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Friction-induced reverse chatter in rigid-body mechanisms with impacts

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Abstract

The focus of this paper is on the possibility of formulating a consistent and unambiguous forward-simulation model of planar rigid-body mechanical systems with isolated points of intermittent or sustained contact with rigid constraining surfaces in the presence of dry friction. In particular, the analysis considers paradoxical ambiguities associated with the coexistence of sustained contact and one or several alternative forward trajectories that include phases of free-flight motion. Special attention is paid to the so-called Painlevé paradoxes where sustained contact is possible even if the contact-independent contribution to the normal acceleration would cause contact to cease. Here, through taking the infinite-stiffness limit of a compliant contact model, the ambiguity in the case of a condition of sustained stick is resolved in favour of sustained contact, whereas the ambiguity in the case of a condition of sustained slip is resolved by eliminating the possibility of reaching such a condition from an open set of initial conditions. A more significant challenge to the goal of an unambiguous forward-simulation model is afforded by the discovery of open sets of initial conditions and parameter values associated with the possibility of a left accumulation point of impacts or reverse chatter – a transition to free flight through an infinite sequence of impacts with impact times accumulating from the right on a limit point and with impact velocities diverging exponentially away from the limit point, even where the contact-independent normal acceleration supports sustained contact. In this case, the infinite-stiffness limit of the compliant formulation establishes that, under a specific set of open conditions, the possibility of reverse chatter in the rigid-contact model is an irresolvable ambiguity in the forward dynamics based at the terminal point of a phase of sustained slip. Indeed, as the existence of a left-accumulation point of impacts is associated with a one-parameter family of possible forward trajectories, the ambiguity is of infinite multiplicity. The conclusions of the theoretical analysis are illustrated and validated through numerical analysis of an example single-rigid-body mechanical model.

1 Introduction

There has been much recent interest in non-smooth formulations of mechanical systems, which can undergo hard impacts with rigid constraining surfaces in the presence of dry friction, see for example [14, 23, 4, 18, 22] and references therein. Much progress has been made in understanding complex nonlinear dynamics of such systems through the study of so-called *discontinuity-induced bifurcations*, changes in system response associated with degenerate interactions with system discontinuities, see e.g. [13, 2, 3, 24]. Unfortunately, where impact can occur in the presence of friction, there are straightforward situations where even basic questions like existence and uniqueness of solutions remain unresolved, despite their being a number of different formalisms, such as Filippov systems [6], complementarity [8], hybrid systems [21], and differential inclusions [17].

This paper continues the study [15] in which the present authors investigated a framework for analysing the dynamics of *planar mechanical systems* with isolated points of contact with rigid constraining surfaces in the presence of friction. Emphasis was there placed on free-flight motions interrupted by brief collisional interactions and on the investigation of an *energetically consistent impact law*, motivated by the work of Stronge [23] (see

also [1]). In particular, [15] focused exclusively on the smoothness properties of the impact law and on possible discontinuity-induced bifurcations of periodic impacting motions due to transitions across discontinuity boundaries in the impact law.

In contrast, the focus of the present paper is on the possibility of formulating a consistent and unambiguous forward-simulation model of the class of systems introduced in [15] under conditions that include sustained contact. We emphasise the elimination, where possible, of ambiguities in the forward time history from given initial conditions. This includes a discussion of the potential non-uniqueness of solutions found in rigid-contact models with friction (cf. [9, 20]) due to the so-called *Painlevé paradox* associated with large coupling between the tangential and normal degrees of freedom during contact [19, 7, 22, 11].

A more significant challenge to the goal of an unambiguous forward-simulation model is afforded by the possibility of *reverse chatter* (sometimes also referred to as a reverse Zeno phenomenon) – an infinite sequence of impacts with impact times accumulating from the right on a limit point and with impact velocities diverging exponentially away from the limit point. As we shall show in this paper, such phenomena cause to a two-fold, insurmountable impediment to the ambition of an unambiguous formulation for all initial conditions. Indeed, not only does reverse chatter offer a viable alternative to continued sustained contact but, more perversely, its existence is associated with a one-parameter family of possible forward trajectories.

The rest of the paper is outlined as follows. Section 2 introduces the general mathematical framework for a planar mechanical system with an isolated point of contact and provides the specific equations of motion for a motivating example of a rigid rod in sustained or intermittent contact with a half-plane. Within the general formulation, explicit evolution equations are derived for different modes of sustained motion – namely sustained slip, sustained stick and sustained free flight. Results from [15] are recalled on explicit analytical expressions for mappings that link incoming velocities to outgoing ones during an instantaneous impact event.

Section 3 considers the possibility of spontaneous transitions from one sustained mode to another without reaching the boundary of the parameter region of existence of that mode. It is argued that such transitions are not possible. In particular, any ambiguity associated with the Painlevé paradox in the cases of sustained slip and sustained stick is here resolved through reference to a limiting behaviour of a family of compliant contact models.

The possible accumulation of impacts, in either forward or reverse time (associated with right- or left-accumulation points, respectively), in the rigid-contact model is considered in Sec. 4. In the context of impact oscillators (see, e.g., [5, 16]), forward-time accumulation of impacts is known to lead to non-uniqueness of dynamics in reverse time. In this paper, we show that in the presence of friction, there exist open regions of initial conditions and parameters for the rigid-contact model, for which an impacting sequence can accumulate in reverse time onto a condition of instantaneous stick. While an ambiguity associated with the coexistence of reverse chatter and sustained stick is eliminated due to the inherent stability of sustained stick found in Sec. 3, we argue that the rigid-contact model is unable to unambiguously adjudicate between continued sustained contact at the conclusion of a phase of sustained slip and a transition to free flight through a rapid sequence of impacts.

The ambiguity introduced through the possibility of reverse chatter at the conclusion of a phase of sustained slip is revisited in Sec. 5 from the perspective of its realisability as a limiting behaviour in the forward simulation of a class of compliant contact models. In particular, the theoretical analysis finds that reverse chatter is indeed the correct limiting behaviour for open sets of initial conditions and parameter values associated with transitions from sustained slip to stick or from sustained negative slip to positive slip, at least for a non-dissipative normal compliance. This affirmative conclusion is subsequently extended by continuity to the case of small amounts of dissipation. For larger amounts of dissipation, the conclusions are found to depend sensitively on the way that damping is introduced.

The results from the previous sections are illustrated through numerical simulations of the model example in Sec. 6. Finally, Sec. 7 discusses possible physical implications of the results and also their wider applicability to alternative formulations of frictional contact problems.

2 Mechanical model

2.1 A Lagrangian formulation

Consider a multibody mechanism whose configuration relative to an inertial reference frame may be described in terms of a column matrix q of generalised coordinates and (possibly) the time coordinate t. Suppose that contact between the multibody mechanism and its environment is mediated through an isolated point P on the mechanism.

Let the row matrices F_c and F_a represent the generalised forces associated with contact interactions and all other generalised forces acting on the mechanism, respectively, with respect to the generalised coordinates. There exist matrices $\beta(q,t)$ and $\gamma(q,t)$, such that $v = \beta(q,t) \cdot \dot{q} + \gamma(q,t)$ describes the velocity of the point P relative to the inertial reference frame, and such that

$$F_c = \lambda \cdot \beta \tag{2.1}$$

for some row matrix λ . From Lagrange's equations of motion, it follows that there exists a positive-definite, symmetric matrix M(q,t), such that

$$\ddot{q} = M^{-1} \cdot \beta^T \cdot \lambda^T + \dots \tag{2.2}$$

and

$$\dot{v} = m^{-1} \cdot \lambda^T + \dots, \tag{2.3}$$

where the omitted terms depend only on F_a , q, \dot{q} , and t, and where

$$m^{-1} = \beta \cdot M^{-1} \cdot \beta^T \tag{2.4}$$

is again symmetric.

Restrict attention to motion constrained to a plane, such that

$$\beta = \begin{pmatrix} \beta_T \\ \beta_N \end{pmatrix}, \quad v = \begin{pmatrix} v_T \\ v_N \end{pmatrix}, \quad \lambda = (\lambda_T, \lambda_N), \quad (2.5)$$

where the subscripts $_T$ and $_N$ refer to components tangential and normal to the common tangent direction at P, respectively. In terms of the notation

$$m^{-1} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}, \tag{2.6}$$

it follows that

$$A > 0, \quad C > 0, \quad AC - B^2 > 0$$
 (2.7)

and thus that m^{-1} is positive definite, provided that β has full row rank.

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2.2 A model example

As an example, consider the planar motion of a homogeneous rod of unit mass and twice unit length, for which contact with the boundary of a stationary half-plane is mediated through the end point P (see Fig. 1).



Figure 1: Sketch of the model example.

Let S_x and S_y be the components of the net external force (excluding contact forces) acting at the centre of mass of the rod. Similarly, let R denote the net external torque (excluding contact forces). In terms of the generalised coordinates x, y, and θ and the generalised velocities

$$v_T := \dot{x}, \quad v_N := \dot{y}, \quad \omega := \theta, \tag{2.8}$$

it follows that

$$\dot{\omega} = 3R - 3\sin\theta\lambda_T - 3\cos\theta\lambda_N \tag{2.9}$$

and

$$\dot{v}_T = a + A\lambda_T + B\lambda_N$$
 and $\dot{v}_N = b + B\lambda_T + C\lambda_N$, (2.10)

where the contact-independent contributions to the tangential and normal acceleration of the point P are given by

$$a = S_x - 3R\sin\theta - \omega^2\cos\theta$$
 and $b = S_y - 3R\cos\theta + \omega^2\sin\theta$ (2.11)

and where

$$A = 1 + 3\sin^2\theta, \quad B = 3\sin\theta\cos\theta, \quad C = 1 + 3\cos^2\theta, \tag{2.12}$$

such that $AC - B^2 = 4$.

While the analysis in what follows is not restricted to this model system, it continues to assume that the separation of the point of contact from its environment is described by the variable y, that the equations of motion in the tangential and normal directions take the form (2.10), and that (2.7) holds.

2.3 A rigid-body contact model

For the moment, restrict attention to a Signorini-Coulomb model of rigid contact with friction, for which

$$\lambda_N y = 0, \quad \lambda_T \in -\mu \lambda_N \operatorname{Sign}(v_T), \quad \text{and} \quad \lambda_N, y \ge 0,$$
(2.13)

where $\mu \ge 0$ is an assumed coefficient of friction and where Sign (·) denotes the set-valued signum function.



Figure 2: The Signorini-Coulomb model of rigid contact with friction.

Consider the rescaled variables

$$\tilde{v}_N := v_N, \quad \tilde{v}_T := \sqrt{\frac{C}{A}} v_T, \quad \tilde{\lambda}_N := C\lambda_N, \quad \tilde{\lambda}_T := \sqrt{AC}\lambda_T$$
(2.14)

and let

$$\tilde{a} := \sqrt{\frac{C}{A}}a + \frac{d}{dt}\left(\sqrt{\frac{C}{A}}\right)v_T, \quad \tilde{b} := b, \quad \tilde{B} := \frac{B}{\sqrt{AC}} \quad \text{and} \quad \tilde{\mu} := \sqrt{\frac{A}{C}}\mu.$$
 (2.15)

From (2.7) it follows that $1 - \tilde{B}^2 > 0$. Dropping tildes, the equations of motion (2.10) then become

$$\dot{v}_T = a + \lambda_T + B\lambda_N$$
 and $\dot{v}_N = b + B\lambda_T + \lambda_N$, (2.16)

where, again, the Signorini-Coulomb condition takes the form (2.13).

2.4 Modes of sustained motion

For notational convenience, consider the *rate constants* (cf. [15]):

$$k_T^{\pm} := B \mp \mu, \quad k_N^{\pm} := 1 \mp \mu B, \quad k_T^0 = 0, \quad \text{and} \quad k_N^0 := 1 - B^2,$$
(2.17)

such that

$$k_T^- > k_T^+, \quad k_N^+ > -k_N^-, \quad \text{and} \quad k_N^0 > 0.$$
 (2.18)

Moreover,

$$k_N^+ < 0 \quad \text{or} \quad -k_T^+ < 0 \quad \Rightarrow \quad k_N^-, k_T^- > 0$$
(2.19)

and

$$k_N^- < 0 \quad \text{or} \quad k_T^- < 0 \quad \Rightarrow \quad k_N^+, -k_T^+ > 0.$$
 (2.20)

Finally, since

$$k_N^+ k_T^- - k_N^- k_T^+ = 2\mu k_N^0, (2.21)$$

it follows that at most one of the four quantities k_N^+ , k_N^- , $-k_T^+$, and k_T^- can be non-positive.

In the Signorini-Coulomb model, it is natural to consider three possible modes of sustained motion on open (non-zero) intervals of time:

- A condition of sustained free flight will be said to occur on such a time interval if y > 0 (in which case $\lambda_T = \lambda_N = 0$); in contrast, if y = 0 and $\lambda_N > 0$ we say that sustained contact occurs.
- During sustained contact, a condition of sustained positive (negative) slip occurs on an open interval of time provided that, on this interval, $v_T > 0$ ($v_T < 0$) and, consequently, $\lambda_T = -\mu\lambda_N$ ($\lambda_T = \mu\lambda_N$). In this event, (2.16) implies that for positive slip

$$(\lambda_T, \lambda_N) = \left(\lambda_T^+, \lambda_N^+\right) := -\frac{b}{k_N^+} \left(-\mu, 1\right).$$
(2.22)

Similarly, for negative slip

$$(\lambda_T, \lambda_N) = \left(\lambda_T^-, \lambda_N^-\right) := -\frac{b}{k_N^-}(\mu, 1).$$
(2.23)

By definition, $\lambda_N > 0$ during sustained contact. It follows that if $k_N^+ > 0$ ($k_N^- > 0$), a condition of sustained positive (negative) slip implies that b < 0, i.e., a negative contact-independent contribution to the vertical acceleration of the point P, and vice versa. The Signorini-Coulomb model also allows for b > 0 during sustained positive (negative) slip if and only if $k_N^+ < 0$ ($k_N^- < 0$). In this case, any state during such a phase of sustained contact could serve as the initial condition for a phase of sustained free flight, introducing a potential ambiguity in the forward dynamics known as the *Painlevé paradox with respect to positive (negative) slip*.

• Alternatively, during sustained contact, a condition of sustained stick will be said to exist for an open interval of time provided that the coefficient of friction $\mu > 0$ and, throughout this time interval, $v_T = 0$ and $|\lambda_T| < \mu \lambda_N$. In this case, it follows from (2.16) that

$$(\lambda_T, \lambda_N) = \left(\lambda_T^0, \lambda_N^0\right) := \frac{1}{k_N^0} \left(bB - a, aB - b\right)$$
(2.24)

and

$$ak_N^+ - bk_T^+ < 0 < ak_N^- - bk_T^-.$$
(2.25)

Since $k_N^0 > 0$ and

$$aB - b = \frac{\left(ak_N^- - bk_T^-\right) - \left(ak_N^+ - bk_T^+\right)}{2\mu} > 0, \qquad (2.26)$$

the requirement that $\lambda_N > 0$ is trivially satisfied in sustained stick. On the other hand, (2.25) and the equality

$$-2\mu k_N^0 b = \left(ak_N^- - bk_T^-\right)k_N^+ - \left(ak_N^+ - bk_T^+\right)k_N^-$$
(2.27)

imply that values of $k_N^{\pm} > 0$ are compatible with sustained stick only if b < 0 and that values of b > 0 are compatible with sustained stick only if $k_N^+ < 0$ or $k_N^- < 0$, corresponding to the *Painlevé paradox with respect to stick*.



Figure 3: The region of values of a and b that satisfy the conditions (2.25) for sustained stick, given different combinations of the signs of k_N^+ and k_T^+ with $B \ge 0$.

Given the sign constraints on k_N^+ , k_N^- , $-k_T^+$, and k_T^- one can easily enumerate different qualitative arrangements of the boundaries between stick and slip during sustained contact. Figure 3 shows the region of values of a and bthat satisfy conditions (2.25) for different combinations of the signs of k_N^+ and k_T^+ with $B \ge 0$. Such parameter values are said to lie in the *friction cone*. The remaining sign combinations with B < 0 may be obtained from this figure by application of the reflection

$$(a,B) \to (-a,-B), \qquad (2.28)$$

which leaves the conditions (2.25) invariant.

Ambiguities in the forward dynamics from a given state that arise through the Painlevé paradoxes need to be resolved in order to enable a well-posed formulation of the forward-time system evolution. In the rigid contact model, the quantities λ_T and λ_N capture the aggregate influence at the contact interface of a variety of physical phenomena, including local deformation, adhesion, and dissipation. The individual characteristics of these physical phenomena are lost in the limit of the rigid contact model. In this limit, distinctly different dynamical behaviours are effectively squeezed together, making it, at times, impossible to assert a unique future evolution from a given initial condition or to evaluate the viability of a certain condition of motion as a reflection of a physically realisable condition. A resolution of these ambiguities must therefore come from outside the rigid contact limit, for example through a study of a general class of compliant, but stiff contact models. We return to this question in Sec. 3.

2.5 Impulsive contact

In addition to phases of sustained motion, the Signorini-Coulomb contact model naturally leads to the consideration of impulsive changes in the generalised velocities associated with isolated collisional contact events. In our previous paper [15], following the approach adopted by Stronge [23] and Batlle [1], such isolated *impacts* were represented by changes in the contact point velocities assumed to occur on a time scale an order of magnitude shorter than typical macroscopic time scales of the entire mechanism.

Specifically, assuming negligible changes in configuration and negligible contributions during the impact phase from the contact-independent generalised forces, the total, effectively instantaneous change in the generalised velocities during the impact phase is given by

$$\Delta \dot{q} = M_0^{-1} \cdot \beta_0^T \cdot m_0 \cdot \Delta v, \qquad (2.29)$$

where the subscript $_0$ refers to evaluation at the moment of impact. In what follows, we shall refer to a relationship that yields $v_0 \mapsto g(v_0) = v_0 + \Delta v$ as an *impact mapping* and to $(q_0, \dot{q}_0) \mapsto (q_0, \dot{q}_0 + \Delta \dot{q})$ as the resultant *impact law*.

The results from [15] on the derivation of closed form expressions for the impact mapping for all possible combinations of initial conditions and parameter values can be summarised as follows. Let $r \in [0, 1]$ be an *energetic coefficient of restitution*, such that r = 1 corresponds to the absence of dissipation in the normal interactions during the impact phase, and consider the functions

$$g_I : \begin{pmatrix} v_T \\ v_N \end{pmatrix} \mapsto \begin{pmatrix} v_T - (1+r)\frac{k_T}{k_N}v_N \\ -rv_N \end{pmatrix},$$
(2.30)

$$g_{II} : \begin{pmatrix} v_T \\ v_N \end{pmatrix} \mapsto \begin{pmatrix} \frac{k'_T}{k'_N} \begin{pmatrix} \frac{k_N}{k_T} v_T - v_N + \sqrt{\left(1 - \frac{k'_N}{k_N}\right) \left(\frac{k_N}{k_T} v_T - v_N\right)^2 + r^2 \frac{k'_N}{k_N} v_N^2} \\ \sqrt{\left(1 - \frac{k'_N}{k_N}\right) \left(\frac{k_N}{k_T} v_T - v_N\right)^2 + r^2 \frac{k'_N}{k_N} v_N^2} \end{pmatrix}$$

$$(2.31)$$

$$g_{III} : \begin{pmatrix} v_T \\ v_N \end{pmatrix} \mapsto \begin{pmatrix} \frac{k'_T}{k'_N} \left(\frac{k_N}{k_T} v_T - v_N + r \sqrt{\left(1 - \frac{k'_N}{k_N}\right) \left(\frac{k_N}{k_T} v_T - v_N\right)^2} + \frac{k'_N}{k_N} v_N^2 \\ r \sqrt{\left(1 - \frac{k'_N}{k_N}\right) \left(\frac{k_N}{k_T} v_T - v_N\right)^2 + \frac{k'_N}{k_N} v_N^2} \end{pmatrix} \right).$$
(2.32)

Then, given initial velocities $v_{T,0}$ and $v_{N,0}$, the particular impact mapping in question is obtained from the following steps:

- 1. Follow the decision tree in Fig. 4 to arrive at region index between 1 and 10.
- 2. Locate the corresponding row in Table 1 and apply the corresponding map with the indicated values for k_T , k_N , k'_T , and k'_N .



Figure 4: Decision tree for determining the impact mapping for a given configuration and velocity state at the onset of contact. To interpret this diagram, start at the parent node and follow the branch to the right when the condition is fulfilled and the branch to the left when it is not. The leaf node region indices determine which impact mapping to apply according to the corresponding row of Table 1.

Region	k_T	k_N	$k_{T}^{'}$	k'_N	Impact mapping
1	k_T^+	k_N^+	*	*	\mathbf{g}_{I}
2	k_T^-	k_N^-	*	*	\mathbf{g}_{I}
3	k_T^+	k_N^+	k_T^0	k_N^0	\mathbf{g}_{II}
4	k_T^-	k_N^-	k_T^0	k_N^0	\mathbf{g}_{II}
5	k_T^+	k_N^+	k_T^0	k_N^0	\mathbf{g}_{III}
6	k_T^-	k_N^-	k_T^0	k_N^0	\mathbf{g}_{III}
7	k_T^+	k_N^+	k_T^-	k_N^-	\mathbf{g}_{II}
8	k_T^-	k_N^-	k_T^+	k_N^+	\mathbf{g}_{II}
9	k_T^+	k_N^+	k_T^-	k_N^-	g _{III}
10	k_T^-	k_N^-	k_T^+	k_N^+	gIII

Table 1: Values of the rate constants and the impact map type for the 10 different regions.

It is straightforward to show that the impact law is C^0 across the boundaries defined by the decision tree in Fig. 4 (see also the graphical depiction in Fig. 5). Indeed, the main thrust of [15] was to determine the discontinuity-induced bifurcations that occur due to the degree of non-smoothness of the impact law across these boundaries.

The combination of Signorini-Coulomb contact model and the above energetic impact model will be referred to below as the *Signorini-Coulomb-Stronge* model.

3 Consistency of sustained motion

An instantaneous transition between two distinct conditions of sustained motion that does not correspond to the violation of an associated condition of existence will be referred to in this section as a *spontaneous transition*. The possibility of such a transition introduces an ambiguity in the forward simulation of the rigid contact model that cannot be resolved within the existing formulation. Such ambiguities must, therefore, be eliminated either by axiomatic dicta or by making reference to experimental observations or non-rigid contact models.

There is no ambiguity associated with a condition of sustained free-flight. As seen in Sec. 2.4 ambiguities are present, however, in the Painlevé cases of b > 0, for which $k_N^+ < 0$ or $k_N^- < 0$. To seek a resolution to these, we first return to the non-Painlevé case of sustained stick in the rigid-contact limit.

3.1 A Filippov system

Suppose that the contact point with $v_T = 0$ lies within the friction cone and $b < 0 < k_N^{\pm}$. In this case, the values of λ_T and λ_N in sustained stick can be obtained by a convex combination of the corresponding values in sustained positive and negative slip. Specifically, it follows directly from (2.22), (2.23), and (2.24) that

$$\left(\lambda_T^0, \lambda_N^0\right) = \alpha^+ \left(\lambda_T^+, \lambda_N^+\right) + \alpha^- \left(\lambda_T^-, \lambda_N^-\right), \qquad (3.1)$$

where

$$\alpha^{\pm} = \mp \frac{k_N^{\pm}}{2\mu b k_N^0} \left(a k_N^{\mp} - b k_T^{\mp} \right). \tag{3.2}$$

In particular,

$$\alpha^+ + \alpha^- = 1 \tag{3.3}$$

and, provided that (2.25) holds,

$$\alpha^{\pm} > 0. \tag{3.4}$$

Let a $^+$, $^-$, and 0 superscript denote quantities evaluated in sustained positive slip, sustained negative slip, and sustained stick, respectively. It similarly follows that the vector field in sustained stick is a convex combination of the vector fields in sustained positive and negative slip, respectively, i.e.,

$$\dot{v}_T^0 = \alpha^+ \dot{v}_T^+ + \alpha^- \dot{v}_T^-, \tag{3.5}$$

$$\dot{v}_N^0 = \alpha^+ \dot{v}_N^+ + \alpha^- \dot{v}_N^-. \tag{3.6}$$

Since

$$\dot{v}_T^+ = a + \lambda_T^+ + B\lambda_N^+ = \frac{1}{k_N^+} \left(ak_N^+ - bk_T^+ \right) < 0, \tag{3.7}$$

$$\dot{v}_T^- = a + \lambda_T^- + B\lambda_N^- = \frac{1}{k_N^-} \left(ak_N^- - bk_T^- \right) > 0, \tag{3.8}$$

provided that $k_N^{\pm} > 0$, it follows that in this case, sustained stick (with $v_T = 0$) is motion on an attractive sliding manifold in a Filippov system [6], see also [13, 2], as long as contact is kept. The rigid-contact model thus excludes the possibility of spontaneous transitions from sustained stick to sustained slip as long as the contact point lies within the friction cone and $b < 0 < k_N^{\pm}$.



Figure 5: The distinct regions in the (v_T, v_N) -plane for each of the five possible sign combinations for k_N^{\pm} and k_T^{\pm} . Dashed lines represent the boundaries between the ten regions of initial conditions. Thick solid lines represent the impulsive motion of individual trajectories during the impact mapping process, see [15] for more details.

3.2 Consistency of stick in general

The argument in the previous section fails in the case that either $k_N^- < 0$ or $k_N^+ < 0$. In this case, the $v_T = 0$ surface can no longer be described as a sliding manifold in the Filippov sense, since either α^+ or α^- will be negative. Nor will the surface $v_T = 0$ be attractive because one of \dot{v}_T^+ or \dot{v}_T^- will have changed sign. The consistency of stick in the this case must therefore be treated by means other than those available within the rigid contact formulation.

To this end, consider a compliant contact model, permissive of small violations of the impenetrability condition y = 0 but respectful of the Coulomb frictional constraints, for which $\lambda_N(y, v_N)$ models a continuous restoring force and $\lambda_T \in -\mu \lambda_N \text{Sign}(v_T)$. Suppose that (2.25) holds. For $\delta \lambda_N \ll 1$, it follows that a state of sustained stick with $\lambda_N = \lambda_N^0 + \delta \lambda_N$ is robust to perturbations in the direction of positive or negative slip, respectively. Indeed, in the event of a slight perturbation in the direction of positive slip, $\lambda_T = -\mu \lambda_N^0 + \mathcal{O}(\delta \lambda_N)$ and

$$\dot{v}_T = \frac{1}{k_N^0} \left(a k_N^+ - b k_T^+ \right) + \mathcal{O}\left(\delta \lambda_N \right) < 0.$$
(3.9)

Similarly, in the event of a slight perturbation in the direction of negative slip, $\lambda_T = \mu \lambda_N^0 + \mathcal{O}(\delta \lambda_N)$ and

$$\dot{v}_T = \frac{1}{k_N^0} \left(ak_N^- - bk_T^- \right) + \mathcal{O}\left(\delta\lambda_N\right) > 0.$$
(3.10)

Finally,

$$\lambda_T^0 = \beta^+ \left(-\mu \lambda_N^0 + \mathcal{O}\left(\delta \lambda_N\right) \right) + \beta^- \left(\mu \lambda_N^0 + \mathcal{O}\left(\delta \lambda_N\right) \right), \tag{3.11}$$

where

$$\beta^{\pm} = \pm \frac{ak_N^{\mp} - bk_T^{\mp}}{2\mu \left(aB - b\right)} + \mathcal{O}\left(\delta\lambda_N\right),\tag{3.12}$$

 $\beta^+ + \beta^- = 1$, and $\beta^\pm > 0$. It follows that the surface $v_T = 0$ is again an attracting sliding manifold of the corresponding Filippov system.

It remains to show that flow along the sliding manifold is consistent with sustained $\delta \lambda_N \ll 1$. Indeed, for sustained stick, i.e., with $\dot{v}_T = v_T = 0$, the equation of motion in the normal direction is given by

$$\ddot{y} - k_N^0 \lambda_N \left(y, \dot{y} \right) = -k_N^0 \lambda_N^0, \tag{3.13}$$

where the right-hand side is a function of the overall configuration of the mechanism. Now suppose that $\lambda_N(y, v_N)$ increases rapidly with depth -y and is a dissipative function of the normal velocity $\dot{y} = v_N$. Provided that changes in λ_N^0 occur at a sufficiently slow timescale, it follows from $k_N^0 > 0$ that the dynamics in the normal direction closely shadows the slow manifold given by $\lambda_N(y, 0) = \lambda_N^0$, i.e., that a condition of sustained stick in the compliant model with λ_N close to λ_N^0 is robust to small perturbations within the sliding manifold.

In the limit of the rigid contact model, these observations appear to justify the assertion that a condition of sustained stick may terminate only if the contact point reaches the boundary of the friction cone. Together with the result from the previous section, this eliminates the possibility of a spontaneous transition from sustained stick to sustained slip when b < 0 or from sustained stick to either sustained slip or sustained free flight in the Painlevé case with b > 0.

3.3 Consistency of slip

Without loss of generality, restrict attention to a case of positive slip in the compliant contact model, but with $\lambda_N = \lambda_N^+ + \delta \lambda_N$ for some $\delta \lambda_N \ll 1$. In this case, the equation of motion in the normal direction is given by

$$\ddot{y} - k_N^+ \lambda_N \left(y, \dot{y} \right) = -k_N^+ \lambda_N^+, \tag{3.14}$$

where the right-hand side is a function of the overall configuration of the mechanism. In the non-Painlevé case $b < 0 < k_N^+$, the same argument as in the previous section implies the existence of an attractive slow manifold given by $\lambda_N(y,0) = \lambda_N^+$ and thus the robustness of positive slip against small perturbations.

In contrast, in the Painlevé case when $k_N^+ < 0 < b$, the slow manifold is unstable to perturbations. In particular the assumption on the rapid increase of λ with depth means that a small increase in y causes a large decrease in λ_N , which causes an exponential growth in y with rapid time constant. As y becomes positive, lift off would occur since b > 0. In contrast, if a perturbation in the negative y direction is added, then y decreases rapidly exponentially and an impact must occur. In the limit of the rigid contact model, the effective stiffness of the compliant model grows without bound, and so the time constant associated with this exponential growth tends to infinity. These arguments imply that sustained positive slip in the Painlevé case is thus infinitely unstable and cannot be sustained.

In the limit of the rigid contact model, these observations appear to justify the assertion that a condition of sustained positive slip is only compatible with $b < 0 < k_N^+$ and may, in this case, terminate only when its conditions of existence fail. The possibility of a spontaneous transition from sustained slip to sustained free flight in the Painlevé case with b > 0 is eliminated since such a state of sustained slip could not be reached by the forward time evolution of the mechanical system.

4 Chatter

A moment in time t^* , for which $y = v_N = 0$ and b < 0 may conceivably serve as an accumulation point for an infinite sequence of impacts that either precede (a right-accumulation point) or succeed (a left-accumulation point) t^* . Right-accumulation points of impact in systems without friction have been referred to as points of *chatter* in the literature, and have received extensive treatment in the case of single degree-of-freedom impact oscillators, see e.g. [5, 16] and references therein. The possibility of a left-accumulation point of impacts, referred to henceforth as a point of *reverse chatter*, introduces a paradoxical ambiguity, as such a point would be associated with a loss of forwards-in-time uniqueness of trajectories. In this case, it would not be possible within the Signorini-Coulomb-Stronge model to exclude a spontaneous transition from an initial condition with $y = v_N = 0$ and b < 0 to free flight through an infinite sequence of impacts with exponentially increasing impact velocities.

Consider as a reference the simple dynamical system given by

$$\dot{y} = v$$
 and $\dot{v} = -1$ (4.1)

and such that $g: v \mapsto -ev$ is applied whenever y = 0 and v < 0. Suppose that e < 1. Then, given initial conditions

$$y_0 = \frac{\alpha \left(1 - \alpha\right) \left(1 - e\right)^2 T^2}{2e}, \qquad v_0 = \frac{\left(e\alpha + \alpha - 1\right) \left(1 - e\right) T}{2e}, \qquad \text{for } \alpha \in [0, 1]$$
(4.2)

at some time t_0 , there follows an initial free-flight phase of duration

$$v_0 + \sqrt{v_0^2 + 2y_0} = (1 - e) \,\alpha T \tag{4.3}$$

and a subsequent chatter phase of infinitely many impacts of total duration

$$\frac{2e}{1-e}\sqrt{v_0^2 + 2y_0} = T - (1-e)\,\alpha T. \tag{4.4}$$

A right-accumulation point of impacts thus occurs at $t^* = t_0 + T$. Here, the phase α parametrises a family of trajectories that reach y = v = 0 at t^* . Indeed, as $y_0 (\alpha = 0) = y_0 (\alpha = 1) = 0$ and $v_0 (\alpha = 1) = -ev_0 (\alpha = 0)$, it follows that as α varies from 0 to 1, the chatter sequence is shifted forward until the original trajectory is regained with all impacts having been shifted one step. In the case that e > 1, the above analysis applies to the rescaled system obtained from the time-reflection $y \mapsto y, v \mapsto -v, t \mapsto -t$, for which $g : v \mapsto -e^{-1}v$. The existence of reverse chatter starting from a particular time instant t^* is then associated, not with a single forward trajectory, but with a one-parameter family of trajectories that connect to this point. Thus, even if one can determine with certainty that reverse chatter will occur, there remains a one-parameter uncertainty in the subsequent motion.

The reference dynamical system corresponds to a unit mass falling in a unit gravitational field, such that collisions with a horizontal surface are modelled by a kinematic coefficient of restitution. In this interpretation, only the case that e < 1 is physically relevant, since e > 1 corresponds to a non-physical increase in kinetic energy at each collision. For reverse chatter to be physically realisable, therefore, a necessary condition is the inclusion of an additional degree of freedom, such that a net increase in normal kinetic energy may be accompanied by a decrease in tangential kinetic energy, which could then be replenished due to the imposed tangential acceleration during the free-flight phase.

To investigate the existence of accumulation points in the Signorini-Coulomb-Stronge model, let $v_{T,0}$ and $0 < -v_{N,0} \ll 1$ be given initial velocities at some time t_0 , such that $y(t_0) = 0$ and b < 0. Denote by $v_{T,1}$ and $0 < v_{N,1} \ll 1$ the tangential and normal velocities of the contact point subsequent to the application of the corresponding impact mapping

$$g: \begin{pmatrix} v_{T,0} \\ v_{N,0} \end{pmatrix} \mapsto \begin{pmatrix} v_{T,1} \\ v_{N,1} \end{pmatrix}, \tag{4.5}$$

where g takes one of the forms g_I , g_{II} of g_{III} given by (2.30)–(2.32) depending on the parameter region 1–10 indicated in Table 1 and Fig. 4. The subsequent free-flight phase with initial height y = 0 then returns to y = 0 after a short time-of-flight given by

$$t_f = -\frac{2v_{N,1}}{b} + \mathcal{O}\left(v_{N,1}^2\right)$$
(4.6)

with tangential and normal velocities of the contact point equal to

$$v_{T,2} = v_{T,1} - \frac{2a}{b} v_{N,1} + \mathcal{O}\left(v_{N,1}^2\right)$$
 and $v_{N,2} = -v_{N,1} + \mathcal{O}\left(v_{N,1}^2\right)$ (4.7)

collectively described by the free-flight map

$$h: \left(\begin{array}{c} v_{T,1} \\ v_{N,1} \end{array}\right) \mapsto \left(\begin{array}{c} v_{T,2} \\ v_{N,2} \end{array}\right).$$

$$(4.8)$$

In the discussion below, we analyse the dynamics of the map

$$f := h \circ g, \tag{4.9}$$

with emphasis on the ratio

$$e := \frac{v_{N,2}}{v_{N,0}}.$$
(4.10)

In particular, if there exists an asymptotic ratio $e \neq 1$ between successive pre-impact values of the normal velocity in the vicinity of a point in state space at a given time t^* , this time may be a right-accumulation point of impacts if e < 1 and a left-accumulation point of impacts if e > 1.

4.1 Chatter in slip

Consider Fig. 5. In the limit as $v_{N,0} \to 0$, $v_{T,0} \neq 0$ is compatible with initial conditions in regions 1 and 2 for any values of the rate constants; in region 5 provided that $k_N^+ < 0$ (panel (b) in Fig. 5); and in region 6 provided that $k_N^- < 0$ (panel (c) in Fig. 5). We exclude from consideration the latter two cases, since there $v_{N,1} = \mathcal{O}(v_{T,0})$ is not small. In contrast, in regions 1 and 2,

$$v_{T,2} = v_{T,0} + \mathcal{O}(v_{N,0})$$
 and $v_{N,2} = rv_{N,0} + \mathcal{O}(v_{N,0}^2)$, (4.11)

i.e., $e = r \leq 1$. If r < 1, it follows that there exists a time $t^* \approx t_0$ that is a right-accumulation point of impacts such that, in the limit, $v_T \approx v_{T,0}$. Reverse chatter is not possible in this case.

4.2 Chatter near stick

Suppose, instead, that $v_{T,0} = -\rho_0 v_{N,0}$ for some constant ρ_0 and consider again the limit as $v_{N,0} \to 0$. For initial conditions in regions 1 or 2, (4.5) and (4.8) give

$$e^{2} = r^{2} + \mathcal{O}\left(v_{N,0}\right). \tag{4.12}$$

Similarly, for initial conditions in regions 3, 4, 7, or 8, it holds that

$$e^{2} = \frac{k'_{N}}{k_{N}}r^{2} + \left(1 - \frac{k'_{N}}{k_{N}}\right)\left(1 + \frac{k_{N}}{k_{T}}\rho_{0}\right)^{2} + \mathcal{O}\left(v_{N,0}\right).$$
(4.13)

Finally, for initial conditions in regions 5, 6, 9, or 10, we obtain

$$e^{2} = r^{2} \left(\frac{k'_{N}}{k_{N}} + \left(1 - \frac{k'_{N}}{k_{N}} \right) \left(1 + \frac{k_{N}}{k_{T}} \rho_{0} \right)^{2} \right) + \mathcal{O}\left(v_{N,0} \right).$$
(4.14)

In particular, the latter case implies that $\lim_{v_{N,0}\to 0} e^2 = r^2$ at $\rho_0 = 0$. Clearly, $\lim_{v_{N,0}\to 0} e^2 = r^2$ in regions 1 and 2 showing again that reverse chatter is not possible in these cases. Similarly, reverse chatter is not possible for B = 0, since in this case $\lim_{v_{N,0}\to 0} e^2 = r^2$ everywhere.

For definiteness, suppose that B > 0, thus restricting attention to panels (a), (b), and (e) in Fig. 5. From (2.17) it follows that

$$k_T^- > 0, \quad k_N^- > 0, \quad k_N^- > k_N^0, \quad k_N^- > k_N^+$$

$$(4.15)$$

and, moreover,

$$k_T^+ < 0 \Leftrightarrow k_N^+ < k_N^0. \tag{4.16}$$

In this case,

- 1. e < r in regions 4 and 6 (panels (a) and (b) in Fig. 5) and in regions 8 and 10 (panel (c) in Fig. 5), since in each of these regions e^2 is a convex quadratic function of ρ_0 .
- 2. $r < e < r\sqrt{k_N^0/k_N^+}$ in regions 3 and 5 with $k_T^+ < 0 < k_N^+$ (panel (a) in Fig. 5), since in each of these regions e^2 is a concave quadratic function of ρ_0 and the maximum value is attained at the boundary between these regions, where $\rho_0 = -k_T^+/k_N^+$.
- 3. e > r and e is an increasing function of ρ_0 in region 5 with $k_T^+, k_N^+ < 0$ (panel (b) in Fig. 5), since in this region e^2 is a convex quadratic function of ρ_0 and

$$\lim_{\nu_{N,0}\to 0,\rho_0\downarrow 0} \frac{de^2}{d\rho_0} = 2r^2 \frac{k_N^+ - k_N^0}{k_T^+} > 0.$$
(4.17)

This conclusion also applies to the case where $k_T^+ < 0$ and $k_N^+ = 0$.

Note that

$$B > 0, \quad r^2 > k_N^+ / k_N^0 \Rightarrow k_T^+ < 0.$$
 (4.18)

For B > 0, it follows that e > 1 if and only if $r^2 > k_N^+/k_N^0$,

$$-\frac{k_T^+(1-r^2)}{r^2(k_N^0-k_N^+)} \left[1 + \sqrt{\frac{r^2k_N^0-k_N^+}{r^2(k_N^0-k_N^+)}}\right]^{-1} < \rho_0$$
(4.19)

(this boundary is in region 5) and, if $k_N^+ > 0$,

$$\rho_0 < -\frac{k_T^+}{k_N^+} \left(1 + \sqrt{\frac{r^2 k_N^0 - k_N^+}{k_N^0 - k_N^+}} \right) \tag{4.20}$$

(this boundary is in region 3). In the special case that r = 1, the latter two conditions are equivalent to $0 < \rho_0$ and, if $k_N^+ > 0$, $\rho_0 < -2k_T^+/k_N^+$. Figure 6 shows the graphs of e as a function of ρ_0 for panels (a), (b), and (e) of Fig. 5, respectively, and for discrete values of the energetic coefficient of restitution r.

To arrive at necessary and sufficient conditions for the existence of reverse chatter, let

$$\rho_2 := -\frac{v_{T,2}}{v_{N,2}} \tag{4.21}$$

and consider the map $\rho_0 \mapsto \rho_2(\rho_0)$. From (4.5) and (4.8), it follows that $\lim_{v_{N,0}\to 0} \rho_2 = -2a/b$ independently of ρ_0 in regions 3 and 5. In the limit as $v_{N,0} \to 0$, the ρ_0 set defined by (4.19–4.20) is thus positively invariant provided that it contains $\rho_0 = -2a/b$. In this case, any initial condition in this set is mapped to $\rho_0 = -2a/b$ and remains there for all subsequent iterates. It follows that, for B > 0, reverse chatter is possible if and only if b < 0,

$$-\frac{k_T^+(1-r^2)}{2r^2(k_N^0-k_N^+)} \left[1 + \sqrt{\frac{r^2k_N^0-k_N^+}{r^2(k_N^0-k_N^+)}}\right]^{-1} < \frac{a}{-b},$$
(4.22)

$$ak_N^+ - \frac{bk_T^+}{2} \left(1 + \sqrt{\frac{r^2 k_N^0 - k_N^+}{k_N^0 - k_N^+}} \right) < 0, \qquad (4.23)$$

and $r^2 > k_N^+/k_N^0$, where the latter two inequalities are trivially satisfied when $k_N^+ \leq 0$. In the special case that r = 1, these conditions are equivalent to b < 0, $k_T^+ < 0$, and $ak_N^+ - bk_T^+ < 0 < a$. As an example, for each value of r, the region to the right of the corresponding curve in Fig. 7(a) corresponds to values of a/(-b) and μ that allow for reverse chatter in the case when B = 0.5. Similarly, the region above the corresponding curve in Fig. 7(b) for r > 0 corresponds to values of θ and the unscaled coefficient of friction μ in the case of the model example that allow for reverse chatter for suitably chosen values of a.

A similar analysis applied to the case B < 0 finally yields the following summary of necessary and sufficient conditions for the possibility of reverse chatter in the rigid contact model, namely:

a. $B > 0, r^2 > k_N^+ / k_N^0$, and a and b values within the subset of

$$b < 0$$
 and $ak_N^+ - bk_T^+ < 0 < a,$ (4.24)

given by (4.22) and (4.23);



Figure 6: Graphs of e as a function of $\rho_0 = -v_{T,0}/v_{N,0}$ for panels (a), (b), and (e) of Fig. 5, respectively, and for discrete values of the energetic coefficient of restitution r. (a) $\mu = 1$, B = 1/2; (b) $\mu = 3$, B = 1/2; and (c) $\mu = 1/3$, B = 1/2. Each panel includes the corresponding region labels.



Figure 7: (a) The region in a/(-b) and μ (for fixed B = 0.5 and varying r) for possible reverse chatter (region lies to the right of the curve). The upper limit on a vanishes to infinity as μ approaches 1/B = 2 from below. (b) The region in θ and unscaled μ for varying r, for which there exists values of a that allow for reverse chatter (region lies above the curve) in the model example (2.8)–(2.11). In both cases, the shaded region corresponds to the case when r = 0.7.

b. $B < 0, r^2 > k_N^-/k_N^0$, and a and b values within the subset of

$$b < 0$$
 and $a < 0 < ak_N^- - bk_T^-$, (4.25)

given by (4.22) and (4.23) with a, k_N^+ , and k_T^+ replaced by $-a, k_N^-$, and $-k_T^-$ respectively.

4.3 Triggering reverse chatter

We have argued in Sec. 3 that spontaneous transitions from non-Painlevé slip are ruled out, as is the possibility of Painlevé slip. Likewise, no spontaneous transitions from stick should occur. The onset of reverse chatter may also be ruled out at the end of a free-flight phase, since then $b < 0 \Rightarrow v_N < 0$, and at a right-accumulation point of impacts, since there e < 1. This leaves the possibility of reverse chatter when reaching the border of one of the sustained contact phases, which we shall refer to as being a *triggered transition*.

Since the conditions we have found for possible reverse chatter form a subset of the conditions for sustained stick (2.25) with b < 0 (see also Fig. 3), it follows that $e \leq 1$ at the boundary of sustained stick. This leaves the possibility of triggering reverse chatter when reaching an interior point of the friction cone from a condition of sustained slip. In particular, because of (2.25) the value of $|v_T|$ in slip is diminishing, so if the conditions for possible reverse chatter are fulfilled and the system is in sustained slip with a small slipping velocity, slipping must end shortly. When slipping ends there is the possibility of either going into sustained stick, or going into reverse chatter.

We will now investigate what becomes of these ambiguities when the Signorini-Coulomb-Stronge model is replaced by a compliant contact model.

5 Reverse chatter in a compliant model

In this section we investigate whether the phenomenon of reverse chatter found in the rigid-contact model is compatible with a chatter-like behaviour found for any member of a large class of compliant contact models in the limit that the effective stiffness grows without bound. The bulk of the analysis considers the possibility of triggering reverse chatter in a compliant model, in which dissipation is absent in the normal force (corresponding to the case r = 1 in the rigid limit). A few remarks concerning the dissipative case r < 1 are made at the end of the section.

5.1 A preliminary rescaling

Consider again a compliant contact model, permissive of small violations of the impenetrability condition y = 0but respectful of the Coulomb frictional constraints, for which $\lambda_N(y)$ models a continuous (and stiff) restoring force and $\lambda_T \in -\mu \lambda_N \text{Sign}(v_T)$. Specifically, suppose that

$$\lambda_N(y) = -h\left(y/\varepsilon^2\right),\tag{5.1}$$

where h is a fixed continuous function, such that

$$h(\eta) = 0$$
 if $\eta \ge 0$, $h'(\eta) > 0$ if $\eta < 0$, and $h(\eta) \to -\infty$ when $\eta \to -\infty$. (5.2)

The effective stiffness is here controlled by the parameter ε , and the rigid limit is obtained as $\varepsilon \to 0$. Since the contact force λ_N is conservative in this model, it can only be used to describe non-dissipative impacts (r = 1) in the rigid limit.

Suppose that b < 0 (which we have already seen is a necessary condition for any kind of chatter) and denote by d a representative value of $h'(\eta)$. Consider the following additional rescaling to (2.14-2.15):

$$\tilde{y} = -\frac{Cd}{b\varepsilon^2}y, \quad \tilde{v}_N = -\frac{\sqrt{Cd}}{b\varepsilon}v_N, \quad \tilde{v}_T = -\frac{\sqrt{Cd}}{b\varepsilon}v_T, \quad \tilde{t} = \frac{\sqrt{Cd}}{\varepsilon}t, \quad (5.3)$$

and

$$\tilde{h}\left(\tilde{y}\right) = -\frac{C}{b}h\left(-\frac{b}{Cd}\tilde{y}\right).$$
(5.4)

Provided that all rescaled variables remain $\mathcal{O}(1)$ over phases of motion of $\mathcal{O}(1)$ duration in the rescaled time variable, it follows that a, b, B and C change by $\mathcal{O}(\varepsilon)$ from their initial values and may be considered constant in a leading-order analysis.

Dropping tildes, it follows that $\dot{y} = v_N$. From (2.16) we find to lowest order in ε that

$$\dot{v}_T = -\frac{a}{b}$$
 and $\dot{v}_N = -1$ (5.5)

in free flight;

$$\dot{v}_T = -\frac{a}{b} - k_T^+ h(y) \text{ and } \dot{v}_N = -1 - k_N^+ h(y);$$
(5.6)

in positive slip;

$$\dot{v}_T = -\frac{a}{b} - k_T^- h(y)$$
 and $\dot{v}_N = -1 - k_N^- h(y);$ (5.7)

in negative slip; and

$$\dot{v}_T = 0 \text{ and } \dot{v}_N = -\left(1 - \frac{a}{b}B\right) - k_N^0 h(y),$$
(5.8)

in stick, where sustained stick requires that

i

$$-\frac{a}{b} - k_T^+ h(y) < 0 < -\frac{a}{b} - k_T^- h(y).$$
(5.9)

(recall the expressions for the rate constants $k_{T,N}^{\pm}$ and $k_{T,N}^{0}$ in (2.17)). From (5.2) it follows that the free-flight equations with y > 0 are a special case of (5.6). In all cases, the (y, v_N) -dynamics are independent of v_T and possess a first integral

$$E(y, v_N) := \frac{v_N^2}{2} + V(y).$$
(5.10)

where the potential function V is defined below for the separate cases of positive slip and stick.

5.2 Necessary conditions

We begin by showing that a chatter-like behaviour is possible in the compliant model near the parameter region identified in Sec. 4. Without loss of generality, restrict attention to the case when B > 0 from which it follows that $k_T^-, k_N^- > 0$. Necessary conditions for reverse chatter in the Signorini-Coulomb-Stronge model are then

$$b < 0, \quad k_T^+ < 0, \quad ak_N^+ - bk_T^+ < 0 < a.$$
 (5.11)

Assuming the inequalities (5.11) hold, let us consider what kinds of triggered transitions can occur in the compliant model. Since $-(a/b) - k_T^- h(y) > -(a/b) > 0$, the system will leave negative slip in finite time, and no transition *into* negative slip is possible. Let $y^* < 0$ be uniquely defined by

$$h(y^*) = -\frac{a}{bk_T^+},$$
(5.12)

such that the condition for sustained stick (17) reads

$$y < y^*. \tag{5.13}$$

A transition from stick to positive slip occurs at points where $v_N > 0$, $v_T = 0$, and $y = y^*$. A transition from positive or negative slip to stick occurs at points where $y < y^*$, $v_T = 0$, and a transition from negative to positive slip occurs at points where $y > y^*$, $v_T = 0$.

Let

$$H(y) = \int_{y^*}^{y} h(\eta) d\eta.$$
(5.14)

The potential function V(y) may then be chosen as

$$V_{+}(y) := y - y^{*} + k_{N}^{+} H(y)$$
(5.15)

in positive slip and as

$$V_0(y) := \left(1 - \frac{a}{b}B\right)(y - y^*) + k_N^0 H(y).$$
(5.16)

in stick. These choices ensure that $V_+(y^*) = V_0(y^*) = 0$, from which it follows that the two corresponding first integrals E_+ and E_0 given by (5.10) are equal at a transition point between stick and positive slip.

With reference to Fig. 8, consider an initial condition $p_0: (y^*, v_{N,0})$ with $v_{N,0} > 0$, so that $E_0(p_0) = E_+(p_0) = v_{N,0}^2/2$. Starting from p_0 , we shall monitor the value of the difference between the stick and slip potentials,

$$P(y) := V_0(y) - V_+(y).$$
(5.17)

Note in particular that

$$P(y^*) = P'(y^*) = 0, \quad P''(y) = -B \ k_T^+ \ h'(y) \ge 0,$$
(5.18)

and $P''(y^*) > 0$, since $y^* < 0$. It follows that

$$P(y) > 0 \text{ if } y \neq y^*.$$
 (5.19)

Under forward evolution through p_0 there occurs a transition from stick to positive slip, so that we must now solve (5.6) with the additional initial condition $v_T = 0$. Since $V'_+(y) = 1$ for $y \ge 0$, it is clear that the orbit starting at p_0 will first reach $v_N = 0$ at a point y_m (possibly > 0) that is a local maximum in y and for which $V_+(y_m) = v_{N,0}^2/2$. From this point, the trajectory in the (y, v_N) -plane continues symmetrically with respect to the y-axis through the point $y = y^*$, $v_N = -v_{N,0}$. Up to this time v_T has been increasing from 0, but from now on it will decrease.

Now, from the final inequality in (5.11) it follows that the quantity

$$-k_T^+ \dot{v}_N + k_N^+ \dot{v}_T = \frac{ak_N^+ - bk_T^+}{-b}$$
(5.20)

is negative in positive slip, i.e., that $-k_T^+v_N + k_N^+v_T$ decreases linearly in time. Since $-k_T^+v_N + k_N^+v_T|_{p_0} = -k_T^+v_{N,0}$, it follows that for $k_N^+ > 0$ the positive slip orbit cannot reach $y = y^*$ (where $v_N = v_{N,0}$) again before v_T reaches 0. Thus v_T must reach zero and a transition to stick must occur at some point $p_1 : (y_1, v_{N,1})$ for which $y_1 < y^*$. The same conclusion follows in the case that $k_N^+ \leq 0$, since this implies that $\dot{v}_N \leq -1$ and consequently that ycontinues to decrease below y^* at a rate $\leq -v_{N,0}$.

Now, since p_1 lies on the positive slip orbit through the starting point,

$$\frac{v_{N,1}^2}{2} + V_+(y_1) = E_+(p_1) = E_+(p_0) = \frac{v_{N,0}^2}{2}.$$
(5.21)



Figure 8: (a) Time histories for y, v_N , and v_T and (b) the corresponding path in the (y, v_N) plane along the trajectory based at p_0 and terminating at p_2 as described in the text. In both cases, h(y) = y for y < 0, a = 1, b = -1, B = 0.5, $\mu = 1.5$ and $v_{N,0} = 2$.

Moreover,

$$E_0(p_1) = \frac{v_{N,1}^2}{2} + V_0(y_1) \Rightarrow E_0(p_1) - E_+(p_1) = P(y_1) > 0$$
(5.22)

and thus

$$E_0(p_1) > E_+(p_1) = \frac{v_{N,0}^2}{2}.$$
(5.23)

Consider now the evolution of the orbit in stick starting from p_1 . $V_0(y)$ is a convex function with a unique minimum at $y = y_0$, where $y_0 < 0$ is uniquely defined by

$$h(y_0) = \frac{aB - b}{bk_N^0}.$$
 (5.24)

It follows that all stick orbits lie on closed integral curves in the (y, v_N) -plane that surround the point $(y_0, 0)$ (corresponding to the leading-order equilibrium state of the (y, v_N) -dynamics in stick). Hence the trajectory must again reach the line $y = y^*$ at a point $p_2 : (y^*, v_{N,2})$ where a further transition to positive slip occurs. From (5.23) it follows that

$$\frac{v_{N,2}^2}{2} = E_0(p_2) = E_0(p_1) > \frac{v_{N,0}^2}{2}, \quad \text{which implies that } v_{N,2} > v_{N,0}.$$
(5.25)

Iterating this argument, we find there is an infinite sequence of points $p_{2n} : (y^*, v_{N,2n}), n = 0, 1, ...,$ at which transitions take place between stick and positive slip. Moreover, the $v_{N,2n}$ -values constitute an increasing sequence. We further see that this sequence cannot be bounded, since that would mean that the (y, v_N) -orbit would have stick parts over intervals of time that decreased to zero. That is, the orbit would converge to a periodic, all slip orbit. But, since $-k_T^+v_N + k_N^+v_T$ decreases linearly in time in slip, such a periodic orbit cannot exist.

To conclude, if a > 0, B > 0, $k_T^+ < 0$, and $ak_N^+ - bk_T^+ < 0$ and to leading order in ε , any transition from stick to positive slip will lead to an infinite alternation of periods of stick and positive slip (including free flight), with an amplitude that grows without bounds. This corresponds to a left-accumulation point of impacts in the rigid limit as $\varepsilon \to 0$.

From the assumptions on h it follows that, if ε is small enough, as $v_{N,2n}$ grows, we will pass through a region where the portion of time spent in contact (y < 0) is small, while the unscaled velocities also remain small. This is thus a region where the theory of Sec. 4 applies, and the quotient between successive $v_{N,2n}$ -values becomes close to a fixed value e > 1. Given two such consecutive $v_{N,2n}$ values, say $v_{N,2k}$ and $v_{N,2k+1}$, the forward motion in the unscaled variables for a given $\varepsilon = \varepsilon^*$ and based at the first of these transitions is essentially identical to that obtained for $\varepsilon = \varepsilon^*/e$ and based at the second of these transitions. Thus we argue that we can converge on a particular forward motion by taking a sequence of ε values with quotient converging to 1/e. By starting with different ε values, we typically converge on different forward motions. This again supports the one-parameter ambiguity in forward motion associated with reverse chattering.

5.3 Transitions from slip

As argued in Sec. 4.3, if the conditions for possible reverse chatter are fulfilled and the system is in a phase of sustained slip with a small slipping velocity, then when slipping ends there is the possibility of either going into sustained stick, or going into reverse chatter.

Specifically, suppose that we reach the end of positive slip at a moment where a > 0, b < 0, B > 0, $k_T^+ < 0$, and $ak_N^+ - bk_T^+ < 0$. We note that this can only happen when $k_N^+ > 0$, since otherwise positive slip cannot occur. Clearly $ak_N^- - bk_T^- > 0$, so a transition to stick is possible. On the other hand, the conditions for a left-accumulation point of impacts are also fulfilled. Thus, either alternative represents a possible forward motion within the Signorini-Coulomb-Stronge model with r = 1. For the compliant model, however, we will get a definite answer.

To this end, suppose that the transition from positive slip to stick occurs at the point p_+ : $(y_+, 0)$ (corresponding to the leading-order equilibrium state of the (y, v_N) -dynamics in positive slip), where $y_+ < 0$ is uniquely defined by

$$h(y_{+}) = -\frac{1}{k_{N}^{+}}.$$
(5.26)

Here,

$$h(y_0) - h(y_+) = B \frac{ak_N^+ - bk_T^+}{bk_N^0 k_N^+} > 0 \quad \text{and} \quad h(y^*) - h(y_0) = -\frac{ak_N^+ - bk_T^+}{bk_T^+ k_N^0} > 0, \tag{5.27}$$

imply that $y_+ < y_0 < y^*$. If the stick orbit starting at p_+ intersects the line $y = y^*$ transversally, a transition back to positive slip occurs. Then, as shown above, to leading order in ε , this must lead to chatter-like behaviour. If, on the other hand, the stick orbit does not intersect the $y = y^*$ transition line transversally, the system stays in stick, oscillating around the stick equilibrium point with constant amplitude. Since $V_0(y^*) = 0$, it follows that $V_0(y_+) > 0$ corresponds to the case of chatter-like behaviour, whereas $V_0(y_+) \leq 0$ corresponds to the case in which stick persists. These two cases are illustrated in Fig. 9(a) and (b), respectively. For example, if h(y) = y for y < 0, we find that

$$V_0(y_+) = -\frac{\left(2k_N^+ - k_N^0\right) \left(ak_N^+ - bk_T^+\right)^2}{2b^2(k_N^+)^2(k_T^+)^2}.$$
(5.28)

In this case, and in the limit as $\varepsilon \to 0$, reverse chatter thus occurs provided that

$$k_N^+/k_N^0 < 1/2. (5.29)$$

Condition (5.29) is thus satisfied by values of μ , for which reverse chatter occurs in the rigid-body model (recall the condition $r^2 > k_N^+/k_N^0$) with $r = 1/\sqrt{2} \approx 0.7$ (corresponding to the shaded regions in Fig. 7). On the other hand, $k_N^+ > 0$ is satisfied by values of μ , for which reverse chatter occurs in the rigid-body model with r > 0 but not r = 0.



Figure 9: Illustrating the conditions on the relative positions of y_0 , y_{\pm} and y^* for deciding whether reverse chatter or stick occurs. See text for details.

Suppose instead that we reach the end of negative slip (where there is no sign restriction on k_N^+) under the same conditions and at a point $p_-: (y_-, 0)$ (corresponding to the leading-order equilibrium state of the (y, v_N) -dynamics in negative slip), where $y_- < 0$ is uniquely defined by

$$h(y_{-}) = -\frac{1}{k_{N}^{-}}.$$
(5.30)

Here,

$$h(y_{-}) - h(y_{0}) = -B \frac{ak_{N}^{-} - bk_{T}^{-}}{bk_{N}^{-}k_{N}^{0}} > 0$$
(5.31)

implies that $y_0 < y_-$. If $y_- < y^*$ there will be a transition from negative slip to stick, and the stick orbit will be a closed orbit with $y \leq y_- < y^*$. In this case, there will never be a transition to positive slip, and the system will keep oscillating around the stick equilibrium point with constant amplitude. If $y_- > y^*$ there will first be a transition from negative slip to positive slip. For the same reasons as outlined above, this positive slip orbit must eventually lead to a transition to stick, and then to a transition to positive slip again, again resulting in chatter-like behaviour. Since

$$h(y_{-}) - h(y^{*}) = \frac{ak_{N}^{-} - bk_{T}^{+}}{bk_{N}^{-}k_{T}^{+}}$$
(5.32)

independently of the form of the function h, it follows that, in the limit as $\varepsilon \to 0$, reverse chatter occurs provided that

$$ak_N^- - bk_T^+ > 0, (5.33)$$

and that the system continues in stick if this condition is violated. These two distinct cases are illustrated in Fig. 9(c) and (d), respectively. It is straightforward to show that condition (5.33) only restricts the allowable values of a but imposes no further constraint on μ .

To summarise, we have found that for a conservative normal force corresponding to r = 1, reverse chatter is compatible with a limiting behaviour of the compliant model as $\varepsilon \to 0$ triggered at the conclusion of slip under conditions that are a subset of the conditions found for the Signorini-Coulomb-Stronge model in Sec. 4. Specifically, the extra condition is (for a piecewise linear normal force) (5.29) when coming form positive slip, which also requires $k_N^+ > 0$. Coming from negative slip, the extra condition is (5.33). As an example, the shaded regions in Fig. 10 correspond to values of a/(-b) and μ that allow for reverse chatter in the case when B = 0.5 (cf. the r = 1 curve of Fig. 7 shown in light grey in Fig. 10).



Figure 10: The region (shaded) for triggering reverse chatter when coming from (a) positive slip (h taken as piecewise linear) and (b) negative slip. In both cases, for B = 0.5.

5.4 The inclusion of damping

The case when r < 1 in the Stronge impact law corresponds to the presence of dissipation in the normal interactions. In order to achieve the impact mapping expressions of Sec. 2.5 by taking the limit $\varepsilon \to 0$ in a compliant model, some form of damping must thus be introduced into the expression for the normal force. Typically such damping enters in the form of a dependence of λ_N on the normal velocity v_N and turns the first integrals of the previous sections into decaying functions of time. Thus, in the cases where we have found that the undamped compliant system stays in the stick phase with constant amplitude after the end of slip, we would expect the inclusion of damping to result in a convergent motion to the quasi-static stick equilibrium point at $(y_0, 0)$.

For the cases where the undamped compliant system analysis predicts a chatter-like behaviour with increasing amplitude, things are not as clear. To yield r < 1 in the limit as $\varepsilon \to 0$, the damping must be fully active at the small (but finite) velocity scale considered in Sec. 4, but may not be so at the even smaller ε velocity scale considered in the present section. Arguing by continuity, if the overall damping is small, corresponding to an r-value close to unity in the rigid limit, and if e > 1 in the rigid model, we would expect the compliant system with damping to also show reverse chatter in the limit as $\varepsilon \to 0$.

In contrast, consider a rigid model with 0 < r < 1 in a case for which the analysis in Sec. 4 predicts e < 1, but which would have e > 1 if r = 1 and all other parameters and initial conditions were held the same. Now, it is perfectly consistent to produce a compliant model that approaches this case in the limit $\varepsilon \to 0$ but for which the damping is sufficiently small at the ε velocity scale, so that the results in the present section derived with a conservative normal force still apply. This would give an initial growth in velocity through a rapid sequence of passages between stick and slip in contact. On the other hand, at the intermediate scale where we may consider the impacts to be hard and the damping sufficient to give the rigid value of r < 1, we would find the decay of impact velocities since e < 1 in the rigid theory. The system must then settle on a periodic motion while still at the ε scale, and the full system would be in a slow creeping mode where there is a very rapid ε scale oscillation between stick, slip, and free flight. The forward dynamics of a model with such a behaviour is clearly going to depend sensitively on the way in which dissipation is introduced into the compliant model.

6 Numerical results

Let us now illustrate the preceding analysis through numerical simulation of the example system introduced in Sec. 2.2 and given by

$$\dot{x} = v_T, \tag{6.1}$$

$$\dot{y} = v_N, \tag{6.2}$$

$$\theta = \omega,$$
 (6.3)

$$\dot{\omega} = 3R - 3\sin\theta\lambda_T - 3\cos\theta\lambda_N, \tag{6.4}$$

$$\dot{v}_T = a + A\lambda_T + B\lambda_N, \tag{6.5}$$

$$\dot{v}_N = b + B\lambda_T + C\lambda_N, \tag{6.6}$$

where

$$a = S_x - 3R\sin\theta - \omega^2\cos\theta$$
 and $b = S_y - 3R\cos\theta + \omega^2\sin\theta$ (6.7)

and where

$$A = 1 + 3\sin^2\theta, \quad B = 3\sin\theta\cos\theta, \quad C = 1 + 3\cos^2\theta,$$

in the original unscaled variables. First, we shall verify numerically that reverse chatter can indeed occur in a stiff but compliant approximation to the Signorini-Coulomb model in the absence of dissipation in the normal force (i.e., corresponding to r = 1 in the rigid limit). Second, we will illustrate the inherent ambiguity that results when using a compliant formulation in order to approximate solutions that can lead to reverse chatter with r < 1. We shall see that one can get different outcomes depending on the ratio between the damping and stiffness length scales as one passes to the rigid limit via both scales tending to zero.

6.1 Pure elastic normal force impacts

Let

$$\lambda_N = -y/\varepsilon^2 \quad \text{for} \quad y \le 0 \tag{6.8}$$

and suppose, as before, that $\lambda_T \in -\mu \lambda_N \operatorname{Sign}(v_T)$. Restrict attention to the case when R = 0, $S_y = -1$ and S_x is constant, which corresponds to a gravitational force acting on the rod, with the stationary half-plane being tilted clockwise an angle $\operatorname{arctan} S_x$ from the horizontal.

Using Filippov's convex method to solve for λ_T when in contact, for $\varepsilon > 0$ we thus have a well defined hybrid system of differential equations for $(x(t), y(t), \theta(t), v_T(t), v_N(t), \omega(t))$. According to standard results there is thus a unique forward evolution from all bounded initial conditions. As $\varepsilon \to 0$, however, the system will be highly numerically stiff. All computations reported below were carried out using the original unscaled variables and were performed using Matlab's stiff ODE solver ode15s. We choose initial conditions such that

$$v_N(0) = \omega(0) = 0, \quad v_T(0) = v_0, \quad \theta(0) = \theta_0, \quad \text{and} \quad y(0) = y_{\pm},$$
(6.9)

where y_{\pm} are the unscaled quasi-static equilibrium values of penetration in positive and negative slip respectively (cf. (5.26) and (5.30)). At such initial points,

$$A = 1 + 3\sin^2\theta_0, \quad B = 3\sin\theta_0\cos\theta_0, \quad C = 1 + 3\cos^2\theta_0, \quad a = S_x, \quad b = -1, \tag{6.10}$$

and these values will change only by a small amount when the end of slip is reached, provided that v_0 is small.

For a first example, consider the initial conditions and parameters for an orbit in positive slip:

 $\theta_0 = 1.18, \quad \mu = 0.9, \quad S_x = 0.5, \quad \varepsilon = 10^{-6}, \quad v_0 = 0.1, \quad \text{and} \quad y(0) = y_+.$ (6.11)

In this case, the conditions

$$k_T^+ < 0, \quad B > 0, \quad b < 0, \quad ak_N^+ - bk_T^+ < 0 < a$$
(6.12)

for possible reverse chatter are fulfilled at the starting point. Moreover, since $k_N^+ > 0$, positive slip is stable. Finally, the unscaled version of (5.29), $AC - 2\mu AB + B^2 < 0$, is satisfied, so the system satisfies the theoretical requirements for reverse chatter when positive slip ends at $v_T = 0$, rather than a continuation in stick. The forward simulation of the system from this point up to t = 2 is illustrated in Fig. 11(a). From the figure we see that at the conclusion of an initial phase of positive slip, the system breaks contact through a large number (60 within the simulated timespan) of "impacts" (short distinct intervals of time for which y < 0). In agreement with the theoretical treatment in Sec. 5.2, during each impact, the tangential velocity v_T returns to 0.



Figure 11: (a) Simulation of reverse chatter in the rod system (6.1-6.6) with a compliant, stiff normal force (6.8) and parameter values and initial conditions given by (6.9), (6.11) which correspond to positive slip. (b) Similar results but with $v_0 = -0.1$, $S_x = 1$, and $y = y_{-}$.

Next, consider the initial conditions and parameters for an orbit in negative slip:

$$\theta_0 = 1.18, \quad \mu = 0.9, \quad S_x = 0.5, \quad \varepsilon = 10^{-6}, \quad v_0 = -0.1, \quad \text{and} \quad y(0) = y_-.$$
 (6.13)

The conditions (6.12) for possible reverse chatter in the rigid-contact model are again fulfilled at the starting point. However, since $ak_N^- - bk_T^+ < 0$, the extra condition (5.33) from Sec. 5 for reverse chatter is not fulfilled. This prediction is confirmed by simulations (not depicted), for which sustained stick results upon reaching the point $v_T = 0$. However, after increasing the horizontal force S_x to 1, we find that $ak_N^- - bk_T^+ > 0$ and reverse chatter now follows after the end of negative slip. A simulation confirming this outcome is shown in Fig. 11(b).

To illustrate the influence of variations of ε , we have simulated the system with the same fixed initial conditions (6.11) as in Fig. 11(a) (except for the unscaled quasi-static value y_+ , which depends on ε) for a range of ε values between $10^{-5.5}$ and 10^{-5} . The results are shown in figure 12(a). Note that the terminal tangential velocity has an almost periodic dependence on $\log_{10}(\varepsilon)$, suggesting that there is no unique value of v_T (t = 2) in the limit $\varepsilon \to 0$.

Figure 12(b) similarly shows the terminal values of y and θ , which very nearly lie on a closed curve, parametrised by the values of ε . We note that the number of impacts in the time interval [0, 2] increases by 6 as ε is decreased from $\approx 10^{-5}$ to $\approx 10^{-5.5}$. From the spacing between impacts we obtain an estimate of 1.23 for e. This agrees closely with the theoretical value of e = 1.24, computed using the theory in Sec. 4, and supports the reasoning at the end of Sec. 5.2.



Figure 12: The effect of variations in ε for the simulation in Fig. 11(a). (a) Tangential velocity at the end of the simulation v_T (t = 2) versus $\log_{10}(\varepsilon)$. (b) θ (t = 2) versus y (t = 2).

Consider a third set of parameter and initial condition values for an orbit in negative slip given by

$$\theta_0 = 1.18, \quad \mu = 1.5, \quad S_x = 2, \quad \varepsilon = 10^{-6}, \quad v_0 = -0.1, \quad \text{and} \quad y(0) = y_{-}.$$
 (6.14)

The conditions (6.12) for possible reverse chatter in the rigid-contact model are again fulfilled at this starting point. Also in this case, $ak_N^- - bk_T^+ > 0$, and reverse chatter is thus expected for the compliant model at the conclusion of the initial phase of negative slip. This is confirmed by the simulations in Fig. 13. Since here $k_N^+ < 0$, the impact mapping in region 5 of Fig. 5(b) predicts an initial decrease in the normal velocity following the onset of impact until the end of positive slip. This is confirmed by the simulation results which exhibit an undershoot in each impact, as the normal velocity first decreases from its value at the beginning of each impact, until stick is reached. Finally, we see from Fig. 13(b) that although the increase in normal impact velocity between impacts is eventually substantial (e = 1.88), the initial rate is much smaller than this.

6.2 Incorporation of damping

There are many ways in which damping can be introduced into a compliant model for the normal force, to give the same effect as the restitution parameter r for impact velocities that are not too small. Here we choose to take

$$\lambda_N = -g \left(v_N / \varepsilon_N \right) \, y / \varepsilon^2 \quad \text{for} \quad y \le 0, \tag{6.15}$$

for some smooth function g(v) that asymptotes to a value that is r^2 lower as $v \to +\infty$ than as $v \to -\infty$, thus introducing a velocity-dependent stiffness. In particular, in the limit $\varepsilon_N \to 0$, we find that the normal force work done in restitution is $-r^2$ times that in compression, which is precisely the definition of the energetic coefficient of restitution used in [15] to derive the rigid impact laws given in Sec. 2.5. Specifically, in what follows we choose

$$g(v) = \frac{(1+r^2) - (1-r^2)\tanh(v)}{2},$$
(6.16)

Now, there is a subtlety in the choice of the relative size of ε_N and ε as both tend to zero. Specifically, on the ε scale, for normal velocities larger than a few multiples of ε_N , the hard impact model is accurate, but for velocities much smaller that ε_N , the damping is small. To illustrate this subtlety, we will take again the initial and parameter values (6.14) and illustrate the effects of introduction of damping. Specifically, let us take r = 0.8. From Sec. 4, this gives a value e = 1.50, so that if the normal velocity can grow out of the ε scale, it will continue to grow. On



Figure 13: (a) Similar to Fig. 11(a) but for initial and parameter values given by (6.14). (b) The same results plot as y versus v_N (note the difference in scale from (a)).

the other hand, we saw from Fig. 13 that the initial rate of increase in v_N was quite small, so a small amount of damping at the ε scale can make the system go into stick instead. If we take $\varepsilon_N = 10^{-4} \gg \varepsilon = 10^{-6}$, the damping at the ε scale is small, and just as before, the system goes into reverse chatter (not depicted). Changing to ε_N to 10^{-6} , however, gives a substantial damping at the ε scale, and now the simulations in Fig. 14 show that the system settles into stick instead. However, increasing the horizontal force S_x to 12 (in which case e becomes a massive 3.68), is found once again to cause the system to go into reverse chatter. But even in this case, the initial rate of increase in the normal velocity is quite small, and much smaller than e.



Figure 14: (a) Similar to Fig. 13(b) for initial and parameter values given by (6.14) and r = 0.8 and a damped compliant model with $\varepsilon_N = \varepsilon$. (b) Similar results when $S_x = 12$.

7 Conclusion

The main achievements of this paper have been to show, through taking the infinite stiffness limit of a compliant formulation, that within a Signorini-Coulomb model of rigid contact in the presence of dry friction:

1. There is no ambiguity in the forward dynamics starting from the interior of any set of conditions that define sustained motion (stick, slip or free flight). Consequently, no *spontaneous transitions* occur between such motions.

2. Under certain open conditions on initial conditions and parameters, there is an irresolvable ambiguity in the forward dynamics from the terminal point of a condition of sustained slip. This is associated with a possible *triggered transition* to free flight through reverse chatter — a divergent sequence of exponentially growing impact velocities. The latter introduces an infinite multiplicity of possible forward trajectories associated with a one-parameter indeterminacy.

In order to reach the latter conclusion, the Signorini-Coulomb model has been complemented by an energetically consistent rigid-body impact law due to Stronge [1, 23, 15] relating incoming velocities to outgoing ones during an instantaneous collisional event. Specifically, by considering the composition of a brief free-flight phase and an instantaneous impact, the analysis has documented open sets of initial conditions and parameter values, for which the contact-independent contributions to the tangential acceleration over-compensate for energy losses associated with collisional dissipation. This has been shown to imply the existence of solutions based at initial conditions of zero tangential and normal velocity, that undergo a transition to free flight through a divergent sequence of exponentially growing impact velocities, even with a downwards contact-independent normal acceleration.

The Stronge impact law is derived by decomposing the impact phase into distinct trajectory segments of positive slip, negative slip, and stick during which the relative normal velocity is either negative (compression) or positive (restitution). Within this formulation, the impact phase terminates when the normal work in restitution equals $-r^2$ times the normal work in compression, where r is the energetic coefficient of restitution. An alternative formulation, related to the kinetic (Poisson) impact law is to have the impact phase terminate when the normal impulse in restitution equals r times the normal impulse during compression, see e.g [10, 18]. Although not presented here, such a formulation can be shown to lead to identical conclusions on regions of initial conditions and parameter values where reverse chatter can occur in the case when r = 1 and qualitatively similar conclusions for r < 1.

The apparent ambiguity in the Signorini-Coulomb-Stronge model between sustained contact and reverse chatter, and the one-parameter indeterminacy of the latter, disappear when the rigid contact model is replaced with a compliant contact model. For such compliant models, specific initial conditions and parameter values give rise to unique forward trajectories with no possibility of accumulation points of impacts. However, as shown through an asymptotic analysis in the effective normal compliance, chatter-like behaviour again occurs for open sets of initial conditions and parameter values (a subset of those found in the rigid contact model), whereas sustained contact results for other open sets. In the zero-compliance, rigid limit the regions of existence of these distinctly different dynamic behaviours are squeezed together, making it impossible to assert a unique future evolution for the rigidcontact model. The apparent ambiguity in the Signorini-Coulomb-Stronge (or Poisson) model is therefore of an essential, irresolvable kind, an existential threat to the analysis of mechanical contact using rigid contact models.

Recall that the Painlevé paradoxes associated with sustained contact occur in cases where b > 0 while, simultaneously, $\lambda_N > 0$. In contrast, the reverse chatter ambiguity considered here occurs in the case where b < 0 and is therefore a fundamentally different phenomenon from the classical Painlevé paradox. Indeed, whereas the latter has here been uniquely resolved through the consideration of a compliant contact model, the ambiguity associated with reverse chatter is reproduced and reinforced by the study of the compliant model. Also, while the classical Painlevé ambiguity is between two distinct, smooth forward time histories, the reverse chatter scenario includes accumulation points of impacts, a highly non-smooth and degenerate alternative.

A further comparison is afforded by the requirement that, for the Painleveé paradoxes in the case that B > 0, it must hold that $k_N^+ < 0$, which implies that $\mu > B^{-1} > 1$ in terms of the rescaled variables. In the model example, this implies a minimum value of the unscaled μ of 4/3 for $\theta = \arccos(\sqrt{5}/2)$. In contrast, for reverse chatter triggered at the conclusion of a phase of negative slip, it must hold that $\mu > B$ in terms of the rescaled variables. In the model example, this implies a lower bound of 0 for the unscaled μ , where the infimum is approached as $\theta \to 0$ or $\pi/2$. For reverse chatter triggered at the conclusion of a phase of positive slip in the case of a linear compliance, it must hold that $\mu > (1 + B^2)/2B$ in terms of the rescaled variables. In the model example, this implies a lower bound of $\pi \approx 0.828$ for $\theta \approx 1.183$.

Finally, we note that reverse chatter initiated with a given $k_N^+ > 0$ can only occur for r in some subset of (0, 1], for values of a/(-b) in some finite range, and for values of e below some upper bound. In contrast, in the case that $k_N^+ < 0$, there is no restriction on the value of r, only a lower bound on a/(-b), and the possibility of arbitrarily large values of e by choosing a sufficiently large. It follows that, in the case of reverse chatter triggered at the conclusion of a phase of negative slip, there is no upper bound on the rate of divergence of impact velocities.

It is interesting to consider the physical implications and viability of reverse chatter. A good mental picture is to think of the model rod as a piece of chalk being either "pushed" or "dragged" across a chalkboard (recall that the constant acceleration S_y points into the board). Specifically, if the angle θ is between 0 and $\pi/2$, then the case when $S_x < 0$ corresponds to the chalk being dragged, and $S_x > 0$ to it being pushed. In the numerical simulations in Sec. 6 $S_x > 0$ so that the chalk is being pushed. It is commonly observed that pushing a piece of chalk in this sense is more difficult than dragging it, because under constant tangential force the slipping point of contact of the chalk with the board can come to rest. The ambiguity associated with reverse chatter now corresponds to two distinct possibilities from this point of rest; either the chalk subsequently pivots about the point of contact mimicking the action of a pole vaulter, or the chalk enters into a sequence of tiny positive slip phases interrupted by instantaneous impacts with exponentially increasing elapsed times during slip and exponentially increasing impact velocities. The latter appears to accord with the everyday experience of a juddering motion of chalk that can occur as it is pushed across a chalkboard with sufficient force. Note too from 7(b) that even though the coefficient of friction is sufficiently large. Moreover, the figure shows that the for a fixed r-value the phenomenon is most likely to be triggered when axis of the chalk is close to perpendicular to the board (θ between 1 and $\pi/2$ radians), rather than close to tangential.

Let us conclude by a consideration of open questions for further study. In particular, there would be value in exhaustively investigating all possible triggered transitions associated with trajectories reaching the boundaries of existence of distinct modes of sustained motion. A particularly interesting case is the example problem discussed by Génot and Brogliato [7], where it is shown that open sets of trajectories reach, in finite time, a singular triggered transition point where $b = k_N^+ = 0$. At such a point, the forward evolution again appears to lead to an ambiguity, a detailed study of which will form the subject of future work by the present authors using similar methods to those presented here. In a broader context, it would further be useful to understand possible discontinuity-induced bifurcations that occur when the event sequence of triggered transitions changes for a distinguished trajectory (such as a periodic orbit) as a parameter is varied. Finally, our analysis has been specific to the Coulomb friction law with equal dynamic and static coefficients of friction and with no variation of this coefficient. Preliminary calculations (not presented) show that similar conclusions on the occurrence of reverse chatter and the unavoidable resulting ambiguity hold for more general dry friction characteristics. A detailed analysis is left for future work.

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