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Stabilizing effect of delay distribution for a class of second-order systems without instantaneous feedback

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Abstract

In many situations in physics, engineering and biology time delays arise naturally due to the time needed to transport information from one part of the system to another and/or to react to incoming information. When differential equations are used in the mathematical modeling, then incorporating time delays leads to a description by a delay differential equation. We consider here a class of second-order scalar delay equations without instantaneous feedback, where the delays enter according to a distribution function. This is a natural description whenever there are more than one delay.

In this paper we show that for this class of systems one can derive stability information about the distributed-delay system by considering the one delay system where the delay is the mean delay of the distribution function. More specifically, we prove that the asymptotic stability of the zero solution of the second-order delay equation with symmetric delay distribution is implied by the stability of the associated mean-delay equation. Our proof is based on the comparison of stability charts of the two equations.

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1 Introduction

Many deterministic real-world processes are modelled by a second-order scalar ordinary differential equation

$$\ddot{x}(t) = f(\dot{x}(t), x(t)), \tag{1}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a sufficiently smooth function. Local stability analysis at an equilibrium point leads to the equation

$$\ddot{x}(t) = -a\dot{x}(t) - bx(t), \;\; a, b \in \mathbb{R}. \tag{2}$$

Apart from arising as a linearized equation, (2) also plays a central role in engineering as the equation describing damped mass-spring oscillators, which are central building blocks of many systems. In this framework, $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are the damping coefficient and the stiffness parameter, respectively. The long-term behaviour of (2) and its role for the dynamics of (1) is indeed well understood.

Delays generally arise in applications due to the time it takes for information to be processes and/or to flow between different components of the system. Hence, when such delays are sufficiently large, they need to be incorporated into the mathematical model, which then takes the form of a delay differential equation (DDE) — a special case of the wider class of functional differential equation (FDEs). More specifically, our starting point is the delayed form of (2), given by the family of second-order scalar delay diffential equations

$$\ddot{x}(t) = -a\dot{x}(t) - bx(t - E), \tag{3}$$

where $E > 0$ is a single fixed delay. Since the position variable x appears only in delayed form, one also refers to (3) as a systems without instantaneous feedback. When the process under consideration is subject to several delays, the most natural and general formulation of the delayed problem is given in the form

$$\ddot{x}(t) = -a\dot{x}(t) - b\int_0^h x(t - \tau)d\mu(\tau). \tag{4}$$

Here the integral is of Stieltjes-type, and the distribution function $\mu : \mathbb{R} \to \mathbb{R}$ is non-decreasing, right-continuous and satisfies

(A1) $\mu(\tau) = 1, \text{ if } \tau \geq h$ and

(A2) $\mu(\tau) = 0, \text{ if } \tau < 0,$

for $h \geq 0$. Conditions (A1) and (A2), together with the monotonicity of μ, imply that

$$\int_0^h d\mu(\tau) = 1.$$

Observe that (3) is a special case of (4) for the special choice of distribution function

$$\mu_E(x) = \begin{cases} 1, & \text{if } x \geq E; \\ 0, & \text{if } x < E. \end{cases} \tag{5}$$

An important difficulty is that, in many situations, the delays and, hence, the distribution function μ are not known exactly; instead, the main information about the problem is the *average* or *mean* or *expectation* of the delay, given by

$$\int_0^h \tau d\mu(\tau).$$

Is this useful information? In other words, what knowledge can we gain about the stability of (4) if we have information about the stability properties of (3) when E is the corresponding mean delay of the distribution μ in (4)? This paper addresses this question about the role of delay distribution for the class (4) of second-order systems without instantaneous feedback.

Indeed, the impact of delay distribution on stability was investigated by quite a number of authors. For instance, in [1] a general method was established to approximate the bound of the stability region for an arbitrary distribution function. An ecological system given by a nonlinear DDE, the linearization of which is (4), was investigated in [2] for specific distributions. In [3] the effect of delay distribution was investigated from a control theoretic point view; it was shown that if the feedback is stabilizing (respectively, destabilizing), then a discrete delay is locally the most stabilizing (respectively, destabilizing) among delay distributions with the same mean. Also from a control theoretic point view, delay distribution has been used in [4] to approximate the behavior of systems with time varying delay. Recently, some traffic flow models with distributed delays were investigated in [5], where the delay distribution models driver reaction times. In [6] a symmetry condition was posed on $\eta : [-r, 0] \to \mathbb{R}$, under which it was shown that the zero solution, $x \equiv 0$, of

$$\dot{x}(t) = \int_{-r}^{0} x(t+s) d\mu(s) \tag{6}$$

is asymptotically stable if and only if

$$\int_{-r}^{0} \sin\left(-\frac{s}{r}\pi\right) d\mu(s) < \frac{\pi}{r}. \tag{7}$$

When assuming that (A1) and (A2) hold and that $\mu : \mathbb{R} \to \mathbb{R}$ is a monotonically nondecreasing function with expectation value E, then the symmetry condition of [6] is equivalent to saying that μ is *symmetric* about its mean E, in the sense that

$$\mu(E - x) = 1 - \mu(E + x - 0). \tag{8}$$

We remark that if $y = E - x,\ y \in [0, 2E]$ then

$$\mu(y) = 1 - \mu(2E - y - 0),\ y \in [0, 2E].$$

Considering $2E = h$, we get $\mu(y) = 1 - \mu(h - y - 0),\ y \in [0, h]$; furthermore, $\mu(\tau) = 0$, if $\tau < 0$ and $\mu(\tau) = 1$, if $\tau \geq h$.

In [7] the first-order DDE

$$\dot{x}(t) = -ax(t) - b\int_{-h}^{0} x(t+s)\mu(s) \tag{9}$$

was considered in the same spirit of determining its stability from the corresponding equation for the mean delay, and a sufficient condition for the stability of the zero solution was derived when μ is a symmetric delay distribution. In [8] we presented a slightly different and complete proof of this result. In this paper we assume that the delay distribution is given by a symmetric distribution function and we carry out a study of the stability region of the zero solution of (4) in the spirit of [8]. More precisely, we establish the following main result.

Theorem 1.1. *Let μ be symmetric about its expectation E. Then the zero solution of* (4) *is asymptotically stable if the zero solution of* (3) *is asymptotically stable.*

Hence, we show that the asymptotic stability of the zero solution, $x \equiv 0$, of (3) for parameters $a, b \in \mathbb{R}$ and $E \in \mathbb{R}^+$ — the mean delay associated with μ in (4) — implies the stability of the zero solution of (4) (for the same values of a and b). In other words, a symmetric delay distribution has a stability preserving effect on the zero solution. Our work can be considered as a generalization of the finding in [2] that replacing the single delay in (3) by two symmetrically distributed discrete delays increases the stability of the system. Note that this result was also derived in [9] with another proof, namely the method of proof also used here. More concretely, the underlying idea in [9, 8] for the study of stability properties of the single and the distributed delay equations is the comparison of stability charts. The purpose of the present paper is to formulate and proof in a more general setup the result of [9], where particular second-order equations were studied that can be considered as special cases of (4) with two delays.

One could paraphrase Theorem 1.1 by saying that symmetric delay distribution not only increases the stability for the second-order scalar DDEs in the class given by (4), but its effect is also similar to the first-order DDEs given by (9). By this we mean that, in a certain part of the parameter space, the mean value of delays contains enough information to decide about the stability of the zero solution of (4). In other words, in the modelling process, even if the distribution is known, the model with only the mean delay gives useful practical information about stability. We remark that this statement is not obvious, because it may not hold for other classes of second-order equations. In fact, we showed in [8] that for the equation

$$\ddot{x}(t) = -\dot{x}(t) - ax(t) - b\int_0^h x(t-\tau)d\mu(\tau) \tag{10}$$

symmetric delay distribution is *not* stability preserving without further assumptions.

We finally mention that, from an application point of view, our work is motivated by the relatively new field of *substructuring* or *hybrid testing* of engineering structures. In this testing approach the system under consideration is split into two main parts: a critical part of interest is tested in the laboratory, and the remainder of the system is run via a model on the computer. The two subsystems are mutually coupled by feeding measurements from the tested part into the computer model, and by driving the laboratory test with output from the model, for example, via electric or hydraulic actuators. Delays arise naturally in this setting, both from running the computer model and due to a delay before actuation is achieved. In practice, the delay from the model computation can often be neglected. On the other hand, experiments show that the delay of the actuators is generally quite large (on the order of a few hundred milliseconds) and may influence the stability of the overall test [10, 11]. Hence, the field of hybrid testing provides a rich class of DDEs; see also [12, 13]. In this context, the single-delay equation (3) could be interpreted as describing damped mass-spring oscillators where information of the position of the mass is not available instantaneously. Similarly, the distributed-delay equation (4) could be interpreted as describing the damped mass-spring oscillators when one performs different independent measurements of the same state variable (in this case the displacement of the mass) but subject to delays as given by the weight function μ. In a different interpretation of (4), one may consider not only the delay due to the actuator, but also that arising from running the computer model which yields a model with distributed delays. The paper is organized as follows. In Section 2 we first introduce some notation and recall some facts on DDEs. We then determine in Section 3 the stability chart of (3); this also allows us to set up the theoretical framework of studying curves of purely imaginary solution to the characteristic equation. This setup is then used in Section 4 to proof Theorem 1.1. Finally, in Section 5 we draw some conclusion and point to future work.

2 Background and notation

This section serves to introduce notions that will play a crucial role in the proof of our main result in Sec. 4. We start by recalling some general facts of the general theory of DDEs; see, for example, [14, 15]. Our object of study are linear autonomous equations of the general form

$$\dot{x}(t) = \int_0^h d\eta(\theta) x(t-\theta) \tag{11}$$

where $\eta(\theta),\ 0 \leq \theta \leq h$, is an $n \times n$ matrix of normalized functions of bounded variation, so that η is continuous from the right on $(0, h)$ and $\eta(h) = 1$. A solution $x : \mathbb{R}^n \to \mathbb{R}^n$ of (11), for a given η and $h > 0$, is a differentiable function satisfying (11). Let $C = C([-h, 0], \mathbb{R}^n)$ denote the Banach space of continuous functions mapping the interval $[-h, 0]$ into $\mathbb{R}^n$, with the supremum norm, and define $x_t \in C$ as $x_t(\theta) = x(t+\theta),\ \theta \in [-h, 0]$. A solution $x^\phi : [0, \infty) \to \mathbb{R}$ is uniquely determined by $x_0^\phi = \phi \in C$. The unique solution with initial function $\phi \in C$ determines a map

$$F(t, \phi) : \mathbb{R}^+ \times C \ni (t, \phi) \mapsto x_t^\phi \in C$$

and the *solution operator* is

$$T(t)\phi : C \ni \phi \mapsto F(t, \phi) \in C,\ t \geq 0.$$

The solution operator is a strongly continuous semigroup with an infinitesimal generator A, the spectrum $\sigma(A) \subset \mathbb{C}$ of which is formed by its point spectrum. Furthermore, $\lambda \in \sigma(A)$ if and only if λ satisfies the *characteristic equation*

$$\det\left(\lambda I - \int_0^h e^{-\lambda\theta} d\eta(\theta)\right) = 0. \tag{12}$$

The roots of (12) are called *characteristic roots*. Stability analysis of (11) is based on the following result.

Theorem 2.1. *[15] The zero solution of equation* (11) *is asymptotically stable if and only if the real part of all characteristic root of* (12) *is negative.*

Hence, local stability investigations can be carried out by finding the zeros of the *characteristic function*

$$\Delta(\lambda) : \mathbb{C} \ni \lambda \mapsto \det\left(\lambda I - \int_0^h e^{-\lambda\theta} d\eta(\theta)\right) \in \mathbb{C}.$$

A difficulty for the stability analysis is that the characteristic function is an analytic function possessing countably infinitely many zeros; see, for example, [16] for examples of stability analysis based on characteristic roots. The following lemma is a useful tool in the stability analysis of parameter-dependent systems; throughout $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ denote the real and imaginary parts of a $\lambda \in \mathbb{C}$, respectively.

Lemma 2.1. *[17] Let $f(\lambda, \alpha) = \lambda^n + g(\lambda, \alpha)$ be an analytic function with respect to λ and α, where $\alpha \in \mathbb{R}^m$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\beta$ for a positive constant β. Assume that*

$$\limsup\{|\lambda^{-n} g(\lambda, \alpha)| : \mathrm{Re}(\lambda) \geq 0,\ |\lambda| \to \infty\} < 1.$$

Then, as α varies, the sum of the roots of $f(\lambda, \alpha) = 0$ in the open right half-plane can change only if a root appears on or crosses the imaginary axis.

3 Stability of the single-delay system

We now determine the stability region of the zero solution of (3), where we make use of a method that can be found in [14]. This section also serves to introduce notions that will play a crucial role in the proof of our main result in Sec. 4.

Recall that (3) is a special case for (4) for $\mu = \mu_E$. The corresponding characteristic function and equation are

$$\Delta(\lambda) : \mathbb{C} \ni \lambda \mapsto \lambda^2 + a\lambda + be^{-\lambda E} \in \mathbb{C} \tag{13}$$

and

$$\lambda^2 + a\lambda + be^{-\lambda E} = 0. \tag{14}$$

The following two propositions exclude certain part of the parameter plane from the stability region of (3).

Proposition 3.1. *For $b \leq 0$ the zero solution of* (3) *is not asymptotically stable.*

Proof. If $b = 0$, then $\lambda = 0$ is always a characteristic root.

For $b < 0$, one considers

$$f(x) : \mathbb{R} \ni x \mapsto x^2 + ax + be^{-xE} \in \mathbb{R},$$

the restriction of the corresponding characteristic function to the real line. Then the fact that $f(0) = b < 0$, together with the continuity of f and the fact that $\lim_{x\to\infty} f(x) = \infty$, implies the existence of an $x^* > 0$ such that $f(x^*) = 0$. $\square$

Proposition 3.2. *For $a < 0$ the the zero solution of* (3) *is not asymptotically stable.*

Proof. The proof of the following proposition is based on Pontryagin's method; see [18]. Let us assume that the zero solution is asymptotically stable. Then, because of Theorem 2.1, all characteristic roots lie to the left of the imaginary axis. Consider now (13) in the following equivalent form

$$H(\lambda) : \mathbb{C} \ni \lambda \mapsto (\lambda^2 + a\lambda)e^{\lambda E} + b \in \mathbb{C}. \tag{15}$$

The assumption on the real part of the zeros of (15) implies that there is no zero in the rectangle $P_{k\alpha} = \{(x, y) : 0 \leq x \leq \alpha, -2k\pi + \varepsilon \leq y \leq 2k\pi + \varepsilon\}$. We show for the vector $w = H(iy)$ that $v(-2k\pi, 2k\pi) = 4k\pi + 2\pi + \delta_1$, where $v(\alpha, \beta)$ is the change in the vector w when $y \in (\alpha, \beta]$ and $\lim_{k\to\infty} \delta_1 = 0$. To this end, first notice that $v(\alpha + \varepsilon, \beta + \varepsilon) = v(\alpha, \beta) + \delta_2$, where $\varepsilon \in \mathbb{R}$ is fixed and $\lim_{\alpha\to\pm\infty} \delta_2 = 0$. Indeed, the fact

$$\frac{H(i(\alpha + \varepsilon))}{H(i\alpha)} = \frac{\left(-(\alpha + \varepsilon)^2 + a(\alpha + \varepsilon)i\right) e^{i\varepsilon} e^{i\alpha}}{(-\alpha^2 + \alpha ai)e^{i\alpha} + b} + \frac{b}{(-\alpha^2 + \alpha ai)e^{i\alpha} + b} \tag{16}$$

implies

$$\lim_{\alpha\to\pm\infty} v(\alpha,\alpha+\varepsilon)=\varepsilon+\delta_3.$$

Applying the latter, together with the evident fact that $v(\alpha,\beta)=v(\alpha,\gamma)+v(\gamma,\delta)$, we obtain

$$v(\alpha+\varepsilon,\beta+\varepsilon)=v(\alpha+\varepsilon,\alpha)+v(\alpha,\beta)+v(\beta,\beta+\varepsilon)=v(\alpha,\beta)+\delta_4.$$

(Here we also used the fact that $v(x,y)=-v(y,x)$.)

Next we show that $v_k(-2k\pi+\varepsilon,2k\pi+\varepsilon)=4k\pi+2\pi+\delta_k$, where $\lim_{k\to\infty}\delta_k=0$. As a consequence of the Argument principle, the angle variation of the vector w under z^2e^z is the sum of the variation for its factors. Because $\frac{\left(e^{i\pi/2}\right)^2}{\left(e^{-i\pi/2}\right)^2}=e^{i2\pi}$, for the function z^2 the variation on the given three sides of $P_{k\alpha}$ is 2π. Since $e^{a+(2k\pi+\varepsilon)i}=e^ae^{\varepsilon i}$, on the intervals $[(2k\pi+\varepsilon)i,\alpha+(2k\pi+\varepsilon)i]$ and $[\alpha-(2k\pi-\varepsilon)i,-(2k\pi-\varepsilon)i]$ the angle variation of the vector w under the function e^z is zero.

On the other hand, the total number N_k of zeros of the function $H(z)$ inside the rectangle $P_{k\alpha}$ is equal to the number of full revolutions of the vector $w=H(z)$, when z traverses all sides of the rectangle. Thus $N_k=T_3-v_k(-2k\pi,2k\pi)$. On the other hand, because of the assumption on the number of zeros in $P_{k\alpha}$, we have that $N_k=0$; that is, $T_3=v_k(-2k\pi,2k\pi)$.

Let

$$\begin{aligned} H(iy) &= F(y)+G(y)i &(17)\\ &= -y^2\cos(y)-a\sin(y)+b+(-y^2\sin(y)+ay\cos(y))i. &(18)\end{aligned}$$

The argument above implies the existence of distinct points $y_j\in(-2k\pi+\varepsilon,2k\pi+\varepsilon)$, where $j=1,2,\dots,l$ and $4k+2\le l$, such that the curve $w=H(iy)$ and the line $-y^2\cos(y)-a\sin(y)+b$ in the complex plane have an intersection at each y_j. This fact implies that the function $\mathbb{R}\ni y\mapsto -y^2\sin(y)+ay\cos(y)\in\mathbb{R}$ has at least $4k+2$ zeros; or equivalently, the equation

$$a\cot y=y \tag{19}$$

has at least $4k+1$ roots. However, one can readily see that (19) has at most $4k$ solutions for $a<0$. This contradiction implies that there is at least one root in the rectangle $P_{k\alpha}$, so that the zero solution of (3) is indeed not asymptotically stable. □

The following lemma tells us that there are parameter values in the (a,b)-plane such that the zero solution of (3) is asymptotically stable.

Proposition 3.3. *If $a>0,\ b>0$ and $bE\le a$ then the zero solution of* (3) *is asymptotically stable.*

Proof. We consider the equation

$$\ddot{x}(t)=-a\dot{x}(t)-bx(t-\varepsilon E),\ \varepsilon\in[0,1] \tag{20}$$

and its characteristic equation

$$\lambda^2 + a\lambda + be^{-\lambda\varepsilon E} = 0. \tag{21}$$

For $\varepsilon = 0$ it reduces to

$$\lambda^2 + a\lambda + b = 0, \tag{22}$$

with roots

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

having negative real parts. Now, assume that (21) with $b \leq a$ has a root with non-negative real-part. Then, because of Lemma 2.1, there is an $\varepsilon_0 \in (0, 1]$ such that (21) has a pair of roots $\pm i\omega_0$, where $\omega_0 > 0$.

Substitution of $i\omega_0$ into (20) results in

$$-\omega_0^2 + ai\omega_0 + b(\cos(\omega_0\varepsilon_0 E) - i\sin(\omega_0\varepsilon_0 E)) = 0, \tag{23}$$

which gives

$$b\cos(\omega_0\varepsilon_0 E) \quad = \omega_0^2, \tag{24}$$

$$b\sin(\omega_0\varepsilon_0 E) \quad = a\omega_0. \tag{25}$$

Clearly (27) has no positive root if $b \leq 0$. $\square$

From Lemma 3.3, we know that the line $b = \frac{a}{E}$ does not bound the stability region from above. To find the upper bound we follow the method of [14, Chapter 11]. For equation (3), the assumption of the existence of a characteristic root $i\omega, \omega > 0$, after separating the real and imaginary parts of the left–hand side of (14), results in the two equations:

$$b\cos(\omega E) - \omega^2 \quad = 0, \tag{26}$$

$$b\sin(\omega E) - a\omega \quad = 0. \tag{27}$$

With the aid of these equations, we can define the two functions

$$a_k(\omega) : I_k^{\pm} \to \mathbb{R},\ \omega \mapsto \frac{\omega\sin(\omega E)}{\cos(\omega E)}, \tag{28}$$

$$b_k(\omega) : I_k^{\pm} \to \mathbb{R},\ \omega \mapsto \frac{\omega^2}{\cos(\omega E)}, \tag{29}$$

where $I_k^+ = (0, \pi/2)$, for $k = 0$ and $I_k^- = ((4k-3)\pi/2, (4k-1)\pi/2), I_k^+ = ((4k-1)\pi/2, (4k+1)\pi/2)$ for $k \in \mathbb{N} \setminus \{0\}$. Finally, for $k \in \mathbb{N}$, we can define the following parametrized curves

$$\Gamma_k^{\pm} = \left\{ \left(\frac{\omega\sin(\omega E)}{\cos(\omega E)}, \frac{\omega^2}{\cos(\omega E)} \right) \middle| \omega \in I_k^{\pm} \right\}.$$

Notice that the functions a_k and b_k are even so it suffices to consider the case $\omega > 0$.

We now define an order (denoted by the symbol $\prec$) on a collection of non-intersecting plane curves, where our interest is in curves $\Gamma_k^{\pm}$ as defined above. We consider the *graph*

$$\mathrm{Gr}(\Gamma) = \{(f(x), g(x)) : x \in I\}$$

of a curve

$$\Gamma(x) : I \to (f(x), g(x)) \in \mathbb{R}^2$$

defined on an interval $I \subset \mathbb{R}$. Consider now two curves $\Gamma_1 = \{(f_1(x), g_1(x)) : x \in I_1\}$ and $\Gamma_2 = \{(f_2(x), g_2(x) : x \in I_2\}$ on I_1 and I_2, respectively, and such that $\mathrm{Gr}(\Gamma_1) \cap \mathrm{Gr}(\Gamma_2) = \emptyset$. Then Γ_1 is said to be *below* Γ_2 — denoted $\Gamma_1 \prec \Gamma_2$ — if there are $x_1 \in I_1$ and $x_2 \in I_2$ such that $f_1(x_1) = f_2(x_2)$ and $g_1(x_1) < g_1(x_2)$. Alternatively, we say that Γ_2 is above Γ_1.

The curve $\Gamma_0^+ = \{(a(\omega), b(\omega)) | \omega \in I\}$, together with the line $b = 0$, forms the boundary of the stability region in the parameter (a, b)-plane. This follows, because, if we set $a = a(\omega)$, $b = b(\omega)$ where $\omega \in I$, then the characteristic equation of (3) has a purely imaginary root. Proposition 3.3 implies that there is no characteristic root with positive real part below this curve and, hence, the zero solution $x \equiv 0$ of equation (3) is asymptotically stable.

Figure 1 shows the curves $\Gamma_k^{\pm}$ for $k = 0, 1, 2$ and the line $b = 0$ in the (a, b)-plane for $E = 1$. The figure illustrates that, for $i < j$, one has $\Gamma_i^+ \prec \Gamma_j^+$ and $\Gamma_j^- \prec \Gamma_i^-$, which is a fact that is not hard to prove. The number of unstable characteristic roots is indicated for each region; the zero solution is stable in the grey shaded region. This can be validated after introducing the following functions.

$$\begin{aligned} F(a, b; s) : \mathbb{R}^2 \times \mathbb{C} \ni (a, b; s) &\mapsto \mathrm{Re}\left(s^2 + as + be^{-sE}\right) \in \mathbb{R}, \\ G(a, b; s) : \mathbb{R}^2 \times \mathbb{C} \ni (a, b; s) &\mapsto \mathrm{Im}\left(s^2 + as + be^{-sE}\right) \in \mathbb{R}, \end{aligned}$$

and the matrix

$$M = \begin{pmatrix} D_a F & D_b F \\ D_a G & D_b G \end{pmatrix}\Bigg|_{(a,b;s)=(a_0,b_0,i\omega_0)}. \tag{30}$$

Here (a_0, b_0) is a point on one of the curves defined via (28) and (29), and ω_0 is the corresponding parameter value. The determinant of M determines how the critical roots in the complex plane depend on two parameters; namely, we will use the following result.

Theorem 3.1 ([14, Chapter 11, Proposition 2.13] **).** *The critical roots are in the parameter region to the left of the curve $(a(\omega), b(\omega))$, when we follow this curve in the direction of increasing ω, whenever* $\det M < 0$ *and to the right when* $\det M > 0$.

Here we have

$$M = \begin{pmatrix} 0 & \cos(\omega E) \\ \omega & -\sin(\omega E) \end{pmatrix}\Bigg|_{(a,b;s)=(a_0,b_0,i\omega_0)}. \tag{31}$$

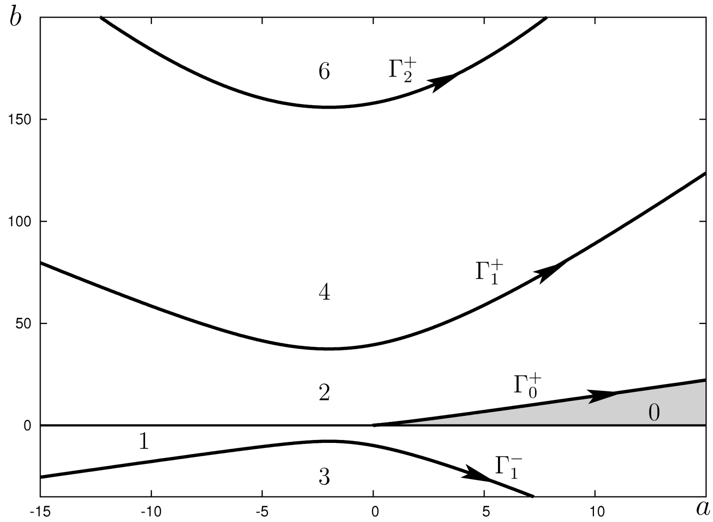

Figure 1: The curves Γ_k for $k = 0, 1, 2, 4$ and the line $b = 0$ of (3). The number of characteristic roots with positive real parts are indicated in each region; the shaded part of the image is the stability region of the zero solution of (3).

This statement means that, because of (28), $a(\omega)$ is a monotone increasing function of ω on each of the intervalls $I_k^{\pm}$. Furthermore, because of (29), the effect of changing parameters on the purely imaginary roots depends only on the sign of $b(\omega)$.

To conclude this section, we derive a sufficient and necessary condition for the asymptotic stability of the zero solution of (3).

Proposition 3.4. *If $a > 0$, $b > 0$ in (3) then its zero solution is asymptotically stable if and only if*

$$E < \frac{2\arccos\left(\frac{-a^2+\sqrt{a^4+4b^2}}{2b}\right)}{\sqrt{\sqrt{a^4+4b^2}-a^2}}.$$

By a simple rescaling of time, (3) takes the form

$$\frac{1}{E^2}\ddot{x}(t) = -a\frac{1}{E}\dot{x}(t) - bx(t-1), \tag{32}$$

which is clearly equivalent to

$$\ddot{x}(t) = -aE\dot{x}(t) - bE^2x(t-1). \tag{33}$$

Assuming that the characteristic equation

$$\lambda^2 + aE\lambda + bE^2e^{-\lambda}$$

of (33) has a pair of purely imaginary roots $\pm i\omega$ with $\omega > 0$, one obtains

$$\omega^2 = bE^2 \cos(\omega), \tag{34}$$

$$E\omega a = bE^2 \sin(\omega). \tag{35}$$

Squaring and adding the last two equations result in the quadratic equation for ω^2:

$$\left(\omega^2\right)^2 + E^2 a^2 \omega^2 - E^4 b^2 = 0. \tag{36}$$

The only solution of (36) satisfying the positivity assumption on ω is

$$\omega^2 = \frac{-E^2a^2 + \sqrt{E^4a^4 + 4E^4b^2}}{2}, \tag{37}$$

which gives

$$\omega = \sqrt{\frac{-E^2a^2 + \sqrt{E^4a^4 + 4E^4b^2}}{2}}, \tag{38}$$

on the one hand, and – after substituting (37) and (38) into (34) –

$$E = \frac{2\arccos\left(\frac{-a^2+\sqrt{a^4+4b^2}}{2b}\right)}{\sqrt{\sqrt{a^4+4b^2} - a^2}} \tag{39}$$

on the other hand.

4 Proof of the main result

We now extend the approach presented in the previous section to obtain information about the upper bound of the stability region of the zero solution of the distributed delay equation (4). More specifically, to prove Theorem 1.1 we show that the stability region of (3) is included in the corresponding stability region of (4). Our method of proof effectively follows, with suitable modifications, the steps taken in [8] in our proof for the first-order case. We present the argument in the form of several Propositions that lead to the statement of Theorem 1.1.

In this section the corresponding characteristic function

$$\Delta_E(\lambda) : \mathbb{C} \ni \lambda \mapsto \lambda^2 + a\lambda + b \int_0^h e^{-\lambda\tau} d\mu(\tau) \in \mathbb{C} \tag{40}$$

and equation

$$\lambda^2 + a\lambda + b \int_0^h e^{-\lambda\tau} d\mu(\tau) = 0 \tag{41}$$

of (4) will play the crucial role.

With a slight modification of the argument used in the proof of Lemma 3.1, we obtain the following.

Proposition 4.1. *For $b \leq 0$ the zero solution of* (3) *is not asymptotically stable.*

To make the notation shorter, we adopt the notation of [7] of writing $C(\omega) = \int_0^h \cos(\omega\tau)d\mu(\tau)$ and $S(\omega) = \int_0^h \sin(\omega\tau)d\mu(\tau)$.

We will need the following.

Proposition 4.2. *If μ is a symmetric distribution then*

$$C(\omega) = 2\cos(\omega E)\int_0^E \cos(\omega\tau)d\mu(\tau)$$

and

$$S(\omega) = 2\sin(\omega E)\int_0^E \cos(\omega\tau)d\mu(\tau).$$

Proof. Let

$$\nu_E : \mathbb{R} \ni \tau \mapsto \frac{\mu(\tau)}{2} \in \mathbb{R}.$$

Because of the symmetry assumption on μ, $2\nu_E$ is symmetric around E. Thus

$$\begin{aligned} C(\omega) &= 2\int_0^{2E} \cos(\omega\tau)d\nu_E(\tau) \\ &= 2\int_0^E \cos(\omega(E-\tau)) + \cos(\omega(E+\tau))d\nu_E(\tau) \\ &= 2\cos(\omega E)\int_0^E \cos(\omega\tau)d\mu(\tau). \end{aligned}$$

The statement for $S(\omega)$ could be shown in the same way. □

We can now prove the analogue of Proposition 3.3.

Proposition 4.3. *If $a > 0$, $b > 0$ and $bE \leq a$ then the zero solution of* (4) *is asymptotically stable if μ is symmetric.*

Proof. We consider the following equation

$$\ddot{x}(t) = -a\dot{x}(t) - b\int_0^{2E} x(t-\varepsilon\tau)d\mu(\tau),\ \varepsilon \in [0,1] \tag{42}$$

and its characteristic equation

$$\lambda^2 + a\lambda + b\int_0^{2E} e^{-\lambda\varepsilon\tau}d\mu(\tau) = 0. \tag{43}$$

For $\varepsilon = 0$ it reduces to

$$\lambda^2 + a\lambda + b = 0, \tag{44}$$

which has the roots

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2-4b}}{2}$$

with negative real parts. Now, assume that (43) with $bE \leq a$ has a root with non-negative real-part. Then, because of Lemma 2.1, there is an $\varepsilon_0 \in (0,1]$ such that (43) has a pair of roots $\pm i\omega_0$, where $\omega_0 > 0$. Substitution of $i\omega_0$ into (42) results in

$$-\omega_0^2 + ai\omega_0 + b(C(\omega_0\varepsilon_0) - iS(\omega_0\varepsilon_0)) = 0. \tag{45}$$

But for the imaginary part of the left–hand side we have

$$\begin{aligned} a\omega_0 - bS(\omega_0\varepsilon_0) &= a\omega_0 - 2b\sin(\omega_0\varepsilon_0 E)\int_0^E \cos(\omega_0\varepsilon_0\tau)d\mu(\tau) \\ &\geq a\omega_0 - b\sin(\omega_0\varepsilon_0 E) \\ &\geq \omega_0 a\left(1 - \frac{\sin(\varepsilon_0\omega_0 E)}{\omega_0 E}\right) > 0. \end{aligned}$$

□

We continue this section with the adaptation of the method applied in Section 3 It is clear that if $i\omega,\ \omega > 0$ is a root of (41), then $-i\omega$ is a root as well. Thus we restrict our attention to $\omega \geq 0$. If we assume that (43) has a root of the form $i\omega,\ \omega > 0$ then this leads to the system of equations:

$$bC(\omega) - \omega^2 \quad = 0, \tag{46}$$

$$bS(\omega) - a\omega \quad = 0. \tag{47}$$

Let $\Omega = \{\omega : C(\omega) = 0,\ \omega \geq 0\}$ be the *zero-set* of $C(\omega)$ associated with (4). It is easy to see that $\Omega \neq \emptyset$.

Although, in general, $C(\omega)$ is not periodic, the following proposition nevertheless gives the possibility to define a curve segment on $I_k^{\pm}$.

Proposition 4.4. *Let $E > 0$; then $\frac{\pi}{2E} + \frac{k\pi}{E} \in \Omega,\ k \in \mathbb{Z}$.*

Proof. With the aid of Proposition 4.2 we get that

$$C(x+y) = 2\left(\cos(xE)\cos(yE) - \sin(xE)\sin(yE)\right)\int_0^E \cos\left((x+y)\tau\right)d\mu(\tau),$$

which validates the statement. □

From Proposition 4.4 we know that $I_k^{\pm} \cap \Omega^c \neq \emptyset$. Thus we can define the subintervals $\hat{I}_{k,l}^{+}$ and $\hat{I}_{k,m}^{-},\ 1 \leq l \leq i,\ 1 \leq m \leq j$ of $I_k^{\pm}$ as the results of intersecting $I_k^{\pm}$ with Ω^c, where Ω^c is the complement of Ω. Here i and j are the (k-dependent) numbers of those subintervals. We can now define the curves

$$\hat{\Gamma}_{k,l}^{+} : \hat{I}_{k,l}^{+} \ni \omega \mapsto (\hat{a}_{k,l}(\omega), \hat{b}_{k,l}(\omega)) \in \mathbb{R}^2,$$

with $\hat{\Gamma}_{k,m}^{-}$ defined in the same way, where $\hat{a}_{k,l}(\omega)$ and $\hat{b}_{k,l}(\omega)$ are determined by (46) and (47). Throughout, if a statement depends on an interval I_k, but independent from any of it subintervals corresponding to (4), we drop the second subindex in our notation. Furthermore, if the statement is independent of the interval of definition, we drop both the subindexes.

The following lemma is the key for the comparison of the stability regions of (4) and (3).

Lemma 4.1. *If $C(\omega_0) = 0$ then*

$$\lim_{\omega\to\omega_0} \hat{a}_k(\omega) = \lim_{\omega\to\omega_0} a_k(\omega).$$

Furthermore, $\hat{a}_k \equiv a_k|_{\hat{I}_k}$.

Proof. Using Proposition 4.2, we get the following

$$\lim_{\omega\to\omega_0} \hat{a}_k(\omega) = \lim_{\omega\to\omega_0} \frac{\omega S(\omega)}{C(\omega)} = \lim_{\omega\to\omega_0} \frac{\omega \sin(\omega E)}{\cos(\omega E)} = \lim_{\omega\to\omega_0} a_k(\omega).$$

□

The relative positions of curves defined via the functions $a(\omega)$ and $b(\omega)$ may be quite complicated, but the following lemma shows an important feature of them. To formulate it we introduce the notation that, for an arbitrary function $\Gamma(x) : I \to (f(x), g(x)) \in \mathbb{R}^2$, the symbol $|\Gamma|$ denotes the function $|\Gamma(x)| : I \to (f(x), |g(x)|) \in \mathbb{R}^2$.

Lemma 4.2. *Let $\mu : \mathbb{R} \to \mathbb{R}$ in (4) be symmetric about its mean $E > 0$. Then $\Gamma_0 \prec |\Gamma^+_{k,l}|$ and $\Gamma_0 \prec |\Gamma^-_{k,m}|$ on $\tilde{I}^+_k$ and $\tilde{I}^-_k$, respectively, for $1 \le l \le i,\ 1 \le m \le j$.*

Proof. Using (46) and then applying Proposition 4.2, we obtain the function

$$\hat{b}(\omega) = \frac{\omega^2}{C(\omega)} = \frac{\omega^2}{2\cos(\omega E)\int_0^E \cos(\omega\tau)d\mu(\tau)} = \frac{b(\omega)}{\int_0^E \cos(\omega\tau)d\mu(\tau)}.$$

That is,

$$\mathrm{Gr}(\Gamma^{\pm}_{k,l}) = \left\{ \left(\omega\frac{\sin(\omega E)}{\cos(\omega E)}, \frac{\omega^2}{2\cos(\omega E)\int_0^E \cos(\omega\tau)d\mu(\tau)} \right) \Big| \omega \in I^{\pm}_{k,l} \right\}.$$

Since $2\int_0^E \cos(\omega\tau)d\mu(\tau) \le 1$, it follows that $|\hat{b}(\omega)| > b(\omega),\ \omega \in I^{\pm}_{k,l}$. Further, if $I^{\pm}_{k,l} = (\omega_L, \omega_R)$ then

$$\lim_{\omega\to\omega_L} |\hat{b}(\omega)| = \lim_{\omega\to\omega_R} |\hat{b}(\omega)| = \infty$$

when $\omega_L \ne 0$.

Thus $\Gamma^+_k \prec \Gamma^+_{k,l}$, $\Gamma^+_k \prec |\Gamma^-_{k,m}|$, $\Gamma^-_{k,m} \prec \Gamma^-_k$ and $|\Gamma^-_k| \prec \Gamma^-_{k,m}$, on the associated intervals. Notice that k, l, m were arbitrary. Hence, because of the fact that $\Gamma_0 \prec \Gamma^+_k$ and $\Gamma^-_k \prec \Gamma_0,\ k \in \mathbb{N}$, the proof is complete.

Note that

$$\lim_{\omega\to 0+} a(\omega) = \lim_{\omega\to 0+} \hat{a}(\omega) = -\lim_{\omega\to 0+} \hat{b}(\omega) = \lim_{\omega\to 0+} b(\omega) = \frac{1}{E}.$$

□

We are now able to prove Theorem 1.1. The matrix corresponding to (4) that is used in Theorem 2.1 takes the form

$$\hat{M} = \begin{pmatrix} 0 & C(\omega) \\ \omega & -S(\omega) \end{pmatrix} \Bigg|_{(a,b;s)=(a_0,b_0,i\omega_0)} . \tag{48}$$

With the aid of M from (31) and $\hat{M}$ above, one can see that the behavior of the critical roots corresponding to (3) and (4) depends only on $b(\omega)$ and $\hat{b}(\omega)$. In either case the roots are above the corresponding curve in the upper half and below the corresponding curve in the lower half of the (a, b)-plane. From Proposition 4.3 we know that there are parameter pairs for which the zero solution of (4) is asymptotically stable. Proposition 4.2 tells us the parameter pairs for which the zero solution of (4) is not asymptotically stable are above Γ_0. This arguments validates our main statement, Theorem 1.1.

4.1 An example with two delays

To illustrate the difficulties arising when comparing the relative positions of the curves $\hat{\Gamma}_{k,l}$, we consider the DDE

$$\ddot{x}(t) = -a\dot{x}(t) - b\frac{1}{2}\left(x\left(t - \frac{1}{3}\right) + x\left(t - \frac{5}{3}\right)\right), \tag{49}$$

which has two delays that are (symmetrically) distributed around $E = 1$.

Figure 2 (a) shows the curve $b = 0$ and the curves $\hat{\Gamma}^{+}_{k,l}$, $k = 0, 1$, for (49); notice that there are substantially fewer curves in Fig. 2(a) compared to Figure 1 for (3) with $E = 1$ (which is on the same scale). The shaded region shows the stability region of the zero solution of (49). The dashed curve is the upper stability boundary Γ_0 for (3) with $E = 1$; the fact that Γ_0 lies well inside the stability region of (49) illustrates our main result that the distributed-delay system (49) has increased stability.

As the larger view of the (a, b)-plane in panel Figure 2 (b) shows, the curves $\hat{\Gamma}_{k,l}$ for (49) are further away from the origin. One can think of these curves as moving as the measure describing the delay distribution is changed. How these curves move is very difficult to say; see also the case study of a two-delay example in [9] for more details.

5 Summary

We considered the effect of symmetric delay-distribution on the stability of the zero solution of a second-oder scalar DDE of the form (4) when the expectation of the delays $E > 0$ is fixed. Our main result is that for this class of equations the stability region of (4) is contained in the stability region of the single-delay DDE (3) for the given expectation value E. This result allows one to make statements about the stability of the zero solution of (4) by considering the corresponding stability properties of (3), which is clearly an easier problem.

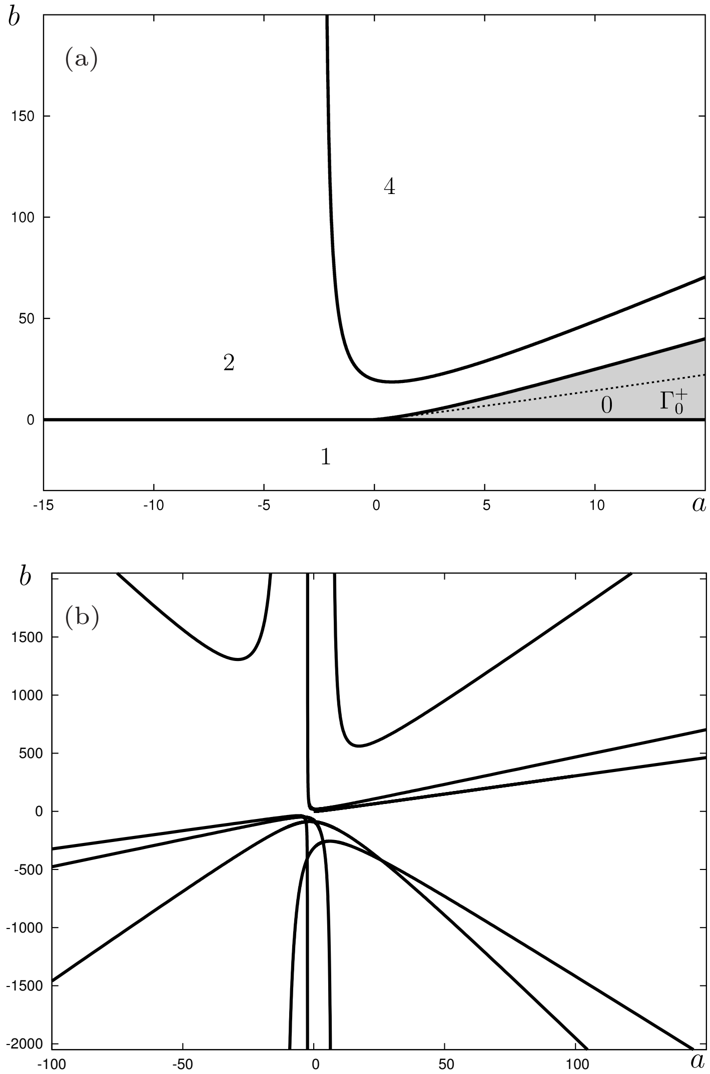

Figure 2: The curves $\hat{\Gamma}_k$ for $k = 0, 1, 2, 4$ and the line $b = 0$ of (49) in the (a, b)-plane. The number of characteristic roots with positive real parts are indicated in each region; the stability region of the zero solution of (3) is shaded. The dashed curve is the stability boundary Γ_0 for (3) with $E = 1$. Panel (a) shows the (a, b)-plane on the scale of Figure 1, and panel (b) shows it on a larger scale.

Namely, the stability condition of Proposition 3.4 becomes a sufficient stability condition for the zero solution of (4). Our result encompasses the special case of corresponding second-order DDEs with a finite number of fixed delays, such as the case of two delays considered in [2].

The overall conclusion is that, in terms of the stability enhancing property of symmetric delay distribution, the class of second-order equations (4) is similar to the first-order case considered in [7, 8]. Thus, for the class of systems given by (4) the average delay gives sufficient information for stability investigation. We stress that this statement is special, because it not true for all second-order equations, such as the ones considered in [8]. The characterization of other classes of equations in which the higher moments of the delay distribution do not play role is ongoing research.

We finally briefly mention how our result could be of use in the application context of hybrid testing. For instance, in an experiment with a single actuator the introduction of an artificial time delay between the simulation and the actuator, together with a substantial delay in the computation time, leads to a mathematical model with distributed delays. This could give the opportunity of an experimental verification of our theoretical results. Furthermore, one may explore how the choice of a particular distribution could be used to keep the experiment away from delay-induced instabilities.

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