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Unstable manifolds of a limit cycle near grazing

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Abstract. We study the local dynamics of an impacting system near a grazing bifurcation point. In particular, we investigate local invariant manifolds of grazing periodic orbits. At a grazing bifurcation point the local return map does not have a Jacobian nor is of Lipschitz continuous, so that the classical theory does not apply. Nevertheless, we are able to use the Graph Transform technique and show that under certain conditions a local Lipschitz unstable manifold of the periodic orbit exists at the grazing bifurcation point.

In this paper we investigate the local unstable invariant manifolds of periodic points of square root type maps. Invariant manifolds are organizing centers in every dynamical system because they form typically basins of attraction or separate different dynamical behaviors in phase space [1, 7, 18]. In smooth invertible systems the existence of local stable and unstable manifolds of a fixed point is guaranteed if the eigenvalues of the fixed point are not on the complex unit circle. However, non-smooth systems are different in this respect, because they cannot be linearized at the switching manifold. Square root maps are even more particular, since they are not Lipschitz continuous at the switching boundary. Here we give conditions and prove the existence of a local unstable manifold of a grazing periodic orbit.

Maps that include square root terms arise as local return maps of periodic orbits in hybrid systems, where the dynamics is determined by a smooth vector field and a map on the associated codimension-one bounding hyperplane. These systems occur as mathematical models of rotor dynamics [8], control [20], gear rattle [16], and several other mechanical systems [17]. To analyze impacting periodic orbits Nordmark [15] introduced a so-called discontinuity mapping, which is a correction to the smooth Poincaré map to account for the low-velocity impact. The discontinuity mapping is usually an approximation and can be obtained as a power series containing fractional powers such as the square root. The idea of the discontinuity map was further extended to other types of non-smooth systems and the coefficients of higher-order terms in the power series were calculated by di Bernardo et al. [4, 6].

In the one-dimensional case the dynamics of the square-root map is well understood [5, 10]. If the dynamics is non-trivial (e.g., the system has a globally stable fixed point or diverging orbits) then one of the following three scenarios are possible at a grazing bifurcation point [5]: 1) immediate chaos arises without periodic windows; 2) a reversed period-adding cascade occurs in which the period of the stable orbit tends to infinity as the system approaches the bifurcation point; or 3) the period-adding cascade is interspersed with chaotic windows. If the dimension of the local mapping is greater than one, similar phenomena can be observed under certain conditions, e.g., when the eigenvalues of the linear part of the map are strictly positive, real, and less than one. In this case the dynamics remains essentially one dimensional and it is organized on an invariant set that consist of several lines referred to as *fingers* [5, 11, 14].

Already in two-dimensional impacting systems other phenomena can be found when the linear part of the map has a pair of complex eigenvalues. In this case an unstable periodic orbit arises around the grazing point. Often this periodic orbit coexists with a stable periodic orbit that emerged in a smooth fold bifurcation [2]. In higher dimensions a lot less is known about this bifurcation and systems require a case-by-case analysis.

We focus in this paper on the n-dimensional case, where a grazing bifurcation occurs. The paper is organized as follows. In the first section we recall preliminary results on low-velocity impacts and introduce the normal form map. We also perform some basic transformations to bring the normal form into a more convenient format. In section 2 we discuss the existence and stability of fixed points for our transformed normal form map. Section 3 describes the main result of the paper and shows that an invariant unstable manifold of the grazing fixed point exists even if the system cannot be linearized at the switching boundary. Finally, we illustrate our theory in section 4 with a two-dimensional example.

1. Background

We study a dynamical system with impacts. Our model consists of two parts, a smooth vector field and a smooth impact map. The vector field is given by the equation

$$\dot{x}(t) = g(x(t), \mu), \tag{1}$$

where $x \in \mathbb{R}^m$, μ is a real parameter and the dot denotes differentiation with respect to time. We assume that $g : S \times \mathbb{R} \to \mathbb{R}^n$ is C^q smooth with $q \geq 1$, where $S \subset \mathbb{R}^m$ is a half space defined as $S = \{x \in \mathbb{R}^m : h(x, \mu) \leq 0\}$ and $h : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ is a C^q smooth function. We define the boundary of S as $\Sigma = \{x \in \mathbb{R}^m : h(x, \mu) = 0\}$. Trajectories of our system can live only in S, which means that if they reach its boundary Σ an impact occurs and they will be reflected back by the impact map $p : \Sigma \to \Sigma$.

Here, we are only interested in the phenomenon, called *grazing*, which happens when a periodic orbit of (1) is tangent to Σ. We denote the family of periodic orbits of (1) by Γ_μ and parameterize it such that periodic orbits for $\mu < 0$ do not leave $S\backslash\Sigma$, the periodic orbit Γ_0 has a tangency with Σ at $\bar{x}$, and for $\mu > 0$ periodic orbits leave $S\backslash\Sigma$, which induces impacts. Locally near $\bar{x}$ we may assume that Σ is flat and approximate h by a linear functional.

Since we are interested in the dynamics local to an orbit Γ_μ, we introduce an approximate Poincaré return map. The Poincaré section is chosen to be transversal both to Σ and Γ_0. A natural choice for the Poincaré section is the manifold $\Pi = \{x \in \mathbb{R}^m : \frac{\mathrm{d}}{\mathrm{d}t}(h(x(t), \mu))|_{t=0} = 0\}$. Thus, the return map is composed of the smooth dynamics $P : \Pi \to \Pi$ induced by (1), disregarding the boundary Σ, and a non-smooth correction called the *discontinuity mapping* P_D. Assuming that periodic orbits in Γ are hyperbolic, P can be replaced by its Jacobian about $\bar{x}$, because the nearby dynamics is structurally stable. The impact map p maps Σ into itself and not from Π to Π. Hence, the discontinuity mapping P_D should account for a backward trajectory from Π to Σ, include the impact defined by p, and follow a trajectory of (1) from Σ to Π. Clearly, if a trajectory around Γ_μ does not impact, no correction is necessary and $P_D = I$.

Therefore, the local Poincaré mapping can be expressed as $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $n = m-1$, $F = P \circ P_D$, or more precisely,

$$x \to F(x, \mu) = \begin{cases} Nx + M\mu + E\sqrt{H(x, \mu)}, & H(x, \mu) \geq 0, \\ Nx + M\mu, & H(x, \mu) < 0, \end{cases} \tag{2}$$

where $H(x, \mu)$

$$H(x, \mu) = C^T(Nx + M\mu) + D\mu,$$

with $N \in \mathbb{R}^{n\times n}$, $C, E, M \in \mathbb{R}^n$ and $\mu, D \in \mathbb{R}$. This mapping is called a normal form because many bifurcations persist with respect to higher-order perturbations. Due to the strongly nonlinear nature of (2) it is difficult to obtain a complete understanding of the dynamics. Our goal is to show the existence of a local unstable manifold for a saddle periodic orbit near a grazing bifurcation. To this end we assume that the periodic orbit is stable before grazing, that is, $\|N\| < 1$.

It is convenient to perform a series of transformations. First we translate the grazing fixed point at $\mu = 0$ to the origin

$$x \mapsto \begin{cases} Nx + K\mu + E\sqrt{C^T N x}, & C^T N x \geq 0, \\ Nx + K\mu, & C^T N x < 0, \end{cases} \tag{3}$$

where $K = -C\,D/(C^T C)$. Then, by appropriately selecting new orthogonal coordinates via a transformation matrix T_E, we transform system (3) into

$$x \mapsto \begin{cases} \hat{N}x + \hat{K}\mu + e_1\hat{\alpha}\sqrt{C^T N T_E x}, & C^T N T_E x \geq 0, \\ \hat{N}x + \hat{K}\mu, & C^T N T_E x < 0, \end{cases} \tag{4}$$

where $\hat{N} = T_E^{-1} N T_E$, $\hat{K} = T_E^{-1} K$ and $e_1 = (1, 0, 0, \ldots)$. We assume that $C^T N T_E e_1 \neq 0$, so that there exists another linear transformation that brings (4) into the form

$$x \mapsto \begin{cases} \tilde{N}x + \tilde{K}\mu + e_1\tilde{\alpha}\sqrt{e_1^T x}, & e_1^T x \geq 0, \\ \tilde{N}x + \tilde{K}\mu, & e_1^T x < 0. \end{cases} \tag{5}$$

Note that the stability of the non-impacting periodic orbit was preserved because $\|\tilde{N}\| = \|N\| < 1$. In what follows we use system (5) where we drop the tilde from our notation.

2. Fixed points and their stability

Assuming that $\|N\| < 1$, the map (5) has a non-impacting stable fixed point $x_1^* = (N - I)^{-1} K\mu$ for $e_1^T x_1^* < 0$, and at most two impacting fixed points x_2^*, x_3^* for which the existence and first coordinate $y_{2,3} = e_1^T x_{2,3}^*$ is determined by the scalar equation

$$y + e_1^T (N - I)^{-1} (K\mu + e_1 \alpha \sqrt{y}) = 0. \tag{6}$$

Because x_1^* is an admissible non-impacting fixed point for $\mu < 0$ and virtual for $\mu > 0$ we have $e_1^T (N - I)^{-1} K > 0$. Depending on the sign of $\alpha e_1^T (N - I)^{-1} e_1$ and μ, equation (6) has a different number of solutions. If $\alpha e_1^T (N - I)^{-1} e_1 < 0$ there are two solutions, $y_2^* = e_1 x_2^*$, which is admissible for $\mu \leq 0$ and $y_3^* = e_1 x_3^*$, which is admissible in a neighbourhood of $\mu = 0$. In this case x_2^* is created at the grazing bifurcation point, where $x_2^* = x_1^*$, and collides with the other fixed point x_3^* at a smooth fold bifurcation for some $\mu^* < 0$; this case is shown in Fig. 1.(a). If $\alpha e_1^T (N - I)^{-1} e_1 > 0$, there is only one admissible fixed point x_2^* which arises at the grazing bifurcation and exists for $\mu > 0$; see Fig. 1.(b). In both cases x_2^* arises as an unstable fixed point, while x_3^* is stable if it exists. The stability change at the bifurcation point $\mu = 0$ is due to the infinite stretching in the e_1-direction.

If $\mu \neq 0$ (but small) none of the fixed points are on the switching boundary and the Jacobian of (5) at x_2^* exists. The Stable Manifold Theorem [12] then implies that, x_2^* has a unique one dimensional unstable manifold with a tangent at x_2^* close to the e_1 direction. As μ approaches zero, this direction will tend to e_1. It is important to realize that this limit is not continuous, that is, e_1 is not an eigenvector of the grazing periodic orbit at $\mu = 0$. Therefore, we cannot conclude that x_2^* has a local unstable manifold for $\mu = 0$.

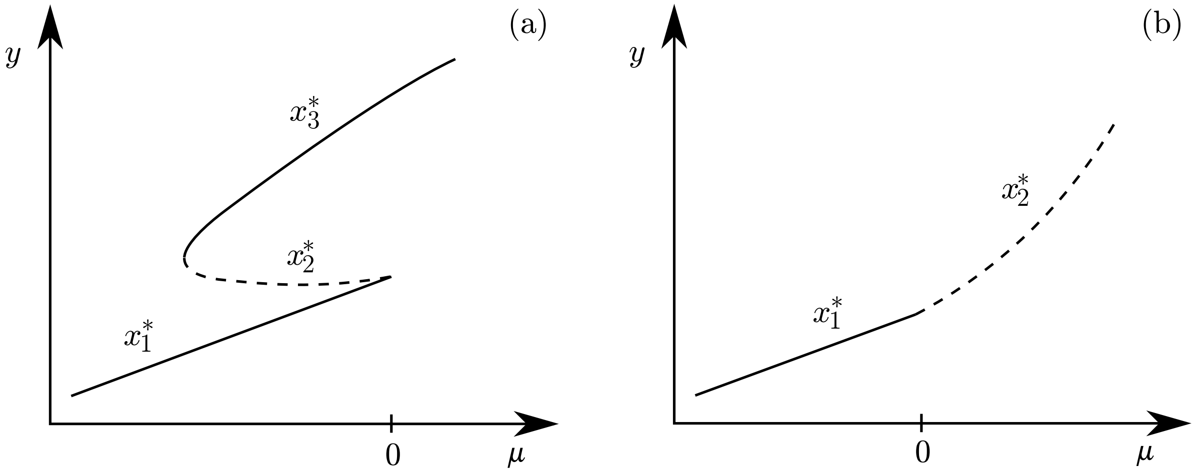

Figure 1. The generic grazing bifurcation leads to a non-smooth fold (a) or persistence of the fixed point branch (b).

In order to find out what happens when $\mu = 0$ we separate system (5) into two parts using the projections $\pi_1 x := e_1^T x$ and $\pi_2 := (I - e_1 e_1^T)$. We also use a new coordinate system in the impacting region of the phase space defined by $\pi_1 x \mapsto \xi^2$ and $\pi_2 x \mapsto \eta$; this makes our analysis simpler in the next section. System (5) can be written in (ξ, η) as

$$f : \begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{N_{11}\xi^2 + N_{12}\eta + \alpha\xi + K_1\mu} \\ N_{21}\xi^2 + N_{22}\eta + K_2\mu \end{pmatrix}, \tag{7}$$

where $N_{ij} = \pi_i N \pi_j$ and $K_i = \pi_i K$. Note that N_{11} and K_1 are scalars. Also, note that the local dynamics around almost every *maximal* period-k orbit, that is a periodic orbit that consists of $k-1$ linear and one nonlinear iterations can be described by (7).

3. The unstable manifold

Under certain conditions we can show that the unstable fixed point x_2^* has a one-dimensional unstable manifold. This is formulated in the following theorem.

Theorem 1 *Consider system (2) with* $\|N\| < 1$. *Assume that* $C^T N T_E e_1 \neq 0$ *so that (2) can be transformed into (5). Then system (5) has a unique Lipschitz-continuous local unstable manifold* $W^u(x_2^*)$ *at* $\mu = 0$, *provided* $\alpha > 0$.

The proof of Thm. (1) uses the Graph Transform operator. The idea is to represent $W^u(x_2^*)$ as the graph of a Lipschitz function

$$\sigma^* \in S_{r,\delta} := \{\sigma \in \mathrm{Lip}_\delta([0,r], \mathbb{R}^{n-1}) \, : \, \sigma(0) = 0\}.$$

Note that $\sup_{x\in[0,r]} \|\sigma(x)\| < r\delta$, which we will use in the proof. The Graph Transform operator $G_f : S_{r,\delta} \to S_{r,\delta}$ maps $\sigma \in S_{r,\delta}$ to a function $G_f(\sigma) \in S_{r,\delta}$ for which $\mathrm{gr}(G_f(\sigma)) = f(\mathrm{gr}(\sigma))$. Figure (2) sketches the action of G_f. The crucial step is to consider this definition pointwise, that is, we wish to express $(G_f(\sigma))(\xi)$ as a computable expression. As illustrated

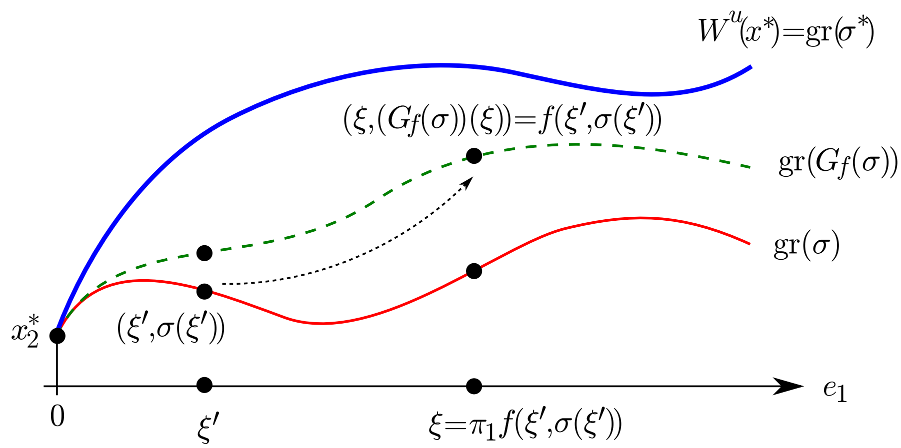

Figure 2. The Graph Transform operator G_f maps a Lipschitz function σ to a Lipschitz function with a graph that is the image under f of the graph of σ.

in Fig. 2, we need to find a point ξ' on the e_1 line such that $\pi_1 f(\xi', \sigma(\xi')) = \xi$. If such a point ξ' can be found then the value of the $G_f(\sigma)$ at ξ is $\pi_2 f(\xi', \sigma(\xi'))$, that is,

$$(G_f(\sigma))(\pi_1 f(\xi', \sigma(\xi'))) = \pi_2 f(\xi', \sigma(\xi')),$$

or equivalently,

$$(G_f(\sigma))(\xi) = \pi_2 f([\pi_1 f(\cdot, \sigma(\cdot))]^{-1}(\xi), \sigma([\pi_1 f(\cdot, \sigma(\cdot))]^{-1}(\xi))).$$

Our goal is to show that G_f is well defined and a contraction on $S_{r,\delta}$. The Banach fixed point theorem [13] then implies that G_f has a unique fixed point σ^*. By definition of G_f, this means that $\mathrm{gr}(\sigma^*)$ is f invariant.

The first step is to show that G_f is well defined. We must show that for every $\xi \in [0, r]$, with $r > 0$ sufficiently small, there is a unique $\xi' \in [0, r]$ with $G_f(\sigma) \in S_{r,\delta}$. For any $\xi \in [0, r] \implies \xi' < \xi$ it is not hard to show that the solution ξ' of $\xi = \pi_1 f(\xi', \sigma(\xi'))$ satisfies $\xi' < \xi$. Indeed, $\xi' < \xi$ is equivalent to

$$(\xi')^2 \leq N_{11}(\xi')^2 + N_{12}\sigma(\xi') + \alpha\xi'.$$

Since $\alpha > 0$, $\alpha\xi'$ has a positive slope so that $(\xi')^2 \leq \alpha\xi$ for ξ' small enough. Clearly, we can choose and r, δ small enough so that that the additional term $N_{11}(\xi')^2 + N_{12}\sigma(\xi')$ does not alter the inequality.

The second step is to make sure that $G_f(\sigma)$ is δ-Lipschitz continuous. We estimate

$$\begin{aligned}\|(G_f(\sigma))(\xi_1) - (G_f(\sigma))(\xi_2)\| &= \|N_{21}((\xi_1')^2 - (\xi_2')^2) + N_{22}(\sigma(\xi_1') - \sigma(\xi_2'))\| \\ &\leq (2r\|N_{21}\| + \delta\|N_{22}\|)|\xi_1' - \xi_2'|.\end{aligned}$$

By Cauchy's Mean Value Theorem we find that

$$\begin{aligned}\frac{|\xi_1' - \xi_2'|}{|\xi_1 - \xi_2|} &\leq \sup_{z\in[0,r]} 2\frac{\sqrt{N_{11}z^2 + N_{12}\sigma(z) + \alpha z}}{|2N_{11}z + \alpha + N_{12}(d\sigma/dz)(z)|} \\ &\leq 2\frac{\sqrt{|N_{11}|r^2 + \|N_{12}\|\epsilon + \alpha r}}{\alpha - \|N_{12}\|\delta},\end{aligned}$$

which implies that

$$\|(G_f(\sigma))(\xi_1) - (G_f(\sigma))(\xi_2)\| \le (2r\|N_{21}\| + \delta\|N_{22}\|) \\ \times 2\frac{\sqrt{|N_{11}|r^2 + \|N_{12}\|\epsilon + \alpha r}}{\alpha - \|N_{12}\|\delta}|\xi_1 - \xi_2|.$$

If r is small enough such that

$$2r\|N_{21}\| + \delta\|N_{22}\| < \delta \tag{8}$$

and

$$\frac{\sqrt{|N_{11}|r^2 + \|N_{12}\|r\delta + \alpha r}}{\alpha - \|N_{12}\|\delta} < \frac{1}{2} \tag{9}$$

then we have $G_f(\sigma) \in S_{\epsilon,\delta}$. Note that (8) and (9) can always be satisfied if $\|N_{22}\| < 1$, provided r and δ are chosen small enough.

We now show that G_f has a unique fixed point. For arbitrary σ_1, σ_2 and $\xi \in [0, r]$, we find $\xi_1, \xi_2 \in [0, r]$ such that $\xi = \pi_1 f(\xi_1, \sigma_1(\xi_1)) = \pi_1 f(\xi_2, \sigma_2(\xi_2))$. Then we have

$$\begin{aligned}\|(G_f(\sigma_1))(\xi) - (G_f(\sigma_2))(\xi)\| &= \|N_{21}(\xi_1^2 - \xi_2^2) + N_{22}(\sigma_1(\xi_1) - \sigma_2(\xi_2))\| \\ &\le 2r\|N_{21}\|\,|\xi_1 - \xi_2| + \|N_{22}\|\,\|\sigma_1(\xi_1) - \sigma_2(\xi_1)\| + \|N_{22}\|\,\|\sigma_2(\xi_1) - \sigma_2(\xi_2)\| \\ &\le (2r\|N_{21}\| + \delta\|N_{22}\|)|\xi_1 - \xi_2| + \|N_{22}\|\|\sigma_1 - \sigma_2\|,\end{aligned}$$

To estimate $|\xi_1 - \xi_2|$ in terms of $\|\sigma_1 - \sigma_2\|$ we introduce the notation $g_1(\xi, \eta) = (f_1(\xi, \eta))^2$. Note that

$$|g_1(\xi_1, \sigma_1(\xi_1)) - g_1(\xi_2, \sigma_1(\xi_1))| \ge |N_{11}^{-1}|\,|\xi_1 - \xi_2|$$

and also

$$\begin{aligned}|g_1(\xi_2, \sigma_2(\xi_2)) - g_1(\xi_2, \sigma_1(\xi_1))| &= |N_{12}(\sigma_2(\xi_2) - \sigma_1(\xi_1))| \\ &\le \|N_{12}\|(\|\sigma_2(\xi_2) - \sigma_2(\xi_1)\| + \|\sigma_2(\xi_1) - \sigma_1(\xi_1)\| \\ &\le \|N_{12}\|(\delta|\xi_1 - \xi_2| + \|\sigma_1 - \sigma_2\|)\end{aligned}$$

Since $g_1(\xi_1, \sigma_1(\xi_1)) = g_1(\xi_2, \sigma_2(\xi_2))$ and, therefore, the left-hand-side of the above two inequalities are the same, we get

$$|\xi_1 - \xi_2| \le \frac{\|N_{12}\|}{|N_{11}^{-1}| - \delta\|N_{12}\|}\|\sigma_1 - \sigma_2\|,$$

as in [9], which yields

$$\|(G_f(\sigma_1))(\xi) - (G_f(\sigma_2))(\xi)\| \le \left(\frac{\|N_{12}\|(2r\|N_{21}\| + \delta\|N_{22}\|)}{|N_{11}^{-1}| - \delta\|N_{12}\|} + \|N_{22}\|\right)\|\sigma_1 - \sigma_2\|.$$

Recall the assumption that $\|N_{22}\| < 1$. Hence, r, δ can be chosen small and we can conclude that G_f is a contraction. □

Remark: Theorem 1 shows that only the coefficient of the square root term and one part of the linear part plays a role in the existence of the unstable manifold. This is due to the infinite stretching of the square root term, which ultimately determines the direction of the local unstable manifold.

It may seem that our conditions are not too restrictive, but, in fact, in most physical systems $\alpha < 0$. Because of the nature of impacts, the map brings every orbit back into the linear region until it starts to impact again. In most cases this problem can be resolved by iterating the linear part as many times as necessary to get a positive α. However, when N has only positive real eigenvalues this method does not work. Simulations show that system (5) has, in this case, one of the three kinds of dynamics similar to the one-dimensional square root map, but the invariant set (the fingers) is rather interspersed; see [5] for examples.

4. Example

We illustrate the theorem with a two-dimensional example. Consider the impacting system

$$(x,y) \mapsto \begin{cases} (\beta x + y + \mu, -\gamma x), & x \le 0, \\ (y + \mu - \sqrt{x}, -\gamma x), & x > 0, \end{cases}$$

which is the local Poincaré map of a periodic orbit in a forced impacting linear oscillator taken from [15]. In its original form $\alpha = -1$, but we consider period-three maximal orbits and perform the transformations described in section 2. We find that

$$\begin{cases} x \mapsto -2\beta\gamma x - (\beta^4 + \beta^2\gamma(\gamma-2) + \gamma^2)y \\ \qquad\qquad + \mu(1 + \beta + \beta^2 - \gamma) + (\gamma - \beta^2)\sqrt{x} \\ y \mapsto \dfrac{\gamma^3 x}{\beta^4 + \beta^2\gamma(\gamma-2) + \gamma^2} - \dfrac{(\beta+\gamma)\mu}{\beta^4 + \beta^2\gamma(\gamma-2) + \gamma^2} \end{cases} \tag{10}$$

for $x > 0$, and $\alpha = (\gamma - \beta^2)$. Hence, in certain regions of the parameter space α is positive. Here, we use $(\gamma, \beta) = (0.5, 0.02)$ as in [2] where the authors found that a grazing bifurcation induces an unstable period-three orbit for $\mu < 0$. This unstable orbit then undergoes a smooth fold bifurcation and continues to exists for $\mu > 0$. In this case $\alpha = 0.46$ and Thm. 1 guarantees the existence of the unstable manifold of the period-three orbit at the grazing bifurcation point for $\mu = 0$.

At the moment of grazing, that is, when $\mu = 0$, two period-three orbits exists that are fixed points of (10). The grazing orbit corresponds to the fixed point $(x_2^*, y_2^*) = (0, 0)$ and the other period-three orbit corresponds to the attracting fixed point $(x_3^*, y_3^*) \approx (9.6 \times 10^{-5}, 6.0 \times 10^{-5}$. As shown in Fig. 3, the unstable manifold $W^u(x_2^*, y_2^*)$ accumlates onto (x_3^*, y_3^*).

We computed $W^u(x_2^*, y_2^*)$ using the Graph Transform operator G_f. Indeed, the initial stretching along the manifold is rather large, so that it is very difficult to find an approximation of $W^u(x_2^*, y_2^*)$ by other means. We found that it is possible to consider the space of Lipschitz functions $y = \sigma(x)$, $x \in [0, r]$ for $r = x_3^*$, that is, G_f is a contraction on the maximal interval and the graph of its fixed point σ^* is the global manifold $W^u(x_2^*, y_2^*)$. An approximation of $\mathrm{gr}(\sigma^*)$ is obtained by iterating the function $\sigma \equiv 0$ eight times, namely, then the graph of the

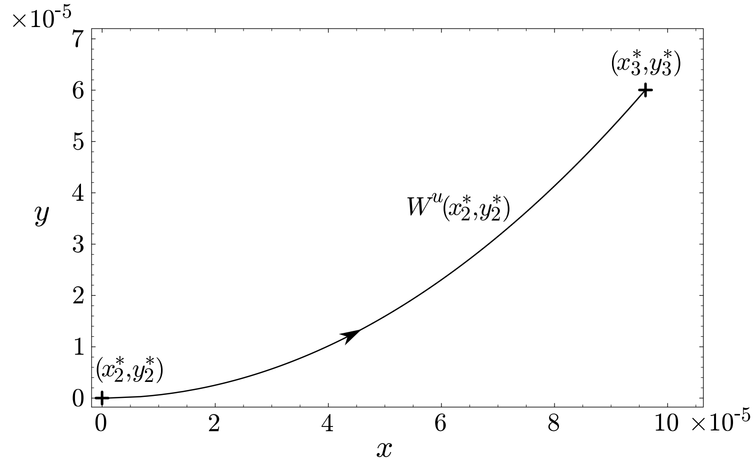

Figure 3. The unstable manifold of the grazing fixed point (x_2^*, y_2^*) of (10) at $\mu = 0$ accumlates on the attracting fixed point (x_3^*, y_3^*). The Graph Transform operator is a contraction along the entire interval $[0, x_3^*]$.

result lies within line thickness of the previous iteration. Figure 3 shows the manifold in the (x, y)-plane along with the fixed points (x_2^*, y_2^*) and (x_3^*, y_3^*).

5. Discussion

We have shown that a one-dimensional invariant unstable manifold of an impacting fixed point persists at a grazing bifurcation point, provided the coefficient of the square-root term in the expression of the local Poincaré map points back to its own domain after a finite number of iterations, which is the case for period-k maximal orbits. If this condition is not satisfied, e.g., if the linear part has strictly positive real eigenvalues, the situation is different, because of the ensuing period-adding cascade, which results in an infinite-period orbit at the grazing bifurcation point. Exactly at the this point the square-root term maps any point to the linear region where it will stay forever. In this case we were unable to prove the existence of the invariant manifold of the grazing orbit. Beyond the grazing bifurcation point the system will have a finite Jacobian at the fixed point. Hence, the classical Stable Manifold Theorem applies and one can globalize the manifold and obtain either a strange attractor or an attractor that contains a large-period orbit.

In the proof of our result we successfully applied the Graph Transform technique, despite the fact that the governing map is not Lipschitz continous. We are hoping that a similar approach can reveal more information about grazing invariant tori in hybrid or Filippov systems. Initial investigations by Dankowicz and Piiroinen [3] and Thota and Dankowicz [19] show that persistence of grazing tori is a subtle problem. It appears that after the grazing bifurcation an attractor persists, even though it may not be a torus in the classical sense.

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