical study of the nonlinear vibration amplitudes of an inclined cable that is excited at its lower attachment point. Our analysis of the response in pure  $Z_2$ -modal amplitudes produced a set of solution curves that present the  $Z_2$  vibration amplitude and the bifurcation curves as functions of the frequency and amplitude of the end displacement. In this way, bifurcation curves were presented without intersections, even though they may intersect and overlap when shown in the  $(\Delta/L, \mu)$  parameters plane. A detailed bifurcation analysis for  $\mu = 0.01$ , a value of the detuning parameter close to the resonant frequency, was performed. Stable branches were identified and the modal amplitude contributions to those branches was shown. Overall, the continuation analysis results in an informative representation of the behaviour of the cable system in a region where the dynamics are extremely sensitive to changes of its driving parameters.

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